On the Joint Dynamics of Inequality and Growth.

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Abstract

In this paper we investigate the joint dynamics of the segmentation of society into communities and the growth process using a simple human capital growth model. Using coalition theory, we prove that in each period, "growth clubs" form. We investigate the socio-economic dynamics of society over time, characterize it and prove that there exists a steady state partition of society, which may be segmented. Then, there is no absolute and general convergence in income levels. We then study these process for different initial inequality patterns.

1 Introduction

The relationship between inequality and the growth process is now well researched by economists, in particular after the development of endogenous growth theory (see Aghion, Caroli and Garcia-Penalosa, 1999, for a survey on this). The dynamics between the two is less understood. The Kuznets curve as a stylized fact makes clear that there is some interdependence between the two and that at some early stage of the growth process, inequality acts as a fuel for growth, whereas at latter stages, a reduction in the growth process tends to homogenize society and reduce inequality. But this view is still in need of a theory.

The aim of the present paper is to develop a theory of the joint dynamics of inequality and growth. We want to prove how the endowment distribution and the growth pattern of an economy interdependently change over time. The growth pattern relates to the distribution of individual growth rates in a given period. In a given period, the existing inequality schedule impinges on the growth pattern; in turn this pattern affects the way agents accumulate capital and therefore the endowment distribution for the next period. This interaction comes through the existence over time of “temporary” growth clubs, coming to life because the growth mechanism relies on the productive feature of a club good: The individual accumulation of human capital for an individual in a given club at a given period depends on the level of the club good. Hence the characteristics of a temporary club affect the individual accumulation of human capital and matters for the individual transmission by members of the club of human capital to their forebears. Growth clubs may grow at different rates because each club produces a specific amount of the club good. This implies that the endowment distribution changes over time, because of the way growth clubs form and human capitals differentially accumulate in each period.

The way growth clubs form in each period appears crucial. Assuming that there is no intertemporal commitment over club formation, clubs form when a period opens and dissolve at the end of each period. In each period, agents have an incentive to form clubs. Here we explicitly endogenize the formation of these clubs by individual agents, relying on coalition theory. This means that the economy is continuously segmented and that the process of segmentation itself evolves over time. In this sense we can talk about a social dynamics, that is the evolution of segmentation into communities or clubs over time because individuals are free to form new communities at each period.

Here we develop a model where the economic dynamic properties of the economy (the evolution of the endowment distribution and the growth pattern) is intertwined with the social dynamics, both being grounded on explicit individual optimizing behavior.

The functioning of the economy is as follows. Agents live for one period, and care about the legacy left to their offspring, because of a "joy-of-giving" motive (there is no population growth, nor uncertainty: each agent is succeeded by another agent). In each period, agents inherite an individual amount of human capital from their forebears which affects their own income. The production of human capital in this period depends on the production of a "club good". Hence agents are induced to form clubs. As generations do not last, clubs do not last either. At the beginning of each period clubs form, and are disbanded at the end of it. As a result, in each period society is segmented into clubs and that segmentation does not last. Given the accumulation of human capital through time, there is growth in the sense that individual incomes grow over time. Since there is heterogeneity among agents, tthe growth patterns are specific to each family and the growth process itself is marked by heterogeneity. The formation of clubs depends on the rational interest of individuals: individuals must be willing to enter a club and be accepted into this club by its other members. These decisions depends on two factors: congestion costs and wealth. Any additional member creates a congestion effect as the more numerous a club is, the less efficient the provision process of the club good is; therefore an agent must be able to contribute enough to the provision of the club good to be accepted . Given the inequality in individual human capitals, it means that not any agent is welcome in a given club: he must be rich enough to overcome the congestion effect and be accepted by its members. Hence inequality leads to social segmentation into clubs. As clubs produce a club good, which matters for the growth of human capital, the growth process in any period depends on the human capital distribution inherited from the previous period. In turn, the growth process leads to a new distribution which itself will impact on the next period's growth process.

Because of the formation of clubs, that is social clusters of individual and their economic effects on the growth process, we witness a joint dynamics of growth, social segmentation and inequality.

Given the interaction over time of the social and the economic dimensions developed in the model, we can address several issues:

- The partition of the economy and the growth pattern for a given period: how clubs form in each period? Can we characterize the partition? What are its impacts on the growth process?
- The dynamics of the partition and of the growth pattern over time: how do they evolve? Can we have periods during which there is a reduction in the fragmentation of the society and a correlated reduction in growth differentials over agents, followed by periods of increasing fragmentation and widening of the economic gaps between agents? Under which circumstances, do we obtain a monotonous process of joint reduction in social fragmentation and inequality? What can we say about the convergence process? Is there a steady state in this economy.

We are able to address these questions and prove that in each period, an equilibrium exists with a partition belonging to the core of the economy. It does not imply that the grand coalition form nor that there be a global and systematic reduction in human capital. It makes unlikely that except under special circumstances, a convergence process takes place over time.

Despite the complex dynamics of club formation and human capital accumulation, we are able to prove that any economy whatever its initial human capital distribution eventually establishes a permanent partition, that is that clubs' borders do not move, even though clubs are recomposed again and again. Then, this leads to full convergence within a club but not between clubs (again, except for some initial distributions and parameter values). If there is more than one club, it must be that there is income and human capital divergence among agents.

Finally, after studying particular types of initial human capital accumulation which helps us to better understand the functioning of this economy, we cannot establish a monotonous relationship between inequality and growth. We explain this absence of a non ambiguous relationship between more inequality and more/less growth, now well documented empirically (see Benabib, 2003), by the fact that inequality is not the sole factor affecting growth, but that other factors like congestion costs and therefore social segmentation play a crucial role in the growth process.

The present paper shares a common perspective with a previous paper co-written with Fernando Jaramillo (Jaramillo, Kempf, Moizeau, forthcoming). In this paper too, we addressed the link between inequality and growth, offering the notion of "growth clubs" and proving their existence, when it is considered that they endogenously form. A growth club is a cluster of individuals bonding together as they share a common resource, which makes them grow together. It was assumed that individuals would live forever and clubs would form at the beginning of time and forever, and that within each club, a club good would be provided once-and-for-all. Hence we proved that the initial distribution of capital affects the entire growth process over time, through the partitioning of the society, and that convergence could not be taken for granted once we consider this segmentation. But there was no feedback from growth to inequality as there is no periodic reshuffling of the partition of society.

Here we enrich the picture as we are able to exhibit a much more complex and two-way process. In the present paper, we exploit and prove the existence of growth clubs. However here, they are temporary clubs as they form and disband in each period. It is this precise characteristics which uncovers the joint dynamics of inequality and growth that we could not obtain in the previous paper: as the borders of clubs (a priori) move over time, the human capital accumulation distribution is reshuffled over time. As a result, inequality dynamics is affected by growth and vice-versa.

The plan of the paper is as follows. In the following section we set up the model. In section 3, we prove, characterize the equilibrium attained in period t and study its properties. Then in section 4, we study the dynamic sequence of these equilibria, proving that a steady state exists, with both a permanent core partition of the economy and a steady-state growth pattern. In section 5, we study this dynamics for various types of endowment distributions. Section 6 concludes.

2 The economy

We consider a model of successive generations of individuals. There is no demographic growth and we assume that each individual lives one period and has a unique offspring so that the population is of constant size N. In each period, the society is formed of N individuals $S = \{1, ..., N\}$. At date $t = 0$, each individual is endowed with a level of human capital h_0^i. Agents are ordered so that $h_0^1 > h_0^2 > .... > h_0^N$. As we shall see, since agents leave bequests to their child, we define h_t^i as the human capital endowed to agent i living in t by her parent.

Individuals differ only according to their human capital endowment. Agents' preferences are the same. For any individual born at date t, preferences depend on private consumption c_t^i and the bequest left to her offspring h_{t+1}^i:

$$U(c_t^i, h_{t+1}^i) = \ln c_t^i + \rho \ln h_{t+1}^i \tag{1}$$

Each individual is endowed by 1 unit of time. $(1 - \mu)$ is the fraction of time devoted to education and μ is the fraction of time devoted to work. For simplicity, μ is assumed to be constant. We assume an aggregate production function such that $Y_t = \sum_{i=1}^{N} \mu h_t^i$. This implies h_t^i equals the hourly wage w_t^i and thus the income of individual i is $y_t^i = \mu h_t^i$. The consumption level is equal to the after-tax net income.

Agents are willing to form or join a club because a club provides a productive club good. More precisely, for each individual i belonging to the $j-$ club C_t^j, the human capital technology is defined by:

$$h_{t+1}^i = \kappa \left((1-\mu) h_t^i \right)^{1-\beta} \left(G_t^j \right)^{\beta} \tag{2}$$

with G_t^j the level of public good in club C_t^j. Moreover, we assume $\kappa > 0$, $\beta \in (0,1)$ so that all factors exhibit diminishing returns. There is no inter-club externality and G_t^j is actually a club good. The amount of human capital left to any offspring depends both on the amount of education provided by her parents and the level of the club good. The quality of education itself depends on the available individual human capital. The club good (a "school") helps a member of the club to form more human capital. The club good is financed by a proportional tax rate τ_t^j which is levied on members' incomes. The tax rate τ_t^j is specific to each club and is endogenously chosen in a manner to be developed later. It can be expressed as follows:

$$G_t^j = \frac{\tau_t^j \mu \sum_{i \in C_t^j} h_t^i}{A(n_t^j)}. \tag{3}$$

The public good is financed through a tax levied on the labor income but the provision technology is hampered by congestion effects, captured by $A(n_t^j)$. We assume that $A'(n_t^j) > 0$ and that $A(n_t^j)$ is a log-convex function.[1] Importantly, congestion costs are anonymous. The harm inflicted by other members of a club to any individual member is related to the mere number of them, not to their precise identity or characteristics.[2]

As agents may willingly form clubs, at any date t the society S may be partitioned:

Definition 1. *A nonempty subset C_t^j of S is called a club and $\mathcal{C}_t = \{C_t^1, ..., C_t^j, ..., C_t^J\}$ for $j = 1, ..., J$ is called a partition of S if:*

(i) $\bigcup_{j=1}^{J} C_t^j = S$;

(ii) $C_t^j \bigcap C_t^{j'} = \emptyset$ *for* $j \neq j.'$

We do not impose that clubs form for more than one period. Hence a partition is defined for a given period. We will denote by $n_t^j = card\left(C_t^j\right)$ the number of individuals belonging to C_t^j.

Finally, capital markets are incomplete: Agents cannot borrow and lend freely so as to alter their decision to accumulate human capital.

3 The core partition at time t.

In each period, given the endowment vector $\left\{h_t^1, ..., h_t^i, ..., h_t^N\right\}$, the functioning of the economy is sequential:

[1] This property will prove useful for the proof of uniqueness.

[2] Given the fixed mode of financing the club good, there is no free-rider effect, contrarily to the model we previously developed. See JKM (2003 and forthcoming).

1. In the first stage, clubs form. This implies some agreement over the membership, the amount of public good to be provided by its members and the tax rate to be chosen by the constituency formed by the sole members of the club. The tax rate chosen within a club is decided through a simple majority rule.

2. Then, in the second stage, individuals produce by allocating their time between labor and education and they leave bequests to their forebear in the next period. Here the individual behaviour is rather passive as the fraction of time devoted to education is exogenous.

