Money and Capital

S. Borağan Aruoba
University of Pennsylvania

Christopher J. Waller
University of Notre Dame

Randall Wright
University of Pennsylvania

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Abstract

Recent advances in monetary theory incorporate some decentralized and some centralized trade. These models have an essential role for money and also allow one to easily add key ingredients from more standard macro models. However, existing papers consider only cases that dichotomize: allocations in centralized and decentralized markets are independent, which implies monetary policy has no effect on consumption, investment, employment, or output in the centralized market. We analyze natural generalizations of the model without this special property, and hence with more interesting positive and normative policy implications. We also compare different mechanisms for monetary exchange, including bargaining and competitive markets.

1 Introduction

We believe that much progress has been made over the last 15 years or so in modeling explicitly the microfoundations of monetary exchange. There is now a large literature analyzing models that go beyond previously prominent reduced-form approaches, such as imposing a cash-in-advance constraint, which says people simply "have to" use money to acquire certain goods, or sticking money into preferences or technology, which says people are "happier or more efficient" when they use money. A representative paper in the microfoundations literature provides details about the underlying environment – preferences (over consumption goods, not assets), technology, the pattern of meetings, information, and so on – that give rise to outcomes where agents may choose endogenously to use certain objects as media of exchange, and attempts to derive conditions under which certain institutions, like monetary exchange per se or certain monetary policies, lead to higher output and welfare. Modeling explicitly the frictions in a model that can make money essential seems like progress.

It is still the case, however, that many mainstream macroeconomists continue to use the reduced-form approach. This was clearly understandable in the early days of the microfoundations literature, for a variety of reasons – not least of which was that papers in this literature needed (or at least used) some very strong assumptions about things like the amount of money and goods agents were allowed to inventory, and also because they were so focused on the process of exchange they abstracted from many of the ingredients that more standard macro models routinely incorporate, like physical capital, labor markets, competitive firms, trends or shocks in productivity, etc. These features looked not only unconventional and perhaps aesthetically unpleasing to some economists, but more importantly they seemed to preclude analyses of many macroeconomic issues, including monetary policy as it is usually conceived.[1]

[1] As Azariadis (1993) describes the situation, "Capturing the transactions motive for holding money balances in a compact and logically appealing manner has turned out to be an enormously complicated task. Logically coherent models such as those proposed by Diamond (1982) and Kiyotaki and Wright (1989) tend to be so removed from neoclassical growth theory as to seriously hinder the job of integrating rigorous monetary theory with the rest of macroeconomics." And as Kiyotaki and Moore (2001) more recently put it, "The matching models are without doubt ingenious and beautiful. But it is quite hard to integrate them with

More recent work in monetary theory has gone some way towards reducing the distance between monetary models with microfoundations and mainstream macro. Examples include the models in Shi (1997) and Lagos and Wright (2003) that do away with the artificial restrictions on inventories in the earlier models, with a minimum loss (perhaps a gain) in tractability. Some details in these two models differ a lot – in particular, Shi assumes that the fundamental decision-making unit is a family with a continuum of members that provide intrahousehold insurance against the luck of the trading process, which by the law of large numbers implies the useful result that every household of the same type starts each trading round with the same real balances, while Lagos and Wright assume individuals have periodic access to centralized markets, which by the assumption of quasi-linear utility delivers the same result. But either approach allows us to much more easily analyze standard questions concerning, say, optimal monetary policy and the welfare cost of inflation.

Still, the base-line models in Shi (1997) and Lagos and Wright (2003) do not look much like mainstream macro, as represented by, e.g., the neoclassical growth model and its many applications to business cycles, public finance, development, and so on. One reason is that those models use a very different price-determination mechanism: since the literature on the microfoundations of money has long been based on the notion that bilateral (or at least relativey small group) trade is a key element contributing to the essentiality of a medium of exchange, rather than competitive Walrasian pricing, this literature adopted one of the mechanisms commonly used in search-theory, usually bargaining or price posting. Another reason is that those models are still missing some of the staple ingredients in standard macro models, including labor markets, capital investment, etc. So while these newer models do allow us to address some more conventional issues, they are still pretty far removed from the mainstream, and hence most practitioners continue to ply the reduced-form approach.

The goal of this project is to continue the integratation of monetary theory with mainstream macro, in two ways. First, following up on a line in Rocheteau and Wright (2003), we explore the implications of using competitive pricing rather than, say, bargaining in the Lagos-Wright model, not only in the centralized market but in all markets. This allows one

the rest of macroeconomic theory – not least because they jettison the basic tool of our trade, competitive markets."

to disentangle which results come from explicitly incorporating frictions into the physical environment (e.g. from assumptions on specialization, information, etc.) and which come from imposing a particular non-competitive price-determination mechanism. Moreover, it turns out that using competitive pricing dramatically simplifies the workings of the model, and this allows us to pursue our second line – which is that given the basic Lagos-Wright structure, one can without much difficulty add firms, labor, and capital markets, basically integrating a prototypical monetary model with the neoclassical growth model.[2]

This second line was also pursued in Aruoba and Wright (2003), but the results there are quite special because the way that model was specified implies a very strong *dichotomy*: one can solve independently for the allocations in the centralized and decentralized markets. This dichotomy result is problematic for several reasons. First, in some sense it means that the model has really not integreated monetary theory and standard macro at all – at best, it shows that they may under certain assumptions coexist without getting in each other's way. Second, it has stark policy conclusions: changing monetary policy affects prices and quantities in the decentralzied market, but has no impact on any variable in the centralized market. In particular, aggregate employment and investment are independent of money. We show here that the dichotomy is *not* general: small and natural changes in the specification lead to versions of the model with rich feedback between the centralized and decentralized markets, and hence where monetary policy has interesting implications for aggregate consumption, employment and investment.

The rest of the paper is organized as follows. In Section 2 we describe the basic model and derive the equilibrium under two different pricing structures: bilateral bargaining and competitive pricing. Optimal monetary policy is discussed as is the impact of changes in the money growth rate on consumption, investment and output. Section 3 extends the basic model by introducing market specific capital and by changing the production technology of capital. Section 4 outlines our calibrtion and Section 5 presents our welfare results. Finally, Section 6 concludes.

[2]It is also possible to add capital to the basic Shi model, as in Shi (1999) or Faig (2001), e.g., but it seems to us slightly easier and perhaps more natural to do so in the Lagos-Wright version because the centralized markets are already up and running.

2 The Basic Model

The environment is similar in spirit to the framework introduced in Lagos and Wright (2003) – hereafter denoted LW. There is a $[0, 1]$ continuum of infinite-lived agents. Time is discrete, and each period is divided into two subperiods called day and night. The differences between these subperiods is as follows. First, at night agents trade in frictionless markets, while by contrast during the day trade occurs in markets with various degrees of frictions, depending on the version of the model. One friction that is present in all versions is a double coincidence problem, generated here by taste and technology shocks. Another such friction is that agents are assumed to be anonymous in day markets, which precludes standard credit arrangements, because they cannot be enforced (Kocherlakota 1998; Wallace 2001). These two frictions make money essential. Additionally, while the night market is always perfectly competitive, we will consider two alternative mechanisms for the day market: competitive price taking, and bilateral bargaining.

At night goods can be either consumed or invested as capital, and productive capital and labor services are rented to firms in competitive markets. During the day labor is not traded in the market, because the technology used by firms at night does not operate during the day; however, agents' own labor effort e may be used as an input into an individual technology in the day market. In the base model capital is also not traded in the day market (but it is in one extension considered below). The assumption is that once put in place capital cannot be physically moved to the location where the day market meets. Although capital is not physically present, agents individual technologies for producing during the day still depend in general on k.[3] We write $q = f(k, e)$ for the individual technology during the day, and $Q = F(K, H)$ for the production function operated by firms at night.

To generate a double coincidence problem we adopt the following specification for tastes

[3] As an example of capital that enters the production function even though it is physically not present and hence not tradable at a given location, think about logging on to your computer from a remote site. The only reason for making capital immobile here is to preclude it from serving as a medium of exchange in the day market; an even simpler alternative would be to interpret k as human capital, but this would obviously change the empirical implications. See Waller (2004) and Lagos and Rocheteau (2002) for models in which capital can be used as money.

and technology during the day: for each agent, with probability σ he wants to consume but cannot produce, with probability σ he can produce but does not want to consume, and with probability $1-2\sigma$ he can neither produce nor consume. This is equivalent for many purposes to the standard specification in the search literature of random bilateral matching, where there is a probability σ of wanting to consume a good produced by a random partner. We frame things here in terms of random tastes and technology rather than random matching simply because it helps some of the discussion to follow, especially the comparison across the different pricing mechanisms. In any case, due to the double coincidence problem and anonymity, money is essential.[4] The supply of money is M and changes according to $M_{+1} = (1+\tau)M$, where we use a subscript $+1$ to denote next period. New money injected via lump sum transfers (or taxes if $\tau < 0$) in the night market.

Instantaneous utility at night is $U(x) - Ah$ where x is consumption, h is labor hours and A is a constant.. Utility during the day is random: with probability σ an agent wants to consume and has utility $u(q)$ where q is consumption; with probability σ an agents is able to produce and has utility $-\eta(e)$ where e is labor effort; and with probability $1-2\sigma$ utility is 0. We assume that $U(x)$, $u(q)$, and $\eta(e)$ have the usual properties. Linearity in h is not important, in principle, but it does generate a huge gain in tractability: as in LW, it allows us to derive nice analytical results.[5] Separability across (x, q, e) facilitates the presentation somewhat, but is not otherwise important, as we show in the Appendix. The discount factor across periods is $\beta \in (0,1)$; to reduce notation there is no discounting between subperiods, but this is easy to relax (see Rocheteau and Wright 2003).

In the analysis below it is convenient to write the agent's disutility of effort as the utility cost of producing goods using capital. Let $c(q,k)$ denote the cost in terms of utility from producing q units of output using k units of capital. The cost function is obtained as follows: for a given k, solve $q = f(e,k)$ for $e = \psi(q,k)$ and let $c(q,k) = \eta[\psi(q,k)]$. Notice $c_q > 0$, $c_k < 0$, $c_{qq} > 0$, and $c_{kk} > 0$ under the usual monotonicity and convexity assumptions on f

[4] We mean essential in the technical sense, that (desirable) allocations can be achieved with money that cannot be achieved without money, subject to the relevant resource and incentive feasibility conditions (again see Kocherlakota 1998 or Wallace 2001).

