The equity premium implied by production

Urban J. Jermann

The Wharton School of the University of Pennsylvania and NBER

Incomplete version - Not for circulation

January 31, 2005

Abstract

We study the implications of producers' first-order conditions for the link between investment and aggregate asset prices. Calibrated to the U.S. postwar economy, the model can generate a sizeable equity premium, with reasonable volatility for market returns and risk free rates. The market's Sharpe ratio and the market price of risk are very volatile. Contrary to most models, our model generates a negative correlation between conditional means and standard deviations of excess returns.

Twenty years after Mehra and Prescott's paper on the equity premium puzzle there is still no widely accepted replacement for the standard time-separable utility specification. Clearly, representing consumer and investor preferences, as well as interpreting consumption data has turned out to be a very difficult task. Contrary to the consumption side, the production side of asset pricing has received considerably less attention. Focusing on the production side shifts the burden towards representing production technologies and interpreting production data. While a number of asset pricing studies have considered nontrivial production sectors, these have generally been studied jointly with some specific preference specification. Thus, the analysis could not escape the constraints imposed by the preference side. A *pure* production asset pricing literature has emerged from the Q-theory of investment. However, typically, these studies focus on the link between investment and realized stock returns. We are not aware of any study that has explicitly determined the equity premium independently from the consumption side. This is the object of our analysis.

In this paper we are interested in studying the macroeconomic determinants of asset prices given by a multi-input aggregate production technology. We focus exclusively on the producers' first-order conditions that link production variables and state-prices, with sectoral investment playing a crucial role. We are interested in two sets of questions. First: what properties of investment and production technologies are important for the first and second moments of risk free rates and aggregate equity returns? Second: does a model plausibly calibrated to the U.S.

economy have the ability to replicate first and second moments of risk free rates and aggregate equity returns?

The work most closely related to ours is Cochrane's work on production based asset pricing (1988, 1991). Some of the features that differentiate our work are that we focus explicitly on the equity premium, we use more general functional forms for adjustment cost, and we base our empirical evaluation on the two main sectoral aggregates of U.S. capital investment, namely equipment & software as well as structures.

We consider the problem of a representative producer that selects multiple fixed input factors. In order to be able to pin down the state-price process, this problem needs to have two related properties. First, markets need to be complete and the producer has to face a full set of state-prices. Second, there needs to be as many predetermined state variables (fixed production factors) as there are states of nature. This assumption of "complete technologies" is necessary in order to be able to read off the full set of state-contingent prices from the production side. In most studies with nontrivial production sectors this property is not satisfied; of course, in a general equilibrium environment it doesn't usually play an important role.

We calibrate our model to a two-sector representation. We use U.S. data on investment for equipment and software, as well as for structures. This sectoral representation is convenient because these two sectors have natural asymmetries. Indeed, we use the plausible assumption that the capital stock for structures is more difficult to adjust than for equipment and software. As becomes clear below, asymmetries across sectors are needed if we want to derive well-behaved–that is positive–state prices.

We characterize sectoral asymmetries that ensure that state-prices are positive and that generate positive and sizeable equity premium. Our key quantitative findings are the following. For unconditional moments, we can plausibly generate an equity premium of several percentage points with risk free rates having a reasonable mean and volatility. For conditional moments, the expected excess equity return is quite volatile, usually more volatile than the risk free rate. Also concerning excess returns, the correlation between conditional means and volatilities is negative.

The paper is organized as follows. In section 1, we present the model and in section 2 the main asset pricing elements. Section 3 introduces functional forms. In section 4 we analyze the theoretical links between asset prices and production data. Section 5 contains our calibration and Section 6 the quantitative analysis.

1 Model

The model represents the producer's choice of capital inputs for a given state price process. Key ingredients are capital adjustment cost and stochastic productivity.

Assume an environment where uncertainty is modelled as the realization of s, one out of a finite set $S = (\mathfrak{s}_1, \mathfrak{s}_2, ...\mathfrak{s}_N)$, with s_t the current period realization and $s^t \equiv (s_0, s_1, ...s_t)$ the history up to and including t. Probabilities of s^t are denoted by $\pi\left(s^t\right)$. Assume an aggregate production function

$$Y\left(s^t\right) = F\left(\left\{K_j\left(s^{t-1}\right)\right\}_{j\in J}, s^t, N\left(s^t\right)\right),$$

s^t introduces possible stochastic technology in the production of the final good, K_j the j-th capital stock, N labor. Capital accumulation for capital good of type j is represented by

$$K_j\left(s^t\right) = K_j\left(s^{t-1}\right)(1-\delta_j) + Z_j\left(s^t\right) I_j\left(s^t\right),$$

where δ_j is the depreciation rate and $Z_j\left(s^t\right)$ is the (stochastic) technology for producing capital goods. The total cost of investment in capital good of type j is given by

$$H_j\left(K_j\left(s^{t-1}\right), I_j\left(s^t\right), Z_j\left(s^t\right)\right).$$

This specification will be further specialized below to allow for capital stocks to exhibit some form of balanced growth.

The representative firm solves the following problem taking as given state prices $P\left(s^t\right)$

$$\max_{\{I,K',N\}} \sum_{t=0}^{\infty}\sum_{s^t}^{\infty} P\left(s^t\right)\left[F\left(\left\{K_j\left(s^{t-1}\right)\right\}, s^t, N\left(s^t\right)\right) - w\left(s^t\right)N\left(s^t\right) - \sum_j H_j\left(K_j\left(s^{t-1}\right), I_j\left(s^t\right), Z_j\left(s^t\right)\right)\right]$$

$$\text{s.t.} \quad : \quad \left[P\left(s^t\right) q_j\left(s^t\right)\right] : K_j\left(s^{t-1}\right)(1-\delta_j) + Z_j\left(s^t\right) I_j\left(s^t\right) - K_j\left(s^t\right) \geq 0, \ \forall s^t, j$$

with s_0 and $K_j(s_{-1})$ given and $P(s_0) = 1$, without loss of generality. The scaling of the multipliers is chosen so that we get intuitive values. Indeed, q represents the marginal value of one unit of installed capital in terms of current period numeraire; in equilibrium it is also the cost of installing one unit of capital including adjustment cost. Note that if Z has a growth trend, as seems to be required by US data on equipment and software, then q in this sector will be trending down–reflecting the fact that equipment and software become cheaper over time. Also note that q is not the ratio of the market value over the book value of capital (in units of the final good), but qZ is. Indeed, the market value of the firm (assuming one capital stock) is qK. However, the book value (or replacement cost) of the capital stock is K/Z, where K is number of units of the capital good and $1/Z = p_I$ the price/value of a unit of capital in terms of the final good. For equipment, this price p_I, as well as q, will be downward trending, while qZ doesn't trend.

First-order conditions are summarized by

$$0=-H_{j,2}\left(K_j\left(s^{t-1}\right),I_j\left(s^t\right),Z_j\left(s^t\right)\right)+Z_j\left(s^t\right)q_j\left(s^t\right),$$

$$\begin{aligned} q_j\left(s^t\right) &= \sum_{s_{t+1}}\frac{P\left(s^t,s_{t+1}\right)}{P\left(s^t\right)} \\ &\cdot\left\{F_{K_j}\left(\left\{K_j\left(s^t\right)\right\},s^t,s_{t+1},N\left(s^t,s_{t+1}\right)\right)-H_{j,1}\left(K_j\left(s^t\right),I_j\left(s^t,s_{t+1}\right),Z_j\left(s^t,s_{t+1}\right)\right)+\left(1-\delta_j\right)q_j\left(s^t,s_{t+1}\right)\right\},\end{aligned}$$

and for N,

$$F_N\left(\left\{K_j\left(s^{t-1}\right)\right\},s^t,N\left(s^t\right)\right)-w\left(s^t\right)=0$$

so that substituting out shadow prices we have

$$\begin{aligned} 1 \;=\; & \sum_{s_{t+1}}P\left(s_{t+1}|s^t\right)\left[\begin{array}{c} F_{K_j}\left(\left\{K_j\left(s^t\right)\right\},s^t,s_{t+1},N\left(s^t,s_{t+1}\right)\right) \\ -H_{j,1}\left(K_j\left(s^t\right),I_j\left(s^t,s_{t+1}\right),Z_j\left(s^t,s_{t+1}\right)\right)+\left(1-\delta_j\right)\frac{H_{j,2}\left(K_j\left(s^t\right),I_j\left(s^t,s_{t+1}\right),Z_j\left(s^t,s_{t+1}\right)\right)}{Z_j\left(s^t,s_{t+1}\right)}\end{array}\right] \\ & \cdot\left[\frac{Z_j\left(s^t\right)}{H_{j,2}\left(K_j\left(s^{t-1}\right),I_j\left(s^t\right),Z_j\left(s^t\right)\right)}\right],\end{aligned}$$

for each j, where the notation $P\left(s_{t+1}|s^t\right)$ shows the price of the numeraire in s_{t+1} conditional on s^t and in units of the numeraire at s^t. From this condition we define the investment return $R^I_j\left(s^t,s_{t+1}\right)$ implicitly through $\sum_{s_{t+1}}P\left(s_{t+1}|s^t\right)R^I_j\left(s^t,s_{t+1}\right)=1$. $R^I_j\left(s^t,s_{t+1}\right)$ is the return we get in s_{t+1} from adding one (marginal) unit of capital of type j in state s^t. The first-order condition shows that in equilibrium adding one marginal unit of a given type of capital produces a change in the profit plan that is worth one unit.

We will specialize the model to have 2 capital inputs and 2 states of nature in each period. In addition to complete markets, that is the producers ability to sell contingent output for each state of nature, we also need to satisfy the requirement of "complete technologies", that is the ability to move resources independently between all states of nature. The complete technology requirement is needed if we want to be able to recover all state prices from the producers first-order conditions.