The game is in the same spirit as the games studied by JKM (2003 and forthcoming).[3] However, here there is no uncooperative behavior, given that individual behaviour is rather passive and the fraction of time devoted to education is exogenous. As the human capital technology does not allow for any economy-wide spillovers, the club-formation game we will focus on is thus a club-formation game without spillovers between clubs.

We shall prove the existence of an equilibrium when agents form clubs and will use the following definitions:

Definition 2: *A core partition* $\widehat{\mathcal{C}}_t = \left\{\widehat{C}_t^1, ..., \widehat{C}_t^j, ...\right\}$ *is such that:*

$$\nexists \pounds \subset S \text{ such that } \forall i \in \pounds, V^i(\pounds) > V^i\left(\widehat{\mathcal{C}}_t\right) \tag{4}$$

where $V^i\left(\widehat{\mathcal{C}}_t\right)$ *denotes the utility for agent* i *associated with partition* $\widehat{\mathcal{C}}_t$.

According to this definition, a partition belongs to the core when it is immune against any club deviation, i.e., no member of the deviating group obtains more than what he is currently getting in the partition.

Definition 3. *At date t,* $\left[\widehat{\mathcal{C}}_t = \{\widehat{C}_t^1, ..., \widehat{C}_t^j, ..., \widehat{C}_t^{J_t}\}; \left(\widehat{\tau}_t^j\right)_{j\in\{1,...,J_t\}}\right]$ *is an equilibrium if it satisfies:*

(i) $\widehat{\tau}_t^j$ is chosen in club $\widehat{C}_t^j$ according to the simple majority rule and such that:

$$\widehat{G}_t^j = \frac{\widehat{\tau}_t^j \mu \sum_{z\in\widehat{C}_t^j} h_t^z}{A(\widehat{n}_t^j)}; \tag{5}$$

(ii) $\widehat{\mathcal{C}}_t$ *belongs to the core of the coalition-formation game.*

According to this definition, the equilibrium we are looking for is such that in each club, the provision of the public good is fully financed, and no agent has any interest to propose or to accept a defection from any club, as the partition belongs to the core. A strong implication of this is that the equilibrium is Pareto-optimal.

Solving backwards allows us to characterize the equilibrium as follows:

Proposition 1. *At date t, the equilibrium exists and is characterized by the following:*

(i) The tax rate in club is chosen by the median voter and is equal to:

$$\widehat{\tau}_t^j = \widehat{\tau} = \frac{\rho\beta}{1+\rho\beta}.$$

[3] It has also been used by Barham et al. (1997).

(ii) The indirect utility $V^i(\widehat{C}_t^j)$ for individual i belonging to $\widehat{C}_t^j$ is equal to:

$$V^i(\widehat{C}_t^j) = F_t^i + \rho\beta \ln \left(\frac{\sum\limits_{z \in \widehat{C}_t^j} h_t^z}{A(\widehat{n}_t^j)} \right) \tag{6}$$

with $F_t^i = \ln\left(\left(\frac{1}{1+\rho\beta}\right)\mu h_t^i\right) + \rho \ln \kappa \left((1-\mu) h_t^i\right)^{1-\beta} + \rho\beta \ln \mu(\frac{\rho\beta}{1+\rho\beta})$.

(iii) The core partition $\widehat{\mathcal{C}}_t$ is unique and consecutive, that is if i and $\widetilde{i}$ both belong to $\widehat{C}_t^j$, then $\forall i^*$, $i > i^* > \widetilde{i}$, $i^* \in \widehat{C}_t^j$.

(iv) Welfare ordering: Consider two individuals such that $i' > i$, $i' \in \widehat{C}_t^{j'}$ and $i \in \widehat{C}_t^j$, then $V^i(\widehat{C}_t^j) > V^{i'}(\widehat{C}_t^{j'})$ and $\widehat{G}_t^j > \widehat{G}_t^{j'}$.

Proof. See Appendix.

Given the consecutivity property of the core partition, it amounts to say that the $\widehat{n}_t^1$ richest agents form the club $\widehat{C}_t^1$, the next $\widehat{n}_t^2$ richest agents form the club $\widehat{C}_t^2$, etc.

Remark that the core partition is obtained for a given period. Given the bequests left by agents to their forebears, the inequality schedule a priori changes from period to period, a priori generating a different core partition at each period. This explains why the various characteristics of a core partition (except the tax rate) are indexed with a subscript t. Now we adopt the convention that clubs are indexed according to the ranking of utilities associated with them. Given the value of $\widehat{G}_t^j = \rho\beta\mu \sum\limits_{z \in \widehat{C}_t^j} h_t^z \left((1+\rho\beta) A(\widehat{n}_t^j)\right)^{-1}$, using (7) the human capital h_{t+1}^i is equal to:

$$h_{t+1}^i = \kappa \left((1-\mu) h_t^i\right)^{1-\beta} \left(\frac{\widehat{\tau}_t^j \mu \sum\limits_{z \in \widehat{C}_t^j} h_t^z}{A(\widehat{n}_t^j)} \right)^{\beta}. \tag{7}$$

It will prove useful to adopt the following convention**.** Consider two clubs $\widehat{C}_t^j$ and $\widehat{C}_t^{j'}$. We rank clubs indexes such that when $V^i(\widehat{C}_t^j) > V^{i'}(\widehat{C}_t^{j'})$, for $i \in \widehat{C}_t^j$ and $i' \in \widehat{C}_t^{j'}$, then $j < j'$.

Proposition 1 and in particular (6) allow us to offer a simple characterization of the core partition valid for date t:

Proposition 2. $\forall t, \widehat{\mathcal{C}}_t = \{\widehat{C}_t^1, ..., \widehat{C}_t^j, ..., \widehat{C}_t^{J_t}\}$ *can be defined as a sequence of pivotal agents* $\{p_t^1, ..., p_t^j, ..., p_t^{J_t}\}$ *where* $\widehat{C}_t^j = \{p_t^{j-1}+1, ..., p_t^j\}$ *and* $h_t^{p_t^j}$ *is such that:*

$$h_t^{p_t^j} \geq \sum_{z=p_t^{j-1}+1}^{p_t^j - 1} h_t^z \left(\frac{A(\widehat{n}_t^j)}{A(\widehat{n}_t^j - 1)} - 1 \right)$$

and

$$h_t^{p_t^j+1} < \sum_{z=p_t^{j-1}+1}^{p_t^j} h_t^z \left(\frac{A(\widehat{n}_t^j + 1)}{A(\widehat{n}_t^j)} - 1 \right)$$

where $p_t^0 + 1 = 1$ *and* $p_t^{J_t} = N$.

A pivotal agent is the poorest agent of a club $\widehat{C}_t^j$. Her human capital endowment is just sufficient for her to contribute minimally but enough to cover the additional congestion costs she inflicts on the other members. The next agent, just after the pivotal agent, who is poorer is unable to cover these costs, and so is not accepted by the members of $\widehat{C}_t^j$. The number of clubs J_t is indexed with time as it may vary over time. The last club $\widehat{C}_t^{J_t}$ is called the "residual" club. Its size is not "optimal" as its pivotal agent is the last agent, so that the last inequality has no meaning for this club.

4 The core partition and the convergence issue.

Given the segmentation put in place in a given period, what are its economic consequences in terms of growth and inequality? This is the first step of the analysis of the socio-economic dynamics of this economy. It is important as, in the next period, the partition will depend on the current income distribution.

4.1 Intra-club convergence

First we can prove that within any club formed in a given period, there is intra-club convergence: the differences between members are reduced. For the equilibrium at t, from (2) and (5), the individual-human-capital growth rate for an agent i who belongs to $\widehat{C}_t^j$ can be written as follows:

$$\frac{h_{t+1}^i}{h_t^i} = \kappa(1-\mu)^{1-\beta}\left(\mu\widehat{\tau}\frac{\sum\limits_{z\in\widehat{C}_t^j} h_t^z}{A(\widehat{n}_t^j)h_t^i}\right)^{\beta}. \tag{8a}$$

This ratio is increasing in the aggregate wealth of the club $\sum\limits_{z\in\widehat{C}_t^j} h_t^z$: quite sensibly, for any individual, the richer is the club she belongs to, the higher is the level of the club good she benefits from, and the higher is the human capital she bequests to her child. But remark also that it is increasing in the ratio between the aggregate wealth and the individual current human capital: the poorer is a member, the more she benefits from the club good. Finally, it is decreasing in the size of the club $\widehat{n}_t^j$: this is due to congestion costs which depress the efficiency of the production of the club good for a given amount of collected taxes.

We give some properties of the core-partition of a given period t and its consequences on growth in the following:

Proposition 3. *(i) Intra-club human capital convergence: At any date t, within clubs, there is human capital convergence:*

$$\frac{h_{t+1}^i}{h_{t+1}^{i'}} < \frac{h_t^i}{h_t^{i'}},\ \forall i, i' \in \widehat{C}_t^j, \forall t.$$

(ii) Whatever t, the initial individual ordering remains valid:

$$h_{t+1}^i > h_{t+1}^{i'}, \forall i, i' \in S, \forall t.$$

Proof: See appendix.

The point (i) states that there is convergence in human capital/endowment between members of a given club. This comes directly from the fact that inputs in the human capital technology exhibit diminishing returns. This leads a poorer agent in a club to benefit more from the club good than a richer member and thus to accumulate relatively more rapidly human capital. However this is not true between individuals belonging to different clubs as there may be a divergence among clubs. Consider two individuals i and i' with $i' > i$, $i \in \widehat{C}_t^j$ and $i' \in \widehat{C}_t^{j'}$. Thus from (2) we may have:

$$\frac{h_{t+1}^i}{h_{t+1}^{i'}} > \frac{h_t^i}{h_t^{i'}}. \tag{9}$$

This is true when

$$\frac{h_{t+1}^i/h_t^i}{h_{t+1}^{i'}/h_t^{i'}} = \frac{\left(G_t^j/h_t^i\right)^\beta}{\left(G_t^{j'}/h_t^{i'}\right)^\beta} > 1$$

which cannot be ruled out. As a consequence, even if the gap between two agents who belong to the same club at t reduces, it may increase in the sequel: because their offspring may belong in the future to different clubs. A priori there is no reason to expect that if at t, i and i' belong to a club $\widehat{C}_t^j$, the individual with the same ranking (their offspring) will necessarily belong to the club $\widehat{C}_{t+1}^j$.

Point (ii) states that there may be a catching-up mechanism at work over time, but that it can never lead to an inversion of the ranking. If at 0, i is richer than i', then i ' offsprings will always remain at least as rich as i' 's offspring. This is due to the obvious fact, that in absolute levels, in any period, a richer individual always accumulates more capital than a poorer one, because she benefits from at least the same level of club good.

