Credible Group Stability in Many-to-Many Matching Problems*

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Abstract

It is known that in two-sided many-to-many matching problems, pairwise-stable matchings may not be immune to group deviations, unlike in many-to-one matching problems (Blair 1988). In this paper, we show that pairwise stability is equivalent to credible group stability when one side has responsive preferences and the other side has categorywise-responsive preferences. A credibly group-stable matching is immune to any "executable" group deviations with an appropriate definition of executability. Under the same preference restriction, we also show the equivalence between the set of pairwise-stable matchings and the set of matchings generated by coalition-proof Nash equilibria of an appropriately defined strategic-form game.

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1 Introduction

Following the success of the National Residency Matching Program (NRMP) in stabilizing the United States hospital-intern market (see Roth [17], Roth and Sotomayor [23], and Roth and Peranson [22]), the United Kingdom also adopted centralized matching procedures in the markets for medical internships in the 1960s. However, there are two important differences between the UK programs and their North American counterparts (Roth [21]). First, the UK medical intern markets are organized regionally rather than nationally: in different regions, different algorithms were adopted by central matching programs. Many of those were abandoned after several years and replaced by new algorithms. An intriguing observation here is that the abandoned algorithms all produced pairwise-unstable matchings and their successor algorithms all produced pairwise-stable matchings. One region adopted a pairwise-stable matching algorithm from the start, and it has been used successfully since. Roth [21] suggested that this natural experiment in the UK markets proved the robustness of pairwise-stable matchings.

Second, in the UK markets, each medical student is required to complete two internships, one medical and one surgical, in a period of twelve months, to be eligible for full registration as doctors (no such categories exist in the US). Each internship lasts for six months. Consultants in teaching hospitals seek some number of students to fill internships in either medicine or surgery.[1] Thus, given the requirement of UK interns to experience both medical and surgical positions, each regional market in the UK needs to be modeled as a "special" two-sided many-to-two matching problem, unlike in the US market. Even in this problem, the Gale-Shapley deferred-acceptance algorithms (Gale and Shapley [9]) yield pairwise-stable matchings under a preference restriction (Roth [18,20], Blair [5]). However, this outcome may no longer be group-stable in a many-to-two matching problem (Blair [5], Roth [21]). This shows a clear contrast with a many-to-one matching market like the US hospital intern market. Although a pairwise-stable matching is required to be immune to only one- or two-agent deviations, Roth [17] showed that if a larger size coalition can deviate from a matching, then a coalition of size one or two can also deviate in many-to-one matching problem. Thus, a pairwise-stable matching is also immune to group deviations. Hence, market stabilization by introduction of centralized matching programs is well justified. However, in many-to-two (-many) matching problems, there can be a group deviation from a pairwise-stable matching that improves the payoff of every member of the deviation. Thus pairwise-stable matchings are not even Pareto-efficient. This creates a puzzle: Why is the pairwise-stable matching so robust in the UK markets?

[1] In the UK market, consultants rather than hospitals are the agents who hire medical students.

In this paper, we provide theoretical support for the robustness of pairwise stability allowing for group deviations in many-to-many matching problems. Unlike many-to-one matching problems, a pairwise-stable matching may not be immune to any group deviations in many-to-many matching problems. However, a closer look at possible group deviations from a pairwise-stable matching reveals that these deviations are not credible in a certain way. Even if a group of agents would benefit from deviating by reorganizing their partnerships, some members may not have incentive to follow the suggested reorganization completely. Consider the following situation. A group is somehow organized, and the members of the group communicate with each other about a deviation plan, and they agree on carrying it out the next day without letting outsiders know about the plan. In the plan, it has been suggested to each of the group's members that she should discontinue some existing partnerships while keeping others and forming some new partnerships with other members. Do all the members follow the suggestion? Some members may choose to follow the plan only partially. For example, it may be even more profitable for some of them not to form some of the suggested partnerships, but instead to keep some existing partnerships they were told to discontinue. In such a case, the suggested group deviation cannot be carried out successfully (unless a group can form a binding agreement). In this case, we say that these deviations are not "executable." More precisely: an executable group deviation is a deviation with a proposed matching that specifies each member's partners and is pairwise-stable within the members of the coalition, assuming outsiders of the coalition are passive agents.[2] We say that a matching is *credibly group-stable* if it is immune to any executable group deviation.

The first main result of this paper is that the set of credibly group-stable matchings is equivalent to the set of pairwise-stable matchings when one side has responsive preferences and the other side has categorywise-responsive preferences (Theorem 1). This domain is natural in the sense that it is the simplest preference domain in the UK hospital-intern markets based on agents' preferences over individuals. As in the US hospital-intern market, agents submit their preferences over individual partners (interns submit preference rankings over individual consultants in each category), not over subsets of partners to the central authority that conducts the match.[3]

[2]Pairwise stability within the deviation group with passive outsiders prevents the following two cases of possible further deviations: a member of the coalition may not want to form some of the links she is supposed to form according to the plan, and she may keep some of the links with outsiders she was told to discontinue; or a pair of members of the group, who are supposed to discontinue links with each other according to the plan, may not go along with the recommendation.

[3]The U.K. markets are *regional*. That is, all positions are in the same geographical area. Moreover, the two jobs are not served simultaneously, but consecutively one after another. However, the match

Credible group stability requires only that no group deviation from a matching is executable. However, there is no guarantee that an executable group deviation itself will be immune to further executable deviations. Thus, to be consistent, game theorists may say that credibility of group deviation should be defined recursively: a deviation is said to be credible if it is immune to further credible deviations. In strategic-form games, a strategy profile is said to be a *coalition-proof Nash equilibrium* (Bernheim, Peleg, and Whinston [4]) if it is immune to any credible deviation in this sense.[4] Our second result shows that the set of matchings generated as outcomes of the coalition-proof Nash equilibria of a strategic-form game appropriately generated from a many-to-many matching problem coincides with the set of credibly group-stable matchings of the same matching problem in the same preference domain as in Theorem 1 (Theorem 2). Theorems 1 and 2 provide justifications for Roth's [21] observation of the UK medical intern markets.

The rest of the paper is organized as follows. In Section 2, we introduce the model and define traditional solution concepts in the literature as well as our new solution concept, credible group stability. We provide examples that illustrate the differences between these concepts. In Section 3, we start with a weak preference restriction, substitutability (Kelso and Crawford [13]). We first show that a credibly group-stable matching is pairwise-stable (Proposition 1), while a pairwise-stable matching may not be credibly group-stable as long as one side has substitutable preferences even if the other side has responsive preferences (Example 4). In Section 3, we prove the equivalence between pairwise stability and credible group stability if one side has responsive preferences and the other has categorywise-responsive preferences (Theorem 1). However, when both sides have categorywise-responsive preferences, the equivalence result may fail (Example 5), and even credibly group-stable matching may not exist (Example 6). In Section 4, we consider a natural strategic-form game of many-to-many matching problems and show that the set of the matchings generated through the coalition-proof Nash equilibria of this game, the set of pairwise-stable matchings, and the set of cred-

is done simultaneously. These all reduce the possibility of having *complementary* preferences over the two jobs and support categorywise-responsive preferences assumption.

[4] A coalition-proof Nash equilibrium is a strategy profile that is immune to any credible strategic coalitional changes in the members' strategies, and the credibility of strategic coalitional deviations is defined recursively in a consistent manner (see Bernheim, Peleg, and Whinston [4]). Our equivalence result gives us another reason that our non-characteristic function approach is more preferable than the characteristic function approach in matching problems. The counterpart of a coalition-proof Nash equilibrium in a characteristic function form game is *the credible core* in Ray [16] that checks credibility of coalitional deviations recursively. However, as is shown in Ray [16], the core and the credible core are equivalent in characteristic function form games. Ray's remarkable result also motivates our usage of non-characteristic function form games.

ibly group-stable matchings are all equivalent under the same preference domain as in Section 3 (Theorem 2). Section 5 concludes the paper with an application of our results in non-bipartite matching markets.

1.1 Related Literature

The most closely related paper is an independent work by Echenique and Oviedo [8] on many-to-many matching problems. They use setwise stability as defined by Roth [18] as their solution concept. A setwise-stable matching is a matching that is immune to any group deviations in which participating members have no incentive to discontinue any partnership after the deviation. One of the main results in Echenique and Oviedo [8] is that if one side has substitutable preferences and the other has "strongly substitutable" preferences, then pairwise stability and setwise stability are equivalent. Our main result states that if one side has categorywise-responsive preferences and the other side has responsive preferences, then pairwise-stability and credible group-stability are equivalent. Although these two result may appear similar, they have no logical relationship with each other, since neither solution concepts nor preference domains in these two statements are the same. Setwise stability is a stronger solution concept than our credible group stability, since the executability requirement rules out more group deviations than individual stability. In the general preference domain, we have *group-stable set* $\subseteq$ *setwise-stable set* $\subseteq$ *credibly group-stable set* $\subseteq$ *pairwise-stable set*. Although categorywise-responsive preferences belong to a family of substitutable preferences, strongly substitutable preferences have no logical relationship with responsive preferences (with quotas).[5]

In many-to-one matching problems, Kelso and Crawford [13] showed that the Gale-Shapley deferred-acceptance algorithm still finds (pairwise-)stable matchings under substitutable preferences. Subsequently, Roth [18, 20, 21] and Blair [5] studied the structure of the set of pairwise-stable matchings in a many-to-many setting under substitutable preferences. On the lattice structure of pairwise-stable matchings, Blair [5], Alkan [2], and Echenique and Oviedo [8] provided results in many-to-many matching problems using different definitions of supremum (and infimum) under different preference domains.

In many-to-one matching problems with responsive preferences, a randomized myopic adjustment process also yields a pairwise-stable matching with probability one (see Roth and Vande Vate [24]. In particular, if an initial matching is randomly selected,

[5] Indeed, as Sotomayor [26] pointed out in her example (see Example 3 below), the set of setwise-stable matchings may be empty under separable preferences (which is a *weaker* requirement than responsive preferences).

every pairwise stable matching can realize with a positive probability. In our separate note (Konishi and Ünver [14]), we show that a similar convergence result still holds in many-to-many matching problems if agents have categorywise-responsive preferences. This result justifies our characterization of the whole set of pairwise-stable matchings instead of the optimal matchings generated by the Gale-Shapley deferred acceptance algorithms.

2 The Model

2.1 Many-to-Many Matching Problem

Let F and W be finite sets of firms and workers with $F \cap W = \emptyset$. For any agent $i \in F \cup W$, the **set of potential partners** M_i is the set of agents on the other side: i.e., $M_i = W$ if $i \in F$, and $M_i = F$ if $i \in W$. We define a preference profile by $\succeq = (\succeq_F, \succeq_W) = ((\succeq_i)_{i \in F \cup W})$, where $\succeq_i$ is a preference ordering over 2^{M_i}. We also use notations $\succeq = (\succeq_F, \succeq_W)$, where $\succeq_F$ and $\succeq_W$ denote preference profiles for F and W, respectively. We assume throughout the paper that for any agent $i \in F \cup W$, agent i's preference relation $\succeq_i$ is **strict**: i.e. $\succeq_i$ is a linear order, meaning that for any $S, T \subseteq M_i$, $S \succeq_i T$ implies that $S = T$ or $S \succ_i T$. A **many-to-many matching problem** is a list $(F, W, \succeq)$. We fix a many-to-many matching problem $(F, W, \succeq)$ in the rest of the paper. A **matching** μ is a mapping from the set $F \cup W$ into the set of all subsets of $F \cup W$ such that for all $i, j \in F \cup W$: (i) $\mu(i) \in 2^{M_i}$, and (ii) $j \in \mu(i)$ if and only if $i \in \mu(j)$.

2.2 Preference Restrictions

A commonly used preference restriction in matching theory is responsiveness with quota. Agent i's preference relation $\succeq_i$ is **responsive with quota** if there is a positive integer q_i such that for any $T \subset M_i$ with $|T| < q_i$, and any $j, j' \in M_i \backslash T$, we have[6]

$$\text{(i) } T \cup \{j\} \succ_i T \cup \{j'\} \Leftrightarrow j \succ_i j' \text{ and}$$
$$\text{(ii) } T \cup \{j\} \succ_i T \Leftrightarrow j \succ_i \emptyset,$$

and for any $T \subseteq M_i$ with $|T| > q_i$, we have $\emptyset \succ_i T$ (Roth [19]).[7] A preference profile $\succeq_T$ is responsive if for any $i \in T$, $\succeq_i$ is responsive with some quota q_i.