2 From investment returns to state prices

Representing the first-order conditions in matrix form we have

$$\left[\begin{array}{cc} R^I_1\left(s^t,\mathfrak{s}_1\right) & R^I_1\left(s^t,\mathfrak{s}_2\right) \\ R^I_2\left(s^t,\mathfrak{s}_1\right) & R^I_2\left(s^t,\mathfrak{s}_2\right)\end{array}\right]\left[\begin{array}{c} P\left(\mathfrak{s}_1|s^t\right) \\ P\left(\mathfrak{s}_2|s^t\right)\end{array}\right]=\mathbf{1}\text{, or compactly: } R^I\left(s^t\right)\cdot p\left(s^t\right)=\mathbf{1}$$

so that the state price vector is obtained from matrix inversion

$$p\left(s^t\right)=\left(R^I\left(s^t\right)\right)^{-1}\mathbf{1}.$$

Clearly, it isn't necessarily the case that this matrix inversion is feasible nor that state prices are necessarily positive. In particular, as further discussed below, the requirement for positive state prices forces us to rule our certain parameterizations.

In this environment, the **risk free return** is given by**:**

$$1/R^f\left(s^t\right) = \mathbf{1}p\left(s^t\right) = P\left(\mathfrak{s}_1|s^t\right) + P\left(\mathfrak{s}_2|s^t\right).$$

It is easy to check the matrix algebra to see that if for one of the investment returns the realized return is not state-contingent $R_j^I\left(s^t, s_{t+1}\right) = R_j^I\left(s^t\right)$, then, as is implied by no-arbitrage, it equals the risk free rate, $R_j^I\left(s^t\right) = R^f\left(s^t\right)$. The close relationship between risk free rates and expected returns to the capital stocks can be illustrated in the following case. Without loss of generality, in the two-state case, we can write

$$R_1\left(s^t, \mathfrak{s}_1\right) = \bar{R}_1\left(s^t\right) - \varepsilon_1\left(s^t\right), \text{ and } R_1\left(s^t, \mathfrak{s}_2\right) = \bar{R}_1\left(s^t\right) + \varepsilon_1\left(s^t\right)$$

$$R_2\left(s^t, \mathfrak{s}_1\right) = \bar{R}_2\left(s^t\right) - \varepsilon_2\left(s^t\right), \text{ and } R_2\left(s^t, \mathfrak{s}_2\right) = \bar{R}_2\left(s^t\right) + \varepsilon_2\left(s^t\right).$$

Of course, $\varepsilon's$ could be positive of negative, and $\bar{R}_j\left(s^t\right)$ is not necessarily the (probability weighted) mean. With some algebra, it is easy to see that

$$R^f\left(s^t\right) = \frac{\bar{R}_1\left(s^t\right)\varepsilon_2\left(s^t\right) - \bar{R}_2\left(s^t\right)\varepsilon_1\left(s^t\right)}{\varepsilon_2\left(s^t\right) - \varepsilon_1\left(s^t\right)},$$

so that if $\bar{R}_1\left(s^t\right) = \bar{R}_2\left(s^t\right)$ then we have

$$R^f\left(s^t\right) = \bar{R}_1\left(s^t\right) = \bar{R}_2\left(s^t\right).$$

That is to say that if the average realized returns are equal for both capital stocks, then this average return also equals the risk free rate.

Now consider expected capital returns:

$$\begin{aligned} E\left[R_1\left(s^t, s_{t+1}\right)|s^t\right] &= R_1\left(s^t, s_{t+1}\right)\cdot\pi'\left(s_{t+1}|s^t\right) \\ E\left[R_2\left(s^t, s_{t+1}\right)|s^t\right] &= R_2\left(s^t, s_{t+1}\right)\cdot\pi'\left(s_{t+1}|s^t\right), \end{aligned}$$

where $\pi'\left(s_{t+1}|s^t\right)$ is the state-contingent probability vector. Note, so far, we had not needed any probabilities. The **aggregate equity return** [XXX introduce some notation that makes clear this is the market return with dividends etcXXX] (the return to the representative firm) can be obtained as the value weighted average of the individual returns–given the homogeneity assumptions introduced below–and thus

$$E\left[R\left(s^t, s_{t+1}\right)|s^t\right] = \sum_j \frac{q_j\left(s^t\right)K_j\left(s^t\right)}{\left[\sum_i q_i\left(s^t\right)K_i\left(s^t\right)\right]}\cdot E\left[R_j\left(s^t, s_{t+1}\right)|s^t\right].$$

While the two capital stocks will usually grow at different rates, the value share of each capital stock may still be stationary because the $q's$ can be trending too. This is typically implied by the growth restriction discussed below. Alternatively, the aggregate return can be nonstationary even with stationary state prices.

The highest sharpe rate (market price of risk) also has a simple expression. Starting from state prices we introduce the stochastic discount factor $m\left(s_{t+1}|s^t\right)$ by dividing and multiplying through by $\pi\left(s_{t+1}|s^t\right)$, so that

$$P\left(s_{t+1}|s^t\right)=\left(\frac{P\left(s_{t+1}|s^t\right)}{\pi\left(s_{t+1}|s^t\right)}\right)\pi\left(s_{t+1}|s^t\right)=m\left(s_{t+1}|s^t\right)\pi\left(s_{t+1}|s^t\right).$$

Then, ruling out arbitrage implies $E_t\left(m\left(s_{t+1}|s^t\right)R^e\left(s^t,s_{t+1}\right)\right)=0$, for $\forall R^e\left(s^t,s_{t+1}\right)$ defined as excess returns. It is then easy to see that

$$\max\frac{E\left[R^e\left(s^t,s_{t+1}\right)|s^t\right]}{Std\left[R^e\left(s^t,s_{t+1}\right)|s^t\right]}=\sqrt{\frac{\sum_{s_{t+1}}P\left(s_{t+1}|s^t\right)^2/\pi\left(s_{t+1}|s^t\right)}{\left[\sum_{s_{t+1}}P\left(s_{t+1}|s^t\right)\right]^2}-1}.$$

3 Functional Forms

In this section, we present the functional forms and the simulation strategies.

A. The production function

We start with a basic production function

$$Y\left(s^t\right)=F^{K1}\left(K_1\left(s^{t-1}\right),s^t\right)+F^{K2}\left(K_2\left(s^{t-1}\right),s^t\right)+F^N\left(N\left(s^t\right),s^t\right),$$

which we restrict further below to accommodate balanced growth.

B. The investment cost function

We will use the following specification

$$H\left(K,I,Z\right)=H\left(K/Z,I\right)=H\left(1,ZI/K\right)\cdot\left(K/Z\right)$$

so that it is homogenous of degree 1 in I and K/Z, as is needed for balanced growth. We use $Z_j\left(s^t\right)=Z_j\left(s^{t-1}\right)\cdot\lambda^{Z_j}\left(s_t\right)$, with $\lambda^{Z_j}\left(s_t\right)$ following a two-state Markov process.[1]

The functional form is assumed to be

$$H\left(1,ZI/K\right)\cdot\left(K/Z\right)=\left\{\frac{b}{\nu}\left(ZI/K\right)^{\nu}+c\right\}\left(K/Z\right),$$

[1] Given the capital accumulation equation, and as we further discuss below, IZ and K are cointegrated, and so are I and K/Z.

with $b, c > 0$, $v > 1$ and $ZI/K \geq 0$.[2]

As can easily seen, this function is (1) convex in I for $v > 1$. (2) adjustment cost and the direct cost for additional capital goods are separable, trivially so because $H(1, ZI/K) \cdot (K/Z) = [H(1, ZI/K) - ZI/K + ZI/K] \cdot (K/Z) = [H(1, ZI/K) - ZI/K] \cdot (K/Z) + I \equiv C(1, ZI/K) \cdot (K/Z) + I$. And we impose that $C(1, ZI/K) \geq 0$–that is, the pure adjustment cost is nonnegative.

The different parameters have roughly the following functions. v determines the cost of choosing volatile production investment plans. For a given investment process, it determines the volatility of the market price of capital. The parameters b and c are used to obtain target average values for the average qZ (marginal cost) and for the total cost.

With this specification, we have the marginal cost given as

$$H_I(K, I, Z) = H_2(1, ZI/K) = b(ZI/K)^{\nu-1}$$

and we can easily check convexity with respect to investment

$$H_{II}(K, I, Z) = b(v-1)(ZI/K)^{\nu-2}(Z/K) > 0 \text{ for } b > 0 \text{ and } v > 1.$$

The case of no-adjustment cost is given by setting $v = b = 1$, $c = 0$, so that

$$H(1, ZI/K) \cdot (K/Z) = I.$$

From the first-order condition we obtain a relationship between the investment rate and the marginal cost of capital

$$qZ = H_I(K, I, Z) = b(ZI/K)^{\nu-1}.$$

Normalizing Z to 1,the elasticity (quantity elasticity of a price) of q with respect to I/K is

$$\frac{\partial q}{\partial(I/K)} \frac{I/K}{q} = v - 1.$$

C. Stationarity of returns and simulation strategy

We want returns to be stationary. This imposes some restrictions on technologies and the growth processes. We also describe here the state space used for our simulations. Consider the investment return for capital stock j given our functional forms

$$\begin{aligned}
& R_j^I\left(s^t, s_{t+1}\right) \\
= \quad & Z_{jt} \cdot \frac{F_j^{Kj}\left(K_{j,t+1}, s^{t+1}\right)}{b\left(Z_{jt} I_{j,t}/K_{j,t}\right)^{v-1}} + \left(Z_{j,t}/Z_{j,t+1}\right) \cdot \frac{b\left(1-\frac{1}{v}\right)\left(Z_{j+1t} I_{jt+1}/K_{jt+1}\right)^{\nu} + c}{b\left(Z_{jt} I_{j,t}/K_{j,t}\right)^{v-1}} \\
& + \left(Z_{j,t}/Z_{j,t+1}\right) \cdot \left(1-\delta_j\right) \cdot \frac{b\left(Z_{jt+1} I_{j,t+1}/K_{j,t+1}\right)^{v-1}}{b\left(Z_{jt} I_{j,t}/K_{j,t}\right)^{v-1}},
\end{aligned}$$

[2] In order to allow for $ZI/K < 0$, we can introduce a cofficient d such that $H(1, ZI/K) = \frac{b}{v}(ZI/K + d)^v + c$. For our empirical implementations this has not been useful sofar.

where we have used a more compact notation. The first term represents the marginal product, the second the adjustment cost reduction next period (growth option) and the third, the leftover capital. In sum, this return can be written as a function of the following

$$R_j^I\left(s^t, s_{t+1}\right) = R_j^I\begin{pmatrix} Z_{jt}I_{j,t}/K_{j,t}, Z_{jt+1}I_{j,t+1}/K_{j,t+1} \\ Z_{jt}F_j^{Kj}\left(K_{j,t+1}, s^{t+1}\right) \\ Z_{j,t+1}/Z_{j,t}, \end{pmatrix}.$$

A sufficient set of conditions for $R_j^I\left(s^t, s_{t+1}\right)$ to be stationary is that each of the 3 different composite variables is stationary. Let us consider each of these terms separately.