The ratio h_{t+1}^i/h_t^i differs between individuals, even if they belong to the same club. Denoting by $\overline{h}_t^j$ the average level of human capital for club $\widehat{C}_t^j$, we denote by γ_t^j the growth rate of the average level of human capital of the club $\widehat{C}_t^j$:

$$\gamma_t^j \equiv \frac{\overline{h}_{t+1}^j}{\overline{h}_t^j} = \frac{\sum_{z \in \widehat{C}_t^j} h_{t+1}^z}{\sum_{z \in \widehat{C}_t^j} h_t^z}$$

is such that:

$$\gamma_t^j = \kappa \left(\frac{1-\mu}{\widehat{n}_t^j}\right)^{1-\beta} \left(\frac{\mu\widehat{\tau}}{A(\widehat{n}_t^j)}\right)^\beta \frac{\sum_{z \in \widehat{C}_t^j} (h_t^z)^{1-\beta}}{(\overline{h}_t^j)^{1-\beta}}. \tag{10}$$

For sake of simplicity, we refer to γ_t^j as the club j's growth rate.[4] We use γ_t^j as a convenient index of the whole process of growth characterizing a club in the core partition. Given what we said above on the inexistence of a systematic process of individual catching-up in human capital over time, it is immediate to deduce that a priori there cannot be either a systematic process of catching-up between clubs. That is, we do not know a priori whether $\frac{\gamma_{t+1}^j}{\gamma_{t+1}^{j'}} > \frac{\gamma_t^j}{\gamma_t^{j'}}$, $\forall j < j'$. Actually, we have no reason to suspect that the number of clubs will decrease or increase or remain constant.

Phases of catching up and increasing gaps can alternate. If there is a long enough period of inter-club catching-up, sooner or later, the partition will change. But in turn this will affect the dynamics of accumulation and may end the catching-up process, at least temporarily.

[4] γ_t^j is not the average of individual growth rates for individuals belonging to $\widehat{C}_t^j$ due to the non-linearities at work in this economy. But we will see later that, in the long run, the growth rate γ_t^j and the average of individual growth rates for members of $\widehat{C}_t^j$ are similar.

Still, interestingly, despite the complexity of the dynamics in this model, there is a direction towards stationarity, as we shall see in the next section.

4.2 Permanent core-partition

We want to know whether given the above characteristics of the socio-economic dynamics, there is still some form of stationary state?

Such a steady state has both a social and an economic dimensions. On the one hand, it must be characterized by some form of permanency in the fragmentation of society: we have seen that a change in the partition over time significantly alters the dynamics of accumulation, of inequality and of growth. On the other hand, it must also be characterized by some form of balancedness in the growth pattern, which in turn implies some permanency in human capital distribution, that is inequality. It is important to insist on the fact that the steady state is not inconsistent with fragmentation, that is the presence of several clubs. If it turns out that the the steady state partition involves more than one club, then clubs grow at different rate depending on the level of human capital of their members.

As the stability of the partition is a precondition for obtaining a steady state growth pattern, we can say that it will be reached before any steady state growth pattern. There may exist a given date t^* such that from t onwards, the core partition does not change anymore: all pivotal agents remain the same over time. We refer to this partition as the $t^* - permanent$ partition and we offer the following:

Definition 3. *A core partition* $\widehat{\mathcal{C}}$ *is said to be* $t^* - permanent$ *when starting at a given date* t^*, $\widehat{\mathcal{C}}_t = \widehat{\mathcal{C}}, \forall t \geq t^*$.

Addressing the issue of the convergence toward a dynamically stable segmentation of the economy, we can prove the following:

Proposition 4. *For any society, there always exists a unique finite date* t^* *such that a permanent core partition forms.*

Proof: See appendix.

This is a rather surprising result, given the a priori complexity of the dynamics. Actually, we cannot say much about the transition path toward the $t^* - permanent$ core partition: clubs may increase or decrease in size over time, except the first club which can only weakly expand. The intuition underlying is as follows. First, we know that at any date t, due to the welfare ordering property of the core partition (see item (iv) of Proposition 1), individuals in the first club get the highest level of welfare. Second, from the intra-club human capital convergence process at work (see item (i) of Proposition 3), individuals in the richest club become progressively more similar, thus increasing their willingness to interact in the same club. Hence, the first club can be modified over time only because new members are accepted.

However, the proposition makes clear that this process, unless special circumstances, does not go toward the eventual disappearance of any social fragmentation, that is the creation of the grand coalition: at a given period, the society necessarily reaches a stable partition, which in general, will imply several clubs. This will become apparent in the sequel.

Importantly, once the $t^* - permanent$ core partition has formed, this does not mean that the economic side as such has reached the steady-state. At t^*, once the memberships have stabilized, individuals will still differ and there will still be a growing process which has no reason to be steady. Actually, we can prove that the economy will ultimately converge to a steady-state growth pattern.

In the following, variables without a subscript t refer to the $t^* - permanent$ core partition. Let us characterize the $t^* - permanent$ core partition by the following:

Proposition 5. *(i) Whithin clubs belonging to the $t^* - permanent$ core partition, there is eventually perfect homogeneity between members:*

$$\lim_{t\to\infty} h_t^i = \overline{h}_t^j, \ \forall i \in \widehat{C}^j.$$

(ii) The growth rate of the average level of human capital of the club $\widehat{C}^j$, is constant over time and such that:

$$\gamma^j = \kappa(1-\mu)^{1-\beta} (\mu\widehat{\tau})^\beta \left(\frac{\widehat{n}^j}{A(\widehat{n}^j)}\right)^\beta. \tag{11}$$

(iii) If the $t^* - permanent$ core partition is composed by more than one club (i.e. the grand coalition does not form), then there is no inter-club convergence in growth rate:

$$\frac{\gamma^j}{\gamma^{j'}} \geq 1, \forall j < j' < J.$$

Proof: See appendix.

Point (i) is easy to understand. Once the core partition has formed, the intra-club convergence logically leads to full homogeneity: this is due to the fact that the club good, club knowledge, is relatively more efficient for lesser endowed agents than for richer ones. Of course, eventually, within a club, the growth rate for any individual capital accumulation is equal to the club growth rate γ^j.

Point (ii) states that, even if the growth pattern is balanced, the growth rates of clubs are not necessarily equal across clubs. This is due to the fact that in the $t^* - permanent$ core partition, the sizes of clubs may differ. When time passes, given the intra-club convergence property, these differences in sizes imply that the efficiency of the club good in fostering capital accumulation and therefore growth are in the limit the only factor of differenciation. If they do differ, the steady-state growth rates differ.

Point (iii) states that the fragmentation of the core partition is inconsistent with any general and aggregate catching up process. This is understandable: if there were a steady state catching up process of richer clubs by poorer ones, eventually the poorer agents would be at least as rich as the richer agents, and the condition of fragmentation, given by the inequalities defining the pivotal agents would not be met. Hence, in a fragmented $t^* - permanent$ core partition, the richer clubs grow at least as fast as the poorer ones. Despite the intra-club homogeneity, there is a tendency to inter-club, that is aggregate, divergence, at least in endowments (if the growth rates are equal), and even in growth rates.

5 Particular inequality schedules

In this section, we emphasize the fact that in such a framework, inequality dynamics may exhibit different history-dependent steady states. However, along the transitional path, the interplay between human capital accumulation and fragmentation can lead to complex inequality dynamics. Potentially, as long as human distribution evolves, the core partition can change. Multiple trajectories can then arise making difficult an analysis aiming to predict the $t^* - permanent$ core partition that would emerge depending on the initial pattern of human capital inequality. So we shall consider particular inequality schedules and show the characteristics of the resulting steady state core partition.

To this aim, it is useful to define the human capital ratio between two different individuals $i, j \in S$ at date t, $\lambda_t^{i,j} = \frac{h_t^i}{h_t^j}$. Then we define an inequality schedule as follows:

Definition 4. *At date t, a society S is characterized by an inequality schedule* $\mathcal{S}_t = \{\lambda_t^{1,2}, ..., \lambda_t^{i,i+1}, ..., \lambda_t^{N-1,N}\}$.

Rewriting conditions defining pivotal agents will prove useful:

$$h_t^{p_t^j} \geq \sum_{z=p_t^{j-1}+1}^{p_t^j-1} h_t^z \left(\frac{A(\widehat{n}_t^j)}{A(\widehat{n}_t^j - 1)} - 1 \right) \Leftrightarrow 1 \geq \sum_{z=p_t^{j-1}+1}^{p_t^j-1} \prod_{x=z+1}^{p_t^j} \lambda_t^{x-1,x} \left(\frac{A(\widehat{n}_t^j)}{A(\widehat{n}_t^j - 1)} - 1 \right) \tag{12}$$

and

$$h_t^{p_t^j+1} < \sum_{z=p_t^{j-1}+1}^{p_t^j} h_t^z \left(\frac{A(\widehat{n}_t^j + 1)}{A(\widehat{n}_t^j)} - 1 \right) \Leftrightarrow 1 < \sum_{z=p_t^{j-1}+1}^{p_t^j} \prod_{x=z+1}^{p_t^j+1} \lambda_t^{x-1,x} \left(\frac{A(\widehat{n}_t^j + 1)}{A(\widehat{n}_t^j)} - 1 \right) \tag{13}$$

where $p_t^0 + 1 = 1$ and $p_t^{J_t} = N$.

5.1 The case of initial constant inter-individual inequality ratio.

First, let us focus on an inequality schedule such that $\lambda_0^{i,i+1} = \lambda$, $\forall i \in S$. This special case means that the initial endowment ratio between two successive individuals does not depend on their exact ranking in the distribution. In the case of static core-partition, JKM remark that this implies that the arbitrage condition is the same for any individual, and that this leads to equal sized clubs. Here we can enlarge the analysis and prove the following

Proposition 6. *(i) When at date* $t = 0$, *a society S is characterized by an inequality schedule* $\mathcal{S} = \{\lambda_0, ..., \lambda_0, ..., \lambda_0\}$, *then the* $0 - permanent$ *core partition exists.*

(ii) It is such that $n_t^j = \widehat{n}, \forall j, t,$ *and* $\gamma_t^j = \widehat{\gamma}_t, \forall j;$

(iii) An increase of the ratio λ *leads a core-partition wih clubs the size of which cannot increase. Similarly, an increase in congestion (a steeper* $A(.)$*) leads a core-partition wih clubs the size of which cannot increase.*

Proof. See appendix.

The two first properties mean that the core-partition immediately forms with equal sized clubs. At period 0,the initial partition forms and due to the fact that the arbitrage conditions for any individual are the same, clubs of equal size form in the core-partition.

Since any individual benefits in the same way of the club externality, each individual human capital grows at the same rate, and following Proposition 5 club growth rates are equal. This precludes any catching up process. The inequality schedule keeps the same property of constant gaps across agents from period 0 to period 1.

Hence in period 1, the problem of the equilibrium is just the same as in period 1. Levels of human capital have changed but the decisions to form clubs are based on the ratio λ. Therefore this explains why the same partition forms again in period 1.

This argument can be reproduced sequentially, period after period, and clubs after clubs. This explains why the permanent core-partition forms immediately, in period 0.

As about the third property, it is difficult to assess the impact of an increase in λ as we do not reason with a continuum of agents. However the result is easy to understand. Any agent is willing to accept a poorer agent if she is able to contribute enough to the club good. The less difference in human capital and thus, in taxing capabilities, the better it is. An increase in inequality (in λ) means that it is more difficult for a relatively poor agent to be accepted by richer agents. Consequently, it leads to a (weakly) more segmented society.

Similarly, an steeper congestion cost function may be interpreted as an increase in congestion: an increase in the size of a club, ceteris paribus, makes the production process of the club good less efficient, and reduces the appeal of accepting an additional agent in the club. This means that an increase in the curvature of the cost function leads to a (weakly) more segmented society.

Given the Proposition 5, it is immediate to remark that: $\lim \gamma_t = \kappa(1-\mu)^{1-\beta} (\mu\widehat{\tau})^{\beta} \left(\frac{\widehat{n}}{A(\widehat{n})}\right)^{\beta}$. Immediate from (11).