[6]Without confusion, we abuse notations: $j \succeq_i j'$, $\emptyset \succeq_i j$ and $j \succeq_i \emptyset$ denote $\{j\} \succeq_i \{j'\}$, $\{j\} \succeq_i \emptyset$ and $\emptyset \succeq_i \{j\}$, respectively, for any $j, j' \in M_i$.

[7]Note that under a strict preference ordering, Condition (ii) implies $T \succ_i T \cup \{j\} \Leftrightarrow \emptyset \succ_i j$ as well. Also note that Condition (ii) is commonly referred to as "separability" in the literature.

Substitutability is a weaker preference restriction than responsiveness, yet some of the important results obtained with responsive preferences are preserved under substitutability: it still guarantees the existence of pairwise-stable matchings and the validity of the polarization results in many-to-many matching problems (Roth [18]). For any $i \in F \cup W$, and any $S \subset M_i$, let $Ch_i(S) \subseteq S$ be such that $Ch_i(S) \succeq_i T$ for any $T \subseteq S$. Agent i's preference relation $\succeq_i$ is **substitutable** if for any $S \subseteq M_i$ and any distinct $j, j' \in Ch_i(S)$, we have $j \in Ch_i(S \backslash \{j'\})$ (Kelso and Crawford [13]). For any $T \subseteq F \cup W$, a preference profile $\succeq_T$ is substitutable if for any $i \in T$, $\succeq_i$ is substitutable.

We now introduce a new preference restriction that is stronger than substitutability but weaker than responsiveness with quota. This preference restriction retains the virtues of responsive preferences yet makes it possible to analyze a market like the UK hospital-intern market. We first introduce the notion of categories of partners. For each agent $i \in F \cup W$, let K_i be a finite set called the **set of categories** for i, and let $\{M_i^k\}_{k \in K_i}$ be a partition of M_i. Agent i's preference relation $\succeq_i$ is **separable across categories** with respect to $\left(K_i, \{M_i^k\}_{k \in K_i}\right)$ if for any category $k \in K_i$, any $S, T \subseteq M_i^k$, and any $I, J \subseteq M_i \backslash M_i^k$, we have

$$S \cup I \succeq_i T \cup I \Leftrightarrow S \cup J \succeq_i T \cup J.$$

Agent i's preference relation $\succeq_i$ is **categorywise-responsive with quotas** if there are a set of categories K_i, a partition $\{M_i^k\}_{k \in K_i}$ of M_i, and a vector of quotas $q_i = (q_i^k)_{k \in K_i}$ such that (i) $\succeq_i$ is separable across categories with respect to $\left(K_i, \{M_i^k\}_{k \in K_i}\right)$, and (ii) in each category $k \in K_i$, the restriction of $\succeq_i$ to $2^{M_i^k}$ is responsive with quota q_i^k. A preference profile $\succeq_T$ is categorywise-responsive if for any $i \in T$, $\succeq_i$ is categorywise-responsive with some quota vector $q_i = (q_i^k)_{k \in K_i}$. Note that categorywise-responsive preferences are substitutable.[8] A regional UK medical intern market can be modeled as a many-to-many matching problem where one side has responsive preferences (consultants), whereas the other has categorywise-responsive preferences with quotas (interns).[9] Let F and W denote consultants and interns, respectively. Each consultant $j \in F$ specializes either in medicine or surgery, i.e., F is partitioned into F^m and F^s. No consultant j categorizes interns, and thus she can have responsive preferences with

[8] For agent i, let $\succeq_i$ be categorywise-responsive and for $T \subseteq M_i$, let $\{j, h\} \subseteq Ch_i(T)$.

(i) If j and h are partners in the same category k for agent i: By separability of $\succeq_i$ across categories $\{j, h\} \subseteq Ch_i\left(T \cap M_i^k\right)$. Since $\succeq_i$ is responsive on M_i^k, we have $h \in Ch_i\left((T \backslash \{j\}) \cap M_i^k\right)$. Separability of $\succeq_i$ across categories implies $h \in Ch_i(T \backslash \{j\})$.

(ii) If j and h are partners in different categories for agent i: By separability of $\succeq_i$ across categories, we have $h \in Ch_i(T \backslash \{j\})$ completing the proof that $\succeq_i$ is substitutable.

[9] In the UK markets, matching mechanisms utilize students' preference orderings over individual consultants in each category. Given these mechanisms, the simplest assumption on preference domain of students is categorywise-responsive preferences.

quota q_j that is the number of positions j has. On the other hand, each intern $i \in I$ has category set $K_i = \{m, s\}$ with $M_i^m = F^m$ and $M_i^s = F^s$, and she also has a unit quota for each category, i.e. $q_i^m = q_i^s = 1$. Using substitutability, the UK medical intern markets can be formulated as a many-to-two matching problem without introducing two categories (see Roth [21]). However, to use this formulation, we need to give up the equivalence between pairwise stability and credible group stability (see Section 3).

In independent work, Echenique and Oviedo [8] introduced another preference restriction. Agent i's preference $\succeq_i$ is **strongly substitutable** if for any $S, T \subseteq M_i$ with $S \succ_i T$, $j \in Ch_i(S \cup \{j\})$ implies $j \in Ch_i(T \cup \{j\})$. There is no logical relationship between responsiveness and strong substitutability (Echenique and Oviedo [8]).[10] A preference profile $\succeq_T$ is strongly substitutable if for any $i \in T$, $\succeq_i$ is strongly substitutable.

2.3 Solution Concepts

In this subsection, we discuss solution concepts used in this paper. First, for any agent $i \in F \cup W$, we say that set $S \subseteq M_i$ is **individually rational** for i if $S \succeq_i \emptyset$, and is **individually stable** for i if $Ch_i(S) = S$. Obviously, individual stability implies individual rationality, but not vice versa. We also say that a matching μ is **individually rational** (**individually stable**) if $\mu(i)$ is individually rational (individually stable) for any $i \in F \cup W$. We say that for any agent $i \in F \cup W$, $j \in M_i$ is **acceptable** if $j \succeq_i \emptyset$. Under substitutable preferences, although an individually stable set contains only acceptable partners, an individually rational set may contain unacceptable partners. The central solution concept in the (two-sided) matching literature is pairwise stability. A matching μ is **pairwise-stable** if (i) for any $i \in F \cup W$, $Ch_i(\mu(i)) = \mu(i)$, i.e. $\mu(i)$ is individually stable, and (ii) for any $i, j \in F \cup W$ with $i \in M_j$, $j \in M_i$, and $j \notin \mu(i)$, we have $j \in Ch_i(\mu(i) \cup \{j\})$ implies $i \notin Ch_j(\mu(j) \cup \{i\})$. For any matching μ, if there is an agent i with $Ch_i(\mu(i)) \neq \mu(i)$, then we say that individual i **blocks** μ, and if there is a firm $f \in F$ and worker $w \in W \backslash \mu(f)$ with $w \in Ch_f(\mu(f) \cup \{w\})$ and $f \in Ch_w(\mu(w) \cup \{f\})$, then we say that pair (f, w) **blocks** μ.

We will introduce two group stability concepts in characteristic function form games. A matching μ' **dominates** a matching μ **via coalition** $T \subseteq F \cup W$ if (i) for all $i \in T$, $j \in \mu'(i)$ implies $j \in T$, and (ii) $\mu'(i) \succ_i \mu(i)$ for all $i \in T$. Condition (i) requires that after deviation, members of T can be matched only with other members of T (characteristic function form game). The **core** of the problem is the set of matchings

[10] Imagine that $f \in F$ has preference ordering $\{w_1, w_2\} \succ_f \{w_1, w_3\} \succ_f \{w_1\} \succ_f \{w_2\} \succ_f \{w_3\} \succ_f \emptyset$. This preference ordering is strongly substitutable, while it is not responsive with quota two. For the other direction, see Example 3 below.

that are not dominated by any other matching. A matching μ' **weakly dominates** a matching μ **via coalition** $T \subseteq F \cup W$ if (i) for any $i \in T$, $j \in \mu'(i)$ implies $j \in T$, (ii) we have $\mu'(i) \succeq_i \mu(i)$ for all $i \in T$, and (iii) $\mu'(i) \succ_i \mu(i)$ holds for some $i \in T$. The **weak core** of the problem is the set of matchings that are not weakly dominated by any other matching.

As we will see below, the characteristic function approach has a limitation in the many-to-many matching problem. Other solution concepts do not assume that deviators need to discontinue all partnerships with outsiders. Let μ be a matching. A matching μ' is **obtainable from** μ **via deviation by** T if for any $i \in F \cup W$ and any $j \in M_i$, (i) $j \in \mu'(i)\backslash\mu(i)$ implies $\{i, j\} \subseteq T$, and (ii) $j \in \mu(i)\backslash\mu'(i)$ implies $\{i, j\} \cap T \neq \emptyset$. A **group deviation** from μ is a group and a matching pair (T, μ') such that (i) μ' is obtainable from μ via T, and (ii) for any $i \in T$ we have $\mu'(i) \succ_i \mu(i)$. We say a matching μ is **group-stable** if μ is immune to any group deviation from μ.[11]

We now discuss two notions of credibility of group deviations. The first notion is setwise stability introduced by Roth [20] and Sotomayor [26]. A group deviation (T, μ') from μ is **individually stable** if μ' is an individually stable matching. A matching μ is **setwise-stable** if μ is immune to any individually stable group deviation. The second notion, which is newly introduced in this paper, is a stronger credibility requirement than setwise stability. A group deviation (T, μ') from μ is **executable** if

(i) for any $i \in T$, $Ch_i(\mu'(i) \cup (\mu(i)\backslash T)) = \mu'(i)$, and

(ii) for any $i, j \in T$ with $j \in M_i\backslash\mu'(i)$, $j \in Ch_i(\mu'(i) \cup (\mu(i)\backslash T) \cup \{j\})$ implies $i \notin Ch_j(\mu'(j) \cup (\mu(j)\backslash T) \cup \{i\})$.

This requires that μ' is pairwise-stable within the members of T assuming that outsiders are passive players. That is, individual stability requires only that no member of T has an incentive to discontinue some of partnerships after a deviation, whereas executability requires that after the deviation, the new matching is pairwise-stable within T assuming that the outsiders are passive agents. A matching μ that is immune to any executable group deviation is called a **credibly group-stable** matching. Credible group stability is a weaker solution than setwise stability, since credibility requirements on group deviations are more demanding in the case of executability.

[11] Group stability is originally defined for many-to-one matching problems (see definition 5.4 in Roth and Sotomayor [23]). We extend this definition to many-to-many matching problems. Group stability is also the same concept as *strong stability* in network games as defined in Jackson and van den Nouweland [11].

2.4 Core and Weak Core

It is well known that in one-to-one matching problems the core and the pairwise-stable set coincide, i.e., the set of pairwise-stable matchings is equivalent to the core and to the weak core. It is also true that in many-to-one matching problems, the set of pairwise-stable matchings and the weak core coincide, although the core may be bigger. This equivalence result no longer holds in many-to-many matching problems. The following simple example (a simplified version of Example 2.6 in Blair [5]) illustrates the difference between the set of pairwise-stable matchings and the weak core in many-to-many matching problems.

Example 1 *Consider a many-to-many matching problem with* $F = \{f_1, f_2\}$ *and* $W = \{w_1, w_2\}$*. Quota for the number of matches for each agent is two. Their preferences are given as follows:*

f_1	f_2	w_1	w_2
$\{w_1\}$	$\{w_2\}$	$\{f_2\}$	$\{f_1\}$
$\{w_1, w_2\}$	$\{w_2, w_1\}$	$\{f_2, f_1\}$	$\{f_1, f_2\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{w_2\}$	$\{w_1\}$	$\{f_1\}$	$\{f_2\}$

In this game, the unique pairwise-stable matching is matching μ *with* $\mu(i) = \emptyset$ *for all* $i \in F \cup W$*, and the unique weak core matching is a complete matching* μ' *with* $\mu'(i) = M_i$ *for all* $i \in F \cup W$*. It is easy to see that empty matching* μ *is the unique pairwise-stable matching, since for each pair* (i, j) *we have either* $\emptyset \succ_i j$ *or* $\emptyset \succ_j i$ *and preferences are responsive with quota 2. It is also easy to see that the complete matching* μ' *is the only weak core matching, since* μ' *is strictly individually rational, and no group deviation can improve upon* μ'*.* □

In many-to-many matching problems, the weak core does not make much sense. This can be seen from the fact that in the above example the weak core matching μ' is not even pairwise-stable. This is because, in the definition of weak core or core, a group deviation T (including a single agent deviation) has to act within T, and the members have to discontinue all the partnerships with members of $(F \cup W)\backslash T$. For example, consider f_1. Under μ', f_1 is matched with w_1 and w_2. She wants to discontinue a partnership with w_2, but wants to keep a partnership with w_1. In the definition of weak core, if f_1 alone wants to deviate, f_1 needs to discontinue all partnerships. But why should w_1 need to discontinue her partnership with f_1 in response to f_1's discontinuing her partnership with w_2? It is not clear, especially because w_1 does not care what happens to a match between f_1 and w_2: there is no such spillover or externality in this game. Actually, this is precisely why the weak core and the core are not the same in many-to-one matching problems even under strict preference orderings. Without including unaffected agents in a group deviation, a pair of agents cannot form a new partnership. However, in the many-to-one matching problems, it is still possible to argue that pairwise stability is a relevant game-theoretic concept, since we can keep the equivalence between the set of pairwise-stable matchings and the weak core. In many-to-many matching problems, the problem with the weak core is more severe, as we have seen. Our observation points out the limitation of describing a matching problem as a characteristic function form game.