First, $Z_{j,t+1}/Z_{j,t}$, as seen above, is stationary by assumption. Second, given the specification of the productivity growth rates as finite elements Markov chains, and assuming that sectoral investment growth rates also follow finite element Markov chains, that is, $I_j\left(s^t, s_{t+1}\right) = I_j\left(s^t\right)\lambda^{I_j}\left(s_{t+1}\right)$, it can easily be shown that for appropriate starting points, $Z_{jt}I_{j,t}/K_{j,t}$ is bounded. Third, we will make the assumption that

$$Z_{jt}F_j^{Kj}\left(K_{j,t+1}, s^{t+1}\right) \equiv A_j.$$

This assumption guarantees stationarity and allows us to focus our analysis on investment dynamics. The implication of this assumption is that to the extent that capital gets cheaper to produce over time, that is as Z_j increases, it also becomes less productive at the margin in physical terms, so that in value terms, the marginal product remains constant.[3]

Stationarity of sectoral investment returns is not sufficient for stationarity of aggregate asset returns. Indeed, as shown in equation, the aggregate return equals a weighted average of the sectoral returns. For stationarity, the weights need to be stationary too. Aggregate returns are given by

$$R\left(s^t, s_{t+1}\right) = \sum_j \frac{\frac{b(Z_{jt}I_{j,t}/K_{j,t})^{v-1}}{Z_{j,t}}K_{j,t+1}}{\sum_i \frac{b(Z_{it}I_{i,t}/K_{i,t})^{v-1}}{Z_{i,t}}K_{i,t+1}} R_j\left(s^t, s_{t+1}\right).$$

A sufficient (and necessary) condition for stationarity, given our previous assumptions, is that $K_{1,t+1}/Z_{1,t}$ and $K_{2,t+1}/Z_{2,t}$ are cointegrated. Given that the investment capital ratios $Z_{jt}I_{j,t}/K_{j,t}$ are stationary, this is equivalent to $I_{1,t}$ and $I_{2,t}$ being cointegrated. Setting investment expenditure growth rates equal across sectors, that is $\lambda^{I1}\left(s_{t+1}\right) = \lambda^{I2}\left(s_{t+1}\right)$, guarantees that $I_{1,t}$ and $I_{2,t}$ are cointegrated. To summarize, because individual quantities have stochastic trends, we end up choosing identical investment expenditure growth realizations across sectors to guarantee stationarity of aggregate equity returns. However, we are free to choose the realizations for λ_t^{Z1} and λ_t^{Z2} independently. This is less restrictive than it might appear for several reasons. As seen above,

[3] This is related to one of the properties implied by Greenwood, Hercowitz and Krusell's balanced growth path.

what matters for the investment returns is the behavior of the product $\lambda_t^{I1}\lambda_t^{Z1}$ and not λ_t^{I1} individually. That is to say that the important element in the calibration is to fit the process of real investment growth rather than the growth in investment expenditure. Moreover, for our empirical counterparts, as shown below, the historical volatilities of $\lambda^{I1}$and λ^{I2} are nearly identical, and realizations of the two growth rates are strongly positively correlated. Alternatively, we could introduce additional components for each process that have purely transitory effects and would thus not need to be restricted to ensure balanced growth. However, given the requirement to keep the number of states small, the additional flexibility introduced in this way would be rather limited

To summarize, we have

$$\begin{aligned} R_j^I\left(s^t, s_{t+1}\right) &= \frac{A_j}{b\left(Z_{jt}I_{j,t}/K_{j,t}\right)^{v-1}} \\ &+ \left(Z_{j,t}/Z_{j,t+1}\right)\cdot\frac{b\left(1-\frac{1}{v}\right)\left(Z_{j+1t}I_{jt+1}/K_{jt+1}\right)^{\nu}+c}{b\left(Z_{jt}I_{j,t}/K_{j,t}\right)^{v-1}} \\ &+ \left(Z_{j,t}/Z_{j,t+1}\right)\cdot\left(1-\delta_j\right)\cdot\frac{b\left(Z_{jt+1}I_{j,t+1}/K_{j,t+1}\right)^{v-1}}{b\left(Z_{jt}I_{j,t}/K_{j,t}\right)^{v-1}}. \end{aligned} \tag{3.1}$$

that is to say

$$R_j^I\left(s^t, s_{t+1}\right) = R_j^I\begin{pmatrix} Z_{jt}I_{j,t}/K_{j,t}, \\ Z_{1t+1}I_{1,t+1}/K_{1,t+1}, \\ Z_{j,t+1}/Z_{j,t}, \end{pmatrix}.$$

For simulation the, following representation is useful:

$$R_j^I\left(s^t, s_{t+1}\right) = R_j^I\left(s_t, K_j\left(s^{t-1}\right), I_j\left(s^t\right), Z_j\left(s^t\right); \lambda^I\left(s_{t+1}\right), \lambda^{Z_j}\left(s_{t+1}\right)\right) \text{ for } j=1,2.$$

Indeed, returns are function of state variables and shock realizations. Seven variables are a sufficient statistic for the current state of the economy s^t, namely $s_t, K_1\left(s^{t-1}\right), K_2\left(s^{t-1}\right), I_1\left(s^t\right), I_2\left(s^t\right),$ $Z_1\left(s^t\right), Z_2\left(s^t\right)$. Clearly $K_j\left(s^t\right)$ matters too, but it is a function of the state variables. The probability distribution of the shocks is summarized by s_t, the realization of the return does not depend on s_t. With a law of motion for the exogenous state s_{t+1}, the law of motion for the rest of the variables follows

$$\begin{aligned} I_j\left(s^t, s_{t+1}\right) &= I_j\left(s^t\right)\lambda^I\left(s_{t+1}\right) \text{ for } j=1,2 \\ Z_j\left(s^t, s_{t+1}\right) &= Z_j\left(s^t\right)\lambda^{Z_j}\left(s_{t+1}\right) \text{ for } j=1,2 \\ K_j\left(s^t\right) &= K_j\left(s^{t-1}\right)\left(1-\delta_j\right) + Z_j\left(s^t\right)I_j\left(s^t\right) \text{ for } j=1,2. \end{aligned}$$

As initial conditions, we will set $K_2\left(s^{-1}\right) = Z_1\left(s^0\right) = Z_2\left(s^0\right) = 1$, and $K_1\left(s^{-1}\right)$ is set equal to the historical average of the ratio of the value of capital in this sector relative to the other sector. Initial investment levels are set at their implied steady state values.

4 What determines the equity premium?

Here we examine the model elements that are key in generating a positive equity premium. We also discuss the properties needed to guarantee that in our simulations state prices are always positive, given the processes for investment and productivity. In this analysis we focus on the ability of sectoral differences in adjustment cost parameters v_j, and depreciation rates δ_j to deliver interesting properties. This focus is guided by the fact that for these two parameters, differences are first-order given the chosen sectoral calibration with equipment & software on one side and structures on the other. We examine additional asymmetries in our quantitative evaluation. The main findings of this section is that the asymmetry in the adjustment cost parameter v is crucial to generate a positive equity premium and to guarantee positive state prices. Sectoral differences in depreciation rates seem less important.

It is easy to show that the relative state prices in the two-state case are given by

$$\frac{P\left(s^t, \mathfrak{s}_1\right)}{P\left(s^t, \mathfrak{s}_2\right)} = \frac{R_2^I\left(s^t, \mathfrak{s}_2\right) - R_1^I\left(s^t, \mathfrak{s}_2\right)}{R_1^I\left(s^t, \mathfrak{s}_1\right) - R_2^I\left(s^t, \mathfrak{s}_1\right)}. \tag{4.1}$$

We start with some basic properties of the model:

1. As is clear from equation 4.1, each capital stock has to do (absolutely) better in one specific state. If it weren't the case, then one of the derived state prices would be negative, which is inconsistent with equilibrium. Indeed, if one type of investment were to generate a higher return in both states, then resource would be moved into this type of capital from the other.

2. If more of a given type of investment is added, its (marginal) return goes down. This is because installation costs increase due to the assumption of convex adjustment cost.[4] In case the marginal product has decreasing returns, there is an additional effect that generates diminishing returns.

3. As is clear from equation 4.1, the state with the higher price has less dispersed returns. Assume that from a relative price of one, the price of one state increases. Capital is then allocated to the sector that has an absolute advantage in this state from the other sector. By doing this, the higher return capital lowers its return and the lower return capital increases its return. Both of these contribute to lowering the dispersion in the considered state.

To focus on risk premiums, the price ratio has to be compared to the ratio of the probabilities. Given that a reasonable empirical specification is not far from iid we consider here the case where

[4] Moreover, next periods capital stock is larger leading to a reduction in the investment capital ratio and thus a reduction of the value of the having installed capital and also a reduction in the value attached to growth option, as can easily be seen.