Finally, we remark that, when at date $t = 0$, a society S is characterized by an inequality schedule $S = \{\lambda_0, ..., \lambda_0, ..., \lambda_0\}$, and $\lambda_0 < \lambda_0^*$, then the grand coalition forms. λ_0^* is such that:

$$1 \geq (\lambda_0^*)^N \left(\frac{A(N)}{A(N-1)} - 1\right)$$

If inequality, as measured in this case by λ_0, is not too large, then there is no segmentation and the entire economy forms a single club. Again, this illustrates the fact that what matters for any individual is the relative ability to contribute of any one poorer than himself. Hence if the poorest agent in the economy is not "too far down" the richest agent, given the congestion costs, he will be able to overcome these costs and be accepted by the richest agent in his club. Hence the grand coalition forms.

5.2 The case of increasing inter-individual inequality ratios

Second, let us consider now an inequality schedule such that $\lambda_0^{i,i+1} \leq \lambda_0^{i+1,i+2}$, $\forall i \in S$. Such a series of inequalities corresponds to an increasing gap between two successive agents with their ranking in the initial distribution: two successive rich agents are "closer" than two successive poorer agents. GKM study this case in the static environment sharing some crucial properties with the present one. They prove that a richer clubs are larger than a poorer club: the club size decreases with the ranking of a club. Again this comes from the fact that the more alike two agents are, the more likely they are to accept each other, because the richer of them knows that the poorer has enough capacity to contribute to his own benefit. The same property is at work here as well and so we can show the following

Proposition 7. (i) When at date $t = 0$, a society S is characterized by an inequality schedule $S = \{\lambda_0^{1,2}, ..., \lambda_0^{i,i+1}, ..., \lambda_0^{N-1,N}\}$, with $\lambda_0^{i,i+1} \leq \lambda_0^{i+1,i+2}$, $\forall i$, then the $0-$permanent core partition exists.

(ii) It is characterized by $n_t^j > n_t^{j+1}, \forall j, t.$

(iii) More congestion implies lower club size at any period.

Proof. See appendix.

Again the striking result is that the permanent core-partition forms immediately. This can be explained as follows/; As in JKM, at period 0,a partition forms with the first club being the largest and the richest. The second club can therefore not compensate by a larger size, meaning coalizing more people, the fact that it is formed of poorer people. Hence it grows less rapidly in period 0, and the inequality ratio between the club 1's pivotal agent and the richest agent in club 2 increases from period 0, to period 1. As a consequence, being farther away from this agent, he still cannot meet the condition for being accepted in club 1. And club 1 does not increase in size from period 0 to period 1. Can it shrink in size. Again no, because inside this club, there is a catching-up process: the pivotal agent in period 1 gets closer to the richest agent than he was in period 0. Hence, he still meets the condition for being accepted in the first club (remember that they are based on relative human capitals, nor on the absolute levels).

Repeat the argument for any two successive clubs at any period and this explains why the permanent core-partition forms immediately.

The last property is explained as before and does not need any further comment.

Of course, the first club grows more rapidly than any other club in any period but it is not true that the growth rate is constant over time and across clubs, as in the previous case. This comes from the fact that given the . But by direct application of Proposition 1, we know that $\lim \gamma_t^j = \kappa(1-\mu)^{1-\beta} (\mu\widehat{\tau})^{\beta} \left(\frac{\widehat{n}^j}{A(\widehat{n}^j)}\right)^{\beta}$ and $\gamma^j > \gamma^{j+1}$, $\forall j$, from (11). Not only is there no convergence in levels of human capital i.e. in income, but there is a strict ranking in growth rates as well, unlike in the previous case. The steady-state implies ever-diverging trajectories in human capital accumulation. This comes from the fact that, since richer clubs are bigger, the growth engine is more efficient for these clubs.

5.3 The case of decreasing inter-individual inequality ratios

Third, let us consider now an initial inequality schedule such that $\lambda_{i,i+1,0} > \lambda_{i+1,i+2,0}$, $\forall i \in S$. Such a series of inequalities means that the further "up" in the initial distribution we are, the more heterogeneity there is between two successive persons. The more dissimilar (in terms of available human capital) two successive agents are, the richer they are. In the static case studied by JKM, they proved that the poorer clubs were the larger. Again, this is an application of the fact that in this type of environment, the clubs form according to a logic of homogeneity. A decrease in human capital differences, corresponding to more homegeneity, leads to larger clubs. So now, the two elements, size and individual wealth, have a conflicting role, whereas in the previous case, they played in the same direction. Now a poorer club may overcome the fact that it is formed with poorer agents by the fact that it coalizes many of them. As a result, it may happen that the club good is larger in a poorer club (a club formed of poorer agents) and then the growth engine played by this good is more powerful: the growth rate of a poorer club may be larger than the growth rate of a richer club. Hence, it may happen that from one period to another, the richest agent in a given club who could not be accepted in a richer club, now is able to be accepted. Hence, there is no reason to believe that the permanent core-partition can be formed immediately when the initial inequality schedule exhibits this characteristics of increasing inequality ratios. Actually we can formally prove the following

Proposition 8. *When at date $t = 0$, a society S is characterized by an inequality schedule $\mathcal{S} = \{\lambda_0^{1,2}, ..., \lambda_0^{i,i+1}, ..., \lambda_0^{N-1,N}\}$, with $\lambda_0^{i,i+1} > \lambda_0^{i+1,i+2}$, $\forall i$, the date t^* at which the permanent core partition establishes may be bigger than 0.*

Proof: see Appendix.

The fact that some catching-up process can take place over time, does not entail that the grand coalition necessarily forms when the inequality schedule satisfies this property of decreasing inequality ratios. This can be seen by a continuity argument from the case of constant inequality ratios. Basically, we have to conclude that for this case, the dynamics may be quite complex, given the contradicting roles played by club size and individual wealth of the membership.

5.4 Comparing economic performances for differently unequal societies

Up to now, we studied the consequences on segmentation and growth of the steepness of the initial human capital distribution. We are also interested in the comparison both in terms of segmentation and growth of two societies differing in terms of "inequality". In other words, we want to address the hotly debated issue of the relationship between inequality and growth, but with the idea of introducing social fragmentation as a key causal element in this relationship: is a more unequal growing more (or less) rapidly, because it is more (or less) segmented?

Before answering this question, several difficulties have to be overcome.

First, inequality is a multi-dimensional concept and it is impossible to say that "one society is more unequal than another one", without further restrictions. Here we restrict our analysis so that we can use the concept of Lorenz-dominance. We shall consider two equal sized societies S and $\widetilde{S}$,with two sequences of initial human capital endowment $L_0 = \left\{h_0^1, ..., h_0^N\right\}$ and $L_0 = \left\{\widetilde{h}_0^1, ..., \widetilde{h}_0^N\right\}$ and such that L_0 is Lorenz-dominating $\widetilde{L}_0$. Because of this Lorenz-dominance, we can say that $\widetilde{S}$ is more unequal than S.

Second, how to define (aggregate) growth in a segmented society, with autonomous clubs that may grow at different rates? Here we use the following definition: we define the aggregate growth rate of an economy S as $\Gamma_t(S) \equiv \frac{1}{N}\sum_{i=1}^{N}\gamma_t^i$. We are interested in the comparison of the aggregate growth rates for two economies, when they have reached their permanent partition and the steady states have been reached. The limit growth rate is defined as follows:

$$\Gamma(S) \equiv \frac{1}{N}\sum_{j=1}^{J} n^j \gamma_\infty^j. \tag{14}$$

Third, we have to define an increase in segmentation. Following HJKM, we we shall use the following

Definition 1 *A society S is (weakly) more segmented in period t than a society $\widetilde{S}$ if the number of non-residual clubs in the core partition $\widehat{\mathcal{C}}_t$ associated with S, $J-1$, is at least equal to the number of non-residual clubs in the core partition $\widetilde{\widehat{\mathcal{C}}}_t$ associated with $\widetilde{S}$, $\widetilde{J}-1$, and the j-th club in S^{E_2} is never larger than the j-th club in S^{E_1}, for $j < J_1$.*

Can we say something about $\Gamma(S)$ and $\Gamma\left(\widetilde{S}\right)$? Can we order these two aggregate growth rates? Actually the answer to this question is no, as we are able to make the following

Claim 9. *Assuming two different inequality schedules such that $\mathcal{S}$ Lorenz-dominates $\widetilde{\mathcal{S}}$, a society S associated with $\mathcal{S}$ does not necessarily grow more rapidly than $\widetilde{S}$ associated with $\widetilde{\mathcal{S}}$.*

Proof. See appendix.

It appears that even we take into consideration the fact that agents form clubs and therefore that society is segmented due to the presence of congestion costs, we cannot obtain an unambigous relationship between inequality and growth.

The proof rests on the simple case where the two inequality schedules are such that the permanent core-partitions are immediately reached, with equal sized clubs, with S being weakly more segmented than $\widetilde{S}$. In this simple case, it is obvious to show that the congestion function plays a role such that it may overcome the impact of different club sizes. This explains the claim. The lesson is that inequality is not the sole cause of differences in (long term) growth rates: the way agents form communities and the resulting congestion effects cannot be neglected. It is unlikely that empirical studies can find an unambigous undisputable link between inequality and growth.

It would be interesting to obtain the same ambiguous result in less restrictive cases. Our view is that it should be supported in these more complex cases.

6 Conclusion

In this paper, we offer a method to study the socio-economic dynamics of societies, based on coalition theory. We develop a view on the growth process, taking fully into account the fact that agents form communities, and that the borders nor the shape of a growing economy are given. Weith explicit and rigourous micro-foundations, we are able to cast a new light where the segmentation of an economy and its dynamic play a critical role in the growth process itself. The economic and social dynamics constantly interact and cannot be separated: growth alters the way society is segmented, segmentation itself affects the way the durable factor accumulates and growth develops.

We prove the existence of a socio-economic equilibrium period after period, leading to a steady state. This steady-state is not necessarily characterized by economic convergence unless the society eventually forms a unique community.

Complex as it may appear, the model we use is relatively simple. Several extensions are worth investigating.

The first one should be to introduce interclub externalities, such as aggregate human capital, for example. Inter-coalitions spillovers are a difficult issue in coalition theory. However, here this could be done without too much difficulties, as we can play on different dates.[5]

A limitation of the model is that the ranking of individuals/families is not modified over time. In the real world, we witness much less stability. And family trajectories may cross over time. An interesting approach

[5] We owe this suggestion to Fernando Jaramillo.

to this question would be to introduce shocks to human capital and look at the amount of shocks necessary to ensure that there is eventual convergence over time.

We relied on a simple endogenous growth mechanism, based on human capital à la Lucas (1988). Other mechanisms are able to generate endogenous growth, linked to various externalities, and can be linked to the gathering of individuals, working together or sharing some resources. It would be interesting to apply the endogenous formation of communities to these alternative frameworks. In particular, R&D clusters play a role in Schumpeterian growth theory and appear to be quite empirically effective: the consensus is that part of the growth gap between the US and European countries is due to differences in the financing and the use of innovations. The relationship between technological communities and growth could be investigated using the type of analysis developed here.

Here financial markets play no role. This is crucial as it assures that an individual can only rely on his club's mates for growing. Finance in many ways can be seen as a way to overcome physical barriers. At the same time, we know how segmented is the financial sphere and that financial clubs exist.

Finally, congestion is linked to the mere number of agents in a club, not to their characteristics. It would be worth to explore the dynamic consequences of non anonymous crowding out.