Before closing this subsection, we provide an example that has an empty core in a many-to-many matching problem: the core may be empty in the characteristic function form game even under responsive preferences.

Example 2 *Consider a many-to-many matching problem with* $F = \{f_1, f_2, f_3, f_4, f_5\}$ *and* $W = \{w_1, w_2, w_3, w_4, w_5\}$*. Quotas are all two. The preference profile is responsive and given as follows:*

f_1	f_2	f_3	f_4	f_5
$\{w_2, w_3\}$	$\{w_3, w_1\}$	$\{w_1, w_2\}$	$\{w_2\}$	$\{w_1\}$
$\{w_2, w_4\}$	$\{w_3, w_5\}$	$\{w_1\}$	$\{\mathbf{w}_2, \mathbf{w}_1\}$	$\{\mathbf{w}_1, \mathbf{w}_2\}$
$\{w_3, w_4\}$	$\{w_1, w_5\}$	$\{w_2\}$	$\emptyset$	$\emptyset$
$\{w_2\}$	$\{w_3\}$	$\{\mathbf{w}_1, \mathbf{w}_3\}$	$\vdots$	$\vdots$
$\{w_2, w_5\}$	$\{w_3, w_4\}$	$\{\mathbf{w}_2, \mathbf{w}_3\}$		
$\{w_3\}$	$\{w_1\}$	$\emptyset$		
$\{w_3, w_5\}$	$\{w_1, w_4\}$	$\vdots$		
$\{\mathbf{w}_2, \mathbf{w}_1\}$	$\{\mathbf{w}_3, \mathbf{w}_2\}$			
$\{w_4\}$	$\{w_5\}$			
$\{\mathbf{w}_4, \mathbf{w}_5\}$	$\{\mathbf{w}_5, \mathbf{w}_4\}$			
$\{\mathbf{w}_3, \mathbf{w}_1\}$	$\{\mathbf{w}_1, \mathbf{w}_2\}$			
$\emptyset$	$\emptyset$			
$\vdots$	$\vdots$			

w_1	w_2	w_3	w_4	w_5
$\{f_1, f_4\}$	$\{f_2, f_5\}$	$\{f_3\}$	$\{f_2\}$	$\{f_1\}$
$\{f_1\}$	$\{f_2\}$	$\{\mathbf{f}_3, \mathbf{f}_1\}$	$\{\mathbf{f}_2, \mathbf{f}_1\}$	$\{\mathbf{f}_1, \mathbf{f}_2\}$
$\{f_1, f_5\}$	$\{f_2, f_4\}$	$\{\mathbf{f}_3, \mathbf{f}_2\}$	$\emptyset$	$\emptyset$
$\{\mathbf{f}_1, \mathbf{f}_2\}$	$\{\mathbf{f}_2, \mathbf{f}_3\}$	$\emptyset$	$\vdots$	$\vdots$
$\{f_4\}$	$\{f_5\}$	$\vdots$		
$\{\mathbf{f}_4, \mathbf{f}_5\}$	$\{\mathbf{f}_5, \mathbf{f}_4\}$			
$\{\mathbf{f}_1, \mathbf{f}_3\}$	$\{\mathbf{f}_2, \mathbf{f}_1\}$			
$\emptyset$	$\emptyset$			
$\vdots$	$\vdots$			

Choices in bold characters are the relevant choices that compose individually rational matchings. Note that for each $k \in \{1, 2, 3\}$, firm f_k does not want to be matched with $\{w_k\}$, but for each $\ell \in \{1, 2, 3\} \setminus \{k\}$, $\{w_\ell, w_k\}$ is individually rational for f_k. However, for each $k \in \{1, 2, 3\}$, worker w_k wants to be matched with $\{f_k\}$, and for each $\ell \in \{1, 2, 3\} \setminus k$, worker w_k does not mind being matched with $\{f_k, f_\ell\}$ (which is a strictly worse match than $\{f_k\}$), but she does not want to be matched with $\{f_\ell\}$. Note also that firms f_1 and f_2 (workers w_1 and w_2) do not want to be matched with $\{w_5\}$ and $\{w_4\}$ ($\{f_5\}$ and $\{f_4\}$), respectively, but each of them does not mind being matched with the partner set $\{w_4, w_5\}$ ($\{f_4, f_5\}$), although this is a less favorable partner set. We will show that the core of this problem is empty. Inspecting individually rational matchings will be sufficient for determining the core, since a core matching is individually rational. There are nine individually rational matchings ($\mu_1, ..., \mu_9$) in this example.[12] *We list them as follows:*

$\mu_1(f_1) = \{w_2, w_1\}, \mu_1(f_2) = \{w_1, w_2\}, \mu_1(f_3) = \mu_1(f_4) = \mu_1(f_5) = \emptyset;$
$\mu_2(f_2) = \{w_3, w_2\}, \mu_2(f_3) = \{w_2, w_3\}, \mu_2(f_1) = \mu_2(f_4) = \mu_2(f_5) = \emptyset;$
$\mu_3(f_1) = \{w_3, w_1\}, \mu_3(f_3) = \{w_1, w_3\}, \mu_3(f_2) = \mu_3(f_4) = \mu_3(f_5) = \emptyset;$
$\mu_4(f_1) = \{w_3, w_1\}, \mu_4(f_2) = \{w_1, w_2\}, \mu_4(f_3) = \{w_2, w_3\}, \mu_4(f_4) = \mu_4(f_5) = \emptyset;$
$\mu_5(f_1) = \{w_2, w_1\}, \mu_5(f_2) = \{w_3, w_2\}, \mu_5(f_3) = \{w_1, w_3\}, \mu_5(f_4) = \mu_5(f_5) = \emptyset;$
$\mu_6(f_1) = \{w_4, w_5\}, \mu_6(f_2) = \{w_5, w_4\}, \mu_6(f_3) = \mu_6(f_4) = \mu_6(f_5) = \emptyset;$
$\mu_7(f_4) = \{w_2, w_1\}, \mu_7(f_5) = \{w_1, w_2\}, \mu_7(f_1) = \mu_7(f_2) = \mu_7(f_3) = \emptyset;$
$\mu_8(f_1) = \{w_4, w_5\}, \mu_8(f_2) = \{w_5, w_4\}, \mu_8(f_3) = \emptyset, \mu_8(f_4) = \{w_2, w_1\}, \mu_8(f_5) = \{w_1, w_2\};$
$\mu_9(f_1) = \mu_9(f_2) = \mu_9(f_3) = \mu_9(f_4) = \mu_9(f_5) = \emptyset;$

None of the above matchings is in the core, although matching μ_9 is the unique

[12] The proof is available upon request.

pairwise-stable matching. For each individually rational matching, there is a matching that dominates it via a coalition: $\mu_1 \rightarrow_{\{f_2,f_3,w_2,w_3\}} \mu_2$, $\mu_2 \rightarrow_{\{f_1,f_3,w_1,w_3\}} \mu_3$, $\mu_3 \rightarrow_{\{f_1,f_2,w_1,w_2\}} \mu_1$, $\mu_4 \rightarrow_{\{f_1,f_2,w_4,w_5\}} \mu_6$, $\mu_5 \rightarrow_{\{f_4,f_5,w_1,w_2\}} \mu_7$, $\mu_6 \rightarrow_{\{f_1,f_2,f_3,w_1,w_2,w_3\}} \mu_5$, $\mu_7 \rightarrow_{\{f_1,f_2,f_3,w_1,w_2,w_3\}} \mu_4$, $\mu_8 \rightarrow_{\{f_2,f_3,w_2,w_3\}} \mu_2$, *and* μ_9 *is dominated by any other individually rational matching via the coalition of matched agents. Thus, the core (and the weak core) is empty.* □

2.5 Group Stability, Setwise Stability, and Credible Group Stability

The main problem of using a solution concept in a characteristic function form game is that the ability of a coalition is limited to the set of matchings within the coalition. Group deviations give more power to deviators by allowing them to keep existing partnerships if they wish.

Although group stability is a natural concept, unfortunately, the set of group-stable matchings may be empty in many-to-many matching problems. It is indeed empty in Example 1, although it is a very simple setup. A pair $(F \cup W, \mu')$ is a group deviation from the unique pairwise-stable matching μ, and since a group-stable matching must be pairwise-stable, there is no group-stable matching in this problem. Thus, we need to discuss credibility of group deviations (see Section 2.3 for definitions).

It is easy to see that the group deviation $(F \cup W, \mu')$ from μ is not individually stable: agents are matched with unacceptable partners. This implies that, in Example 1, the unique pairwise-stable matching is setwise-stable, and we can get around the nonexistence problem of a group-stable matching. However, it is not always the case under responsive preferences. The following example (a simplified version of Example 3 in Sotomayor [26]) illustrates the difference between executability and individual stability.

Example 3 *Consider the following many-to-many matching problem. Quotas are all two. Let* $F = \{f_1, f_2, f_3, f_4\}$ *and* $W = \{w_1, w_2, w_3, w_4\}$ *with responsive preferences* *stated as*

f_1	f_2	f_3	f_4	w_1	w_2	w_3	w_4
$\{w_1, w_3\}$	$\{w_2, w_3\}$	$\{\mathbf{w}_1, \mathbf{w}_2\}$	$\{\mathbf{w}_1, \mathbf{w}_2\}$	$\{f_2, f_3\}$	$\{f_1, f_3\}$	$\{\mathbf{f}_1, \mathbf{f}_2\}$	$\{\mathbf{f}_1, \mathbf{f}_2\}$
$\{w_1, w_4\}$	$\{w_2, w_4\}$	$\{w_1\}$	$\{w_1\}$	$\{f_2, f_4\}$	$\{f_1, f_4\}$	$\{f_1\}$	$\{f_1\}$
$\boxed{\{w_1, w_2\}}$	$\boxed{\{w_2, w_1\}}$	$\{w_2\}$	$\{w_2\}$	$\boxed{\{f_2, f_1\}}$	$\boxed{\{f_1, f_2\}}$	$\{f_2\}$	$\{f_2\}$
$\{\mathbf{w}_3, \mathbf{w}_4\}$	$\{\mathbf{w}_3, \mathbf{w}_4\}$	$\emptyset$	$\emptyset$	$\{\mathbf{f}_3, \mathbf{f}_4\}$	$\{\mathbf{f}_3, \mathbf{f}_4\}$	$\emptyset$	$\emptyset$
$\{w_3, w_2\}$	$\{w_3, w_1\}$	$\vdots$	$\vdots$	$\{f_3, f_1\}$	$\{f_3, f_2\}$	$\vdots$	$\vdots$
$\{w_4, w_2\}$	$\{w_4, w_1\}$			$\{f_4, f_1\}$	$\{f_4, f_2\}$		
$\{w_1\}$	$\{w_2\}$			$\{f_2\}$	$\{f_1\}$		
$\{w_3\}$	$\{w_3\}$			$\{f_3\}$	$\{f_3\}$		
$\{w_4\}$	$\{w_4\}$			$\{f_4\}$	$\{f_4\}$		
$\{w_2\}$	$\{w_1\}$			$\{f_1\}$	$\{f_2\}$		
$\emptyset$	$\emptyset$			$\emptyset$	$\emptyset$		
$\vdots$	$\vdots$			$\vdots$	$\vdots$		

The unique pairwise-stable matching μ is described by bold characters in the above table. Now consider a group deviation (T, μ') from μ with $T = \{f_1, f_2, w_1, w_2\}$ and μ' fully matched up within T only (in rectangles in the above table). This is beneficial for each agent in T, and it blocks μ. Moreover, since all partners of deviators are individually stable and preferences are responsive, (T, μ') is an individually stable deviation from μ, in turn implying that there is no setwise-stable matching in this example.[13] *In contrast, μ' is not pairwise-stable with passive outsiders, since, say, agent f_1 follows the suggested deviation plan only partially. She is willing to establish partnerships with w_1, yet she would not be willing to establish her partnership with w_2: instead, she keeps her partnership with w_3. Thus, it can be shown that the unique pairwise-stable matching μ is also a credibly group-stable matching.* $\square$

In the next section, we investigate credibly group-stable matchings under various preference restrictions.