$\frac{\pi(.,\mathfrak{s}_1)}{\pi(.,\mathfrak{s}_2)} = 1$, so that ratio of state prices equals the ratio of the stochastic discount factor

$$\frac{P\left(s^t, \mathfrak{s}_1\right)}{P\left(s^t, \mathfrak{s}_2\right)} = \frac{m\left(s^t, \mathfrak{s}_1\right)}{m\left(s^t, \mathfrak{s}_2\right)}.$$

Without loss of generality, we take state 2 to be the high growth state. State 2, is then also the state where realized returns are higher. This is because with convex adjustment cost, installed capital has become more valuable. Therefore to have a positive equity premium, the relative price of a state 2 payout has to be low, this is achieved by having high dispersion in state 2. That is to say, to have a positive equity premium, the high growth states needs to generate more dispersed returns than the low growth state.

We now examine which parameter configurations can contribute to a positive equity premium and generate positive state prices. We focus on differences across sectors in the adjustment cost curvature and the depreciation rates–everything else being symmetric across the two sectors. For this analysis we consider a second-order Taylor approximation of the investment return at the steady-state around the mean of investment growth. We assume that only investment expenditure growth is allowed to vary, but not the marginal product nor investment specific technology shock. As a starting point, we rewrite equation 3.1 for the investment return more compactly as

$$R\left(I'\right) = \frac{1}{qZ}\left\{A_1 + \left(Z/Z'\right)\left[b\left(1 - \frac{1}{v}\right)\left(Z'I'/K'\right)^{\nu} - c + (1-\delta)\, b\left(Z'I'/K'\right)^{v-1}\right]\right\}.$$

A second-order Taylor approximation is obtained by assuming that all investment capital ratios are at their steady state values $ZI_j/K_j = Z'I_j'/K_j' = \overline{ZI_j/K_j} = (\lambda_I\lambda_Z - 1) + \delta$ and with $qZ = b\left(\overline{ZI/K}\right)^{v-1}$:

$$R\left(\Delta\lambda^I\right) = \bar{R} + C\cdot\Delta\lambda^I + \frac{B}{2}\cdot\left(\Delta\lambda^I\right)^2 + o\left(\left(\Delta\lambda^I\right)^2\right)$$

where $\Delta\lambda \equiv \left(\lambda' - \bar{\lambda}'\right)$, with $\bar{\lambda}'$ the center point of the approximation, here the mean, $\bar{R}$ a constant given below, and

$$\begin{aligned} C &= v - 1 \\ B &= \left(\frac{1}{\lambda^I}\right)^2\left(\frac{1}{\lambda_Z}\right)\cdot\left\{\lambda_I\lambda_Z\,(v-1)^2 - (1-\delta)\,(v-1)\right\}. \end{aligned}$$

Because we have already assumed iid shocks, the size of the up and down deviations from the mean in a two-state setting are identical, that is, we have

$$\Delta\lambda_j\left(\mathfrak{s}_2\right) = -\Delta\lambda_j\left(\mathfrak{s}_1\right) \equiv \Delta\lambda_j, \text{ for each } j \in (1,2),$$

moreover, assuming equal investment growth volatility in the two sectors, we have

$$\Delta\lambda_1 = \Delta\lambda_2 = \Delta\lambda.$$

Finally, we introduce these approximations into the ratio determining relative state prices,

$$\begin{aligned}\frac{P(.,\mathfrak{s}_1)}{P(.,\mathfrak{s}_2)} &= \frac{R_2^I(.,\mathfrak{s}_2)-R_1^I(.,\mathfrak{s}_2)}{R_1^I(.,\mathfrak{s}_1)-R_2^I(.,\mathfrak{s}_1)} = \frac{[C_2-C_1]\,\Delta\lambda+\left[\left(\bar{R}_2-\bar{R}_1\right)+\frac{1}{2}\left(B_2-B_1\right)\left(\Delta\lambda\right)^2\right]+o(.)}{[C_2-C_1]\,\Delta\lambda-\left[\left(\bar{R}_2-\bar{R}_1\right)+\frac{1}{2}\left(B_2-B_1\right)\left(\Delta\lambda\right)^2\right]+o(.)} \\ &= \frac{CC+BB+o(.)}{CC-BB+o(.)}\end{aligned} \tag{4.2}$$

As shown by equation 4.2, in order to have positive prices, we need $CC >> 0$ or $<< 0$. CC needs to be sufficiently fare away from 0 so that despite the dispersion induced by BB, the numerator and the denominator always have the same sign. Moreover, CC should not be too sensitive to the state of the economy, to ensure that this property is satisfied everywhere. For a positive equity premium, if $CC > 0$, we need $BB > 0$, so that the price of the low state, that is state 1, is higher (mutatis mutandum for $CC < 0$).

We will now check the conditions on v and δ needed to generate these properties. In our calibration we will have equipment and software as sector 1 and structures as sector 2. Our calibration will be one where structures depreciate less and are more costly to install, that is, $\delta_2 < \delta_1$ and that $v_2 > v_1$.

Positive prices. With equally volatile investment growth across sectors we need to check what type of asymmetry has the ability to generate positive prices. Given

$$C = v - 1,$$

clearly, asymmetry in v is needed. Moreover, the level of δ has no effect.

Positive equity premium. Given that $CC > 0$, if $B_2 > B_1$and $\bar{R}_2 > \bar{R}_1$, then the equity premium is positive. Thus we check if our specification of the asymmetry for δ and v contributes to this outcome. Remember, in our calibration, $\delta_2 < \delta_1$ and $v_2 > v_1$; and let us therefore consider the derivatives with respect to these two parameters. Using again $\overline{ZI/K} = (\lambda_I\lambda_Z - 1) + \delta$, we have

$$\begin{aligned} B &= \left(Z/Z'\right)(v-1)\left[(v-1)\left(\overline{IZ/K}\right)+(v-2)(1-\delta)\right] \\ &= \left(\frac{1}{\lambda_Z}\right)(v-1)\,b\left[(v-1)\left((\lambda_I\lambda_Z-1)+\delta\right)+(v-2)(1-\delta)\right] \\ &= \left(\frac{v-1}{\lambda_Z}\right)\left[(v-1)\left((\lambda_I\lambda_Z-1)+\delta\right)+(v-2)(1-\delta)\right] \\ &= (v-1)^2\lambda_I-\left(\frac{v-1}{\lambda_Z}\right)(1-\delta)\,. \end{aligned}$$

Now take the derivatives. First for v

$$\begin{aligned}\frac{\partial B}{\partial v} &= \left(\frac{1}{\lambda^I}\right)^2\left(\frac{1}{\lambda_Z}\right)\cdot\left[\lambda_I\lambda_Z(v-1)^2-(1-\delta)(v-1)\right] \\ &= \left(\frac{1}{\lambda^I}\right)^2\left(\frac{1}{\lambda_Z}\right)\cdot\left[2\lambda_I\lambda_Z(v-1)-(1-\delta)\right]\end{aligned}$$

if $v > \left[\frac{1}{2}\left[(1-\delta)/\lambda_I \lambda_Z\right] + 1 < 1.5\right]$, then

$$\frac{\partial B}{\partial v} > 0.$$

So that it is enough for $v > 1.5$ for this to hold. Then for δ

$$\frac{\partial B}{\partial \delta} = \left(\frac{1}{\lambda^I}\right)^2 \frac{v-1}{\lambda_Z} > 0 \text{ , if } v > 1.$$

Given the signs of the derivatives, the asymmetry in v helps generate a positive equity premium, while the asymmetry in δ does not. What about quantitative importance of the two opposing effects? The following example shows that moving v from 2 to 5 is quantitatively more important than moving δ between .05 and .15. Indeed, $\frac{\partial B}{\partial \delta}\Delta\delta = (v-1).1 = .2$, (if $v = 3$), while $\frac{\partial B}{\partial v}\Delta v \approx [3]\, 3 = 9$. Thus, we conclude that the selected asymmetry should contributed positively to the equity premium.

In our calibration we set $\bar{R}_2$ and $\bar{R}_1$ independently from other parameters. As discuss below, this implicitly consists in picking values of A_j so that target values of $\bar{R}_j$ are achieved. Given the limited amount of direct information about A_j, we choose this approach.Alternatively, we could set A_j at some given value, and then let $\bar{R}_2$ be determined endogenously. In this case, an analysis like the one performed here for B can be carried out assuming that A_j are equal across sectors.

5 Calibration

The majority of the parameter values are picked to be consistent with quantity data alone, without including any information from asset prices. For the rest of the parameters, asset pricing moments are used for calibration. We first present a short summary of our baseline calibration, details are provided thereafter.

A. Summary

The table lists the parameters to calibrate and the chosen baseline values:

$$\begin{aligned}
\rho, fr &= [0.2, 0.9] \\
\begin{bmatrix} \lambda^I(\mathfrak{s}_1) \\ \lambda^I(\mathfrak{s}_2) \end{bmatrix} &= \begin{bmatrix} 0.9601 \\ 1.1058 \end{bmatrix}, \begin{bmatrix} \lambda^{Ze}(\mathfrak{s}_1) \\ \lambda^{Ze}(\mathfrak{s}_2) \end{bmatrix} = \begin{bmatrix} 1.0069 \\ 1.0069 \end{bmatrix}, \begin{bmatrix} \lambda^{Zs}(\mathfrak{s}_1) \\ \lambda^{Zs}(\mathfrak{s}_2) \end{bmatrix} = \begin{bmatrix} 1.0069 \\ 1.0069 \end{bmatrix} \\
\delta_e, \delta_s &= [0.112, 0.031] \\
(K_e/Z_e)/(K_s/Z_s) &= 0.6 \\
\nu_e &= 2.5, \nu_s = 4.5, b_e, b_s, c_e, c_s : \ q_{ss} = 1.5 \\
A_1, A_2 &: \text{ so that } R_{ss} = 1.021
\end{aligned}$$

ρ and fr are the first-order autocorrelation and the frequency of realizations that are below the mean relative to above the mean. A set of parameters is chosen based on direct empirical counterparts; namely, $\left(\lambda^I, \lambda^{Ze}, \lambda^{Zs}, \rho, fr\right)$, (δ_e, δ_s), $(K_e/Z_e)/(K_s/Z_s)$. For the baseline calibration, the variance of the investment specific productivity shocks is set to zero, we present results with non-zero variance below. In order to replicate steady-state values for q, we pick (b_e, b_s), with the lowest possible cost, which determines (c_e, c_s). For the curvature parameters, based on casual empiricism, we assume: $\nu_e < \nu_s$; with the exact values picked to maximize the model's fit. Finally, A_1, and A_2 are picked to minimize the impact of hard to measure quantities and to help the model.