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APPENDIX

A Proof of Proposition 1.

1. Proof of (i).

Let us compute the preferred tax rate by individual i in club C_j. The first-order conditions are:

$$-\frac{1}{1-\tau_t^i} - \frac{\rho\beta}{\tau_t^i} = 0$$

which leads to the most preferred fraction of working time:

$$\widehat{\tau}_t^i = \frac{\rho\beta}{1+\rho\beta}. \tag{15}$$

There is unanimity across time, clubs and agents. Hence the solution is $\widehat{\tau} = \frac{\rho\beta}{1+\rho\beta}$.

2. Proof of (ii).

Replacing (15) in (1), and using (2) and (3), the indirect utility function for agent i in coalition $\widehat{C}_t^j$ can be expressed as follows:

$$V^i(\widehat{C}_t^j) = \ln\left(\left(\frac{1}{1+\rho\beta}\right)\mu h_t^i\right) + \rho\ln\kappa\left((1-\mu)\,h_t^i\right)^{1-\beta} + \rho\ln\left(\frac{\sum_{z\in\widehat{C}_t^j}\mu(\frac{\rho\beta}{1+\rho\beta})h_t^z}{A(\widehat{n}_t^j)}\right)^{\beta}.$$

The core partition exists following Farrell and Scotchmer's (1985) proof of their theorem.[6]

3. Proof of (iii).

Since (6) is increasing in the individual human capital and decreasing in the size of the clubs, the proof of the consecutivity property is identical to Proposition 1's in JKM (2003).

The proof of uniqueness is as follows. Consider a consecutive club whose richest member $\overline{i}$ is endowed with $\overline{h}_t$ and the poorest $\underline{i}$ with $\underline{h}_t$. An agent i^* is admitted in a consecutive club C_t^j, formed with n_t^j members, as long as the benefit from the tax rate encovers the congestion effect, i.e. given (6):

$$\left[\ln\left(\sum_{z\in C_t^j} h_t^z + h_t^{i^*}\right) - \ln\left(\sum_{z\in C_t^j} h_t^z\right)\right] - \left[\ln\left(A(n_t^j+1)\right) - \ln\left(A(n_t^j)\right)\right] > 0. \tag{16}$$

Since the first term in brackets is monotonously decreasing with the membership in a club and the second term in brackets is monotonously increasing with n_t^j, given the assumption we made on congestion costs, there is a unique individual $\underline{i}$ such that (16) is true for any $i, \overline{i} \leq i \leq \underline{i}$ and untrue for $\underline{i} < i$. The uniqueness of the core partition follows.

4. Proof of (iv).

[6] See also Konishi, Sönmez et al. (2002).

Consider $\widehat{C}_t^1$, $\widehat{C}_t^2$ and the consecutive club C_t^1 defined as: $\{1, ..., i^*\}$ with $i^* = \widehat{n}_t^2$. Since they belong to the core partition, and $\sum_{z \in C_t^1} h_t^z > \sum_{z \in \widehat{C}_t^2} h_t^z$, we deduce that:

$$V^i(\widehat{C}_t^1) > V^i(C_t^1) > V^i(\widehat{C}_t^2).$$

The same argument may be repeated for any $\widehat{C}_t^i$ and $\widehat{C}_t^{i+1}$. This completes the proof of (iv).

B Proof of Proposition 3.

1. Proof of (i). At any date t, consider two individuals i and i' with $i' > i$, $i \in \widehat{C}_t^j$ and $i' \in \widehat{C}_t^j$. Thus from (2) we have

$$\frac{h_{t+1}^i}{h_{t+1}^{i'}} = \left(\frac{h_t^i}{h_t^{i'}}\right)^{1-\beta} < \frac{h_t^i}{h_t^{i'}}.$$

Still, $h_t^i > h_t^{i'}$, $\forall t$.

2. Proof of (ii). Given the welfare ordering property of the core partition, $\widehat{G}_t^j \geq \widehat{G}_t^{j'}$ for $j \leq j'$. This implies that, for $i' > i$, $i \in \widehat{C}_t^j$ and $i' \in \widehat{C}_t^{j'}$, we have

$$\frac{h_{t+1}^i}{h_{t+1}^{i'}} = \left(\frac{h_t^i}{h_t^{i'}}\right)^{1-\beta} \left(\frac{\widehat{G}_t^j}{\widehat{G}_t^{j'}}\right)^{\beta} \geq \left(\frac{h_t^i}{h_t^{i'}}\right)^{1-\beta} > 1. \tag{17}$$

This equation means that the ordering of individuals according to wealth remains unaltered through time.

C Proof of Proposition 4.

Step 1. We focus on the first clubs of the successive (i.e. over time) core partitions. We prove in this step that the first club of the core partition at any $t+1$ is at least as large as the first club in the core partition at t: $n_t^1 \geq n_{t-1}^1, \forall t$.

Consider the "initial" pivotal agent p_0^1 of the "initial" first club $\widehat{C}_0^1$. We can prove that he will always belong to the sequence of first clubs $\left\{\widehat{C}_t^1\right\}_{t>0}$, i.e. the size of the first club can only (weakly) increase. This is true if the two following properties are satisfied:

(i) for any $t > 0$, $V^{p_0^1}(\widehat{C}_t^1) > V^{p_0^1}(\pounds)$, where $\pounds$ is any club in S : this comes directly from the property of the core partition and the welfare ordering.

(ii) for any $t > 0$,

$$h_t^{p_0^1} \geq h_t^{p_t^1}. \tag{18}$$

Let us start from the definition of the initial pivotal agent p_0^1 for $\widehat{C}_0^1$:

$$h_0^{p_0^1} \geq \sum_{z=1}^{p_0^1-1} h_0^z \left(\frac{A(\widehat{n}_0^1)}{A(\widehat{n}_0^1 - 1)} - 1\right) \Leftrightarrow 1 \geq \sum_{z=1}^{p_0^1-1} \frac{h_0^z}{h_0^{p_0^1}} \left(\frac{A(\widehat{n}_0^1)}{A(\widehat{n}_0^1 - 1)} - 1\right).$$

Obviously:

$$1 \geq \sum_{z=1}^{p_0^1-1} \frac{h_0^{\tilde{z}}}{h_0^{p_0^1}} \left(\frac{A(\widehat{n}_0^1)}{A(\widehat{n}_0^1-1)} - 1 \right) > \sum_{z=1}^{p_0^1-1} \left(\frac{h_0^{\tilde{z}}}{h_0^{p_0^1}} \right)^{(1-\beta)} \left(\frac{A(\widehat{n}_0^1)}{A(\widehat{n}_0^1-1)} - 1 \right).$$

Hence, from the individual-human-capital growth (2),

$$h_1^{p_0^1} \geq \sum_{z=1}^{p_0^1-1} h_1^{\tilde{z}} \left(\frac{A(\widehat{n}_0^1)}{A(\widehat{n}_0^1-1)} - 1 \right)$$

and p_0^1 is accepted in $\widehat{C}_1^1$. By recurrence, this is true for $\widehat{C}_t^1$. This implies that once agent p_0^1 lives in the same club with richer agents in the initial period 0, he will be always accepted by them at any date t. This proves (18).

Step 2. We now prove that there exists a $\widetilde{t}_1$ such that $\widehat{n}_0^1 \leq \widehat{n}_t^1 = \widehat{n}_{t+1}^1 \leq N$, $\forall t > \widetilde{t}_1$. From Step 1, we know that $n_t^1 \geq n_{t-1}^1$, $\forall t > 0$. Hence, either there exists $\widetilde{t}_1$ such that $\widehat{n}_t^1 = \widehat{n}_{t+1}^1 < N$, $\forall t > \widetilde{t}_1$, or not. If not, there exist dates t_1^* such that $\widehat{n}_{t_1^*}^1 < \widehat{n}_{t_1^*+1}^1 \leq N$. But since $max(\widehat{n}_t^1) = N$, this implies that there exists a t_1^{**} such that $\widehat{n}_t^1 = N$, $\forall t \geq t_1^{**}$. This completes the proof of Step 2 and proves that at some finite date $\widetilde{t}_1$, a constant $\widehat{C}^1 = \widehat{C}_t^1, \forall t > \widetilde{t}_1$, forms.

Step 3. We now prove that as long as the first club has not reached his permanent configuration, some clubs cannot have reached theirs. From Step 1, if $\widehat{C}_t^1 \neq \widehat{C}^1$, then $\exists \widehat{t}, t \leq \widehat{t} < \widetilde{t}_1$, such that $p_{\widehat{t}}^1 > p_{\widehat{t}-1}^1$. The consecutivity property implies, that $p_{\widehat{t}-1}^1 + 1$, which at time $\widehat{t}-1$ belonged to $\widehat{C}_{\widehat{t}-1}^2$, belongs at date $\widehat{t}$ to $\widehat{C}_{\widehat{t}}^1$. Hence, since $\widehat{C}_{\widehat{t}-1}^2 \neq \widehat{C}_{\widehat{t}}^2$, then $\widehat{C}_{\widehat{t}-1}^2 \neq \widehat{C}^2$.

Step 4. Starting at $t \geq \widetilde{t}_1$, consider $\widehat{C}_t^2$. This club is the first club of the subset $\widehat{\mathcal{C}}_t \backslash \widehat{C}^1$ of the successive core partitions of $\widehat{\mathcal{C}}_t \backslash \widehat{C}^1$. Hence, we can apply the previous steps 1-3, and deduce that there exists a date $\widetilde{t}_2 \geq \widetilde{t}_1$ that $\widehat{n}_{\widetilde{t}_2}^2 \leq \widehat{n}_t^2 = \widehat{n}_{t+1}^2 \leq N$, $\forall t > \widetilde{t}_2$ and therefore $\widehat{C}_t^2 = \widehat{C}^2, \forall t > \widetilde{t}_2$.

[Beware: this does not mean that the individuals belonging to $\widehat{C}^1$ behave in isolation from the rest of society from $\widetilde{t}_1$ on and do not interact anymore with the rest of society. At any date, these agents remake their decisions about accepting new members in the first club, but they always answer negatively. Hence, the size and membership of the second club can only vary by accepting new poorer members from subsequent clubs, as its members have no interest in reducing $C_t^{2\prime}$s size.]

Step 5. Using a similar reasoning for the following clubs, we sequentially obtain permanent clubs $\widehat{C}^j$, until a date t^*, where all clubs have reached their permanent configuration. The $t^* - permanent$ core partition then obtains.

Let us show that t^* is unique. First, let us focus on the first club. Suppose by contradiction that there are two dates $\widetilde{t}_1$ and $\widetilde{\widetilde{t}}_1$, $\widetilde{t}_1 > \widetilde{\widetilde{t}}_1$, such that:

(i) For any $t > \widetilde{t}_1$, $\widehat{C}_t^1 = \widehat{C}^1$, $\widehat{C}^1 \subset \widehat{\mathcal{C}}_t$,

(ii) For any $t > \widetilde{\widetilde{t}}_1$, $\widehat{C}_t^1 = \widehat{C}^1$, $\widehat{C}^1 \subset \widehat{\mathcal{C}}_t$.

Hence, for t with $\widetilde{t}_1 \leq t \leq \widetilde{\widetilde{t}}_1$, there are two permanent first clubs, contradicting the uniqueness property of the core-partition.

Second, the same argument applies for any higher indexed club. Hence, this leads to a unique t^*.■

D Proof of Proposition 5.