[5] Rogerson (1988) shows that having utility linear in h is equivalent having general preferences, indivisible labor, and employment lotteries; the same is true here.

and η, and have $c_{qk} < 0$ under the additional restriction $f_e f_k \eta'' > \eta' (f_k f_{ee} - f_e f_{ek})$, which always holds in the case where k is a normal input, including the case $f_{ke} > 0$.[6]

We analyze the model by first considering the night market and then the day market. At night, if r is the rental rate on capital and w the real wage, profit maximization implies $r = F_K(K, H)$ and $w = F_H(K, H)$, and constant returns implies equilibrium profits are zero. Normalize the price of the capital/consumption good to 1 and let ϕ be the relative price of money. Let $W(m, k)$ and $V(m, k)$ denote the value functions of agents entering the night market and day market, respectively, with money and capital (m, k). Then the problem of an agent in the night market is

$$\begin{aligned} W(m,k) &= \max_{x,h,m_{+1},k_{+1}} U(x) - Ah + \beta V(m_{+1}, k_{+1}) \\ s.t.\ x &= rk + wh + \phi(m + \tau M - m_{+1}) + (1-\delta)\, k - k_{+1}, \end{aligned}$$

where δ is the depreciation rate, (m_{+1}, k_{+1}) is the money and capital taken out of the market, pre-transfer, and τM is the transfer. Eliminating h using the budget equation, we have

$$\begin{aligned} W(m,k) &= \frac{A}{w}[\phi(m + \tau M) + (r + 1 - \delta)\, k] \\ &\quad + \max_{x,m_{+1},k_{+1}} \left[U(x) - \frac{A}{w}(x + \phi m_{+1} + k_{+1}) + \beta V(m_{+1}, k_{+1}) \right]. \end{aligned}$$

[6] Given $q = f(k, e)$ implies $e = \psi(q, k)$, $\partial e/\partial q = \psi_q = 1/f_e > 0$ and $\partial e/\partial k = \psi_k = -f_k/f_e < 0$. Also, $\psi_{qq} = -f_{ee}/f_e^3 > 0$, $\psi_{kk} = -1\left(f_e^2 f_{kk} - 2f_e f_k + f_k^2 f_{ee}\right)/f_e > 0$, and $\psi_{kq} = (f_k f_{ee} - f_e f_{ek})/f_e$. Hence, $c_q = \eta'/f_e > 0$, $c_k = -\eta' f_k/f_e < 0$, $c_{qq} = 1\,(f_e \eta'' - \eta' f_{ee})/f_e^3 > 0$, $c_{kk} = -\left[\eta'\left(f_e^2 f_{kk} - 2f_e f_k + f_k^2 f_{ee}\right) - f_k^2 \eta''\right]/f_e^2 > 0$ and $c_{kq} = [-f_e f_k \eta'' + \eta'(f_k f_{ee} - f_e f_{ek})]/f_e$. Saying k is normal means that in the problem $\min we + rk$ $s.t.$ $f(k, e) \geq q$, the solution satisfies $\partial k/\partial q = -(f_k f_{ee} - f_e f_{ek}) > 0$.

The first order conditions for the choice variables are[7]

$$\begin{aligned} x &: U'(x) = \frac{A}{w} \\ m_{+1} &: \frac{A\phi}{w} = \beta V_m(m_{+1}, k_{+1}) \\ k_{+1} &: \frac{A}{w} = \beta V_k(m_{+1}, k_{+1}). \end{aligned} \tag{1}$$

A key result is that, given prices, W is linear in m and k,

$$W_m(m,k) = \frac{A\phi}{w} \tag{2}$$

$$W_k(m,k) = \frac{A(r+1-\delta)}{w}. \tag{3}$$

Moreover, it should be clear from the above that the choice of (m_{+1}, k_{+1}) is independent of (m,k), and this makes the distribution of money and capital holdings degenerate in equilibrium. Intuitively, the linearity of utility in h in an LW environment eliminates wealth effects, and this makes all agents choose the same (m_{+1}, k_{+1}) regardless of (m,k).[8] While models with nondegenerate distributions are worth studying, for some questions it seems reasonable to abstract from distributional issues and study representative agent models first. This is what we get from the linearity of utility in h.

We now proceed to the day market. The value function is

$$V(m,k) = \sigma V_b(m,k) + \sigma V_s(m,k) + (1-2\sigma)W(m,k) \tag{4}$$

[7] The second order conditions are complicated, and generally ambiguous, since they involve second derivatives of V which can involve third derivatives of u and c, at least under the bargaining mechanism. Following the methods in LW, one can show that V is concave if the bargaining power parameter θ is close to 1, or if we impose additional conditions on preferences and technology (in LW c was normalized to be linear and u' was assumed log concave). We avoid these details and simply assume V is concave in the bargaining model, but again this is always true for θ close to 1.

[8] Actually, in addition to linearity in h, we also require V strictly concave and an interior solution; see LW for technical assumptions to guarantee these results. The assumptions needed for interiority involve initial conditions: if (m,k) is very disperse across people, then the rich remain rich and the poor remain poor for several periods; if we start with (m,k) not too disperse, however, we converge quickly to a degenerate distribution and stay there.

where

$$\begin{aligned} V_b(m,k) &= u(q_b) + W(m - d_b, k) \\ V_s(m,k) &= -c(q_s,k) + W(m + d_s, k) \end{aligned}$$

are the value functions when one is a buyer and seller, respectively, and q_b and d_b are the amounts of output and money agents expect to exchange when buying, and q_s and d_s are the amounts when selling, to be determined below.[9] Using the result in (2) that $W_m = A\phi/w$, we have

$$V(m,k) = \sigma\left[u(q_b) - d_b\frac{A\phi}{w} - c(q_s,k) + d_s\frac{A\phi}{w}\right] + W(m,k).$$

Differentiating with respect to m and k yields the envelope conditions

$$V_m(m,k) = \sigma\left[u'\frac{\partial q_b}{\partial m} - \frac{A\phi}{w}\frac{\partial d_b}{\partial m}\right] + \sigma\left[-c_q\frac{\partial q_s}{\partial m} + \frac{A\phi}{w}\frac{\partial d_s}{\partial m}\right] + \frac{A\phi}{w} \tag{5}$$

$$V_k(m,k) = \sigma\left[u'\frac{\partial q_b}{\partial k} - \frac{A\phi}{w}\frac{\partial d_b}{\partial k}\right] + \sigma\left[-c_q\frac{\partial q_s}{\partial k} - c_k + \frac{A\phi}{w}\frac{\partial d_s}{\partial k}\right] + \frac{A\left(r+1-\delta\right)}{w}. \tag{6}$$

It remains to specify how prices are determined in the day market, so that we can substitute for the derivatives in the above expressions. This will differ across the two versions of the model presented below.

Before pursuing equilibrium, however, as a benchmark we begin with the planner's problem, unconstrained by the assumption that agents are anonymous, so that we can simply enforce whatever exchange we like without using money. The planner's problem is described by

$$\begin{aligned} J(k) &= \max_{x,h,q,k_{+1}} U(x) - Ah + \sigma u(q) - \sigma c(q,k) + \beta J(k_{+1}) \\ s.t.\ x &= F(k,h) + (1-\delta)k - k_{+1} \end{aligned} \tag{7}$$

Substituting for x and differentiating, the first order conditions are

$$\begin{aligned} h: &\quad A = U'(x)F_h(k,h) \\ k_{+1}: &\quad U'(x) = \beta J'(k_{+1}) \\ q: &\quad u'(q) = c_q(q,k) \end{aligned} \tag{8}$$

[9] It should be clear how exactly the same equation would emerge from a random matching model (see LW, for example).

The envelope condition is

$$J'(k) = U'(x)[F_k(k,h) + 1 - \delta] - \sigma c_k(q,k),$$

and the Euler equation is

$$U'(x) = \beta U'(x_{+1})[F_k(k_{+1},h_{+1}) + 1 - \delta] - \beta\sigma c_k(q_{+1},k_{+1}) \tag{9}$$

It is clear that the solution has $q = q^*(k)$ where $q^*(k)$ satisfies $u'(q) = c_q(q,k)$. Given this, the other control variables (k_{+1}, h, x) satisfy relatively standard conditions, the first equation in (8), (9), and the constraint in (7).

2.1 Equilibrium I: Bargaining

Here we consider a mechanism used in much recent work in monetary theory, where agents bargain bilaterally. While the results are more complicated under bargaining than the competitive mechanism presented below, bargaining is arguably a very natural solution concept in models with frictions, and also serves to highlight certain effects that the competitive mechanism masks. Thus, here each agent with a desire to consume is matched with one who can produce. Since they - in particular, the buyers - are anonymous, trade must be quid pro quo meaning they must pay with cash. The buyer transfers d dollars to the seller in exchange for q units of output, where (q,d) are determined via the generalized Nash solution with the bargaining power of the buyer denoted θ and threat points given by continuation values. In general, (q,d) depends on the assets of buyer and seller, (m_b,k_b) and (m_s,k_s).[10] There are two obvious feasibility conditions for the exchange: q cannot exceed the output of the seller, $q \leq f(e,k_s)$, and d cannot exceed the money holdings of the buyer, $d \leq m_b$.