B. Details of different parts of calibration

(i) Investment and productivity processes We consider the empirical counterparts of three quantities

$$
\begin{aligned}
I \cdot Z &\equiv \text{Investment (addition to capital stock in units of capital good)} \\
I &\equiv \text{Investment expenditure in units of numeraire final good (consumption)} \\
Z &\equiv \text{Investment-specific technological change} \\
1/Z &= p_I = P_I/P_C : \text{ Replacement cost (not including adj. cost) for capital, also bookvalue.}
\end{aligned}
$$

We will use the following time series: the quantity index of investment in each sector to give us IZ, the deflator of investment goods and the deflator of nondurables and services to jointly give us Z for each sector. We use annual data for these 3 series, for equipment and structures, covering 1947-2003.

The calibration of the investment growth process is in two steps. First, the probability matrix is determined to match the serial correlation and the frequency of low and high growth states. These two moments do not depend on the shock values themselves but only on the probabilities. Specifically, the two diagonal elements of the probability matrix are given as

$$\pi_{11} = \frac{\rho + fr}{1 + fr}; \; \pi_{22} = \frac{1 + fr \cdot \rho}{1 + fr},$$

where $\rho \equiv$ autocorelation and $fr \equiv p_1/p_2$, that is the relative frequency of state 1 (the recession state). For the relative frequency we have counted the number of realizations of investment growth below its mean: there are 26/56 for equipment and 27/56 for structures. This gives a relative frequency for recessions compared to expansions of .9 (assuming 26.5/56 recessions). The first order serial correlations of the growth rates of investment are 0.13 and 0.28, respectively. The common ρ is set at the average of 0.2.

Second, we select 6 values for the realized growth rates of investment expenditure and the sector specific technological progress. As discussed above, equal growth realizations for λ^I are required for balanced growth, but given that λ^I does not enter any formula alone, this isn't a major constraint. The 6 values are picked to match a weighted average of the empirical counterparts of the means and variances for $I_j Z_j$ and Z_j. This represents a total of 8 moments. Historical means and standard deviations equal

$$\begin{aligned} \mu^{IZe} - 1 &= 5.71\% \qquad \sigma^{IZe} = 7.81\% \\ \mu^{IZs} - 1 &= 2.29\% \qquad \sigma^{IZs} = 6.86\% \end{aligned}$$

$$\begin{aligned} \mu^{Ze} - 1 &= 1.82\% \qquad \sigma^{Ze} = 2.56\% \\ \mu^{Zs} - 1 &= -0.44\% \qquad \sigma^{Zs} = 2.35\%. \end{aligned}$$

With transparency in mind, we chose a baseline calibration that eliminates all asymmetries that are not crucial. In particular, for investment, we set the mean and the standard deviation in both sectors at the average across sectors, so that the model counterparts are $\mu^{IZe} = \mu^{IZs} = 1.04$, and $\sigma^{IZe} = \sigma^{IZs} = 0.0733$. Means and standard deviations for the investment specific technological change are set at $\mu^{Ze} = \mu^{Zs} = 1.0069$, and $\sigma^{Ze} = \sigma^{Zs} = 0$. In this case, all moments can be matched with the values given in the table above. Given that we are limited to a two-state representation, investment growth rates in the model are perfectly correlated; the correlation between the empirical counterparts is 0.6.

We also present results for a calibration that seeks to match as closely as possible the 8 means and standard deviations above. We minimize the equally weighted average of the squared deviations of the model's moments from their empirical counterparts. As shown in the appendix, this can be done reasonably well. The empirical correlations of sectoral investment with its specific technological growth are 0.43 and -0.32, while the correlations of the technological growth across sectors is 0.3. Clearly, due the limited degrees of freedom in our two state process we cannot match all these moments. As we show below, for most quantities of interest, this consideration is quantitatively second order.

(ii) Depreciation rates We need the constant depreciation rates for equipment and software as well as for structures: (δ_e, δ_s). We can directly compute depreciation rates from the Fixed Assets tables, and take the mean. We get 13.06% and 2.7% for 1947-2002. Because NIPA's depreciation includes physical wear as well as economic obsolesence, we adjust the data to take into account that the model depreciation covers only physical depreciation. To do this we add the price increase in the capital good. So that

$$\delta_t = \frac{D_t}{K_t} + p_{I,t}/p_{I,t-1} - 1 = \frac{D_t}{K_t} + (Z_{t-1}/Z_t - 1),$$

with D_t depreciation according to NIPA. This adjustment decreases depreciation by 1.82% for equipment and -0.44% for structures.

<u>(iii) Relative size of capital stocks</u> The ratio of the capital stocks, $(K_{e,t}/Z_{e,t}) / (K_{s,t}/Z_{s,t})$, is needed only for computing aggregate returns, which, as we have seen, are a function of the price weighted sum of the of the two capital stocks. In the model, the ratio of the physical capital stocks $K_{e,t}/K_{s,t}$ is trending, while the ratio of the bookvalues of the capital stocks (in terms of the consumption good) $(K_{e,t}/Z_{e,t}) / (K_{s,t}/Z_{s,t})$ is not trending. For calibration, we will have the ratio of bookvalues replicating roughly the average of the historical values. This will be implemented in the following way. We normalize $Z_{e,0} = Z_{s,0} = K_{s,0} = 1$, and set $K_{e,0} = mean\left((K_{e,t}/Z_{e,t}) / (K_{s,t}/Z_{s,t})\right) \cdot (K_{s,0}/Z_{s,0}) / Z_{e,0} = mean\left((K_{e,t}/Z_{e,t}) / (K_{s,t}/Z_{s,t})\right)$.

We use Current-Cost Net Stocks of Fixed Assets from the BEA. With this data, for the period 1947-2002, $mean\left((K_{e,t}/Z_{e,t}) / (K_{s,t}/Z_{s,t})\right) = 0.6$. This is to say that the value of equipment is 0.6 of the value of structures.

<u>(iv) Adjustment cost and marginal product</u> To parameterize the adjustment cost function, we choose the following procedure sequentially:

1) Pick v to get good results for asset prices under the restriction that $\nu_e < \nu_s$. Specifically, the selected values roughly generate the highest equity premium with the lowest (reasonable) return volatility, by also guaranteeing positive state prices. More formal searches have selected very similar parameter combinations.

2) Pick b so that $qZ's$ are consistent with average values reported in the literature.

3) c is then picked to minimize the overall amount of output lost due to adjustment cost.

In addition, to casual empiricism, there is also other evidence that suggests that adjustment costs are larger for structures than for equipment and software. In particular, the fact that the serial correlation of the growth rates is somewhat higher for structures than for equipment can be interpreted as an expression of the desire to smooth investment over time due to the relatively high adjustment cost.

There are many examples of studies that estimate qZ. Lindenberg and Ross (1981) report averages for two-digit sectors for the period 1960-77 between .85 and 3.08. Lewellen and Badrinath (1997) report an average of 1.4 across all sectors for the period 1975-91, Gomes (1999) reports an average of 1.56. Based on this we will use a steady-state target value for qZ of 1.5 for both sectors. One problem with using empirical studies to infer about the required heterogeneity in the sectoral costs is that most studies consider adjustment costs by sector of activity. For our analysis, we would need information about the adjustment cost by type of capital for a representative firm.

The marginal product coefficients A_1 and A_2 are set implicitly so as to have the steady-state return R_{ss} given by equation **??** equalized in the two sectors and to replicate the mean risk free rate.

(iv).1 Calibrate b and c to fit target qZ at steady state $I/(K/Z)$. From the capital accumulation equation we have

$$IZ/K = K'/K - (1-\delta),$$

For a deterministic steady-state, the growth rate of capital equals the growth rate of investment. Based on this, we define the steady-state investment/capital ratio as

$$\overline{IZ/K} = \left(\overline{\lambda_I \lambda_Z} - 1\right) + \delta,$$

where we use the mean of $\lambda_I \lambda_Z$ implied by the model as the empirical counterpart for $\overline{\lambda_I \lambda_Z}$, and the depreciation rate described above. From the first-order condition in steady-state, we have that

$$\overline{qZ} = b\left(\overline{IZ/K}\right)^{v-1}, \tag{5.1}$$

which we use to fix $\bar{b} = \overline{qZ}/\left(\left(\overline{\lambda_I \lambda_Z} - 1\right) + \delta\right)^{v-1}$.

Finally, we set c so that, roughly speaking, the total cost is minimized. The total cost is increasing in c, so we will pick the lowest possible c. This is done by making the total cost equal to zero at the point where the marginal cost is equal to zero. Clearly, if c where set lower than this, there would be a region where the total cost is negative. Specifically, we first find the IZ/K for which the marginal cost is zero, that is where $qZ = 1$, and using equation 5.1 as

$$ZI/K_{\text{no-cost}} = \left(1/\bar{b}\right)^{1/(v-1)},$$

we set the total cost to zero at this level of $ZI/K_{\text{no-cost}}$

$$\begin{aligned}
\frac{\bar{b}}{v}\left(IZ/K_{\text{no-cost}}\right)^{v} + c &= IZ/K_{\text{no-cost}} \\
c &= IZ/K_{\text{no-cost}} - \frac{\bar{b}}{v}\left(IZ/K_{\text{no-cost}}\right)^{v} \\
c &= \left(1/\bar{b}\right)^{1/(v-1)} - \frac{\bar{b}}{v}\left(\left(1/\bar{b}\right)^{1/(v-1)}\right)^{v}.
\end{aligned}$$

6 Quantitative findings

The key asset pricing moments we are interested in are first and second unconditional moments for equity and risk free returns. We also consider time-varying means and volatilities.