[13] Note that preferences in this example (and the one in Sotomayor 1999) do not satisfy strong substitutability; thus non-existence of a setwise-stable matching does not contradict Echenique and Oviedo's (2003) equivalence result. For example, let $S = \{w_1, w_2\}$ and $T = \{w_3, w_4\}$. Although $S \succ_{f_1} T$ and $w_2 \in Ch_{f_1}(S \cup \{w_2\}) = \{w_1, w_2\}$, we have $w_2 \notin Ch_{f_1}(T \cup \{w_2\}) = \{w_3, w_4\}$.

3 The Results

3.1 Substitutable Preferences

The first result shows that credible group stability implies pairwise stability under substitutable preferences.

Proposition 1 *Every credibly group-stable matching is pairwise-stable, when $\succeq$ is substitutable.*

Proof. We prove the contrapositive of the statement. Let $\succeq$ be substitutable and μ be a pairwise-unstable matching. There are two possibilities: (i) there exists $i \in F \cup W$ with $Ch_i(\mu(i)) \neq \mu(i)$, or (ii) there is a pair $(f, w) \in F \times W$ such that $w \in Ch_f(\mu(f) \cup \{w\})$ and $f \in Ch_w(\mu(w) \cup \{f\})$. We inspect these two cases separately:
Case (i): A deviation $(\{i\}, \mu')$ with $\mu'(i) = Ch_i(\mu(i)) \subset \mu(i)$ is executable, since agent i has no incentive to recover any of the discontinued partnerships in μ. Hence, μ is not credibly group-stable.
Case (ii): Since Case (i) does not hold, μ is an individually stable matching. Let $\mu'(f) = Ch_f(\mu(f) \cup \{w\})$, $\mu'(w) = Ch_w(\mu(w) \cup \{f\})$, $\mu'(w') = \mu(w') \backslash \{f\}$ for any worker $w' \in W \backslash Ch_f(\mu(f) \cup \{w\})$, and $\mu'(f') = \mu(f') \backslash \{w\}$ for any firm $f' \in F \backslash Ch_w(\mu(w) \cup \{f\})$. Then group deviation $(\{f, w\}, \mu')$ from μ is executable, since agents f and w have no incentive to recover any partnership that was discontinued in μ or remain single. Hence, μ is not credibly group-stable, completing the proof. $\square$

However, there may be a pairwise-stable matching that is not credibly group-stable even when one side has responsive preferences and the other side has substitutable preferences, as the following example shows.

Example 4 *Consider the following 16-agent many-to-many matching problem. Let*

$$F = \{f_1, f_2, f_3, f_4, \bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4\} \text{ and } W = \{w_1, w_2, w_3, w_4, \bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4\}.$$

Each firm has responsive preferences described as follows: each firm without a bar has quota 3, and her preferences are lexicographic in the order of the ranking of individual partners, that is, for example, for f_1, $\{w_1\}$ is more preferable than $\{\bar{w}_2, \bar{w}_3, \bar{w}_4\}$. Each *firm with a bar has quota 1. Firm preferences over individual partners are as follows:*

f_1	f_2	f_3	f_4	$\bar{f}_1$	$\bar{f}_2$	$\bar{f}_3$	$\bar{f}_4$
$\boxed{w_1}$	$\boxed{w_2}$	$\boxed{w_3}$	$\boxed{w_4}$	$\boxed{w_1}$	$\boxed{w_2}$	$\boxed{w_3}$	$\boxed{w_4}$
$\bar{\mathbf{w}}_2$	$\bar{\mathbf{w}}_3$	$\bar{\mathbf{w}}_4$	$\bar{\mathbf{w}}_1$	$\mathbf{w}_2$	$\mathbf{w}_1$	$\mathbf{w}_4$	$\mathbf{w}_3$
$\bar{\mathbf{w}}_3$	$\bar{\mathbf{w}}_4$	$\bar{\mathbf{w}}_1$	$\bar{\mathbf{w}}_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\bar{\mathbf{w}}_4$	$\bar{\mathbf{w}}_1$	$\bar{\mathbf{w}}_2$	$\bar{\mathbf{w}}_3$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\boxed{\bar{w}_1}$	$\boxed{\bar{w}_2}$	$\boxed{\bar{w}_3}$	$\boxed{\bar{w}_4}$				
$\boxed{w_2}$	$\boxed{w_1}$	$\boxed{w_4}$	$\boxed{w_3}$				
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$				

Workers have substitutable preferences. Their preferences are stated as follows:

$$\{f_2, \bar{f}_2\} \succ_{w_1} \boxed{\{f_2, f_1, \bar{f}_1\}} \succ_{w_1} \{f_2, f_1\} \succ_{w_1} \{f_2, \bar{f}_1\} \succ_{w_1} \{f_2\} \succ_{w_1}$$
$$\{\bar{\mathbf{f}}_2\} \succ_{w_1} \{f_1, \bar{f}_1\} \succ_{w_1} \{f_1\} \succ_{w_1} \{\bar{f}_1\} \succ_{w_1} \emptyset \succ_{w_1} ...,$$

$$\{f_1, \bar{f}_1\} \succ_{w_2} \boxed{\{f_1, f_2, \bar{f}_2\}} \succ_{w_2} \{f_1, f_2\} \succ_{w_2} \{f_1, \bar{f}_2\} \succ_{w_2} \{f_1\} \succ_{w_2}$$
$$\{\bar{\mathbf{f}}_1\} \succ_{w_2} \{f_2, \bar{f}_2\} \succ_{w_2} \{f_2\} \succ_{w_2} \{\bar{f}_2\} \succ_{w_2} \emptyset \succ_{w_2} ...,$$

$$\{f_4, \bar{f}_4\} \succ_{w_3} \boxed{\{f_4, f_3, \bar{f}_3\}} \succ_{w_3} \{f_4, f_3\} \succ_{w_3} \{f_4, \bar{f}_3\} \succ_{w_3} \{f_4\} \succ_{w_3}$$
$$\{\bar{\mathbf{f}}_4\} \succ_{w_3} \{f_3, \bar{f}_3\} \succ_{w_3} \{f_3\} \succ_{w_3} \{\bar{f}_3\} \succ_{w_3} \emptyset \succ_{w_3} ...,$$

$$\{f_3, \bar{f}_3\} \succ_{w_4} \boxed{\{f_3, f_4, \bar{f}_4\}} \succ_{w_4} \{f_3, f_4\} \succ_{w_4} \{f_3, \bar{f}_4\} \succ_{w_4} \{f_3\} \succ_{w_4}$$
$$\{\bar{\mathbf{f}}_3\} \succ_{w_4} \{f_4, \bar{f}_4\} \succ_{w_4} \{f_4\} \succ_{w_4} \{\bar{f}_4\} \succ_{w_4} \emptyset \succ_{w_4} ...,$$

$$\boxed{\{f_1\}} \succ_{\bar{w}_1} \{\mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4\} \succ_{\bar{w}_1} \{f_2, f_3\} \succ_{\bar{w}_1} \{f_2, f_4\} \succ_{\bar{w}_1} \{f_2\} \succ_{\bar{w}_1}$$
$$\{f_3, f_4\} \succ_{\bar{w}_1} \{f_3\} \succ_{\bar{w}_1} \{f_4\} \succ_{\bar{w}_1} \emptyset \succ_{\bar{w}_1} ...,$$

$$\boxed{\{f_2\}} \succ_{\bar{w}_2} \{\mathbf{f}_3, \mathbf{f}_4, \mathbf{f}_1\} \succ_{\bar{w}_2} \{f_3, f_4\} \succ_{\bar{w}_2} \{f_3, f_1\} \succ_{\bar{w}_2} \{f_3\} \succ_{\bar{w}_2}$$
$$\{f_4, f_1\} \succ_{\bar{w}_2} \{f_4\} \succ_{\bar{w}_2} \{f_1\} \succ_{\bar{w}_2} \emptyset \succ_{\bar{w}_2} ...,$$

$$\boxed{\{f_3\}} \succ_{\bar{w}_3} \{\mathbf{f}_4, \mathbf{f}_1, \mathbf{f}_2\} \succ_{\bar{w}_3} \{f_4, f_1\} \succ_{\bar{w}_3} \{f_4, f_2\} \succ_{\bar{w}_3} \{f_4\} \succ_{\bar{w}_3}$$
$$\{f_1, f_2\} \succ_{\bar{w}_3} \{f_1\} \succ_{\bar{w}_3} \{f_2\} \succ_{\bar{w}_3} \emptyset \succ_{\bar{w}_3} ...,$$

$$\boxed{\{f_4\}} \succ_{\bar{w}_4} \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} \succ_{\bar{w}_4} \{f_1, f_2\} \succ_{\bar{w}_4} \{f_1, f_3\} \succ_{\bar{w}_4} \{f_1\} \succ_{\bar{w}_4}$$
$$\{f_2, f_3\} \succ_{\bar{w}_4} \{f_2\} \succ_{\bar{w}_4} \{f_3\} \succ_{\bar{w}_4} \emptyset \succ_{\bar{w}_4} \ldots .$$

Given this preference profile, a matching μ that matches each agent with the partners in bold characters in the above tables is a pairwise-stable matching. However, a matching μ' that matches each agent with the partners in rectangles in the above tables is also a pairwise-stable matching. Matching μ' Pareto-dominates μ and μ' is pairwise-stable in $F \cup W$ together imply that group deviation $(F \cup W, \mu')$ from μ is executable. □

Note that in this example, the number of partners of an agent can differ in different pairwise-stable matchings. This is one of the properties that do not hold under substitutability in many-to-one matching problems, unlike responsiveness.[14]

3.2 Responsive and Categorywise-Responsive Preferences

In the last subsection, we observed that equivalence between pairwise stability and credible group stability cannot be obtained when the preference profile is substitutable. Example 4 showed that this result is true even if one side has a responsive preference profile. However, in the UK markets, matching mechanisms utilize students' preference orderings over individual consultants in each category and consultants' preference orderings over individual students. Thus given the usage of these mechanisms, the simplest assumptions on preference domains are that students' preference profile is categorywise-responsive, and that consultants' preference profile is responsive. Thus, it appears to be important to investigate pairwise stability in this domain. Throughout this subsection, *we assume that F has responsive preferences and W has categorywise-responsive preferences.*

We introduce one more piece of notation. For any agent $i \in F \cup W$, and any $S \subseteq M_i$, let $\beta_i(S) \in S$ be such that $j \succeq_i \beta_i(S)$ for all $j \in S$; i.e., β_i selects *the least preferable element in the set of partners.*

Using β_i, it is easy to see that we can state the following lemma about pairwise-stable matchings and executable deviations in this domain.

[14] Martinez, Masso, Neme, and Oviedo [15] show that the set of single agents may not be the same in pairwise stable matchings in a college admissions problem (many-to-one matching problem) under substitutability, while Roth [17] shows it is the case under responsiveness (a.k.a. rural hospital theorem). This phenomenon of substitutable preferences seems to play an important role in our counterexample, too. See also Hatfield and Milgrom [10] for an extensive discussion of many-to-one matching problems by using an integrating approach.

Lemma 1 *When $\succeq_F$ is responsive with quotas $(q_f)_{f\in F}$, and $\succeq_W$ is categorywise-responsive with categories and quotas being $(K_w,(M_w^k,q_w^k)_{k\in K_w})_{w\in W}$, we have the following:*

(1) A matching μ is ***pairwise-stable*** *if and only if*

(a) (respecting quotas)

(i) for any $f\in F$, $|\mu(f)|\leq q_f$, and

(ii) for any $w\in W$ and any $k\in K_w$, $|\mu(w)\cap M_w^k|\leq q_w^k$;

(b) (no blocking individual) for any $i\in T$, $\beta_i(\mu(i))\succ_i\emptyset$; and

(c) (no blocking pair) for any pair $(f,w)\in F\times W$ with $f\in M_w^k\backslash\mu(w)$ for some $k\in K_w$,

(A) $\emptyset\succ_f w$, or $\beta_f(\mu(f))\succ_f w$ with $|\mu(f)|=q_f$, or

(B) $\emptyset\succ_w f$, or $\beta_w(\mu(w)\cap M_w^k)\succ_w f$ with $|\mu(w)\cap M_w^k|=q_w^k$.