The buyer's payoff from the trade is $u(q) + W(m_b - d, k_b)$ and his threat point $W(m_b,k_b)$. Thus, his surplus is

$$\begin{aligned} S_b &= u(q) + W(m_b - d, k_b) - W(m_b,k_b) \\ &= u(q) - d\phi A/w, \end{aligned}$$

[10]Note that while all agents have the same (m,k) in equilibrium, we still need to ask what happens if a given individual deviates off the equilibrium path.

by virtue of (2). The seller's payoff is $-c(q,k_s)+W(m_s+d,k_s)$ and his threat point $W(m_s,k_s)$. Thus his surplus is

$$\begin{aligned} S_s &= -c(q,k_s)+W(m_s+d,k_s)-W(m_s,k_s) \\ &= -c(q,k_s)+d\phi A/w. \end{aligned}$$

The bargaining problem can be written

$$\max_{q,d} S_b^{\theta} S_s^{1-\theta} \text{ s.t. } d \leq m_b.$$

As in LW, one can show that in equilibrium with $k_s = K$ for all agents the constraint holds with equality, $d = m_b$. Also as in LW, this further implies $q \leq q^*(k_s)$ where $q^*(k_s)$ is the solution to $u'(q) = c_q(q,k_s)$, typically with strict inequality $q < q^*(k_s)$ (here the inequality is strict unless $\theta = 1$ and we follow the optimal monetary policy). To solve the bargaining problem, insert $d = m_b$ and take the first order condition with respect to q to get

$$\theta S_s u'(q) = (1-\theta) S_b c_q(q,k_s).$$

Then insert S_b and S_s and rearrange as $\phi m_b = g(q,k_s)w/A$, where

$$g(q,k_s) \equiv \frac{\theta c(q,k_s)u'(q)+(1-\theta)u(q)c_q(q,k_s)}{\theta u'(q)+(1-\theta)c_q(q,k_s)}. \tag{10}$$

Hence, $q = q(m_b,k_s)$, where the function $q(m_b,k_s)$ is given by the solution to $A\phi m_b/w = g(q,k_s)$ (the dependence on prices w and ϕ as well as the parameter A is implicit). This implies the key derivatives we need in (5) and (6) are given by $\partial q/\partial m_b = A\phi/wg_q > 0$ and $\partial q/\partial k_s = -g_k/g_q > 0$, where

$$g_q = \frac{c_q u'[\theta u' + (1-\theta)c_q] + \theta(1-\theta)[u-c][u'c_{qq} - c_q u'']}{[\theta u' + (1-\theta)c_q]^2} > 0 \tag{11}$$

$$g_k = \theta \frac{c_k u' \left[\theta u' + (1-\theta)c_q\right] + c_{qk}(1-\theta)u'\left(u-c\right)}{\left[\theta u' + (1-\theta)c_q\right]^2} < 0 \tag{12}$$

(we also have $\partial q/\partial m_s = \partial q/\partial k_b = 0$, $\partial d_b/\partial m_b = 1$, and $\partial d_s/\partial m_b = \partial d_b/\partial k_s = \partial d_s/\partial k_s = 0$). Thus, if the buyer brings more cash or the seller brings more capital to a meeting, more output gets traded. Notice that in general the price is non-linear: if the buyer brings half as much money, he does not get half as much q. For $\theta = 1$, $g(q,k_s) = c(q,k_s)$, which makes things a

lot simpler: $g_q = c_q$ and $g_k = c_k$, and so therefore $\partial q/\partial m_b = A\phi/wc_q$ and $\partial q/\partial k = -c_k/c_q$. In this case, if marginal cost c_q is constant, pricing is linear: if you spend another dollar you get another unit of q.[11]

Inserting $(m, k) = (M, K)$ and the derivatives, (5) and (6) become

$$\begin{aligned} V_m(M,K) &= \sigma\frac{u'(q)A\phi}{g_q(q,K)w} + \frac{(1-\sigma)A\phi}{w} \\ V_k(M,K) &= \frac{A\,(r+1-\delta)}{w} - \sigma\gamma(q,K), \end{aligned}$$

where $\gamma(q,K) = \frac{c_k(q,K)g_q(q,K) - c_q(q,K)g_k(q,K)}{g_q(q,K)} < 0$. Substituting these into the first order conditions for m_{+1} and k_{+1} in (1), and inserting the equilibrium prices $\phi = g(q,k_s)w/MA$, $r = F_K(K,H)$, and $w = F_H(K,H)$, we arrive at the equilibrium conditions

$$\frac{g(q,K)}{M} = \beta\frac{g(q_{+1},K_{+1})}{M_{+1}}\left[1-\sigma+\sigma\frac{u'(q_{+1})}{g_q(q_{+1},K_{+1})}\right] \tag{13}$$

$$U'(x) = \beta U'(x_{+1})[F_K(K_{+1},H_{+1}) + 1 - \delta] - \beta\sigma\gamma(q_{+1},K_{+1}). \tag{14}$$

The other equilibrium conditions come from the first order condition for x in (1) and the resource constraint on total output

$$A = U'(x)F_H(K,H) \tag{15}$$

$$x = F(K,H) + (1-\delta)K - K_{+1}. \tag{16}$$

A monetary equilibrium is defined as (positive, bounded) paths for (q, K_{+1}, H, x) satisfying (13)-(16), given the initial K_0. A nonmonetary equilibrium also always exists, which satisfies $q = 0$ instead of (13), (14) with $\gamma(\cdot) = 0$, and (15)-(16), which are simply the equilibrium conditions for the standard nonmonetary growth model (with h entering utility linearly). Returning to monetary equilibria, consider the case where $M_{+1} = (1+\tau)M$ with τ constant, so that it makes sense to focus on a steady state, defined as a constant solution (q, K, H, x) to (13)-(16). Defining the rate of time preference ρ and the nominal interest rate i such that $\beta = \frac{1}{1+\rho}$ and $1 + i = (1+\rho)(1+\tau)$, we can simplify the steady state conditions

[11] We can also simplify the bargaining solution by setting $\theta = 0$, but then $m_b = 0$ and the monetary equilibrium breaks down. The reason $\theta = 1$ does not symmetrically imply $k_s = 0$ is that the same capital is used in the day and night market in this version of the model.

as[12]

$$1 + \frac{i}{\sigma} = \frac{u'(q)}{g_q(q,K)} \tag{17}$$

$$\rho + \delta = F_K(K,H) - \sigma\frac{\gamma(q,K)}{U'(x)} \tag{18}$$

$$A = U'(x)F_H(K,H) \tag{19}$$

$$x = F(K,H) - \delta K. \tag{20}$$

First, one simple special case of our model is the specification in Aruoba and Wright (2003), where capital does not enter the daytime technology, $c(q,K) = c(q)$. In this case $g(q,K) = g(q)$, $\gamma(q,K) = 0$, and the equilibrium conditions are

$$\begin{aligned}
\frac{g(q)}{M} &= \beta\frac{g(q_{+1})}{M_{+1}}\left[1 - \sigma + \sigma\frac{u'(q_{+1})}{g'(q_{+1})}\right] \\
U'(x) &= \beta U'(x_{+1})[F_K(K_{+1},H_{+1}) + 1 - \delta] \\
A &= U'(x)F_H(K,H) \\
x &= F(K,H) + (1-\delta)K - K_{+1}.
\end{aligned}$$

This model displays a strong dichotomy: the first equation determines the path for q and the other three determine the paths for (K_{+1}, H, x) independently. An implication of this feature is that M, which enters only the first equation, affects q but not (K_{+1}, H, x); that is, investment, employment and consumption in the night market is independent of monetary policy.

Of course this does not mean policy is super neutral in Aruoba and Wright (2003): the path of M affects q, and q is a real variable. For example, in steady state q satisfies

$$1 + \frac{i}{\sigma} = \frac{u'(q)}{g'(q)}.$$

>From this it follows that $\partial q/\partial i < 0$ as long as the steady state q is unique (which is true under certain conditions addressed in LW). Moreover, we know that $q < q^*$ in any equilibrium, where q^* is the efficient quantity defined by $u'(q^*) = c'(q^*)$. Hence, we maximize welfare by making i as small as is consistent with equilibrium. This turns out to be the

[12] This expression for i satisfies the Fisher equation, which eliminates arbitrage opportunities from holding nominal versus real assets.

Friedman Rule, $i = 0$, which requires the money growth rate τ^F to satisfy $(1+\tau^F)(1+\rho) = 1$ (for any $\tau < \tau^F$ equilibrium does not exist; see LW). Hence, the optimal policy is $\tau = \tau^F$ and it implies $u'(q) = g'(q)$. However, τ^F does not yield the first best outcome unless $\theta = 1$, since in the case $\theta = 1$, $g(q) = c(q)$ and so $\tau = \tau^F$ implies $u'(q) = c'(q)$. When $\theta < 1$ the Friedman Rule corrects the dynamic wedge associated with impatient agents holding non-interest-bearing money, but monetary policy cannot correct a second distortion identified in LW as a hold-up problem in the bargaining game when $\theta < 1$.

The dichotomy in Aruoba and Wright is very special, and does not hold in the generalization where k enters the cost function since K and q both appear in (13) and (14), and there is no way to solve independently first for q and then the other variables. Naturally, the efficient investment decision not only takes into account the fact that K affects productivity in the night technology, but also productivity in the day technology. A change in the growth rate of M affects q and this in turn affects the return to K. Intuitively, an increase in inflation (nominal interest rates) reduces the return to trading in the day, which affects the value of capital in that market and hence investment. But the same capital is used in both day and night production, and so an increase in inflation affects productivity and hence employment and output in the night markets.

However, in the case $\theta = 1$, notice that $\gamma(q, K) = 0$. This means that, although the model is not dichotomous, it is recursive: (14)-(16) can be solved for (x, K_{+1}, H) independently of q, and the solution is exactly the path from the standard (nonmonetary) model; then, given the path for capital, (13) determines the path for q. In this case, anything that affects capital affects the value of money, but there is no feedback in the other direction from q to K. For example, in steady state we have

$$\frac{\partial q}{\partial K} = \frac{c_{qk}}{c_q u'' - u' c_{qq}} > 0$$

(anything that increases K raises the value of money). An implication is that monetary policy affects q, but not investment, employment or consumption in night markets. Intuitively, what happens when $\theta = 1$ is that sellers get none of the gains from trade, so they realize none of the cost savings from bringing extra capital into the day market (another holdup problem) and hence the investment decision is based solely on returns in night production.