Table 1 presents the model implications from the baseline calibration as well as empirical counterparts for a set of unconditional moments. Model results are based on sample moments of a very long (40000 years) simulated time series. For unconditional moments, the key finding is that the model is able to generate an equity premium of several percentage points with reasonable volatility for the equity return as well as for the risk free rate. The model's mean Sharpe ratio is about one third of the one that is implied in the historic equity premium. Given the higher adjustment cost curvature for structures relative to equipment and software, as expected structures have a higher return volatility and a higher risk premium than equipment and software.

The model is able to generate considerable time-variation in conditional risk premiums. Indeed, the standard deviation of the one-period ahead conditional equity premium is at 4.83%, which is considerably higher than the standard deviation of the risk free rate at 2.97%. There is a variety of empirical studies measuring return predictability, with a variety of conclusions. For instance, Campbell and Cochrane (1999) report $R^{2\prime}s$ of 0.18 and 0.04 for regressions of excess returns on lagged price-dividend ratios at a one-year horizon for the periods $1947-95$ and $1871-1993$, respectively. Combining the R^2 with the volatility of the excess returns, $\sqrt{R^2} std\left(R-R^f\right)$ provides an estimate of the volatility of the conditional equity premium. Setting $R^2=0.1$ this would be $\sqrt{0.1}\times 0.17 = 5.27\%$. Thus, the model's value of 4.83% is close.

What is driving expected excess returns? In general, assuming the absence of arbitrage, we have that

$$E_t\left(R_{t+1}-R_t^f\right)=\frac{\sigma_t\left(m_{t+1}\right)}{E_t m_{t+1}}\sigma_t\left(R_{t+1}\right)\rho_t\left(m_{t+1},R_{t+1}\right).$$

Possibly, return volatility can drive risk premiums, but this doesn't seem empirically relevant. According to Lettau and Ludvigson (2004) this is not the case for the US postwar period. Indeed, they find strong negative correlations between conditional means and volatilities. Our model is consistent with this fact. Indeed, for the baseline calibration the correlation between conditional means and volatilities is -0.34. This negative correlation seems very robust to parameter changes.

Most standard models cannot replicate this finding. With CRRA utility and lognormal consumption, expected returns are given by

$$E_t\left(R_{t+1}-R_t^f\right)=\gamma\cdot\sigma_t\left(\ln C'/C\right)\cdot\sigma_t\left(R_{t+1}\right)\cdot\rho_t\left(m_{t+1},R_{t+1}\right).$$ [5]

In the Mehra-Prescott setup, all terms in the equation are roughly constant, with the correlation, $\rho_t\left(m_{t+1},R_{t+1}\right)$, roughly equal to one. In Campbell and Cochrane's model, $\frac{\sigma_t(m_{t+1})}{E_t m_{t+1}}$ displays considerable variation. However, as is clear from their Figures 4 and 5, conditional means and volatilities are positively correlated.

Let us focus now directly on the Sharpe ratio

$$\frac{E_t\left(R_{t+1}-R_t^f\right)}{\sigma_t\left(R_{t+1}\right)}=\frac{\sigma_t\left(m_{t+1}\right)}{E_t m_{t+1}}\rho_t\left(m_{t+1},R_{t+1}\right). \tag{6.1}$$

[5] The approximation comes from replacing $\gamma\cdot\sigma_t\left(\ln C'/C\right)$ by $\sqrt{\exp\left[\gamma^2 var_t\left(\ln C'/C\right)\right]-1}$.

Given the volatile conditional means and the negative correlation between conditional means and volatilities, Sharpe ratios are very volatile. According to Lettau and Ludvigson (2004), for quarterly data, market implied Sharpe ratios have a mean of 0.39 and a standard deviation of 0.448, which implies a coefficient of variation of $0.448/0.39 = 1.15$. In our model, for the baseline calibration, this ratio equals, $0.26/0.17 = 1.56$. That is, considering that our model generates average Sharpe ratios of roughly 1/3 of the ones implied by the aggregate market, it nevertheless has the ability to generate considerable volatility in Sharpe ratios

What drives the volatility of the Sharpe ratio? Both parts on the right hand side of equation 6.1 contribute. As shown in Table 1, the market price of risk is moving, but its mean and standard deviation differ from those of the market's Sharpe ratio. The mean of the market price of risk is (obviously) larger, while the volatility is lower. Remember the model structure, given that we have perfectly correlated shocks, the correlation $\rho_t\left(m_{t+1}, R_{t+1}\right)$ can only be 1 or -1. Clearly, therefore, $\rho_t\left(m_{t+1}, R_{t+1}\right)$ is switching between values of 1 and -1 as a function of the state of the economy. To further investigate this regime shifting property, note that if we make the shock process IID the mean and standard deviation of the Sharpe ratio and the market price of risk are much closer to each other than in the baseline case with serial correlation (and asymmetric states) as seen in Table 2. That is to say that the occurrence of a negative (conditional) correlation between the market return and the stochastic discount factor, and thus a negative Sharpe ratio is much rarer in the IID case. Table 3 show results for the case with investment specific technology shocks. While there are some quantitative differences, none of our main conclusions are affected. Note that we did not recalibrate the rest of the rest of the model.

To further illustrate model properties, we show here model implications from feeding through the investment realizations for the U.S. for the period 1947-2003. Given that investment growth in our model has only two values, the fit of the driving process is not perfect. Nevertheless, as shown in Figure 1, the fit is very good, with correlations between the model and the data of 0.78 and 0.71 for equipment and structures respectively. Figure 2 shows that the model's generated returns are indeed related to actually realized stock returns, with a correlation of 0.48. Figure 3 shows conditional moments. The two panels on the left show that conditional volatility is more persistent (and thus history dependent) than expected returns. The right hand side panel shows the market price of risk and the market's Sharpe ratio. Considering the 1990s, through the series of 8 high realizations in investment growth, expected returns, and Sharpe ratios are declining over time. The figure also shows that with a low investment growth realization, the market Sharpe ratio becomes negative, and thus the conditional correlation $\rho_t\left(m_{t+1}, R_{t+1}\right)$ becomes negative. It is interesting here to consider again the calibration with IID investment growth to further highlight the persistent component driving risk premiums. Figure 3b, presents the realized conditional moments corresponding to the IID case we presented in Table 2. Here the state of the economy is summarized by the two investment levels and capital stocks $\left(I_j\left(s^t\right)Z_j\left(s^t\right), K_j\left(s^{t-1}\right)\right)_{j=1,2}$. Here, the conditional mean becomes even more persistent than the volatility. Only twice in the postwar period does the market Sharpe ratio become negative. In the 1990s, it is at the 8th successive realization of a high investment rate that the market Sharpe ratio becomes negative. It then stays negative for another 3 periods, where investment growth is low. What are the elements of the state vector that are driving this result? There is a strong negative relationship between investment/capital ratios and Sharpe ratios, see Table 1 and 2. However, the relationship isn't perfect, and thus, investment and capital matter individually.

Finally, Figure 4 compares the model's risk free rate to the realized returns of short term Treasuries. While model and data seem to have a similar upward trend through the postwar period, the high frequency model implications are not close to the data.

A. Sensitivity analysis

tba

7 Conclusion

tba

8 References

Campbell and Cochrane (1999)

Cochrane, John production based (1988, 1991) and book

Greenwood, Hercowitz and Krusell,

Joao F. Gomes, 2001, Financing Investment, The American Economic Review, Vol. 91, No. 5 (Dec.), pp. 1263-1285.

Lettau and Ludvigson (2004)

Lewellen and Badrinath (1997)

Lindenberg and Ross (1981)

Mehra and Prescott (1986)

Table 1
Asset Pricing Implications: Baseline calibration

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market	$R^{E\&S}$	$R^{E\&S}$-R^f	R^S	R^S-R^f
Mean		3.17%	1.11%	0.25	0.17		1.83%		4.33%
Std	19.55%		2.97%	0.18	0.26	11.23%		25.76%	

Std[E(R^M-R^f|t)]	4.83%	Corr(Ep , StdR)	-0.34
Std[Std(R^M-R^f|t)]	1.15%		

	Corr(IKZ , E(R^M-R^f|t))	Corr(IKZ , R^f)	Corr(IKZ , Sharpe)
E&S	0.15	0.16	-0.86
S	-0.16	-0.07	-0.59

	Corr(λ^I, R)
E&S, S	0.98

Real returns 1947-2003	R^M	R^M-R^f	Rf	Sharpe
Mean		8.35%	1.09%	0.49
Std	17.24%		2.07%	

Table 2
Asset Pricing Implications: IIID case; no serial correlation, no asymmetric states

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market	$R^{E\&S}$	$R^{E\&S}$-R^f	R^S	R^S-R^f
Mean		3.11%	1.06%	0.18	0.17		1.79%		4.29%
Std	19.99%		2.55%	0.10	0.12	11.42%		26.13%	

Std[E(R^M-R^f|t)]	2.28%	Corr(Ep , StdR)	-0.74
Std[Std(R^M-R^f|t)]	0.98%		

	Corr(IKZ , E(R^M-R^f|t))	Corr(IKZ , R^f)	Corr(IKZ , Sharpe)
E&S	-0.43	-0.88	-0.39
S	-0.75	-0.63	-0.68

	Corr(λ^I, R)
E&S, S	0.99

Real returns 1947-2003	R^M	R^M-R^f	Rf	Sharpe
Mean		8.35%	1.09%	0.49
Std	17.24%		2.07%	

Table 3

Asset Pricing Implications: with shocks to investment technology

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market	$R^{E\&S}$	$R^{E\&S}$-R^f	R^S	R^S-R^f
Mean		2.05%	1.65%	0.28	0.13		1.08%		2.79%
Std	16.91%		4.14%	0.17	0.30	9.37%		22.72%	