1. Proof of (i). At the $t^* - permanent$ core partition, for two individuals i and i' with $i' > i$ and $i, i' \in \widehat{C}^j$, from (2), we have at $t \geq t^*$

$$\frac{h_{t+1}^i}{h_{t+1}^{i'}} = \left(\frac{h_t^i}{h_t^{i'}}\right)^{1-\beta} < \frac{h_t^i}{h_t^{i'}} \tag{19}$$

and thus, $\lim_{t\to\infty} h_t^i = \lim_{t\to\infty} h_t^{i'} = \overline{h}_t^j$.

2. Proof of (ii). The growth rate of each club $\widehat{C}^j$ derives from (**??**):

$$\gamma_t^j = \frac{\kappa}{\widehat{n}_t^j}(1-\mu)^{1-\beta}\left(\mu\widehat{\tau}\frac{\sum_{z\in\widehat{C}_t^j} h_t^z}{A(\widehat{n}_t^j)(\overline{h}_t^j)}\right)^{\beta}\frac{\sum_{z\in\widehat{C}_t^j}(h_t^z)^{1-\beta}}{(\overline{h}_t^j)^{1-\beta}}.$$

Using (19), (11) obtains.

3. Proof of (iii). Suppose there is more than 1 club in the $t^* - permanent$ core partition. If $\gamma^j < \gamma^{j'}, j < j'$. But, then $\lim\limits_{t\to\infty} h^{j'} = \lim\limits_{t\to\infty} h^j$ which contradict the pivotal agent condition and there cannot be more than one club. Then, given (11), $n^j \geq n^{j'}, j < j'$ implies $\gamma^j \geq \gamma^{j'}, j < j'$.

E Proof of Proposition 6.

Let us first characterize the core partition of a society S with the inequality schedule $\mathcal{S} = \{\lambda_0, ..., \lambda_0\}$.

According to (12) and (13), the core partition $\widehat{\mathcal{C}}_0$ can be characterized by a sequence of pivotal agents, i.e. $\{p_0^1, ..., p_0^j, ..., p_0^J\}$ with:

$$h_0^{p_0^j} \geq \sum_{z=p_0^{j-1}+1}^{p_0^j-1} h_0^z\left(\frac{A(\widehat{n}_0^j)}{A(\widehat{n}_0^j-1)} - 1\right) \Leftrightarrow 1 \geq \sum_{z=p_0^{j-1}+1}^{p_0^j-1} \lambda_0^{p_0^j-z}\left(\frac{A(\widehat{n}_0^j)}{A(\widehat{n}_0^j-1)} - 1\right) \tag{20}$$

and

$$h_0^{p_0^j+1} < \sum_{z=p_0^{j-1}+1}^{p_0^j} h_0^z\left(\frac{A(\widehat{n}_0^j+1)}{A(\widehat{n}_0^j)} - 1\right) \Leftrightarrow 1 < \sum_{z=p_0^{j-1}+1}^{p_0^j} \lambda_0^{p_0^j+1-z}\left(\frac{A(\widehat{n}_0^j+1)}{A(\widehat{n}_0^j)} - 1\right) \tag{21}$$

The conditions for defining a pivotal agent of a given club are identical for any club, as they do not depend on any level of human capital. Hence, whatever $j \in \{1, ..., J-1\}$, $\widehat{n}_0^j = \widehat{n}_0^{j-1}$. Then the proof of this proposition follows from the proof of Proposition 7 using this equality. ■

F Proof of Proposition 7.

1. Proof of (i)

Let us first characterize the core partition of a society S with the inequality schedule $\mathcal{S} = \{\lambda_0^{1,2}, ..., \lambda_0^{i,i+1}, ..., \lambda_0^{N-1,N}\}$ with $\lambda_0^{i,i+1} \leq \lambda_0^{i+1,i+2}, \forall i \in S$.

According to Proposition 3 of JKM [2003], we know that whatever $j \in \{1, ..., J-1\}$, $\widehat{n}_0^j \leq \widehat{n}_0^{j-1}$ if and only if $\lambda_0^{i,i+1}$ increases with the rank, i.e. $\lambda_0^{i,i+1} \leq \lambda_0^{i+1,i+2}$, $\forall i \in S$.

Thus, according to (12) and (13), the core partition can be characterized by a sequence of pivotal agents, i.e. $\{p_0^1, ..., p_0^j, ..., p_0^J\}$ with:

$$h_0^{p_0^j} \geq \sum_{z=p_0^{j-1}+1}^{p_0^j-1} h_0^z \left(\frac{A(\widehat{n}_0^j)}{A(\widehat{n}_0^j - 1)} - 1 \right) \Leftrightarrow 1 \geq \sum_{z=p_0^{j-1}+1}^{p_0^j-1} \prod_{x=z+1}^{p_0^j} \lambda_0^{x-1,x} \left(\frac{A(\widehat{n}_0^j)}{A(\widehat{n}_0^j - 1)} - 1 \right)$$

and

$$h_0^{p_0^j+1} < \sum_{z=p_0^{j-1}+1}^{p_0^j} h_0^z \left(\frac{A(\widehat{n}_0^j + 1)}{A(\widehat{n}_0^j)} - 1 \right) \Leftrightarrow 1 < \sum_{z=p_0^{j-1}+1}^{p_0^j} \prod_{x=z+1}^{p_0^j+1} \lambda_0^{x-1,x} \left(\frac{A(\widehat{n}_0^j + 1)}{A(\widehat{n}_0^j)} - 1 \right).$$

Step 1: We show that $\mathcal{C}_0 = \mathcal{C}_1$. This amounts to show that whatever j the agent p_0^j+1 is still not accepted by club j at date $t = 1$. This is true if:

$$h_1^{p_0^j+1} < \sum_{z=p_0^{j-1}+1}^{p_0^j} h_1^z \left(\frac{A(\widehat{n}_0^j + 1)}{A(\widehat{n}_0^j)} - 1 \right) \Leftrightarrow 1 < \sum_{z=p_0^{j-1}+1}^{p_0^j} \frac{h_1^z}{h_1^{p_0^j+1}} \left(\frac{A(\widehat{n}_0^j + 1)}{A(\widehat{n}_0^j)} - 1 \right) \quad (22)$$

Between date 0 and date 1, for two individuals i and i' ($i' > i$) and $i, i' \in \widehat{C}_0^j$, we have

$$\frac{h_1^i}{h_1^{i'}} = \left(\frac{h_0^i}{h_0^{i'}} \right)^{1-\beta} = \left(\prod_{x=i}^{i'-1} \lambda_0^{x,x+1} \right)^{(1-\beta)}$$

We can thus rewrite the above inequality as follows:

$$1 < \left(1 + \sum_{z=p_0^{j-1}+1}^{p_0^j-1} \left(\prod_{x=z}^{p_0^j-1} \lambda_0^{x,x+1} \right)^{(1-\beta)} \right) \left(\frac{h_0^{p_0^j}}{h_0^{p_0^j+1}} \right)^{1-\beta} \left(\frac{\sum_{z \in \widehat{C}_0^j} h_0^z}{A(\widehat{n}_0^j)} \frac{A(\widehat{n}_0^{j+1})}{\sum_{z \in \widehat{C}_0^{j+1}} h_0^z} \right)^{\beta} \left(\frac{A(\widehat{n}_0^j + 1)}{A(\widehat{n}_0^j)} - 1 \right)$$

$$\Leftrightarrow$$

$$1 < \sum_{z=p_0^{j-1}+1}^{p_0^j} \left(\prod_{x=z}^{p_0^j} \lambda_0^{x,x+1} \right)^{(1-\beta)} \left(\prod_{x=p_0^{j-1}+1}^{p_0^j} \lambda_0^{x,x+1} \right)^{\beta} \left(\frac{\sum_{z \in \widehat{C}_0^j} \frac{h_0^z}{h_0^{p_0^{j-1}+1}}}{\sum_{z \in \widehat{C}_0^{j+1}} \frac{h_0^z}{h_0^{p_0^j+1}}} \frac{A(\widehat{n}_0^{j+1})}{A(\widehat{n}_0^j)} \right)^{\beta} \left(\frac{A(\widehat{n}_0^j + 1)}{A(\widehat{n}_0^j)} - 1 \right) \quad (23)$$

Remark that (23) is satisfied if the following inequality is true:

$$\sum_{z=p_0^{j-1}+1}^{p_0^j} \prod_{x=z}^{p_0^j} \lambda_0^{x,x+1} < \sum_{z=p_0^{j-1}+1}^{p_0^j} \left(\prod_{x=z}^{p_0^j} \lambda_0^{x,x+1} \right)^{(1-\beta)} \left(\prod_{x=p_0^{j-1}+1}^{p_0^j} \lambda_0^{x,x+1} \right)^{\beta} \left(\frac{\sum_{z \in \widehat{C}_0^j} \frac{h_0^z}{h_0^{p_0^{j-1}+1}}}{\sum_{z \in \widehat{C}_0^{j+1}} \frac{h_0^z}{h_0^{p_0^j+1}}} \frac{A(\widehat{n}_0^{j+1})}{A(\widehat{n}_0^j)} \right)^{\beta} \quad (24)$$

Obviously:

$$\frac{\sum\limits_{z\in\widehat{C}_0^j}\frac{h_0^z}{h_0^{p_0^{j-1}+1}}}{\sum\limits_{z\in\widehat{C}_0^{j+1}}\frac{h_0^z}{h_0^{p_0^j+1}}}\frac{A(\widehat{n}_0^{j+1})}{A(\widehat{n}_0^j)}=\frac{\sum\limits_{z\in\widehat{C}_0^j}\frac{h_0^z}{h_0^{p_0^{j-1}+1}A(\widehat{n}_0^j)}}{\sum\limits_{z\in\widehat{C}_0^{j+1}}\frac{h_0^z}{h_0^{p_0^j+1}A(\widehat{n}_0^{j+1})}}$$

By the definition of the core partition and $\widehat{n}_0^{j+1}<\widehat{n}_0^j$, we know that:

$$\sum_{z\in\widehat{C}_0^j}\frac{h_0^z}{h_0^{p_0^{j-1}+1}A(\widehat{n}_0^j)}>\sum_{z=p_0^{j-1}+1}^{p_0^{j-1}+\widehat{n}_0^{j+1}}\frac{h_0^z}{h_0^{p_0^{j-1}+1}A(\widehat{n}_0^{j+1})}$$

Given the inequality schedule $\mathcal{S}=\{\lambda_0^{1,2},...,\lambda_0^{i,i+1},...,\lambda_0^{N-1,N}\}$, such that $\lambda_0^{i,i+1}\leq\lambda_0^{i+1,i+2},\forall i\in S$, we thus deduce that:

$$\sum_{z\in\widehat{C}_0^j}\frac{h_0^z}{h_0^{p_0^{j-1}+1}A(\widehat{n}_0^j)}>\sum_{z=p_0^{j-1}+1}^{p_0^{j-1}+\widehat{n}_0^{j+1}}\frac{h_0^z}{h_0^{p_0^{j-1}+1}A(\widehat{n}_0^{j+1})}>\sum_{z\in\widehat{C}_0^{j+1}}\frac{h_0^z}{h_0^{p_0^j+1}A(\widehat{n}_0^{j+1})}$$

Hence:

$$\frac{\sum\limits_{z\in\widehat{C}_0^j}\frac{h_0^z}{h_0^{p_0^{j-1}+1}}}{\sum\limits_{z\in\widehat{C}_0^{j+1}}\frac{h_0^z}{h_0^{p_0^j+1}}}\frac{A(\widehat{n}_0^{j+1})}{A(\widehat{n}_0^j)}>1$$

Moreover, it is obvious that:

$$\sum_{z=p_0^{j-1}+1}^{p_0^j}\prod_{x=z}^{p_0^j}\lambda_0^{x,x+1}<\sum_{z=p_0^{j-1}+1}^{p_0^j}\left(\prod_{x=z}^{p_0^j}\lambda_0^{x,x+1}\right)^{(1-\beta)}\left(\prod_{x=p_0^{j-1}+1}^{p_0^j}\lambda_0^{x,x+1}\right)^{\beta}$$

Hence, (24) is satisfied.