(2) For each matching μ, a group deviation (T,μ') from μ is ***executable*** *if and only if*

(a) (respecting quotas)

(i) for any $f\in F\cap T$, $|\mu'(f)|\leq q_f$, and

(ii) for any $w\in W\cap T$ and any $k\in K_w$, $|\mu'(w)\cap M_w^k|\leq q_w^k$;

(b) (no blocking individual among insiders possibly with passive outsiders)

(i) for any $i\in T$, $\beta_i(\mu'(i))\succ_i\emptyset$,

(ii) for any $f\in F\cap T$, and any $w\in\mu(f)\setminus(T\cup\mu'(f))$, $\emptyset\succ_f w$, or $\beta_f(\mu'(f))\succ_f w$ with $|\mu'(f)|=q_f$, and

(iii) for any $w\in W\cap T$, any $k\in K_w$, and any $f\in\left(\mu(w)\cap M_w^k\right)\setminus(T\cup\mu'(w))$, $\emptyset\succ_w f$, or $\beta_w\left(\mu'(w)\cap M_w^k\right)\succ_w f$ with $|\mu'(w)\cap M_w^k|=q_w^k$; and

(c) (no blocking pair among insiders) for any pair $(f,w)\in(F\cap T)\times(W\cap T)$ with $f\in\left(T\cap M_w^k\right)\backslash\mu'(w)$ for some $k\in K_w$,

(A) $\emptyset\succ_f w$, or $\beta_f(\mu'(f))\succ_f w$ with $|\mu'(f)|=q_f$, or

(B) $\emptyset\succ_w f$, or $\beta_w(\mu'(w)\cap M_w^k)\succ_w f$ with $|\mu'(w)\cap M_w^k|=q_w^k$.

Since the proof of Lemma 1 is immediate from the definitions of pairwise stability, executability, responsiveness, and categorywise responsiveness, we skip it. The first main result of this paper is as follows:

Theorem 1 *The set of pairwise-stable matchings is equivalent to the set of credibly group-stable matchings, when $\succeq_F$ is responsive, and $\succeq_W$is categorywise-responsive.*

Proof. One direction has been proved in Proposition 1 under substitutable preferences. Thus, we will prove that every pairwise-stable matching is credibly group-stable, when $\succeq_F$ is responsive, and $\succeq_W$is categorywise-responsive. Let $\succeq_F$ be responsive with quotas $(q_f)_{f\in F}$, and $\succeq_W$ is categorywise-responsive with categories and quotas given by $(K_w, (M_w^k, q_w^k)_{k\in K_w})_{w\in W}$. We prove this direction by contradiction. Suppose that μ is a pairwise-stable matching and that (T, μ') is an executable group deviation from μ. This supposition will be made throughout the proof.

First, we investigate the properties of newly created partnerships. Note that for any $f \in F$ and $w \in W$ with $f \in \left(\mu'(w) \cap M_w^k\right) \backslash \mu(w)$ for some $k \in K_w$ (a new partner), we have $f, w \in T$, since μ' is obtainable from μ. Moreover, since (T, μ') is executable, for these f and w, we have $w \succ_f \emptyset$ and $f \succ_w \emptyset$ by Condition 2-b-i of Lemma 1. We first prove the following claims.

Claim 1: For any $w \in W$, $k \in K_w$ and $f \in \left(\mu'(w) \cap M_w^k\right) \backslash \mu(w)$, either $\beta_f\left(\mu(f)\right) \succ_f w$ or $\beta_w\left(\mu(w) \cap M_w^k\right) \succ_w f$.

Proof of Claim 1. We prove the claim by contradiction. Suppose there are $w \in W$ and $f \in \left(\mu'(w) \cap M_w^k\right) \backslash \mu(w)$ for some $k \in K_w$ such that $w \succ_f \beta_f\left(\mu(f)\right)$ and $f \succ_w \beta_w\left(\mu(w) \cap M_w^k\right)$. Since (T, μ') is an executable deviation from μ, by Condition 2-b-i of Lemma 1, we have $w \succ_f \emptyset$ and $f \succ_w \emptyset$. By Condition 1-c of Lemma 1 the last two statements imply that μ is pairwise-unstable, that is because (f, w) blocks μ, contradicting that μ is pairwise-stable. Therefore, such agents f and w do not exist.$\diamondsuit$

Claim 2: For any $w \in W$, $k \in K_w$ and $f \in \left(\mu'(w) \cap M_w^k\right) \backslash \mu(w)$, either $\beta_f\left(\mu(f)\right) \succ_f w$ with $|\mu(f)| = q_f$ or $\beta_w\left(\mu(w) \cap M_w^k\right) \succ_w f$ with $|\mu(w) \cap M_w^k| = q_w^k$.

Proof of Claim 2. Let $w \in W$, $k \in K_w$ and $f \in \left(\mu'(w) \cap M_w^k\right) \backslash \mu(w)$. Since (T, μ') is an executable deviation from μ, by Condition 2-b-i of Lemma 1 we have $w \succ_f \emptyset$ and $f \succ_w \emptyset$. By Claim 1, either $\beta_f\left(\mu(f)\right) \succ_f w$ or $\beta_w\left(\mu(w) \cap M_w^k\right) \succ_w f$. First consider $\beta_f\left(\mu(f)\right) \succ_f w$. There are two cases: $|\mu(f)| = q_f$ or $|\mu(f)| < q_f$:
Case 1. $|\mu(f)| = q_f$: Then the proof of Claim 2 is complete.
Case 2. $|\mu(f)| < q_f$: Since μ is pairwise-stable, there are no blocking pairs. In particular, (f, w) cannot block μ. Since $|\mu(f)| < q_f$, $w \succ_f \emptyset$, and $f \succ_w \emptyset$, we have $\beta_w\left(\mu(w) \cap M_w^k\right) \succ_w f$ and $\left|\mu(w) \cap M_w^k\right| = q_w^k$ by Condition 1-c of Lemma 1.

The case with $\beta_w\left(\mu(w) \cap M_w^k\right) \succ_w f$ can be dealt with in a similar manner. $\diamondsuit$

Claim 2 allows us to introduce a new concept. For any worker w, any of her categories k and any firm $f \in \left(M_w^k \cap \mu'(w)\right) \backslash \mu(w)$, we say that firm f **is pointed by** worker w if $\beta_f(\mu(f)) \succ_f w$ and $|\mu(f)| = q_f$; and that worker w **is pointed by** firm f if $\beta_w(\mu(w) \cap M_w^k) \succ_w f$ and $|\mu(w) \cap M_w^k| = q_w^k$. *Claim 2 says that in any newly created partnership, there is always an agent who is pointed by the other.* Let P_F be the set of pointed firms, i.e.

$$P_F = \{f \in F : \exists w \in \mu'(f) \backslash \mu(f) \text{ such that } \beta_f(\mu(f)) \succ_f w \text{ and } |\mu(f)| = q_f\}.$$

For any $f \in P_F$, since there exists some $w \in \mu'(f) \backslash \mu(f)$, pair (f, w) is a newly created partnership, and $f, w \in T$ must hold.

Claim 3: If a firm f is pointed by $r \geq 1$ workers, then $|\mu(f) \backslash \mu'(f)| > r$.

Proof of Claim 3. Let firm f be pointed by r workers $w_1, w_2, ..., w_r$. This implies that $\beta_f(\mu(f)) \succ_f w_h$ for all $h \in \{1, ..., r\}$ and firm f's quota q_f is binding under μ. The latter statement implies that firm f needs to discontinue partnerships with at least r incumbent partners (each of whom is more preferable than $w_1, w_2, ..., w_r$) in order to have new partnerships with $w_1, w_2, ..., w_r$. Since $\mu'(f) \succ_f \mu(f)$ and $\succeq_f$ is responsive with quota q_f, there should be at least one more new partner $w' \in \mu'(f) \backslash \mu(f)$ such that $w' \succ_f \beta_f(\mu(f))$ for firm f to be compensated. Hence, firm f establishes at least $r + 1$ new partnerships. Since firm f's quota is binding under μ, firm f must discontinue strictly more than r old partnerships to create room for these new partners under μ'.$\diamondsuit$

This claim simply says that if a firm is pointed by r workers, then she needs to discontinue at least one additional partnership to improve her situation.

Claim 4: Let $f \in P_F$ and $w \in \mu(f) \backslash \mu'(f)$ be such that $f \in M_w^k$ for some $k \in K_w$ (i.e., partnership (f, w) is discontinued). Then (i) $w \in T$, (ii) $|\mu'(w) \cap M_w^k| = q_w^k$, and (iii) $\beta_w(\mu'(w) \cap M_w^k) \succ_w f$.

Proof of Claim 4. Let firm $f \in P_F$ be pointed by worker $w' \in T$ and let worker $w \in \mu(f) \backslash \mu'(f)$ be such that $f \in M_w^k$ for some $k \in K_w$, that is, partnership (f, w) is discontinued by the group deviation (T, μ'). Since f is pointed by w', $\beta_f(\mu(f)) \succ_f w'$. Since $w' \in \mu'(f)$ and $w \in \mu(f) \backslash \mu'(f)$, we have $w \succ_f \beta_f(\mu'(f))$. We prove each part separately:

(i) Suppose that $w \notin T$. This implies that $w \in \mu(f) \backslash (T \cup \mu'(f))$. This together with $w \succ_f \beta_f(\mu'(f))$ contradicts executability of (T, μ') by Condition 2-b-ii of Lemma 1. Therefore $w \in T$.

(ii) Suppose that $|\mu'(w) \cap M_w^k| < q_w^k$. Since μ is pairwise-stable, we have $f \succ_w \emptyset$ by Condition 1-b-ii of Lemma 1. Since $f, w \in T$ (see (i)), this together with $w \succ_f \beta_f(\mu'(f))$ contradicts the executability of (T, μ') by Condition 2-c of Lemma 1. Therefore $|\mu'(w) \cap M_w^k| = q_w^k$.

(iii) Suppose that $f \succ_w \beta_w\left(\mu'(w) \cap M_w^k\right)$. Since $f, w \in T$ (see (i)), this together with $w \succ_f \beta_f\left(\mu'(f)\right)$ contradicts the executability of (T, μ') by Condition 2-c of Lemma 1. Therefore $\beta_w\left(\mu'(w) \cap M_w^k\right) \succ_w f$. $\diamondsuit$

We define one more new concept. Let D_W be the set of workers each of whom has discontinued at least one partnership in some category with some firm in P_F, i.e.

$$D_W = \{w \in W : \exists f \in \left(\mu(w) \cap M_w^k \cap P_F\right) \backslash \mu'(w) \text{ for some } k \in K_w\}.$$

By Claim 4 (i), it immediately follows that $D_W \subset T$.

Claim 5: Let $w \in D_W$. If w has discontinued $r \geq 1$ partnerships with firms in P_F in category $k \in K_w$: i.e.

$$\left|\left(\mu(w) \cap M_w^k \cap P_F\right) \backslash \mu'(w)\right| = r,$$

then there are at least r firms in P_F who are pointed by worker w in category k.

Proof of Claim 5. Let $w \in D_W$ be such that she has discontinued $r \geq 1$ partnerships with firms in P_F in category $k \in K_w$. Pick any $f' \in \left(\mu'(w) \cap M_w^k\right) \backslash \mu(w)$. We will show that $f' \in P_F$. Let $f \in \left(\mu(w) \cap M_w^k \cap P_F\right) \backslash \mu'(w)$, that is, firm f is one of the firms in P_F that worker w discontinued partnerships in category k. By Claim 4 (iii), we have $\beta_w\left(\mu'(w) \cap M_w^k\right) \succ_w f$. Since $f' \in \mu'(w) \cap M_w^k$ and $f \in \mu(w) \cap M_w^k$, it follows that $f' \succ_w \beta_w\left(\mu(w) \cap M_w^k\right)$. By Claim 2, we have either (i) $\beta_{f'}\left(\mu(f')\right) \succ_{f'} w$ with $|\mu(f')| = q_{f'}$, or (ii) $\beta_w\left(\mu(w) \cap M_w^k\right) \succ_w f'$ with $|\mu(w) \cap M_w^k| = q_w^k$. Obviously, (ii) does not hold in this case, and (i) follows. Thus, f' is pointed by w, and $f' \in P_F$. Since f' is picked arbitrarily in $\left(\mu'(w) \cap M_w^k\right) \backslash \mu(w)$, every firm in $\left(\mu'(w) \cap M_w^k\right) \backslash \mu(w)$ is pointed by w. By Claim 4 (ii), we have $\left|\mu'(w) \cap M_w^k\right| = q_w^k$. Since w has discontinued r partnerships with firms in $M_w^k \cap P_F$, she must form at least r partnerships as well. Thus, there must be at least r firms in $M_w^k \cap P_F$ that are pointed by w. $\diamondsuit$

Claim 6: The set P_F is non-empty.