This holdup problem in the demand for capital is general (it does not only apply in the extreme case $\theta = 1$) and will cause K to diverge from its efficient level. This represents an additional distortion over and above the usual inefficiency that arises when $\tau > \tau^F$, and the holdup problem in money demand that arises when $\theta < 1$. Normally these holdup problems are resolved if one sets θ correctly (this is the insight of Hosios (1990) and others), but here it cannot be done: $\theta = 1$ is required to resolve the holdup problem in the demand for money, but this is the worst possible case for the holdup problem in the demand for capital.[13] When capital reduces the cost of producing day goods, this should be taken into account when investing in K, but whenever $\theta > 0$ the investor has to share the cost savings with the buyer and hence under-invests. There is obviously no way to set θ to both 1 and 0 to eliminate both holdup problems in the bargaining game. In the next section we consider an alternative pricing mechanism that does.[14]

2.2 Equilibrium II: Competitive Pricing

The idea of using competitive (Walrasian) price-taking behavior as an alternative to bargaining in search-type monetary models was explored in Rocheteau and Wright (2003). There it was assumed that agents were randomly allocated trade opportunities in the sense of access to markets but in these markets, rather than having agents bargain bilaterally, there is an auctioneer who sets prices to equate supply and demand. It is legitimate to consider this pricing mechanism and still assume anonymous traders so as to rule out credit and maintain an essential role for money.[15] In fact, this mechanism can be reinterpreted in terms of "competitive search equilibrium" – an equilibrium concept used by others in nonmonetary search theory. In Rocheteau and Wright (2003), this mechanism actually dominates Walrasian pricing due to a "search externality" at the entry decision; since we do not have an entry decision here the allocations are the same under the two mechanisms - Walrasian pricing and competitive search - we present things in terms of the simpler story.

[13] When $\theta = 0$, we have $\gamma(q, K) = c_k(q, K)$, which yields the efficient investment decision, given q but also yields $q = 0$.

[14] In addition to LW, see Rauch (2000), and Camera, Reed and Waller (2003) for discussions of holdup problems in monetary models.

[15] See also Levine (19xx), Kocherlakota (2003), and Temzilides (19xx) for related models.

The value function for the day market before the shocks are realized has the same form as in (4) except now $V_b(m,k)$ and $V_s(m,k)$ are different. The buyer's problem is

$$\begin{aligned} V_b(m,k) &= \max_{q_b,d} \; u(q_b) + W(m-d,k) \\ s.t. \;\; pq_b &= d \;\text{ and }\; d \leq m \end{aligned}$$

and the seller's problem is

$$V_s(m,k) = \max_{q_s} -c(q_s,k) + W(m+pq_s,k).$$

These are standard competitive demand and supply problems with p taken parametrically. In equilibrium $q_b = q_s = q$ because we have conveniently assumed there are the same number σ of buyers and sellers.

The buyer's choice satisfies $u'(q) = pW_m(M-pq,k) = pA\phi/w$ if the constraint is not binding and $q = M/p$ if it is, where we have inserted the equilibrium condition $m = M$, and $W_m = A\phi/w$ (which we can do because the night market here is exactly the same as before). The seller's choice satisfies $c_q(q,k) = pW_m(M+pq,k) = pA\phi/w$. If the buyer's constraint is not binding, the equilibrium q solves $u'(q) = c_q(q,k)$, or $q = q^*(k)$; if the constraint is binding, the equilibrium solves $c_q(q,k) = A\phi M/wq$. It is again easy to show that the constraint will be binding in equilibrium.

The next step is to differentiate (**??**) with respect to m to get

$$\begin{aligned} V_m(m,k) &= \sigma\left[u'(q) - pA\phi/w\right]\frac{\partial q}{\partial m} + A\phi/w \\ &= \sigma\frac{u'(q)}{p} + (1-\sigma)A\phi/w \end{aligned}$$

where we have used $\partial q/\partial m = 1/p$ since the buyer's constraint is binding. Similarly,

$$V_k(m,k) = -\sigma c_k(q,k) + A(r+1-\delta)/w.$$

Inserting V_m and V_k into the first-order conditions in (1) and rearranging yields the analogs

to (13)-(14) for this model:[16]

$$\frac{c_q(q,K)q}{M} = \beta\frac{c_q(q_{+1},K_{+1})q_{+1}}{M_{+1}}\left[1-\sigma+\sigma\frac{u'(q_{+1})}{c_q(q_{+1},K_{+1})}\right] \tag{21}$$

$$U'(x) = \beta U'(x_{+1})\left[F_K(K_{+1},H_{+1})+1-\delta\right]-\beta\sigma c_k(q_{+1},K_{+1}) \tag{22}$$

The other equilibrium conditions are the same, and we repeat them here for convenience:

$$A = U'(x)F_H(K,H) \tag{23}$$

$$x+K_{+1} = F(K,H)+(1-\delta)K. \tag{24}$$

Monetary equilibrium is now defined by (positive, bounded) paths for (q,x,K_{+1},H) satisfying (21)-(24) given the initial K_0. The difference between the bargaining and competitive pricing models is in the difference between (13)-(14) and (21)-(22). They differ because $g(q,K) \neq c_q(q,K)q$ and $g_q(q,K) \neq c_q(q,K)$ in the first pair of equations and because $\gamma(q,K) \neq c_k(q,K)$ in the second pair. Suppose we concentrate for now on steady states.[17] Then in the competitive pricing model we have

$$1+\frac{i}{\sigma} = \frac{u'(q)}{c_q(q,K)} \tag{25}$$

$$\rho+\delta = F_K(K,H)-\sigma\frac{c_k(q,K)}{U'(x)} \tag{26}$$

while in the bargaining model we have

$$1+\frac{i}{\sigma} = \frac{u'(q)}{g_q(q,K)} \tag{27}$$

$$\rho+\delta = F_K(K,H)-\sigma\gamma(q_{+1},K_{+1}). \tag{28}$$

Competitive pricing significantly alters the model: (25) and (27) are the same iff $\theta=1$; and (26) and (28) are the same iff $\theta=0$. In this way, competitive pricing is able to eradicate the holdup problem in both the money demand and investment decisions. The idea is that in

[16] In this model it is easy to verify the second order conditions must hold; the difference is that now pricing is linear so we do not need any conditions on third derivatives the way we do in the bargaining model with $\theta<1$.

[17] In steady state the difference between $g(q,K)$ and $c_q(q,K)q$ across the two models is irrelevant. This is not so out of steady state. For example, even if $\theta=1$, so that $g(q,K)=c(q,K)$, (13) and (21) differ as long as $c(q,K) \neq c_q(q,K)q$ – i.e. as long as c is nolinear in q.

the competitive model agents take the price as given; their individual choices have no effect on the terms of trade. Since both holdup problems are eliminated under Walrasian pricing, the only distortion remaining is the dynamic wedge associated with discounting, and under the Friedman rule $i = 0$ we get the first best.

Comparing (9) with (22), the investment decision is not distorted in the competitive monetary equilibrium except to the extent that q is wrong. The first order condition for q in (8) says that the efficient solution is $q = q^*(k)$. From (21), for this to be true in the competitive monetary equilibrium we require

$$\frac{M_{+1}}{M} = \beta \frac{c_q(q_{+1}, K_{+1}) q_{+1}}{c_q(q, K) q};$$

in particular, in a steady state we require the Friedman rule. Hence, the steady state of the competitive monetary equilibrium achieves the first best outcome at $i = 0$: the value of money is given by $q = q^*(k)$, and then investment, employment and consumption are all efficient. By comparison, in the bargaining model, even at $i = 0$, q was too low due to the holdup problem in money demand that occurs whenever $\theta < 1$, and k is too low due to the holdup problem in investment that occurs whenever $\theta > 0$.

To close this section, we mention that even though the above equations determine the aggregate variables (q, x, H, K_{+1}), the individual values of these variables differs across agents. First, only a measure σ of the population consume q and have $m = 0$ when they enter the night market. A group also of measure σ are sellers each period and enter the night market with $m = 2M$, while a group of measure $1 - 2\sigma$ did not trade and enter with $m = M$. These agents all choose the same x, k' and m', but supply different amounts of labor,

$$h = \begin{cases} H + \frac{A\phi}{w} M & \text{for buyers} \\ H - \frac{A\phi}{w} M & \text{for sellers} \\ H & \text{otherwise} \end{cases} \tag{29}$$

where H is aggregate hours.

2.3 Example

To obtain more insight on how inflation affects the steady state of the economy, we construct an example using explicit functional forms. Analysis of the general model is contained in

the appendix. For ease of presentation, we focus on the competitive pricing equilibrium.

Consider the following functional forms:[18]

$$\begin{aligned} F(K,H) &= K^{\alpha}H^{1-\alpha} \quad 0<\alpha<1 \\ U(x) &= \ln x \\ u(q) &= \frac{q^{1-\gamma}}{1-\gamma} \quad 0<\gamma<1 \\ c(q,K) &= q^{\varphi}K^{1-\varphi} \quad \varphi>1. \end{aligned}$$

Let $\Bbbk = K/H$ denote the capital-labor ratio. Then equations (19), (20), (25) and (26) can be solved to obtain

$$x = \frac{(1-\alpha)}{A}\Bbbk^{\alpha} \tag{30}$$

$$K = \frac{(1-\alpha)\Bbbk}{A(1-\delta\Bbbk^{1-\alpha})} \text{ with } K>0,\ \frac{\partial K}{\partial \Bbbk}>0 \text{ for } \Bbbk < \left(\frac{1}{\delta}\right)^{\frac{1}{1-\alpha}} \tag{31}$$

$$q = \left[\frac{\sigma}{\varphi(i+\sigma)}\right]^{\frac{1}{\varphi-1+\gamma}}\left[\frac{(1-\alpha)\Bbbk}{A(1-\delta\Bbbk^{1-\alpha})}\right]^{\frac{\varphi-1}{\varphi-1+\gamma}} \tag{32}$$

$$\begin{aligned} \rho+\delta &= \frac{\alpha}{\Bbbk^{1-\alpha}} + \sigma\left[\frac{\sigma}{\varphi(i+\sigma)}\right]^{\frac{\mu}{\gamma}}\left[(1-\alpha)/A\right]^{1-\mu}(\varphi-1)\frac{(1-\delta\Bbbk^{1-\alpha})^{\mu}}{\Bbbk^{\mu-\alpha}} \\ &\equiv N(\Bbbk) \end{aligned} \tag{33}$$

where $\mu = \frac{\varphi\gamma}{\varphi+\gamma-1} < 1$.