Step 2. Now we show that if the core partition is such that $p_{t-1}^j=p_0^j$, $\forall t>2$, and $\forall j=1,...,J-1$, then $p_{t-1}^j=p_t^j$.

Let us consider two individuals i and i' ($i'>i$) both belonging at date $t-1$ to $\widehat{C}_0^j$ since date $t=0$. We can then write the human capital ratio at date t between those individuals:

$$\frac{h_t^i}{h_t^{i'}}=\left(\frac{h_0^i}{h_0^{i'}}\right)^{(1-\beta)^t}=\left(\prod_{x=i}^{i'-1}\lambda_0^{x,x+1}\right)^{(1-\beta)^t}. \tag{25}$$

Now we check whether individual p_0^j+1 who was excluded from club $\widehat{C}_0^j$ at date $t-1$ is still excluded at date t. Hence assuming the following inequality is true:

$$h_{t-1}^{p_0^j+1}<\sum_{z=p_0^{j-1}+1}^{p_0^j}h_{t-1}^z\left(\frac{A(\widehat{n}_0^j+1)}{A(\widehat{n}_0^j)}-1\right)$$

we want to prove that:

$$h_t^{p_0^j+1}<\sum_{z=p_0^{j-1}+1}^{p_0^j}h_t^z\left(\frac{A(\widehat{n}_0^j+1)}{A(\widehat{n}_0^j)}-1\right)$$

is satisfied.

Given (25) and the human capital dynamics, we can rewrite those inequalities as follows:

$$1<\left(1+\sum_{z=p_0^{j-1}+1}^{p_0^j-1}\left(\prod_{x=z}^{p_0^j-1}\lambda_0^{x,x+1}\right)^{(1-\beta)^{t-1}}\right)\left(\frac{h_{t-1}^{p_0^j}}{h_{t-1}^{p_0^j+1}}\right)\left(\frac{A(\widehat{n}_0^j+1)}{A(\widehat{n}_0^j)}-1\right)$$

and

$$1<\left(1+\sum_{z=p_0^{j-1}+1}^{p_0^j-1}\left(\prod_{x=z}^{p_0^j-1}\lambda_0^{x,x+1}\right)^{(1-\beta)^{t}}\right)\left(\frac{h_{t-1}^{p_0^j}}{h_{t-1}^{p_0^j+1}}\right)^{1-\beta}\left(\frac{\sum\limits_{z\in\widehat{C}_0^j}h_{t-1}^z}{A(\widehat{n}_0^j)}\frac{A(\widehat{n}_0^{j+1})}{\sum\limits_{z\in\widehat{C}_0^{j+1}}h_{t-1}^z}\right)^{\beta}\left(\frac{A(\widehat{n}_0^j+1)}{A(\widehat{n}_0^j)}-1\right)$$

The above inequality is equivalent to:

$$1<\left(1+\sum_{z=p_0^{j-1}+1}^{p_0^j-1}\left(\prod_{x=z}^{p_0^j-1}\lambda_0^{x,x+1}\right)^{(1-\beta)^{t}}\right)\left(\frac{h_{t-1}^{p_0^j}}{h_{t-1}^{p_0^j+1}}\right)\left(\frac{h_0^{p_0^{j-1}+1}}{h_{t-1}^{p_0^j}}\right)^{\beta}\left(\frac{\sum\limits_{z\in\widehat{C}_0^j}\frac{h_0^z}{h_0^{p_0^{j-1}+1}}}{\sum\limits_{z\in\widehat{C}_0^{j+1}}\frac{h_0^z}{h_0^{p_0^j+1}}}\frac{A(\widehat{n}_0^{j+1})}{A(\widehat{n}_0^j)}\right)^{\beta}\left(\frac{A(\widehat{n}_0^j+1)}{A(\widehat{n}_0^j)}-1\right)$$

We thus have:

$$\begin{aligned}1 &< \left(1+\sum_{z=p_0^{j-1}+1}^{p_0^j-1}\left(\prod_{x=z}^{p_0^j-1}\lambda_0^{x,x+1}\right)^{(1-\beta)^{t}}\right)\left(\frac{h_{t-1}^{p_0^j}}{h_{t-1}^{p_0^j+1}}\right)\left(\frac{h_0^{p_0^{j-1}+1}}{h_{t-1}^{p_0^j}}\right)^{\beta} \\ &\quad\cdot\left(\frac{\sum\limits_{z\in\widehat{C}_0^j}\frac{h_0^z}{h_0^{p_0^{j-1}+1}}}{\sum\limits_{z\in\widehat{C}_0^{j+1}}\frac{h_0^z}{h_0^{p_0^j+1}}}\frac{A(\widehat{n}_0^{j+1})}{A(\widehat{n}_0^j)}\right)^{\beta}\left(\frac{A(\widehat{n}_0^j+1)}{A(\widehat{n}_0^j)}-1\right)\end{aligned} \tag{26}$$

Remark that (26) is satisfied if the following is true:

$$\begin{aligned}&\left(1+\sum_{z=p_0^{j-1}+1}^{p_0^j-1}\left(\prod_{x=z}^{p_0^j-1}\lambda_0^{x,x+1}\right)^{(1-\beta)^{t-1}}\right)\left(\frac{h_{t-1}^{p_0^j}}{h_{t-1}^{p_0^j+1}}\right) \\ <&\left(1+\sum_{z=p_0^{j-1}+1}^{p_0^j-1}\left(\prod_{x=z}^{p_0^j-1}\lambda_0^{x,x+1}\right)^{(1-\beta)^{t}}\right)\left(\frac{h_{t-1}^{p_0^j}}{h_{t-1}^{p_0^j+1}}\right)\left(\frac{h_{t-1}^{p_0^{j-1}+1}}{h_{t-1}^{p_0^j}}\right)^{\beta}\left(\frac{\sum\limits_{z\in\widehat{C}_0^j}\frac{h_{t-1}^z}{h_{t-1}^{p_0^{j-1}+1}}}{\sum\limits_{z\in\widehat{C}_0^{j+1}}\frac{h_{t-1}^z}{h_{t-1}^{p_0^j+1}}}\frac{A(\widehat{n}_0^{j+1})}{A(\widehat{n}_0^j)}\right)^{\beta}\end{aligned} \tag{27}$$

Obviously:

$$\frac{\sum\limits_{z\in\widehat{C}_0^j}\frac{h_{t-1}^z}{h_{t-1}^{p_0^{j-1}+1}}}{\sum\limits_{z\in\widehat{C}_0^{j+1}}\frac{h_{t-1}^z}{h_{t-1}^{p_0^j+1}}}\frac{A(\widehat{n}_0^{j+1})}{A(\widehat{n}_0^j)}=\frac{\sum\limits_{z\in\widehat{C}_0^j}\frac{h_{t-1}^z}{h_{t-1}^{p_0^{j-1}+1}A(\widehat{n}_0^j)}}{\sum\limits_{z\in\widehat{C}_0^{j+1}}\frac{h_{t-1}^z}{h_{t-1}^{p_0^j+1}A(\widehat{n}_0^{j+1})}}$$

By the definition of the core partition and that $\widehat{n}_0^{j+1}<\widehat{n}_0^j$ we know that:

$$\sum_{z\in\widehat{C}_0^j}\frac{h_{t-1}^z}{h_{t-1}^{p_0^{j-1}+1}A(\widehat{n}_0^j)}>\sum_{z=p_0^{j-1}+1}^{p_0^{j-1}+\widehat{n}_0^{j+1}}\frac{h_{t-1}^z}{h_{t-1}^{p_0^{j-1}+1}A(\widehat{n}_0^{j+1})}$$

Given the inequality dynamics, we thus have:

$$\sum_{z\in\widehat{C}_0^{j+1}} \frac{h_{t-1}^z}{h_{t-1}^{p_0^j+1} A(\widehat{n}_0^{j+1})} = \left(1+\sum_{z=p_0^j+1}^{p_0^{j+1}-1}\left(\prod_{x=z}^{p_0^{j+1}-1}\frac{1}{\lambda_0^{x,x+1}}\right)^{(1-\beta)^{t-1}}\right)\frac{1}{A(\widehat{n}_0^{j+1})}$$

and

$$\sum_{z=p_0^{j-1}+1}^{p_0^{j-1}+\widehat{n}_0^{j+1}} \frac{h_{t-1}^z}{h_{t-1}^{p_0^{j-1}+1} A(\widehat{n}_0^{j+1})} = \left(1+\sum_{z=p_0^{j-1}+1}^{p_0^{j-1}+\widehat{n}_0^{j+1}}\left(\prod_{x=z}^{p_0^{j-1}+\widehat{n}_0^{j+1}}\frac{1}{\lambda_0^{x,x+1}}\right)^{(1-\beta)^{t-1}}\right)\frac{1}{A(\widehat{n}_0^{j+1})}$$

Given the inequality schedule $\mathcal{S}=\{\lambda_0^{1,2},...,\lambda_0^{i,i+1},...,\lambda_0^{N-1,N}\}$, such that $\lambda_0^{i,i+1}\leq\lambda_0^{i+1,i+2},\forall i\in S$, we thus deduce that

$$\sum_{z\in\widehat{C}_0^j}\frac{h_0^z}{h_0^{p_0^{j-1}+1}A(\widehat{n}_0^j)} > \sum_{z=p_0^{j-1}+1}^{p_0^{j-1}+\widehat{n}_0^{j+1}}\frac{h_0^z}{h_0^{p_0^{j-1}+1}A(\widehat{n}_0^{j+1})} > \sum_{z\in\widehat{C}_0^{j+1}}\frac{h_0^z}{h_0^{p_0^j+1}A(\widehat{n}_0^{j+1})}.$$

This implies that:

$$\frac{\sum_{z\in\widehat{C}_0^j}\frac{h_{t-1}^z}{h_{t-1}^{p_0^{j-1}+1}}}{\sum_{z\in\widehat{C}_0^{j+1}}\frac{h_{t-1}^z}{h_{t-1}^{p_0^j+1}}}\frac{A(\widehat{n}_0^{j+1})}{A(\widehat{n}_0^j)} > 1. \tag{28}$$

For any $z=\left\{p_0^{j-1}+1,...,p_0^j-1\right\}$ we have:

$$\left(\prod_{x=z}^{p_0^j-1}\lambda_0^{x,x+1}\right)^{(1-\beta)^{t-1}} < \left(\prod_{x=z}^{p_0^j-1}\lambda_0^{x,x+1}\right)^{(1-\beta)^t}\left(\frac{h_{t-1}^{p_0^{j-1}+1}}{h_{t-1}^{p_0^j}}\right)^\beta.$$

Therefore:

$$\left(1+\sum_{z=p_0^{j-1}+1}^{p_0^j-1}\left(\prod_{x=z}^{p_0^j-1}\lambda_0^{x,x+1}\right)^{(1-\beta)^{t-1}}\right) < \left(1+\sum_{z=p_0^{j-1}+1}^{p_0^j-1}\left(\prod_{x=z}^{p_0^j-1}\lambda_0^{x,x+1}\right)^{(1-\beta)^t}\right)\left(\frac{h_{t-1}^{p_0^{j-1}+1}}{h_{t-1}^{p_0^j}}\right)^\beta$$

and (27) and hence (26) are satisfied. This completes the proof of (i).