Proof of Claim 6. Since (T, μ') is a group deviation from μ, and μ is pairwise-stable (and thus cannot be blocked by an individual), $T \cap W \neq \emptyset$, and for any $w \in W \cap T$, $\mu'(w) \backslash \mu(w) \neq \emptyset$. Suppose that $P_F = \emptyset$. Then, for any $w \in W \cap T$, and any $f \in \mu'(w) \backslash \mu(w)$, w is pointed by f in some category $k \in K_w$ by Claim 2, and thus $\beta_w\left(\mu(w) \cap M_w^k\right) \succ_w f$. This implies $\mu(w) \succ_w \mu'(w)$ by categorywise responsiveness of $\succeq_w$, contradicting (T, μ') is a group deviation from μ. Thus, P_F is non-empty. $\diamondsuit$

We now are ready to complete the proof of the theorem. Set P_F is non-empty by Claim 6. Let $r \geq 1$ be the number of partnerships that have been discontinued by firms in P_F. By the definition of D_W, these discontinued partnerships are with workers in D_W. By Claim 5, workers in D_W who discontinued r partnerships with firms in P_F would establish at least r new partnerships with firms in P_F by pointing them. By Claim 3, those pointed firms in P_F should have discontinued at least $r+1$ partnerships. This is a contradiction. Therefore (T, μ') cannot be executable. $\square$

It is important to have no category in the preferences of one side (here F). If both sides have categorywise-responsive preference profiles, the equivalence between pairwise stability and credible group stability does not hold (since a symmetric argument of Claim 3 is not valid for set W, which has a categorywise-responsive preference profile: agent w's loss in a category may be compensated by a gain in another category). Indeed, the following example shows that our result is no longer true when both sides have categorywise-responsive preferences.[15]

Example 5 *Consider a many-to-many matching problem with $F = \{f_1, f_2, f_3, f_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. There are two categories for each agent, and the partner set in each category is given as odd-indexed partners for the first category and even-indexed partners for the second category. Each agent has a unit quota for each category. The preferences are categorywise-responsive and stated as follows:*

f_1	f_2	f_3	f_4	w_1	w_2	w_3	w_4
$\{w_3, w_2\}$	$\{w_1, w_4\}$	$\{w_1, w_4\}$	$\{w_3, w_2\}$	$\{f_1, f_4\}$	$\{f_3, f_2\}$	$\{f_3, f_2\}$	$\{f_1, f_4\}$
$\boxed{\{w_3, w_4\}}$	$\boxed{\{w_3, w_4\}}$	$\boxed{\{w_1, w_2\}}$	$\boxed{\{w_1, w_2\}}$	$\boxed{\{f_3, f_4\}}$	$\boxed{\{f_3, f_4\}}$	$\boxed{\{f_1, f_2\}}$	$\boxed{\{f_1, f_2\}}$
$\{\mathbf{w}_1, \mathbf{w}_2\}$	$\{\mathbf{w}_1, \mathbf{w}_2\}$	$\{\mathbf{w}_3, \mathbf{w}_4\}$	$\{\mathbf{w}_3, \mathbf{w}_4\}$	$\{\mathbf{f}_1, \mathbf{f}_2\}$	$\{\mathbf{f}_1, \mathbf{f}_2\}$	$\{\mathbf{f}_3, \mathbf{f}_4\}$	$\{\mathbf{f}_3, \mathbf{f}_4\}$
$\{w_1, w_4\}$	$\{w_3, w_2\}$	$\{w_3, w_2\}$	$\{w_1, w_4\}$	$\{f_3, f_2\}$	$\{f_1, f_4\}$	$\{f_1, f_4\}$	$\{f_3, f_2\}$
$\{w_3\}$	$\{w_4\}$	$\{w_1\}$	$\{w_2\}$	$\{f_4\}$	$\{f_3\}$	$\{f_2\}$	$\{f_1\}$
$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$	$\{f_2\}$	$\{f_1\}$	$\{f_4\}$	$\{f_3\}$
$\{w_2\}$	$\{w_1\}$	$\{w_4\}$	$\{w_3\}$	$\{f_1\}$	$\{f_2\}$	$\{f_3\}$	$\{f_4\}$
$\{w_4\}$	$\{w_3\}$	$\{w_2\}$	$\{w_1\}$	$\{f_3\}$	$\{f_4\}$	$\{f_1\}$	$\{f_2\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Let μ be a matching described by bold characters, and let μ' be a matching described by rectangles. Both of them are pairwise-stable matchings. Moreover, $\mu'(i) \succ_i \mu(i)$ for all $i \in F \cup W$. Therefore, $(F \cup W, \mu')$ is an executable group deviation from μ. $\square$

A slightly modified version of the above example shows that there may not exist a credibly group-stable matching when both sides have categorywise-responsive preferences (see Example 6 in the appendix.)

[15] This insightful example has been suggested by a referee.

4 Strategic-Form Games

We can rewrite our matching problem as a strategic-form game in which each agent is a player, each player simultaneously announces a subset of players she wants to be matched with, and a match is made if and only if each of a pair of players announces each other's name. Here, we show that this game is useful to clarify the relationships among the notions of stable matchings in matching problems. A **strategic-form game** is a list $G(F \cup W) = (F \cup W, (S_i, u_i)_{i \in F \cup W})$, where for any player $i \in F \cup W$, her strategy set is $S_i = 2^{M_i}$, and her payoff function is $u_i : \Pi_{j \in F \cup W} S_j \to \mathbb{R}$ such that $u_i(s) \geq u_i(s')$ if and only if $m_i(s) \succeq_i m_i(s')$, where $m_i(s) = \{j \in M_i : j \in s_i \text{ and } i \in s_j\}$ is the list of the sets of players who are matched with i in each category under a matching resulting from strategy profile $s \in \Pi_{j \in F \cup W} S_j$. Let $m = (m_i)_{i \in F \cup W}$ be the vector function such that for any $s \in \Pi_{j \in F \cup W} S_j$, $m(s)$ is the matching resulting from s. For any $I \subseteq F \cup W$, any $s \in \Pi_{j \in F \cup W} S_j$ and any $s'_I \in \Pi_{j \in I} S_j$, the pair (I, s'_I) is a **strategic coalitional deviation from** s if $u_i(s'_I, s_{-I}) > u_i(s)$ for every $i \in I$. A strategy profile $s^* \in \Pi_{j \in F \cup W} S_j$ is a **strong Nash equilibrium** of $G(F \cup W)$ if there exists no strategic coalitional deviation from s^* (Aumann [3]). In fact, it is easy to see that the set of matchings generated by strong Nash equilibria of the strategic-form game is equivalent to the set of group-stable matchings. Thus, if we apply the notion of a strong Nash equilibrium to a many-to-one (and, of course, to a one-to-one) matching game, the set of the matchings generated from strong Nash equilibria and the set of pairwise-stable matchings are equivalent without invoking the *weak* core (by the reason described earlier).[16] However, in a many-to-many matching game, a strong Nash equilibrium may not exist (recall Example 1 and consider the strategic-form game defined for this many-to-many matching problem).

Next we define a weaker solution concept based on credibility of strategic coalitional deviations: coalition-proof Nash equilibrium (Bernheim, Peleg, and Whinston [4]).[17] For $I \subseteq F \cup W$, consider a **reduced game** $G(I, s_{-I})$ that is a strategic-form game with players in I and is created from $G(I)$ by setting each player $j \in (F \cup W) \backslash I$ to be a passive player who plays a given $s_j \in S_j$ no matter what happens. A **coalition-proof Nash equilibrium** (**CPNE**) is recursively defined as follows:

(a) For any $i \in F \cup W$ and any $s_{-i} \in \Pi_{j \in (F \cup W) \backslash \{i\}} S_j$, strategy $s_i^* \in S_i$ is a **CPNE** of reduced game $G(\{i\}, s_{-i})$ if there is no $s'_i \in S_i$ with $u_i(s'_i, s_{-i}) > u_i(s_i^*, s_{-i})$.

[16] One of the results in Kara and Sönmez [12] shows that in a two-sided many-to-one matching problem, the same game form implements pairwise-stable correspondence in strong Nash equilibrium.

[17] In a network formation problem, Dutta, van den Nouweland, and Tijs [6] and Dutta and Mutuswami [7] use CPNE of a strategic-form game to analyze the resulting networks.

(b) Pick any positive integer $r < |F \cup W|$. Let all CPNEs of a reduced game $G(J, s_{-J})$ be defined for any $J \subset F \cup W$ with $|J| \leq r$ and any $s_{-J} \in \Pi_{i \in (F \cup W) \setminus J} S_i$. Then,

(i) for any $I \subseteq F \cup W$ with $|I| = r + 1$, s_I^* is **self-enforcing** in reduced game $G(I, s_{-I})$ if for every $J \subset I$ we have s_J^* is a CPNE of reduced game $G(J, (s_{-I}, s_{I \setminus J}^*))$ of $G(I, s_{-I})$, and

(ii) for any $I \subseteq F \cup W$ with $|I| = r + 1$, s_I^* is a **CPNE** of reduced game $G(I, s_{-I})$ if s_I^* is self-enforcing in reduced game $G(I, s_{-I})$, and there is no other self-enforcing s_I' such that $u_i\left(s_I', s_{-I}\right) > u_i\left(s_I^*, s_{-I}\right)$ for every $i \in I$.

For any $I \subseteq F \cup W$ and any strategy profile s, let $CPNE(G(I, s_{-I}))$ denote the set of CPNE strategy profiles on I for the game $G(I, s_{-I})$. For any strategy profile s, a strategic coalitional deviation (I, s_I') from s is **credible** if $s_I' \in CPNE(G(I, s_{-I}))$. A CPNE is *a strategy profile that is immune to any credible strategic coalitional deviation.*

The second main result of the paper is the following:

Theorem 2 *The set of pairwise-stable matchings, the set of credibly group-stable matchings, and the set of matchings generated from coalition proof Nash equilibria of the strategic-form game $G\left(F \cup W\right)$ are all equivalent, when $\succeq_F$ is responsive, and $\succeq_W$ is categorywise-responsive.*

We know that pairwise stability is equivalent to credible group stability if $\succeq_F$ is responsive and $\succeq_W$ is categorywise-responsive (Theorem 1). Thus, we need to show only that the resulting matching of a CPNE is pairwise-stable (proved below in Lemma 2), and that a credibly group-stable matching is the outcome of a CPNE (proved below in Lemma 3). Although these statements will be proved under substitutability, the equivalence between pairwise stability and credible group stability requires the stronger preference restriction of Theorem 1.[18] We start with Lemma 2. Recall that for any strategy profile s and any agent $i \in F \cup W$, $m_i(s) = \{j \in s_i : i \in s_j\}$ and that $m = (m_i)_{i \in F \cup W}$.