Equation (33) determines the solution for $\Bbbk$ which can then be used to determine the steady state values of x, q, K and H. It is straightforward to show that for $\mu \geq \alpha$ $N(\Bbbk)$ is a monotonically decreasing function in $\Bbbk$ that approaches infinity as $\Bbbk \to 0$ and approaches zero as $\Bbbk \to +\infty$. Thus, a unique equilibrium value of $\Bbbk$ exists. For $\sigma = 0$, we obtain the non-stochastic steady state corresponding to Hansen's (1985) RBC model. With $\sigma > 0$, capital creates additional value in production during the day market which leads agents to accumulate more capital on the margin. An increase in the money growth rate decreases $N(\Bbbk)$ for any given value of $\Bbbk$. Consequently, greater money growth raises i and reduces the steady state value of $\Bbbk$ which in turn lowers x, K, and q. Furthermore, from (31), $H =$

[18] The cost function below is obtained when $\eta(e) = e$ and $q = e^{\Phi}k^{1-\Phi}$ where $0 < \Phi < 1$. As a result, $\varphi = 1/\Phi > 1$.

$(1-\alpha)/[A(1-\delta\Bbbk^{1-\alpha})]$ which is increasing in $\Bbbk$. So agents also work less in the night market when money growth is higher.[19]

The intuition for these results is the following. An increase in inflation lowers the value of money and the quantity of goods traded in the day market. Since production is lower, the marginal value of capital in the day market falls and so agents accumulate less capital in the night market. The reduction in capital reduces the real wage and so agents work less in the night market. Since the planner's problem is replicated only under the Friedman rule, $i = 0$, then any $i > 0$ is clearly welfare reducing.

3 Alternative Specifications

3.1 Two Capital Goods

So far, the same stock of physical capital k was an input to both day and night production. However, it would also seem reasonable to assume that different types of capital are needed to produce each good. In this section we modify the baseline model to allow for two types of capital: k is used to produce goods at night and a new type of capital z is used to produce day goods. Production of both capital stocks requires an investment at night; k and z are both traded solely in the night market and are not mobile. The two capital stocks can also depreciate at different rates, δ for k and ω for z.

The problem in the night market is now

$$\begin{aligned} W(m,k,z) &= \max_{x,h,m_{+1},k_{+1},z_{+1}} U(x) - Ah + \beta V(m_{+1},k_{+1},z_{+1}) \\ s.t.\ x &= \phi(m - m_{+1} + \tau M) + wh + rk + (1-\omega)z + (1-\delta)k - k_{+1} - z_{+1}. \end{aligned}$$

Eliminating h, this can be written as

$$\begin{aligned} W(m,k,z) &= \frac{A}{w}[\phi(m+\tau M) + (r+1-\delta)k + (1-\omega)z] \\ &\quad + \max_{x,m_{+1},k_{+1},z_{+1}} U(x) - \frac{A}{w}(x + \phi m_{+1} + k_{+1} + z_{+1}) + \beta V(m_{+1},k_{+1},z_{+1}). \end{aligned}$$

[19] For $\alpha > \mu$, $N(\Bbbk)$ can be U-shaped implying that multiple equilibria may exist.

The first order conditions are

$$\begin{aligned} x &: U'(x) = \frac{A}{w} \\ m_{+1} &: \frac{\phi}{w} = \beta V_m(m_{+1}, k_{+1}, z_{+1}) \\ k_{+1} &: \frac{A}{w} = \beta V_k(m_{+1}, k_{+1}, z_{+1}) \\ z_{+1} &: \frac{A}{w} = \beta V_z(m_{+1}, k_{+1}, z_{+1}) \end{aligned} \tag{34}$$

and the envelope conditions are given by

$$\begin{aligned} W_m(m,k,z) &= \frac{A}{w}\phi \\ W_k(m,k,z) &= \frac{A}{w}(r+1-\delta) \\ W_z(m,k,z) &= \frac{A}{w}(1-\omega) \end{aligned} \tag{35}$$

As with k, (34) shows that agents take the same amount of z out of the night market. Hence the distribution of (m,k,z) will be degenerate in equilibrium. In the day market, everything is as before except we replace $c(q,k)$ with $c(q,z)$. The bargaining solution is still given by (10) with the substitution of z for k,

$$\frac{A\phi m}{w} = g(q,z_s) \equiv \frac{\theta c(q,z_s)u'(q) + (1-\theta)u(q)c_q(q,z_s)}{\theta u'(q) + (1-\theta)c_q(q,z_s)}.$$

As before it can be shown that buyers spend all of their money balances so that $d = m$.

The value function in the day market is the same as before except there is an extra state variable, and z replaces k. The envelope conditions are

$$\begin{aligned} V_m(m,k,z) &= \frac{A}{w}\phi\left[1-\sigma+\sigma\frac{u'(q)}{g_q(q,z)}\right] \\ V_k(m,k,z) &= \frac{A}{w}(r+1-\delta) \\ V_z(m,k,z) &= \frac{A}{w}(1-\omega) - \sigma\gamma(q,z) \end{aligned}$$

where $\gamma(q,z) = \frac{c_q(q,z)g_z(q,z) - c_z(q,z)g_q(q,z)}{g_q(q,z)} < 0$. Again, if $\theta = 1$ then $\gamma(q,z) = 0$, and if $\theta = 0$, $\gamma(q,z) = -c_z(q,z)$.

The same methods used above to close the model with bargaining reduces the equilibrium

conditions to

$$\frac{g(q,Z)}{M} = \beta\frac{g(q_{+1},Z_{+1})}{M_{+1}}\left[1-\sigma+\sigma\frac{u'(q_{+1})}{g_q(q_{+1},z_{+1})}\right] \tag{36}$$

$$U'(x) = \beta U'(x_{+1})[F_K(K_{+1},H_{+1})+1-\delta] \tag{37}$$

$$U'(x) = \beta U'(x_{+1})\left[-\frac{\sigma\gamma(q_{+1},Z_{+1})}{U'(x_{+1})}+1-\omega\right] \tag{38}$$

$$A = U'(x)F_H(K,H) \tag{39}$$

$$x+K_{+1}+Z_{+1} = F(K,H)+(1-\delta)K+(1-\omega)Z \tag{40}$$

Equation (36) is equivalent to (13) with Z replacing K. Equation (37) is the standard equilibrium condition for k_{+1} in the one-sector growth model. Equation (38) is the equilibrium condition for z_{+1}.

In steady state we get

$$1+\frac{i}{\sigma} = \frac{u'(q)}{g_q(q,Z)} \tag{41}$$

$$\rho+\omega = -\sigma\frac{\gamma(q,Z)F_H(K,H)}{A} \tag{42}$$

$$\rho+\delta = F_K(K,H) \tag{43}$$

$$A = U'(x)F_H(K,H) \tag{44}$$

$$x = F(K,H)-\delta K-\omega Z \tag{45}$$

This model also does not display the dichotomy in Aruoba-Wright, even though k has no direct effect on q production. Since investment in z is done in the night market, it has to be financed by changes in x, h or k_{+1}.[20]

For $\theta=1$, $g_q(q,z)=c_q(q,z)$ and $\gamma(q,z)=0$. Then from (41) we see that the Friedman rule generates the efficient quantity, conditional on z, $q^*=q^*(z)$. However, when $\theta=1$, $z=0$. The reason is that z only has value in q production, and when $\theta=0$ sellers get no surplus from selling q. Since z is costly, agents do not accumulate any. This is an extreme outcome of the holdup problem; if z is a necessary input for q production, then for $\theta=0$ the holdup problem causes q production and the monetary equilibrium to collapse.

[20] However, when z does not depreciate, $\omega=0$, the model is recursive since k, h and x are determined by (43), (44) and (45) independently of q and z. Changes in k, h and x will affect q and z but not vice-versa. Since monetary policy changes q, this will change the steady state level of z but will have no effect on k, h, and x in the night market. In this sense, when $\omega=0$ the dichotomy reappears.

With Walrasian pricing, once again the holdup problems on money and capital are eliminated and we get

$$1+\frac{i}{\sigma} = \frac{u'(q)}{c'(q,Z)} \tag{46}$$

$$\rho+\omega = -\sigma\frac{c_z(q,Z)F_H(K,H)}{A} \tag{47}$$

As with bargaining, the dichotomy is broken. Consequently, changes in the money growth rate will affect the choice of z which affects x, h and k_{+1}. Intuitively, we expect that an increases in the money growth rate τ raise i, which lowers q thereby reducing the incentive to invest in z.

3.2 Example

Again, we use explicit functional forms to gain insight as to how monetary policy affects the economy. We use the same functional forms as before except that Z now replaces K in the cost function. For presentation purposes we look at the equilibrium with Walrasian pricing.

Using the specified functional forms as before, (43), (44) and (46) yield

$$\begin{aligned} K &= H\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \\ x &= \frac{1-\alpha}{A}\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} \\ q &= Z^{\frac{\varphi-1}{\varphi-1+\gamma}}\left\{\frac{\sigma}{\varphi\left(i+\sigma\right)}\right\}^{\frac{1}{\varphi-1+\gamma}} \end{aligned}$$

implying that the capital-labor ratio is uniquely pinned down which in turn determines the equilibrium level of consumption. Using these expressions (47) yields

$$Z=\left[\frac{\sigma}{\varphi\left(i+\sigma\right)}\right]^{\frac{1}{\gamma}}\left[\frac{\sigma\left(\varphi-1\right)\left(1-\alpha\right)}{A\left(\rho+\omega\right)}\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}}\right]^{\frac{1}{\mu}}$$

where $\mu=\frac{\varphi\gamma}{\varphi+\gamma-1}<1$ as before. So Z is pinned down. Finally, (45) yields

$$H=\Theta\left\{1-\alpha+A\omega\left[\frac{\sigma}{\varphi\left(i+\sigma\right)}\right]^{\frac{1}{\gamma}}\left[\frac{\sigma\left(\varphi-1\right)\left(1-\alpha\right)}{A\left(\rho+\omega\right)}\right]^{\frac{1}{\mu}}\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha(1-\mu)}{\mu(1-\alpha)}}\right\}$$

where $\Theta=\frac{\rho+\delta}{A[\rho+(1-\alpha)\delta]}$.