Std[E(R^M-R^f|t)]	4.81%	Corr(Ep , StdR)	-0.11
Std[Std(R^M-R^f|t)]	1.26%		

	Corr(IKZ , E(R^M-R^f|t))	Corr(IKZ , R^f)	Corr(IKZ , Sharpe)
E&S	0.49	-0.94	0.31
S	0.07	-0.64	0.04

	Corr(λ^I, R)
E&S, S	0.97

Real returns 1947-2003	R^M	R^M-R^f	Rf	Sharpe
Mean		8.35%	1.09%	0.49
Std	17.24%		2.07%	

Figure 1

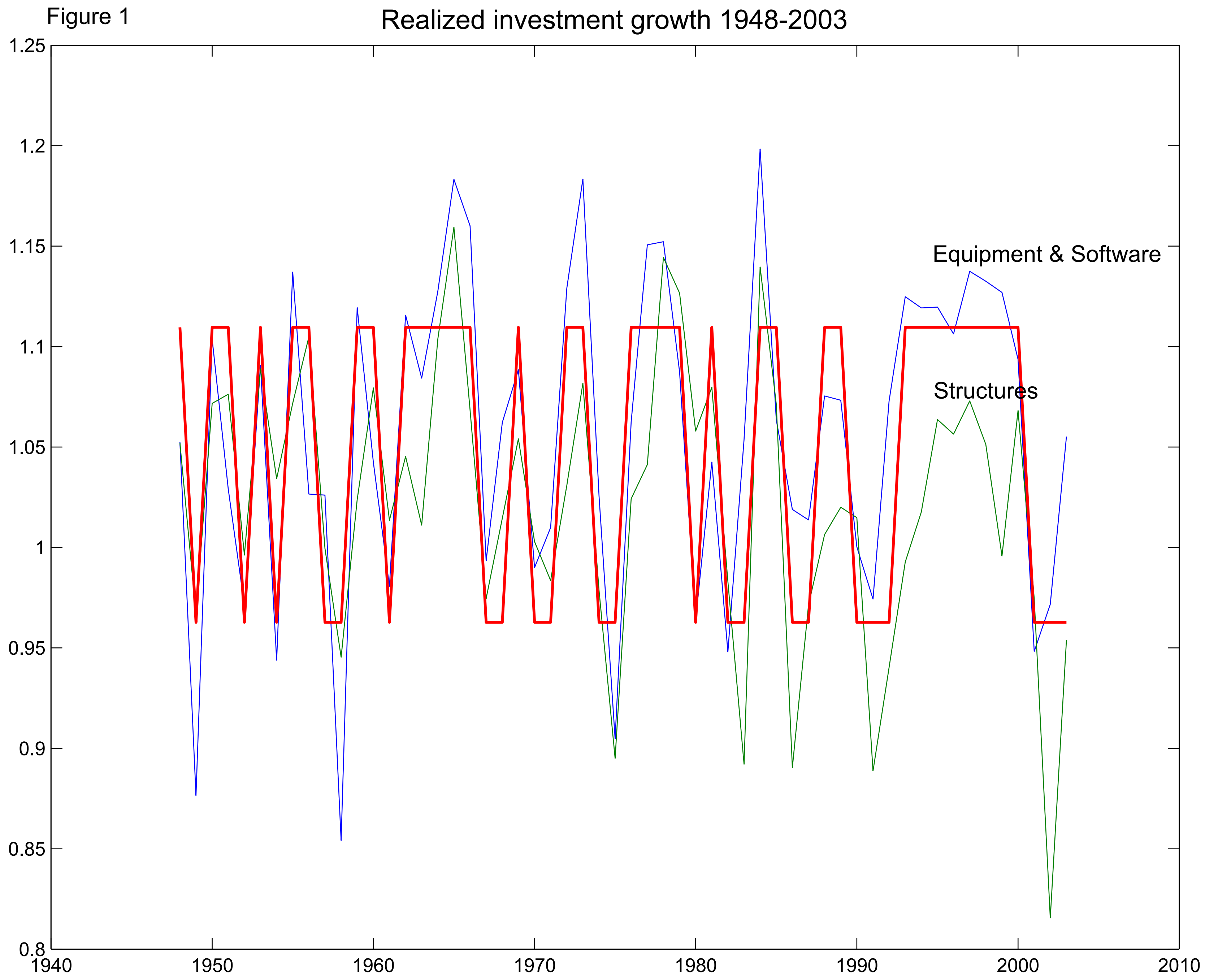
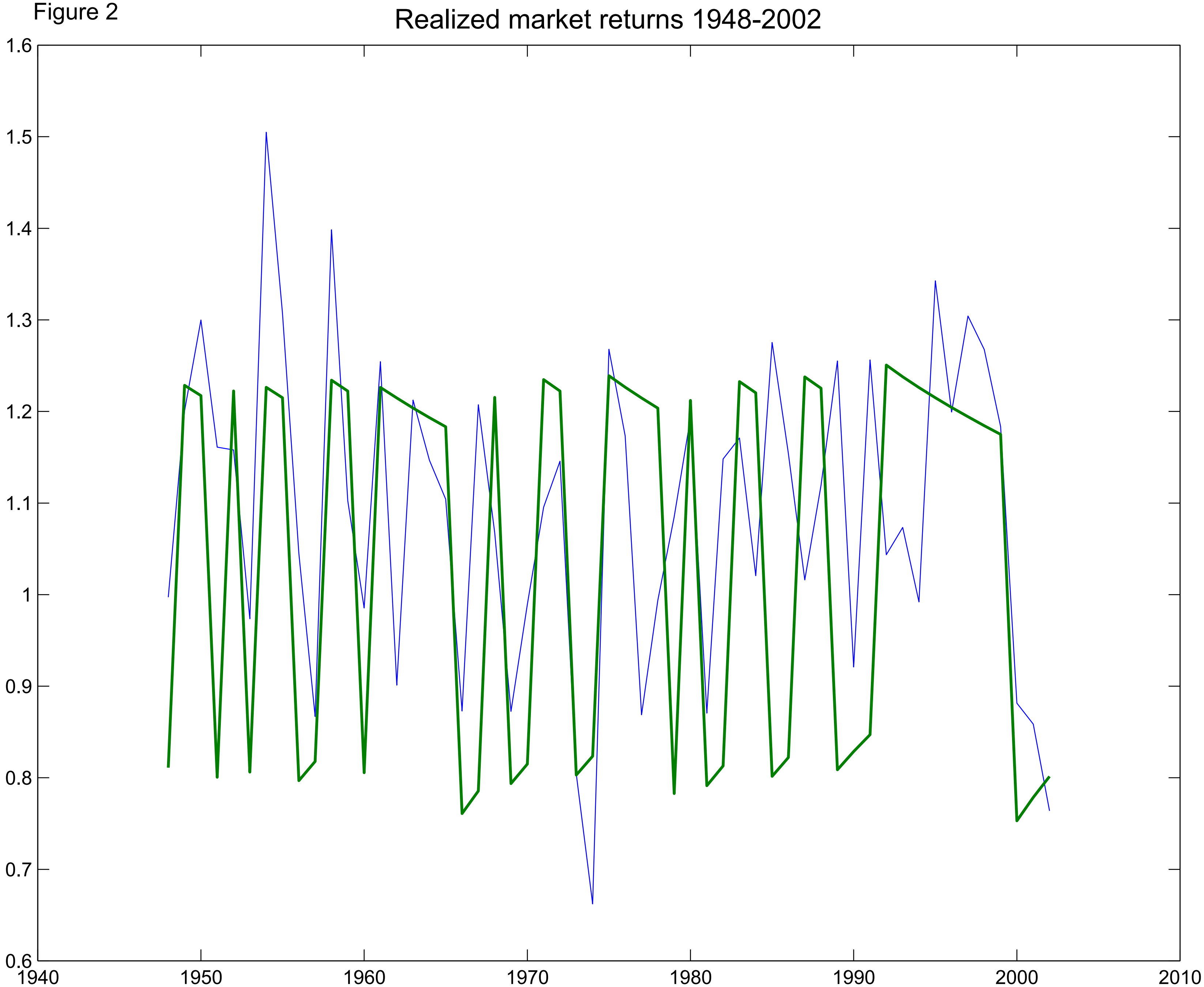

Realized investment growth 1948-2003
1.25
1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
0.8
1940
1950
1960
1970
1980
1990
2000
2010
Equipment & Software
Structures

Figure 2



Realized market returns 1948-2002
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
1940
1950
1960
1970
1980
1990
2000
2010