Lemma 2 *If $s^* \in CPNE(G(F \cup W))$ then $m\left(s^*\right)$, the matching generated from s^*, is a pairwise-stable matching, when $\succeq$ is substitutable.*

Proof. Let $s^* \in CPNE\left(G\left(F \cup W\right)\right)$. Suppose that matching $m(s^*)$ is not pairwise-stable. Then, either (i) there is $i \in F \cup W$ such that $Ch_i(m_i(s^*)) \neq m_i(s^*)$ (matched

[18] Under substitutability, Lemmata 2 and 3 show that *the set of credible group-stable matchings $\subseteq$ the set of matchings generated from CPNEs $\subseteq$ the set of pairwise-stable matchings.*

with an individually unstable agent), or (ii) there is a pair $(f, w) \in F \times W$ such that $w \in Ch_f(m_f(s^*) \cup \{w\})$ and $f \in Ch_w(m_w(s^*) \cup \{f\})$ (pair (f, w) blocks $m(s^*)$). Suppose that case (i) is true. This means that there is a player i who is willing to discontinue some of the partnerships under $m(s^*)$. She can do that in $G(F \cup W)$ by simply not announcing such partners. Considering $G(\{i\}, s^*_{-\{i\}})$, we can easily see that s^*_i is not a CPNE of the reduced game. This is a contradiction. Thus, suppose that case (ii) is true, and there is a pair $(f, w) \in F \times W$ that blocks $m(s^*)$. Consider a strategic coalitional deviation by $\{f, w\}$ with (s'_f, s'_w), where s'_f and s'_w are such that $s'_f = Ch_f(m_f(s^*) \cup \{w\})$ and $s'_w = Ch_w(m_w(s^*) \cup \{f\})$. This deviation is obviously beneficial for both agents f and w, since $m_i(s'_f, s'_w, s^*_{-\{f,w\}}) = Ch_i(m_i(s^*) \cup \{j\}) \succ_i m_i(s^*)$ for each $i \in \{f, w\}$ and $j \in \{f, w\} \setminus \{i\}$ (pair (f, w) blocks $m(s^*)$). Since s^* is a Nash equilibrium (a CPNE is a Nash equilibrium as well), for any $i \in F \cup W$ and any $\tilde{s}_i \in S_i$, we have $m_i(s^*) \succeq_i m_i(\tilde{s}_i, s^*_{-i})$ implying together with $m_i(s'_f, s'_w, s^*_{-\{f,w\}}) \succ_i m_i(s^*)$ that for any $\tilde{s}_i \in S_i$, we have $m_i(s'_f, s'_w, s^*_{-\{f,w\}}) \succ_i m_i(\tilde{s}_i, s^*_{-i})$. Let $\{i, j\} = \{f, w\}$. Since $m_i(\tilde{s}_f, \tilde{s}_w, s^*_{-\{f,w\}}) \subseteq m_i(s^*) \cup \{j\}$ for any $(\tilde{s}_f, \tilde{s}_w) \in S_f \times S_w$, we have $m_i(s'_f, s'_w, s^*_{-\{f,w\}}) = Ch_i(m_i(s^*) \cup \{j\}) \succeq_i m_i(\tilde{s}_f, \tilde{s}_w, s^*_{-\{f,w\}})$. The last two statements imply that agents f and w cannot achieve better matches than their partners under $m(s'_f, s'_w, s^*_{-\{f,w\}})$ by changing their strategies together or alone against $s^*_{-\{f,w\}}$. Hence $(\{f, w\}, (s'_f, s'_w))$ is a credible strategic coalitional deviation from s^*, contradicting that s^* is a CPNE and completing the proof of the lemma. □

Lemma 3 *For every credibly group-stable matching μ, there exists $s \in CPNE(G(F \cup W))$ such that $\mu = m(s)$, when $\succeq$ is substitutable.*

Proof. Recall that a CPNE is immune to *credible strategic coalitional deviations* in the game and a credibly group-stable matching is immune to *executable group deviations* in the problem. Hence, if for any strategy profile s and any credible strategic coalitional deviation (T, s'_T) from s in game $G(F \cup W)$, there exists an executable group deviation from matching $m(s)$ in the many-to-many matching problem, then the proof of the lemma will be complete. We will prove this as follows:

Let s be a strategy profile and (T, s'_T) be a credible strategic coalitional deviation from s. We denote the resulting strategy profile by $s' = (s'_T, s_{-T})$. Let μ be a matching generated from s, i.e. $\mu = m(s)$, and let μ' be the one generated from s', i.e. $\mu' = m(s')$. Note that $s'_j = s_j$ for any $j \in (F \cup W) \backslash T$. We will show that (T, μ') is an executable group deviation from μ. More specifically, we will prove that (i) for any $i \in T$, $Ch_i(\mu'(i) \cup (\mu(i) \backslash T)) = \mu'(i)$, and (ii) for any $i, j \in T$ with $j \in M_i \backslash \mu'(i)$, $j \in Ch_i(\mu'(i) \cup (\mu(i) \backslash T) \cup \{j\})$ implies $i \notin Ch_j(\mu'(j) \cup (\mu(j) \backslash T) \cup \{i\})$.

Condition (i): Suppose, to the contrary, that there exists an agent $i \in T$ with $Ch_i(\mu'(i) \cup (\mu(i) \backslash T)) \succ_i \mu'(i)$. Then profile s'_T is not immune to agent i's credible strategic deviation $s''_i = Ch_i(\mu'(i) \cup (\mu(i) \backslash T))$, since $u_i(s''_i, s'_{F \cup W \backslash \{i\}}) > u_i(s')$, contradicting $s'_T \in CPNE(G(T, s_{-T}))$.

Condition (ii): Suppose, to the contrary, that for some firm $f \in T \cap F$ and worker $w \in T \cap W$ with $w \notin \mu'(f)$, we have $w \in Ch_f(\mu'(f) \cup (\mu(f) \backslash T) \cup \{w\})$ and $f \in Ch_w(\mu'(w) \cup (\mu(w) \backslash T) \cup \{f\})$. This implies that for any $i \in \{f, w\}$ and any $j \in \{f, w\} \setminus \{i\}$, we have $Ch_i(\mu'(i) \cup (\mu(i) \backslash T) \cup \{j\}) \succ_i \mu'(i)$. Coalition $\{f, w\}$ can deviate from s' by setting $s''_i = Ch_i(\mu'(i) \cup (\mu(i) \backslash T) \cup \{j\})$ for each $i \in \{f, w\}$ and $j \in \{f, w\} \setminus \{i\}$, since $m_i(s''_f, s''_w, s'_{-\{f,w\}}) = Ch_i(\mu'(i) \cup (\mu(i) \backslash T) \cup \{j\}) \succ_i \mu'(i) = m_i(s')$. Since f and w have already attained the highest possible payoffs by choosing (s''_f, s''_w) against $s'_{-\{f,w\}}$, neither f nor w nor jointly $\{f, w\}$ can credibly deviate from $(s''_f, s''_w, s'_{-\{f,w\}})$, in turn implying that the strategic coalitional deviation $\left(\{f, w\}, \left(s''_f, s''_w\right)\right)$ from s'_T is credible. This contradicts $s'_T \in CPNE(G(T, s_{-T}))$, completing the proof of the lemma. $\square$

5 Conclusion

This paper establishes a theoretical foundation of pairwise stability in many-to-many matching problems when group deviations are allowed. We define credible group stability by restricting group deviations based on their credibility and prove the equivalence between pairwise stability and credible group stability when one side has responsive preferences while the other side has categorywise-responsive preferences. This domain fits well with the UK hospital-intern markets. Moreover, in the same domain, we show the equivalence between pairwise-stable matchings and the set of matchings generated by coalition-proof Nash equilibria of appropriately defined noncooperative matching games.

We also investigate what happens if the preference domain is expanded. We show by Examples 4 and 5 that if the domain is expanded then the equivalence no longer holds, since some pairwise-stable matchings can be Pareto-ordered.

We conclude noting that our Theorems 1 and 2 hold under *responsive preferences* for general non-bipartite multi-partner matching problems.[19] The proof is almost identical to the ones of Theorems 1 and 2, so it is omitted. A general multi-partner matching problem is a list $(N, (M_i, \succeq_i)_{i \in N})$ such that N is a finite set of agents, and for each $i \in N$, $M_i \subseteq N \backslash \{i\}$ is the set of feasible partners for i, and $\succeq_i$ is a preference ordering

[19] Our results do not apply in Sönmez's [25] *generalized matching problems* (thus, neither in Alkan's [1] k-sided matching problems with $k \geq 3$, nor in housing market problems). Our theorem requires that a partnership can be formed by a bilateral agreement only.

over 2^{M_i}.

Theorem 3 *In general multi-partner matching problems, the set of pairwise-stable matchings, the set of credibly group-stable matchings, and the set of matchings generated from coalition-proof Nash equilibria of the strategic-form game $G(N)$ are all equivalent, when $\succeq_N$ is responsive.*

Appendix

Example 6 *Consider a many-to-many matching problem with $F = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$ and $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$. There are two categories for each agent, and the partner set in each category is given as odd-indexed partners for the first category and even-indexed partners for the second category (the latter four agents in each category have only one acceptable agent each). Each agent has unit quota for each category. The preferences are categorywise-responsive and stated as follows:*

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
$\{w_3, w_2\}$	$\{w_1, w_4\}$	$\{w_1, w_4\}$	$\{w_3, w_2\}$	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$
$\{w_3, w_6\}$	$\{w_5, w_4\}$	$\{w_1, w_8\}$	$\{w_7, w_2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\boxed{\{w_3, w_4\}}$	$\boxed{\{w_3, w_4\}}$	$\boxed{\{w_1, w_2\}}$	$\boxed{\{w_1, w_2\}}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{\mathbf{w}_1, \mathbf{w}_2\}$	$\{\mathbf{w}_1, \mathbf{w}_2\}$	$\{\mathbf{w}_3, \mathbf{w}_4\}$	$\{\mathbf{w}_3, \mathbf{w}_4\}$				
$\{w_1, w_6\}$	$\{w_5, w_2\}$	$\{w_3, w_8\}$	$\{w_7, w_4\}$				
$\{w_1, w_4\}$	$\{w_3, w_2\}$	$\{w_3, w_2\}$	$\{w_1, w_4\}$				
$\{w_3\}$	$\{w_4\}$	$\{w_1\}$	$\{w_2\}$				
$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$				
$\{w_2\}$	$\{w_1\}$	$\{w_4\}$	$\{w_3\}$				
$\{w_6\}$	$\{w_5\}$	$\{w_8\}$	$\{w_7\}$				
$\{w_4\}$	$\{w_3\}$	$\{w_2\}$	$\{w_1\}$				
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$				

w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
$\{f_1,f_4\}$	$\{f_3,f_2\}$	$\{f_3,f_2\}$	$\{f_1,f_4\}$	$\{f_2\}$	$\{f_1\}$	$\{f_4\}$	$\{f_3\}$
$\{f_5,f_4\}$	$\{f_3,f_6\}$	$\{f_7,f_2\}$	$\{f_1,f_8\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\boxed{\{f_3,f_4\}}$	$\boxed{\{f_3,f_4\}}$	$\boxed{\{f_1,f_2\}}$	$\boxed{\{f_1,f_2\}}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{\mathbf{f}_1,\mathbf{f}_2\}$	$\{\mathbf{f}_1,\mathbf{f}_2\}$	$\{\mathbf{f}_3,\mathbf{f}_4\}$	$\{\mathbf{f}_3,\mathbf{f}_4\}$				
$\{f_5,f_2\}$	$\{f_1,f_6\}$	$\{f_7,f_4\}$	$\{f_3,f_8\}$				
$\{f_3,f_2\}$	$\{f_1,f_4\}$	$\{f_1,f_4\}$	$\{f_3,f_2\}$				
$\{f_4\}$	$\{f_3\}$	$\{f_2\}$	$\{f_1\}$				
$\{f_2\}$	$\{f_1\}$	$\{f_4\}$	$\{f_3\}$				
$\{f_1\}$	$\{f_2\}$	$\{f_3\}$	$\{f_4\}$				
$\{f_5\}$	$\{f_6\}$	$\{f_7\}$	$\{f_8\}$				
$\{f_3\}$	$\{f_4\}$	$\{f_1\}$	$\{f_2\}$				
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$				

For example, for agent f_1, in the even category, w_2 is the best, w_6 is the second best, and w_4 is the worst partners respectively. Unlike Example 5, pairwise-stable matching is unique (the F-optimal and the W-optimal matchings are identical): a pairwise-stable matching μ is described by bold characters (the latter four agents in each category is unmatched). Now let μ' be a matching described by rectangles. Note that μ' is not pairwise-stable, since w_6 and f_1 can deviate. However, μ' is pairwise-stable within $T = \{f_1, f_2, f_3, f_4, w_1, w_2, w_3, w_4\}$, and $\mu'(i) \succ_i \mu(i)$ for all $i \in T$. Therefore (T, μ') is an executable group deviation from μ, and there is no credibly group-stable matching.$\square$

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Appendix for the Referees

In this Appendix, we will show that $\mu_1, ..., \mu_9$ are the only individually rational matchings in Example 2. *Let μ be an individually rational matching.*

Claim 1: We have (i) $\mu(f_4) = \mu(f_5) = \emptyset$ or $\mu(f_4) = \mu(f_5) = \{w_1, w_2\}$, and (ii) $\mu(w_4) = \mu(w_5) = \emptyset$ or $\mu(w_4) = \mu(w_5) = \{f_1, f_2\}$.