How does policy affect this economy? An increase in the money growth rate above the Friedman rule increases the nominal interest rate. An increase in i again lowers the value of money and thus the quantity of goods produced in the day market. As before, this reduces the marginal value of a unit of Z so there is less investment. Since agents need fewer resources for investment, they work less in the night market and so there is less K. Aggregate output in the night market falls however, the capital-labor ratio is unaffected which leaves the real wage and consumption unchanged. Since aggregate output falls but consumption stays the same, the saving rate declines.

3.3 Capital Produced in the Day

In the previous models, all investment occurred at night, and so money is not needed to pay for capital goods. It is known that in reduced form models it makes a difference if one has to pay for capital goods with cash; e.g. Stockman (1981). To consider this effect in our model, we modify things by assuming that investment occurs in the day market where agents are anonymous and therefore money is essential for trade. Suppose that agents do not consume the output of the day market at all but instead use it as an intermediate input that can be transformed into capital k for production at night, where without loss of generality we assume q can be transformed one for one into k.[21] As in the previous sections, a fraction σ have the ability to produce the intermediate input, and the same fraction have the ability to transform it into capital, but no agent can do both. Once capital is produced it is immobile, as in the other models, and so it cannot serve as a medium of exchange.

Capital is productive in the night market, where it will be rented to competitive firms, but not the day market – i.e. $c(q,k) = c(q)$. Since trade is anonymous, money is needed to buy capital, as in Stockman's model. The night market problem is

$$\begin{aligned} W(m,k) &= \max_{x,h,m_{+1}} U(x) - Ah + \beta V(m_{+1},k) \\ st \quad x &= wh + (r+1-\delta)\,k + \phi(m + \tau M - m_{+1}). \end{aligned}$$

[21] See Shi (1999) for a related model.

We assume for now that k is not traded in this market. Substituting for h we obtain

$$W(m,k) = \max_{x,m_{+1}} U(x) - \frac{A}{w}\left[x - (r+1-\delta)\,k - \phi(m+\tau M - m_{+1})\right] + \beta V(m_{+1},k).$$

The first-order conditions are given by

$$\begin{aligned} x \quad &: \quad U'(x) = \frac{A}{w} \\ m_{+1} \quad &: \quad \frac{A}{w}\phi = \beta V_m(m_{+1},k). \end{aligned} \tag{48}$$

Note that since individual k is obtained in the day market in this model, individual capital holdings depend on the random shocks. Hence, there is a distribution of k across agents. Since the first-order condition for m_{+1} is not independent of one's capital holdings it is not obvious at this stage if the distribution of money holdings is degenerate. We demonstrate below that it is. The envelope conditions are still given by (2) and (3).

One can assume agents bargain just as in the earlier model, but the surpluses are different. The buyer gives up d units of money and acquires q units of intermediate goods which is transformed into $k = d$ units of capital. Hence his surplus is $S_b = W(m-d,k+q) - W(m,k) = q(r+1-\delta)A/w - \phi dA/w$. Similarly, the seller's surplus is $S_s = -c(q) + W(m+d,k) - W(m,k) = -c(q) + \phi dA/w$. Notice these surpluses and hence (q,d) are independent of the individuals' capital holdings and the seller's money holdings. Again one can show $d = m_b$. Then the first-order condition for q can be written

$$m_b\phi = g(q,r,w) = \frac{\theta c(q) + (1-\theta)qc'(q)}{\theta\,(r+1-\delta)\,A/w + (1-\theta)c'(q)}\,(r+1-\delta)$$

and $\partial q/\partial m_b = \phi/g_q(q,r,w)$.

The value function in the day market is now

$$\begin{aligned} V(m,k) &= \sigma \int \left\{ W\left[m-d, k+q\left(m\right)\right] + W(m+\tilde{d}\left(\tilde{m}\right),k) - c\left[q\left(\tilde{m}\right)\right] \right\} \mathsf{dF}(\tilde{m}) \\ &\quad + (1-2\sigma)\,W(m,k) \\ &= \sigma\left[q\left(m\right)(r+1-\delta)\,\frac{A}{w} - d\phi\frac{A}{w}\right] + \sigma\int\left\{-c\left[q\left(\tilde{m}\right)\right] + \tilde{d}\left(\tilde{m}\right)\phi\frac{A}{w}\right\}\mathsf{dF}(\tilde{m}) + W(m,k) \end{aligned}$$

where $\mathsf{F}(\tilde{m})$ is the distribution of money holdings across agents and $\tilde{d}\left(\tilde{m}\right)$ is the money received by a randomly encountered buyer holding $\tilde{m}$ units of money. The integration is

only with regards to $\tilde{m}$ since capital holdings are irrelevant for the payoffs in bargaining. The envelope condition is

$$\begin{aligned} V_m(m,k) &= \sigma \int \left\{ (r+1-\delta)\frac{A}{w}\frac{\partial q(m)}{\partial m} - \phi\frac{A}{w}\frac{\partial d}{\partial m} \right\} \mathsf{dF}(\tilde{m}) + \frac{A}{w}\phi \\ &= \sigma\,(r+1-\delta)\frac{A}{w}\frac{\partial q(m)}{\partial m} - \sigma\phi\frac{A}{w} + \frac{A}{w}\phi \\ &= \frac{A}{w}\phi\left[1-\sigma+\sigma\frac{r+1-\delta}{g_q(q,r,w)}\right]. \end{aligned}$$

Since $V_m(m,k)$ is independent of the buyer's capital holdings, then it must be the case that the choice of money taken out of the night market according to (48) is the same for everyone – the distribution of m is again degenerate regardless of whether or not the distribution of capital is degenerate.

The first-order condition for m_{+1} implies

$$\frac{g(q,r,w)}{Mw} = \beta\frac{g(q_{+1},r_{+1},w_{+1})}{M_{+1}w_{+1}}\left[1-\sigma+\sigma\frac{r_{+1}+1-\delta}{g_q(q_{+1},r_{+1},w_{+1})}\right]. \tag{49}$$

It is apparent that this model does not dichotomize – we cannnot solve for q without knowing $r = F_K(K,H)$ and $w = F_H(K,H)$. In steady state, we have

$$1+\frac{i}{\sigma} = \frac{F_K(K,H)+1-\delta}{g_q\left[q,F_K(K,H),F_H(K,H)\right]}.$$

If we set $\theta = 1$ then $g(q,r,w) = c(q)w/A$, and $g_q(q,r,w) = c'(q)w/A = c'(q)F_H(K,H)/A$, which reduces the steady state condition to

$$1+\frac{i}{\sigma} = A\frac{F_K(K,H)+1-\delta}{c'(q)F_H(K,H)}.$$

Using (19)-(20) and the steady-state condition $\sigma q = \delta K$, a steady state with $\theta = 1$ is a pair (K,H) solving

$$1+\frac{i}{\sigma} = A\frac{F_K(K,H)+1-\delta}{c'(\delta K/\sigma)F_H(K,H)} \tag{50}$$

$$A = U'\left[F(K,H)+(1-\delta)K\right]F_H(K,H). \tag{51}$$

Using (50) and (51), it is straightforward to show that $\partial K/\partial i < 0$. The intuition behind this result is that an increase in the money growth rate lowers the value of money acquired by sellers of intermediate goods and so they produce less. Since intermediate goods are used

to produce capital, it follows immediately that aggregate K is lower. Thus, we get a similar result to Stockman but for a different reason.

What if agents were allowed to trade k in the night market? Notice that it is merely a secondary market – no investment occurs, only reallocation of k. Let λ denote the price of existing capital. Then the agent's value function in the night market satisfies

$$W(m,k) = \max_{x,m_{+1},k_{+1}} U(x) - \frac{A}{w}\left[x + \lambda k_{+1} - \lambda\left(r+1-\delta\right)k - \phi(m+\tau M - m_{+1})\right] + \beta V(m_{+1},k_{+1}).$$

The first order condition for k_{+1} is

$$\frac{A}{w}\lambda = \beta V_k(m_{+1},k_{+1})$$

Since wealth is linear in capital holdings and capital does not affect the value of intermediate good trades, $V_k(m_{+1},k_{+1}) = W_k(m_{+1},k_{+1}) = \frac{A}{w_{+1}}\lambda_{+1}\left(r_{+1}+1-\delta\right)$ which gives

$$\frac{A}{w}\lambda = \beta\frac{A}{w_{+1}}\lambda_{+1}\left(r_{+1}+1-\delta\right)$$

This expression is independent of individual k and merely pins down the path for the price of capital in the secondary market such that no arbitrage opportunities exist. Agents are indifferent between buying or selling capital at this price and so the distribution of capital is not pinned down without further assumptions on agents' behavior.

With competitive pricing, buyers choose how much of the intermediate good to purchase. As before, $d = m$ so buyers spend all of their money and acquire $q_b = m/p$ units of goods. Sellers set marginal cost equal to the value of a marginal unit of money received in payment, $c'(q_s) = \frac{A}{w}\phi p$. In equilibrium, $q_b = q_s = q$ which solves $c'(q) = \frac{A\phi m}{wq}$. Following the same methods as before, the first-order condition for money becomes

$$\frac{A}{w}\phi = \beta\frac{A}{w_{+1}}\phi_{+1}\left[1-\sigma+\sigma A\frac{r_{+1}+1-\delta}{c'(q_{+1})w_{+1}}\right]$$

Using $\frac{A}{w}\phi = c'(q)/p$ and $p = m/q$ this can be written as

$$\frac{c'(q)q}{M} = \beta\frac{c'(q_{+1})q_{+1}}{M_{+1}}\left[1-\sigma+\sigma A\frac{r_{+1}+1-\delta}{c'(q_{+1})w_{+1}}\right] \tag{52}$$

Comparing (49) and (52) note that the dynamics of the model under bargaining and Walrasian pricing will differ if $g(q,r,w) \neq c'(q)q$ and $g_q(q,r,w) \neq c'(q)w/A$. In steady state, (52) becomes

$$1 + \frac{i}{\sigma} = A\frac{F_K(K,H)+1-\delta}{c'(q)F_H(K,H)}$$

which is the same steady-state expression that arises under bargaining when $\theta = 1$. So an equilibrium with Walrasian pricing is a pair (K, H) solving (50) and (51). Once again, there is no dichotomy and excessive money growth, creates inflation, raises the nominal interest rate and lowers the equilibrium capital stock.