Figure 3a: Baseline Calibration

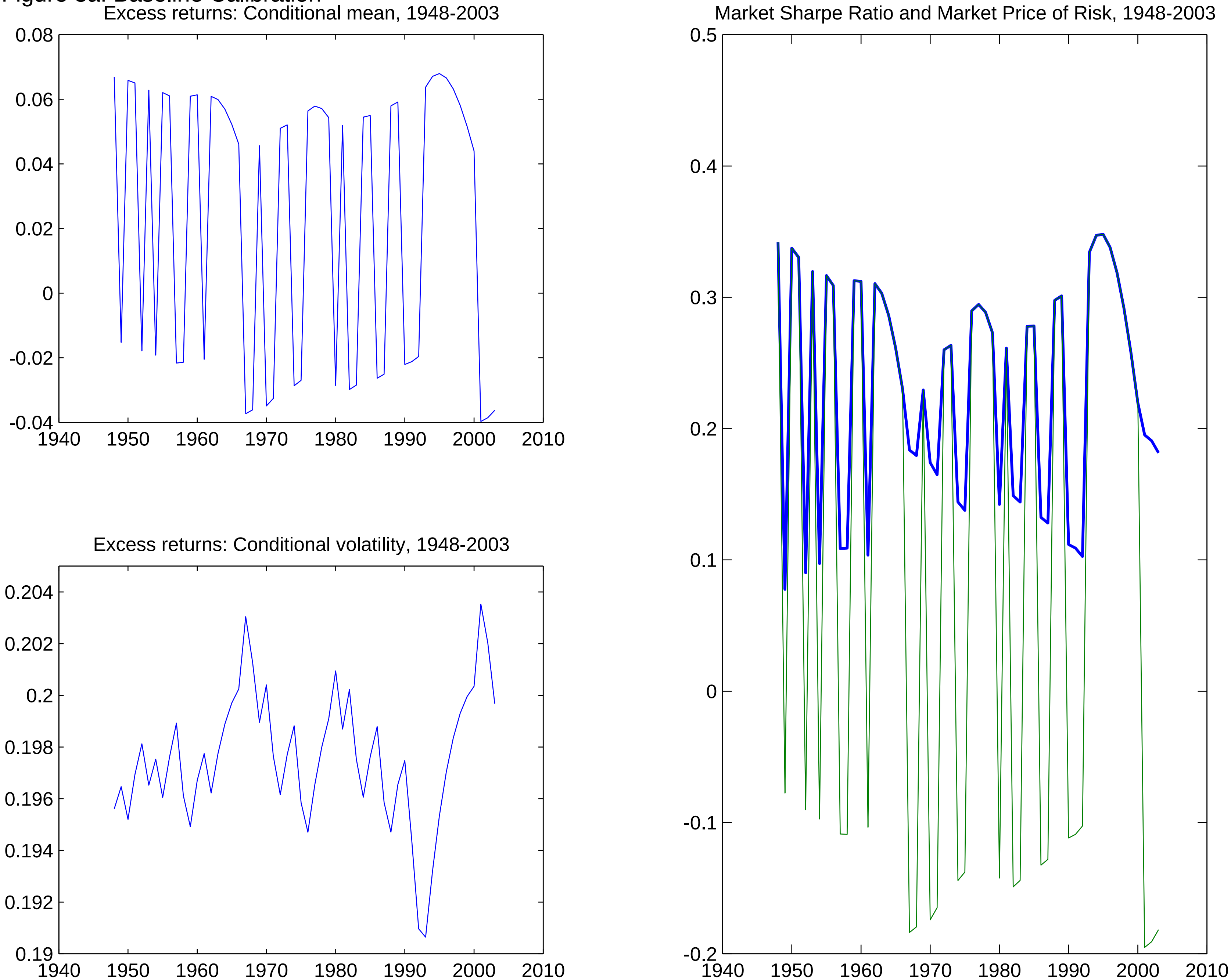


Figure 3b: IID case

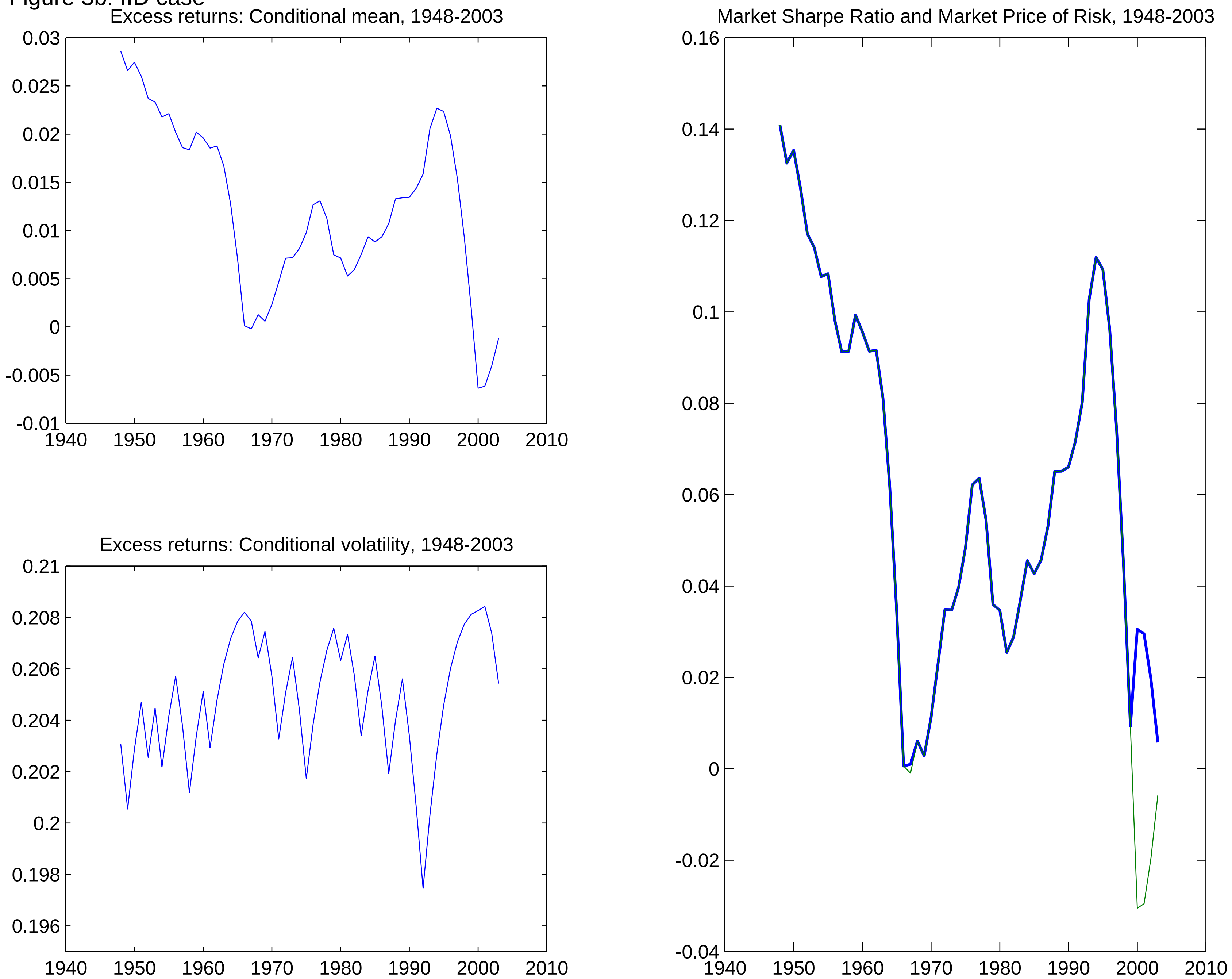
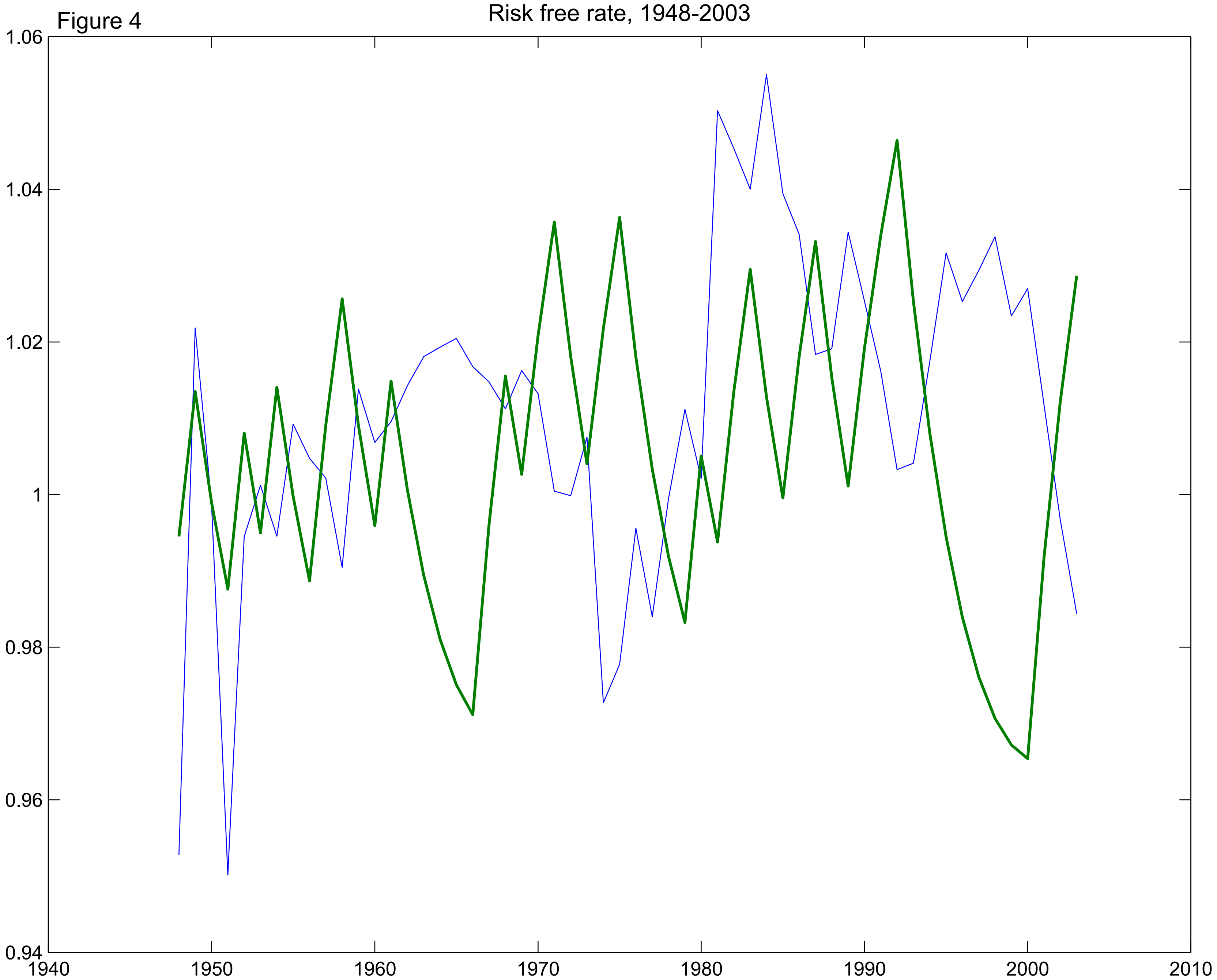


Figure 4


Risk free rate, 1948-2003
1.06
1.04
1.02
1
0.98
0.96
0.94
1940
1950
1960
1970
1980
1990
2000
2010