Proof of Claim 1. We prove two statements separately.

(i) Suppose that $\mu(f_4) \neq \emptyset$. Then, $w_2 \in \mu(f_4)$ must hold. This implies $\mu(w_2) = \{f_2, f_4\}$ or $\{f_5, f_4\}$.

Case 1. $\mu(w_2) = \{f_2, f_4\}$: In this case, $\mu(f_2) = \{w_3, w_2\}$ or $\{w_1, w_2\}$. In the former case, $\mu(w_3) = \{f_3, f_2\}$ holds, and $\mu(f_3) = \{w_1, w_3\}$ or $\{w_2, w_3\}$. Since $\mu(w_2) = \{f_2, f_4\}$, $\mu(f_3) = \{w_1, w_3\}$ must follow. This then implies $\mu(w_1) = \{f_1, f_3\}$, and thus $\mu(f_1) = \{w_2, w_1\}$ or $\{w_3, w_1\}$ must hold. However, neither w_2 nor w_3 has f_1 as a partner. This is a contradiction. Thus, we have the latter case $\mu(f_2) = \{w_1, w_2\}$. This implies $\mu(w_1) = \{f_1, f_2\}$, and $\mu(f_1) = \{w_2, w_1\}$ or $\{w_3, w_1\}$. The former contradicts $\mu(w_2) = \{f_2, f_4\}$, and the latter implies $\mu(w_3) = \{f_3, f_1\}$. Thus, $\mu(f_3) = \{w_1, w_3\}$ or $\{w_2, w_3\}$. However, this contradicts $\mu(w_1) = \{f_1, f_2\}$ and $\mu(w_2) = \{f_2, f_4\}$. There is no individually rational matching for Case 1.

Case 2. $\mu(w_2) = \{f_5, f_4\}$: In this case, $\mu(f_5) = \{w_1, w_2\}$ follows, and $\mu(w_1) = \{f_1, f_5\}$ or $\{f_4, f_5\}$ must hold. In the former case, $\mu(f_1) = \{w_3, w_1\}$ is implied since $\mu(f_1) = \{w_2, w_1\}$ contradicts $\mu(w_2) = \{f_5, f_4\}$. As in Case I, $\mu(f_1) = \{w_3, w_1\}$ implies $\mu(f_3) = \{w_1, w_3\}$ or $\{w_2, w_3\}$, and these contradict $\mu(w_2) = \{f_5, f_4\}$ and $\mu(w_1) = \{f_1, f_5\}$, respectively. Thus, $\mu(w_1) = \{f_1, f_5\}$ cannot happen, and $\mu(w_1) = \{f_4, f_5\}$ must hold. This implies $\mu(f_4) = \{w_1, w_2\}$. Hence, we have shown that $\mu(f_4) \neq \emptyset$ implies $\mu(f_4) = \mu(f_5) = \{w_1, w_2\}$.

Since f_4 and f_5 are totally symmetric, we can repeat exactly the same argument for the case of $\mu(f_5) \neq \emptyset$. If $\mu(f_5) \neq \emptyset$, then we have $\mu(f_4) = \mu(f_5) = \{w_1, w_2\}$. This proves the first statement in the claim.

(ii) Suppose that $\mu(w_4) \neq \emptyset$. Then, $\mu(w_4)$ contains f_2. Then, $\mu(f_2) = \{w_3, w_4\}$, $\{w_1, w_4\}$ or $\{w_5, w_4\}$ holds.

Case 1. $\mu(f_2) = \{w_3, w_4\}$: This implies $\mu(w_3) = \{f_3, f_2\}$, and thus $\mu(f_3) = \{w_1, w_3\}$ or $\{w_2, w_3\}$. In the former case, $\mu(w_1) = \{f_1, f_3\}$ and thus $\mu(f_1) = \{w_2, w_1\}$ ($\mu(f_1) = \{w_3, w_1\}$ contradicts $\mu(w_3) = \{f_3, f_2\}$). This implies $\mu(w_2) = \{f_2, f_1\}$. This contradicts $\mu(f_2) = \{w_3, w_4\}$.

Case 2. $\mu(f_2) = \{w_1, w_4\}$: This implies $\mu(w_1) = \{f_1, f_2\}$, and thus $\mu(f_1) = \{w_3, w_1\}$ (otherwise, $\mu(f_1) = \{w_2, w_1\}$, and $\mu(w_2) = \{f_2, f_1\}$ that contradicts $\mu(f_2) = \{w_1, w_4\}$).

This implies $\mu(w_3) = \{f_3, f_1\}$ and thus $\mu(f_3) = \{w_2, w_3\}$ (otherwise, $\mu(f_3) = \{w_1, w_3\}$ that contradicts $\mu(w_1) = \{f_1, f_2\}$). This implies $\mu(w_2) = \{f_2, f_3\}$. This contradicts $\mu(f_2) = \{w_1, w_4\}$.

Case 3. $\mu(f_2) = \{w_5, w_4\}$: This implies $\mu(w_5) = \{f_1, f_2\}$, and thus $\mu(f_1) = \{w_2, w_5\}$, $\{w_3, w_5\}$, or $\{w_4, w_5\}$.

Case *a*. $\mu(f_1) = \{w_2, w_5\}$: This implies $\mu(w_2) = \{f_2, f_1\}$, which contradicts $\mu(f_2) = \{w_5, w_4\}$.

Case *b*. $\mu(f_1) = \{w_3, w_5\}$: This implies $\mu(w_3) = \{f_3, f_1\}$, thus $\mu(f_3) = \{w_1, w_3\}$ or $\{w_2, w_3\}$. If $\mu(f_3) = \{w_1, w_3\}$, then $\mu(w_1) = \{f_1, f_3\}$ that contradicts $\mu(f_1) = \{w_3, w_5\}$. If $\mu(f_3) = \{w_2, w_3\}$, then $\mu(w_2) = \{f_2, f_3\}$ that contradicts $\mu(f_2) = \{w_5, w_4\}$.

Case *c*. $\mu(f_1) = \{w_4, w_5\}$: This implies $\mu(w_4) = \{f_2, f_1\}$, and this is consistent with $\mu(f_2) = \{w_5, w_4\}$. Hence, we have shown that $\mu(w_4) \neq \emptyset$ implies $\mu(w_4) = \mu(w_5) = \{f_1, f_2\}$.

Since w_4 and w_5 are totally symmetric, we can repeat exactly the same argument for the case of $\mu(w_5) \neq \emptyset$. If $\mu(w_5) \neq \emptyset$, then we have $\mu(w_4) = \mu(w_5) = \{f_1, f_2\}$. This proves the second statement in the claim. $\diamondsuit$

Claim 2: Suppose that $w_1 \in \mu(f_1)$. Then, $\mu \in \{\mu_1, \mu_3, \mu_4, \mu_5\}$.

Proof of Claim 2. Note that $w_1 \in \mu(f_1)$ implies that $\mu(f_4) = \mu(f_5) = \emptyset$ and $\mu(w_4) = \mu(w_5) = \emptyset$ must hold by Claim 1. Since $w_1 \in \mu(f_1)$, $\mu(f_1) = \{w_2, w_1\}$ or $\{w_3, w_1\}$. These two cases are treated as Cases 1 and 2 in order.

Case 1. $\mu(f_1) = \{w_2, w_1\}$: Since $f_1 \in \mu(w_2)$, $\mu(w_2) = \{f_2, f_1\}$ holds. This implies $w_2 \in \mu(f_2)$: i.e., either $\mu(f_2) = \{w_3, w_2\}$, or $\mu(f_2) = \{w_1, w_2\}$.

Case *a*. $\mu(f_2) = \{w_3, w_2\}$: We have $\mu(w_3) = \{f_3, f_2\}$, and $\mu(f_3) = \{w_1, w_3\}$ or $\{w_2, w_3\}$ must follow. In the former case, $\mu(w_1) = \{f_1, f_3\}$ holds, and we have a matching μ_5. The latter case contradicts $\mu(w_2) = \{f_2, f_1\}$, and this cannot be a matching.

Case *b*. $\mu(f_2) = \{w_1, w_2\}$: We have $f_2 \in \mu(w_1)$ and $\mu(w_1) = \{f_1, f_2\}$. This implies that we have matching μ_1.

Case 2. $\mu(f_1) = \{w_3, w_1\}$: This implies $\mu(w_3) = \{f_3, f_1\}$. There are two possibilities: $\mu(f_3) = \{w_1, w_3\}$, or $\mu(f_3) = \{w_2, w_3\}$.

Case *a*. $\mu(f_3) = \{w_1, w_3\}$: We have $\mu(w_1) = \{f_1, f_3\}$, and this creates the individually rational matching μ_3.

Case *b*. $\mu(f_3) = \{w_2, w_3\}$: We have $\mu(w_2) = \{f_2, f_3\}$. Thus, $\mu(f_2) = \{w_3, w_2\}$ or $\{w_1, w_2\}$. The former is not a matching, since $\mu(w_3) = \{f_1, f_3\}$. The latter generates a matching μ_4.

Thus, $w_1 \in \mu(f_1)$ implies that individually rational matching μ must be one of μ_1, μ_3, μ_4, and μ_5. $\diamondsuit$

Claim 3: Suppose that $w_1 \notin \mu(f_1)$. Then, $\mu \in \{\mu_2, \mu_6, \mu_7, \mu_8, \mu_9\}$.

Proof of Claim 3. Suppose that $w_1 \notin \mu(f_1)$. This is equivalent to $f_1 \notin \mu(w_1)$. This implies $\mu(w_1) = \{f_4\}$, $\{f_4, f_5\}$ or $\emptyset$. By Claim 1, $\mu(w_1) = \{f_4\}$ does not occur.

Case 1. $\mu(w_1) = \{f_4, f_5\}$: This implies $\mu(w_2) = \{f_4, f_5\}$ (Claim 1). Note that w_3 does not form any partnership unless f_3 forms partnership with w_3, and that f_3 would not do so without having either w_1 or w_2. Since w_1 and w_2 have binding quotas, $\mu(w_3) = \emptyset$. Now suppose that $\mu(w_4) \neq \emptyset$. Then, by Claim 1, $\mu(f_1) = \mu(f_2) = \{w_4, w_5\}$. This generates μ_8. Instead, suppose that $\mu(w_4) = \emptyset$. Then, by Claim 1, $\mu(w_5) = \emptyset$. This generates μ_7.

Case 2. $\mu(w_1) = \emptyset$: This implies $\mu(f_4) = \mu(f_5) = \emptyset$ (Claim 1). Focus on w_3. There are two cases.

Case *a*. $f_3 \in \mu(w_3)$: In this case, $\mu(f_3) = \{w_2, w_3\}$ must follow, and thus $\mu(w_3) = \{f_2, f_3\}$. This implies $\mu(w_2) = \{f_2, f_3\}$. Since $w_2 \in \mu(f_2)$, we obtain μ_2 (Claim 2).

Case *b*. $\mu(w_3) = \emptyset$: This implies $\mu(f_3) = \{w_2\}$ or $\emptyset$. Suppose that $\mu(f_3) = \{w_2\}$. In this case, $\mu(w_2) = \{f_2, f_3\}$ holds, and thus $\mu(f_2) = \{w_3, w_2\}$ or $\{w_1, w_2\}$. However, either one contradicts $\mu(w_3) = \emptyset$ or $\mu(w_2) = \emptyset$, respectively. Thus, $\mu(f_3) = \emptyset$. Now, focus on w_2. Since $\mu(f_5) = \emptyset$, $f_2 \in \mu(w_2)$ or $\mu(w_2) = \emptyset$ holds. In the former case, we have $\mu(f_2) = \{w_3, w_2\}$ or $\{w_1, w_2\}$. Either case contradicts $\mu(w_3) = \emptyset$ or $\mu(w_2) = \emptyset$ again. Thus, we have shown that $\mu(f_3) = \mu(f_4) = \mu(f_5) = \mu(w_1) = \mu(w_2) = \mu(w_3) = \emptyset$. By applying Claim 1, we conclude that μ is either μ_6 or μ_9.

All cases together prove that $w_1 \notin \mu(f_1)$ implies $\mu \in \{\mu_2, \mu_6, \mu_7, \mu_8, \mu_9\}$. $\diamondsuit$

Claims 2 and 3 show that the set of individually rational matchings is $\{\mu_1, \mu_2, ..., \mu_9\}$ completing the proof.