4 Calibration

To be completed...

5 Welfare Analysis

To be completed...

6 Conclusions

In this paper we have taken another step towards closing the gap between search models of money and standard macro models. We have shown how deriving the demand for money from first principles can be incorporated in the neoclassical growth model and how monetary policy affects aggregate output, employment and consumption. The key point of our paper is that there are many links by which changes in the value of money in the search market spill over to affect real variables in markets that do not require the use of money for exchange.

Appendix A

Here we consider the model with utility nonseparable in (x, q, e), but still linear in h, say $\hat{U}(x, q, e) - Ah$. Since q and e are determined during the day, they are state variables in the night market. For this section we assume that capital is not used for production during the day so $q = f(e)$. so we let $W(m, k, q, e)$ now denote the value function at night,

$$\begin{aligned} W(m,k,q,e) &= \max_{x,h,m_{+1},k_{+1}} \hat{U}(x,q,e) - Ah + \beta V(m_{+1},k_{+1}) \\ s.t.\ x &= rk + wh + \phi(m + \tau M - m_{+1}) + (1-\delta)\,k - k_{+1}. \end{aligned}$$

Substituting for h yields

$$\begin{aligned} W(m,k,q,e) &= \frac{A}{w}[\phi(m+\tau M) + (r+1-\delta)\,k] \\ &\quad + \max_{x,m_{+1},k_{+1}} \left[\hat{U}(x,q,e) - \frac{A}{w}(x + \phi m_{+1} + k_{+1}) + \beta V(m_{+1},k_{+1})\right]. \end{aligned} \tag{53}$$

The first-order conditions are given by:

$$x \ : \ \hat{U}_x(x,q,e) = \frac{A}{w} \tag{54}$$

$$k_{+1} \ : \ \frac{A}{w} = \beta V_k(m_{+1},k_{+1}) \tag{55}$$

$$m_{+1} \ : \ \frac{A}{w}\phi = \beta V_m(m_{+1},k_{+1}) \tag{56}$$

Hence we again have a degenerate distribution of (m, k). More importantly for this section, the choice of x in the night market is affected by how much the agent consumed or produced in the day market. The envelope conditions are

$$W_m(m,k,q,e) = \frac{A}{w}\phi \tag{57}$$

$$W_k(m,k,q,e) = \frac{A}{w}(r+1-\delta) \tag{58}$$

$$W_q(m,k,q,e) = \hat{U}_q(x,q,e) \tag{59}$$

$$W_e(m,k,q,e) = \hat{U}_e(x,q,e). \tag{60}$$

Suppose that during the day agents meet and bargain bilaterally. The bargaining problem is $\max S_b^\theta S_s^{1-\theta}$ subject to $q = f(e)$ and $d \leq m$, where now we have

$$\begin{aligned} S_b &= W(m_b - d, k_b, q, 0) - W(m_b, k_b, 0, 0) \\ S_s &= W(m_s + d, k_s, 0, e) - W(m_s, k_s, 0, 0) \end{aligned}$$

By the usual logic, one can show $d = m_b$. Using this and $e = \psi(q) = f^{-1}(q)$, the first order condition with respect to q can be written

$$\theta S_s \hat{U}_q(x_b, q, 0) + (1-\theta) S_b \hat{U}_e \left[x_s, 0, \psi(q)\right] \psi'(q) = 0. \tag{61}$$

Agents generally choose different values of x in the night market. Letting x_s, x_b, and x_0 be the quantities purchased by day market sellers, buyers and non-traders, we have

$$\begin{aligned} S_b &= \hat{U}(x_b, q, 0) - \hat{U}(x_0, 0, 0) - \frac{A}{w}\left(x_b - x_0 + \phi m_b\right) \\ S_s &= \hat{U}\left[x_s, 0, \psi(q)\right] - \hat{U}(x_0, 0, 0) - \frac{A}{w}\left(x_s - x_0 - \phi m_b\right). \end{aligned}$$

>From the FOC for x,

$$\begin{aligned} \hat{U}_x(x_s, 0, \psi(q)) &= \frac{A}{w} \\ \hat{U}_x(x_b, q, 0) &= \frac{A}{w} \\ \hat{U}_x(x_0, 0, 0) &= \frac{A}{w} \end{aligned}$$

>From these we get the equilibrium choices $x_s = x_s\left[\psi(q), \frac{A}{w}\right]$, $x_b = x_b(q, \frac{A}{w})$ and $x_0 = x_0(\frac{A}{w})$. Then we can solve (61) to obtain

$$\frac{A}{w}\phi m_b = g(q, \frac{A}{w})$$

where

$$\begin{aligned} g(q, \frac{A}{w}) &= \frac{(1-\theta)\left\{U\left[x_0(\frac{A}{w}), 0, 0\right] - U\left[x_b(q, \frac{A}{w}), q, 0\right]\right\} U_e\left\{x_s\left[\psi(q), \frac{A}{w}\right], 0, \psi(q)\right\}\psi'(q)}{\theta U_q\left[x_b(q, \frac{A}{w}), q, 0\right] - (1-\theta) U_e\left\{x_s\left[\psi(q), \frac{A}{w}\right], 0, \psi(q)\right\}\psi'(q)} \\ &+ \frac{\theta\left\{U\left[x_0(\frac{A}{w}), 0, 0\right] - U\left[x_s\left[\psi(q), \frac{A}{w}\right], 0, \psi(q)\right]\right\} U_q\left[x_b(q, \frac{A}{w}), q, 0\right]}{\theta U_q\left[x_b(q, \frac{A}{w}), q, 0\right] - (1-\theta) U_e\left\{x_s\left[\psi(q), \frac{A}{w}\right], 0, \psi(q)\right\}\psi'(q)} \\ &+ \frac{(1-\theta)\frac{A}{w}\left[x_b(q, \frac{A}{w}) - x_0(\frac{A}{w})\right] U_e\left\{x_s\left[\psi(q), \frac{A}{w}\right], 0, \psi(q)\right\}\psi'(q)}{\theta U_q\left[x_b(q, \frac{A}{w}), q, 0\right] - (1-\theta) U_e\left\{x_s\left[\psi(q), \frac{A}{w}\right], 0, \psi(q)\right\}\psi'(q)} \\ &+ \frac{\theta\frac{A}{w}\left\{x_s\left[\psi(q), \frac{A}{w}\right] - x_0(\frac{A}{w})\right\} U_q\left[x_b(q, \frac{A}{w}), q, 0\right]}{\theta U_q\left[x_b(q, \frac{A}{w}), q, 0\right] - (1-\theta) U_e\left\{x_s\left[\psi(q), \frac{A}{w}\right], 0, \psi(q)\right\}\psi'(q)}. \end{aligned}$$

The key observation here is that A/w enters g. If $U = U(x) + u(q) - \eta(e) - Ah$ is separable, then $g(q, \frac{A}{w}) = g(q)$ reduces to the model in the text with no capital used in the

day market – that is, to a model that dichotomizes. Also, for any U, if $\theta = 1$ then the previous equation reduces to

$$\frac{A}{w}\phi m_b = U\left[x_0(\frac{A}{w}), 0, 0\right] - U\left\{x_s\left[\psi(q), \frac{A}{w}\right], 0, \psi(q)\right\} + \frac{A}{w}\left\{x_s\left[\psi(q), \frac{A}{w}\right] - x_0(\frac{A}{w})\right\}.$$

Notice

$$g_q(q, \frac{A}{w}) = -U_e\left\{x_s\left[\psi(q), \frac{A}{w}\right], 0, \psi(q)\right\}\psi'(q) > 0$$

since $U_x\left\{x_s\left[\psi(q), \frac{A}{w}\right], 0, \psi(q)\right\} = \frac{A}{w}$ from the first order condition for x. If $U = U(x, q) - \eta(e) - Ah$, then the first order conditions imply $x_s\left[\psi(q), \frac{A}{w}\right] = x_0(\frac{A}{w})$, and this becomes

$$\frac{A}{w}\phi m = \eta[\psi(q)] = c(q).$$

The value function in the day market is given by

$$V(m, k) = \sigma W(m, k, q, 0) + \sigma W\left[m, k, 0, \psi(\tilde{q})\right] + (1 - 2\sigma)W(m, k, 0, 0) \tag{62}$$

By the usual methods the first order condition for m is

$$\frac{A\phi}{w} = \beta\frac{A\phi_{+1}}{w_{+1}}\left[1 - \sigma + \sigma\frac{U_q\left[x_b(q_{+1}, \frac{A}{w_{+1}}), q_{+1}, 0\right]}{g_q(q_{+1}, \frac{A}{w_{+1}})}\right]$$

or

$$\frac{g(q, \frac{A}{w})}{M} = \beta\frac{g(q_{+1, \frac{A}{w_{+1}}})}{M_{+1}}\left[1 - \sigma + \sigma\frac{U_q\left[x_b(q_{+1}, \frac{A}{w_{+1}}), q_{+1}, 0\right]}{g_q(q_{+1}, \frac{A}{w_{+1}})}\right]$$