Remark that (28) in the long run is consistent with Prop.5 (11). It implies that $\gamma^j>\gamma^{j'}$, $\forall j,j'$.

2. Proof of (*ii*)

Given that $\widehat{\mathcal{C}}_0=\widehat{\mathcal{C}}_j$, it is immediate that $\widehat{n}_t^j\leq\widehat{n}_t^{j-1},\forall t$.

$$\frac{\gamma_t^j}{\gamma_t^{j'}} = \frac{\frac{\sum_{z\in\widehat{C}_t^j}(h_t^z)^{1-\beta}}{(\overline{h}_t^j)^{1-\beta}}}{\frac{\sum_{z\in\widehat{C}_t^{j'}}(h_t^z)^{1-\beta}}{(\overline{h}_t^{j'})^{1-\beta}}} = \frac{\frac{\sum_{z\in\widehat{C}_t^j}(h_t^z)^{1-\beta}}{\left(\sum_{z\in\widehat{C}_t^j}(h_t^z)\right)^{1-\beta}}}{\frac{\sum_{z\in\widehat{C}_t^{j'}}(h_t^z)^{1-\beta}}{\left(\sum_{z\in\widehat{C}_t^{j'}}(h_t^z)\right)^{1-\beta}}} = \frac{\frac{\sum_{z\in\widehat{C}_t^j}\left(\frac{h_t^z}{h^{p_t^j}}\right)^{1-\beta}}{\left(\sum_{z\in\widehat{C}_t^j}\left(\frac{h_t^z}{h^{p_t^j}}\right)\right)^{1-\beta}}}{\frac{\sum_{z\in\widehat{C}_t^{j'}}\left(\frac{h_t^z}{h^{p_t^{j'}}}\right)^{1-\beta}}{\left(\sum_{z\in\widehat{C}_t^{j'}}\left(\frac{h_t^z}{h^{p_t^{j'}}}\right)\right)^{1-\beta}}}$$

Given (**??**), this ratio is equal to 1.

3. Proof of (*iii*).

Assume that for a given λ_0 and a given $A(.)$, the optimal size is n_0. Hence $p_0^j = n_0$ is the first pivotal agent. Assume that $\widetilde{A}(.)$ is sufficiently steeper than $A(.)$ so that the first pivotal agent is changed to $\widetilde{p}_0^j$. Given the inequalities defining the pivotal agent, $\widetilde{p}_0^j$ cannot be higher than p_0^j.

This is true for any club, given the equal size property. Extending the reasoning for any period completes the proof.

A similar reasoning applies to a bigger λ_0. ■

G Proof of Proposition 8.

Let us study the core partition of a society S with the inequality schedule $\mathcal{S} = \{\lambda_0^{1,2}, ..., \lambda_0^{i,i+1}, ..., \lambda_0^{N-1,N}\}$, with $\lambda_0^{i,i+1} \geq \lambda_0^{i+1,i+2}, \forall i \in S$.

According to Proposition 3 of JKM [2003], we know that in this case, whatever $j \in \{1, ..., J-1\}$, $\widehat{n}_0^{j+1} \geq \widehat{n}_0^j$.

According to (12) and (13), the core partition can be characterized by a sequence of pivotal agents, i.e. $\{p_0^1, ..., p_0^j, ..., p_0^J\}$ with:

$$h_0^{p_0^j} \geq \sum_{z=p_0^{j-1}+1}^{p_0^j-1} h_0^z \left(\frac{A(\widehat{n}_0^j)}{A(\widehat{n}_0^j - 1)} - 1 \right) \Leftrightarrow 1 \geq \sum_{z=p_0^{j-1}+1}^{p_0^j-1} \prod_{x=z+1}^{p_0^j} \lambda_0^{x-1,x} \left(\frac{A(\widehat{n}_0^j)}{A(\widehat{n}_0^j - 1)} - 1 \right)$$

and

$$h_0^{p_0^j+1} < \sum_{z=p_0^{j-1}+1}^{p_0^j} h_0^z \left(\frac{A(\widehat{n}_0^j + 1)}{A(\widehat{n}_0^j)} - 1 \right) \Leftrightarrow 1 < \sum_{z=p_0^{j-1}+1}^{p_0^j} \prod_{x=z+1}^{p_0^j+1} \lambda_0^{x-1,x} \left(\frac{A(\widehat{n}_0^j + 1)}{A(\widehat{n}_0^j)} - 1 \right).$$

We show that the case $\widehat{\mathcal{C}}_0 \neq \widehat{\mathcal{C}}_1$ cannot be ruled out. This amounts to show that the agent $p_0^j + 1$ may be accepted by club j at date $t = 1$, that is that we cannot prove that inequality (23) is systematically satisfied.

By the definition of the core partition, we know that:

$$\sum_{z \in \widehat{C}_0^j} \frac{h_0^z}{h_0^{p_0^{j-1}+1} A(\widehat{n}_0^j)} > \sum_{z=p_0^{j-1}+1}^{p_0^{j-1}+\widehat{n}_0^{j+1}} \frac{h_0^z}{h_0^{p_0^{j-1}+1} A(\widehat{n}_0^{j+1})}$$

but, given the inequality schedule $\mathcal{S} = \{\lambda_0^{1,2}, ..., \lambda_0^{i,i+1}, ..., \lambda_0^{N-1,N}\}$, such that $\lambda_0^{i,i+1} \geq \lambda_0^{i+1,i+2}, \forall i \in S$, we cannot say that:

$$\sum_{z=p_0^{j-1}+1}^{p_0^{j-1}+\widehat{n}_0^{j+1}} \frac{h_0^z}{h_0^{p_0^{j-1}+1} A(\widehat{n}_0^{j+1})} > \sum_{z \in \widehat{C}_0^{j+1}} \frac{h_0^z}{h_0^{p_0^j+1} A(\widehat{n}_0^{j+1})}$$

Hence as we cannot prove that:

$$\frac{\sum_{z \in \widehat{C}_0^j} \frac{h_0^z}{h_0^{p_0^{j-1}+1}}}{\sum_{z \in \widehat{C}_0^{j+1}} \frac{h_0^z}{h_0^{p_0^j+1}}} \frac{A(\widehat{n}_0^{j+1})}{A(\widehat{n}_0^j)} > 1$$

we cannot prove that (24) is satisfied. A similar reasoning applies to any period t.

We can construct an example which proves that $\widehat{\mathcal{C}}_0 \neq \widehat{\mathcal{C}}_1$. We consider an economy with 6 agents and the following initial distribution of human capital:

$h_0^6 = \frac{50.142}{1.2} = 41.785, h_0^5 = \frac{65.185}{1.3} = 50.142, h_0^4 = \frac{88}{1.35} = 65.185, h_0^3 = 88, h_0^2 = 120, h_0^1 = 720$. This distribution is such that $\lambda_0^{i,i+1} \geq \lambda_0^{i+1,i+2}, \forall i \in S$

The structural parameters have the following values: $\kappa = 10, \mu = .5, \beta = .5, \rho = .5, a = 1/10$.

1 - We construct the core partition $\widehat{\mathcal{C}}_0$,starting with the first club of the first period $\widehat{C}_0^1$. Using the conditions for the first pivotal agent, we get that agent 2 is accepted by agent 1 as:

$$120 \geq 720 \left(\frac{\exp 0.2}{\exp 0.1} - 1 \right) = 75.723.$$

But agent 3 is not accepted by agents 1 and 2 as:

$$24 < 840 \left(\frac{\exp 0.3}{\exp 0.2} - 1 \right) = 88.344.$$

Agent 3 is then the richest member of the second club $\widehat{C}_0^1$. Agent 4 is accepted by agent 3, agent 5 by agents 3 and 4, agent 6 by agents 3,4 and 6 as:

$$\begin{aligned} 65.185 &\geq 88 \left(\frac{\exp 0.2}{\exp 0.1} - 1 \right) = 9.255 \\ 50.142 &\geq (65.185 + 88) \left(\frac{\exp 0.3}{\exp 0.2} - 1 \right) = 16.111 \\ 41.785 &\geq (65.185 + 88 + 50.142) \left(\frac{\exp 0.4}{\exp 0.3} - 1 \right) = 21.384. \end{aligned}$$

Finally, $\widehat{\mathcal{C}}_0$ is formed of the two clubs {1,2} and {3,4,5,6}.

Now we consider the next core partition $\widehat{\mathcal{C}}_1$. First we compute the individual human capitals available at period 1. Using (7), we get:

$$\begin{aligned} h_1^1 &= 7200(.5)^{.5} \left(.1 \frac{840}{(\exp .2)\, 720} \right)^{.5} = 1573.5 \\ h_1^2 &= 1200(.5)^{.5} \left(.1 \frac{840}{(\exp .2)\, 120} \right)^{.5} = 642.37 \\ h_1^3 &= 880(.5)^{.5} \left(.1 \frac{(41.785 + 50.142 + 65.185 + 88)}{(\exp .4)\, 88} \right)^{.5} = 268.87 \end{aligned}$$

Then we remark that agent 3 is now accepted by agents 1 and 2 in $\widehat{C}_1^1$ as:

$$268.87 > (1573.5 + 642.37) \left(\frac{\exp 0.3}{\exp 0.2} - 1 \right) = 233.05.$$

This suffices to prove that in this economy $\widehat{\mathcal{C}}_0 \neq \widehat{\mathcal{C}}_1$.

H Proof of claim 9.

We consider the special case where $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are characterized by equal endowment ratios: $\lambda_0^{i,i+1} = \lambda_0^{i+1,i+2}, \forall i \in S$, $\widetilde{\lambda}_0^{i,i+1} = \widetilde{\lambda}_0^{i+1,i+2}, \forall i \in \widetilde{S}$. In both societies, the core partitions are characterized by equal size clubs, and since S is weakly less segmented than $\widetilde{S}$, $J \leq \widetilde{J}$ and $\widetilde{n} \leq n$. Moreover, we assume that $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are such that there exists no residual clubs in any of the two core partitions. Hence $nJ = \widetilde{n}\widetilde{J} = N$

From (11), the general formula for $\Gamma(S)$ is:

$$\Gamma(S) \equiv \frac{1}{N}\sum_{j=1}^{J} \widehat{n}^j \gamma^j = \frac{\kappa(1-\mu)^{1-\beta}(\mu\widehat{\tau})^\beta}{N}\sum_{j=1}^{J}\frac{\left(\widehat{n}^j\right)^{1+\beta}}{\left(A(\widehat{n}^j)\right)^\beta}.$$

Then:

$$\frac{\Gamma(S)}{\Gamma\left(\widetilde{S}\right)} = \frac{\left(\frac{n}{A(n)}\right)^\beta}{\left(\frac{\widetilde{n}}{A(\widetilde{n})}\right)^\beta}$$

This ratio is larger than 1 if $A(.)$ has an elasticity lower than 1. This completes the proof.