It is clear from this expression that q cannot be determined independently of w which in turn is a function of K via $w = F_H(K, H)$. A steady-state satisfies

$$\begin{aligned}
1 + \frac{i}{\sigma} &= \frac{U_q\left\{x_b\left[q, \frac{A}{F_H(K,H)}\right], q, 0\right\}}{g_q\left[q, \frac{A}{F_H(K,H)}\right]} \\
\rho + \delta &= F_K(K, H) \\
x &= F(K, H) - \delta K \\
H &= \frac{x - [F_K(K, H) - \delta]K}{F_H(K, H)}
\end{aligned}$$

and

$$\begin{aligned}
H &= \sigma h_s + \sigma h_b + (1 - 2\sigma)h_0 \\
x &= \sigma x_b\left(\psi(q), \frac{A}{w}\right) + \sigma x_s\left[\psi(q), \frac{A}{w}\right] + (1 - 2\sigma)x_0\left(\frac{A}{w}\right)
\end{aligned}$$

where

$$\begin{aligned} h_s &= H + \frac{1}{F_H(K,H)}\left(x_s\left[\psi(q),\frac{A}{w}\right]-x\right)-\phi M\frac{A}{F_H(K,H)} \\ h_b &= H + \frac{1}{F_H(K,H)}\left(x_b\left(q,\frac{A}{w}\right)-x\right)+\phi M\frac{A}{F_H(K,H)} \\ h_0 &= H + \frac{1}{F_H(K,H)}\left(x_0\left(\frac{A}{w}\right)-x\right) \end{aligned}$$

with h_s, h_b, h_0 denoting the hours worked in the night market by day market sellers, buyers and non-traders respectively. It is clear from this equation that unless q disappears from x when aggregating over x_b, x_s and x_0, the dichotomy is broken and changes in i affect x, H and K. So monetary policy affects q and e and this spillsover to affect consumption, hours worked and capital accumulation in the night market. For $\theta = 1$, we have

$$1+\frac{i}{\sigma} = \frac{U_q\left\{x_b\left[q,\frac{A}{F_H(K,H)}\right],q,0\right\}}{-U_e\left\{x_s\left[\psi(q),\frac{A}{w}\right],0,\psi(q)\right\}\psi'(q)}. \tag{63}$$

Under the Friedman rule, this reduces to

$$U_q\left\{x_b\left[q,\frac{A}{F_H(K,H)}\right],q,0\right\} = -U_e\left\{x_s\left[\psi(q),\frac{A}{w}\right],0,\psi(q)\right\}\psi'(q)$$

which is the efficiency condition for producing q in the day market.

Under Walrasian pricing, buyers in the day market solve the following problem

$$\begin{aligned} &\max_{q_b} W(m_b - pq_b, k_b, q_b, 0) \\ &s.t.\ \ pq_b \ \leq m_b \end{aligned}$$

where p is the money price of goods. The seller's problem is

$$\max_{q_s} W\left[m_s + pq_s, k_s, 0, \psi(q_s)\right]$$

The seller's first-order condition is

$$W_m\left[m_s + pq_s, k_s, 0, \psi(q_s)\right]p + W_e\left[m_s + pq_s, k_s, 0, \psi(q_s)\right]\psi'(q_s) = 0$$

or

$$\frac{A\phi}{w}p = -U_e\left[x_s, 0, \psi(q_s)\right]\psi'(q_s)$$

By the usual methods, the first-order condition for m becomes

$$\frac{g\left[q, x_s(\frac{A}{w})\right]}{M} = \beta \frac{g\left[q_{+1}, x_s(\frac{A}{w_{+1}})\right]}{M_{+1}} \left\{ 1 - \sigma + \sigma \frac{U_q\left[x_b(q_{+1}, \frac{A}{w_{+1}}), q_{+1}, 0\right]}{-U_e\left\{x_s\left[\psi(q_{+1}), \frac{A}{w_{+1}}\right], 0, \psi(q_{+1})\right\}\psi'(q_{+1})} \right\}$$

In steady state

$$1 + \frac{i}{\sigma} = \frac{U_q\left\{x_b\left[q, \frac{A}{F_H(K,H)}\right], q, 0\right\}}{-U_e\left\{x_s\left[\frac{A}{F_H(K,H)}\right], 0, \psi(q)\right\}\psi'(q)} \tag{64}$$

Equation (64 is equal to the bargaining steady state under bargaining with $\theta = 1$, (63).

Appendix B

Baseline Model

Assume constant returns to scale production function for general goods. So

$$\frac{F(K,H)}{H} = F(K/H,1) = F(\Bbbk)$$

where $\Bbbk = K/H$. In the steady state of the baseline model with Walrasian pricing we have the following four equations:

$$1+\frac{i}{\sigma} = \frac{u'(q)}{c_q(q,K)} \tag{65}$$

$$\rho+\delta = F_K(\Bbbk) - \sigma\frac{c_K(q,K)}{U'(x)} \tag{66}$$

$$A = U'(x)F_H(\Bbbk) \tag{67}$$

$$x = H\left[F(\Bbbk) - \delta\Bbbk\right], \tag{68}$$

>From (67) we get

$$x = U'^{-1}\left(\frac{A}{F_H(\Bbbk)}\right) \tag{69}$$

which combined with (68) yields

$$H = \frac{U'^{-1}\left(\frac{A}{F_H(\Bbbk)}\right)}{\Bbbk\left[F(\Bbbk)/\Bbbk - \delta\right]} \tag{70}$$

which implies

$$K = \frac{K}{H}H = \frac{U'^{-1}\left(\frac{A}{F_H(\Bbbk)}\right)}{F(\Bbbk)/\Bbbk - \delta} \tag{71}$$

>From (71) we obtain

$$\frac{dK}{d\Bbbk} = \frac{1}{F(\Bbbk)/\Bbbk - \delta}\frac{1}{U''\left(\frac{A}{F_H(\Bbbk)}\right)}\left(\frac{-AF_{HK}(\Bbbk)}{F_H(\Bbbk)^2}\right) - \frac{U'^{-1}\left(\frac{A}{F_H(\Bbbk)}\right)}{\left[F(\Bbbk)/\Bbbk - \delta\right]^2}\left[\frac{F_K(\Bbbk)\Bbbk - F(\Bbbk)}{\Bbbk^2}\right]$$

Equation (65) yields

$$\frac{\partial q}{\partial K} = \frac{\left(1+\frac{i}{\sigma}\right)c_{qK}(q,K)}{u''(q) - \left(1+\frac{i}{\sigma}\right)c_{qq}(q,K)} > 0$$

$$\frac{\partial q}{\partial i} = \frac{c(q,K)}{\sigma\left[u''(q) - \left(1+\frac{i}{\sigma}\right)c_{qq}(q,K)\right]} < 0$$

Using (71) and (65) we get

$$\begin{aligned} u'(q) &= \left(1+\frac{i}{\sigma}\right) c_q \left(q, \frac{U'^{-1}\left(\frac{A}{F_H(\Bbbk)}\right)}{F(\Bbbk)/\Bbbk - \delta}\right) \\ &\Rightarrow \quad q = q(\Bbbk, i) \end{aligned}$$

Finally, we can rewrite (66) as

$$\rho + \delta = F_K(\Bbbk) - \frac{\sigma}{A} F_H(\Bbbk) c_K \left(q(\Bbbk, i), \frac{U'^{-1}\left(\frac{A}{F_H(\Bbbk)}\right)}{F(\Bbbk)/\Bbbk - \delta}\right) \equiv N(\Bbbk) \tag{72}$$

A steady state for the baseline model is a value $\Bbbk$ that solves (72). From this expression we have that

$$\begin{aligned} \frac{\partial N(\Bbbk)}{\partial \Bbbk} &= F_{KK}(\Bbbk) \\ &\quad -\frac{\sigma}{A} F_{HK}(\Bbbk) c_K \left(q(\Bbbk, i), \frac{U'^{-1}\left(\frac{A}{F_H(\Bbbk)}\right)}{F(\Bbbk)/\Bbbk - \delta}\right) \\ &\quad -\frac{\sigma}{A} F_H(\Bbbk) \left(c_{qK} \frac{\partial q}{\partial K} + c_{KK}\right) \frac{\partial K}{\partial \Bbbk} \end{aligned}$$

The first term is negative. The second term is positive if $F_{HK}(\Bbbk)$ is positive. The third term is ambiguous. Thus without further restrictions on the properties of the cost function, it is not possible to say anything about existence or uniqueness of the equilibrium.

Two types of capital

For this model with Walrasian pricing, replace $c(q, K)$ with $c(q, Z)$ where Z is special capital. The steady-state conditions are

$$1 + \frac{i}{\sigma} = \frac{u'(q)}{c'(q, Z)} \tag{73}$$

$$\rho + \delta = F_K(\Bbbk) \tag{74}$$

$$\rho + \omega = -\sigma \frac{c_z(q, Z)}{U'(x)} \tag{75}$$

$$A = U'(x) F_H(\Bbbk) \tag{76}$$

$$x = H\left[F(\Bbbk) - \delta \Bbbk\right] - \omega Z \tag{77}$$

>From (74), the steady state value of $\Bbbk$ is given by

$$\Bbbk = F_K^{-1}(\rho + \delta) \tag{78}$$

As before (76) gives us (69) which in conjunction with (78) gives

$$x = U'^{-1}\left(\frac{A}{F_H\left[F_K^{-1}(\rho+\delta)\right]}\right) \tag{79}$$

Equation (73) can be written to obtain

$$\begin{aligned} u'(q) &= \left(1+\frac{i}{\sigma}\right) c_q(q,Z) \\ &\Rightarrow \quad q = q(Z,i), \text{ with } q_z(Z,i) > 0 \text{ and } q_i(Z,i) < 0 \end{aligned}$$

where q is unique given Z. Consequently, (75) becomes

$$\rho + \omega = -\frac{\sigma}{A} F_H\left[F_K^{-1}(\rho+\delta)\right] c_z\left[q(Z,i), Z\right] \tag{80}$$

which pins down Z if a solution exists. Finally, using (77), (79) and (80) we get

$$x = H\left[F\left[F_K^{-1}(\rho+\delta)\right] - \delta F_K^{-1}(\rho+\delta)\right] - \omega Z$$

which reduces to

$$\bar{H} = \frac{x + \omega Z}{\left\{F\left[F_K^{-1}(\rho+\delta)\right] - \delta F_K^{-1}(\rho+\delta)\right\}} \tag{81}$$

Thus, if a solution to (80) exists, then q, x, K, and H are all uniquely determined.

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