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IDENTITIES FOR HOMOGENEOUS UTILITY FUNCTIONS*

Miguel A. Espinosa[†] and Juan D. Prada-Sarmiento[‡]

Abstract

Using a homogeneous and continuous utility function that represents a household's preferences, this paper proves explicit identities between most of the different objects that arise from the utility maximization and the expenditure minimization problems. The paper also outlines the homogeneity properties of each object. Finally, we show explicit algebraic ways to go from the indirect utility function to the expenditure function and from the Marshallian demand to the Hicksian demand and vice versa, without the need of any other function, thus simplifying the integrability problem avoiding the use of differential equations.

Keywords: Identities, homogeneous utility functions and household theory.

JEL Classification: D10, D11

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IDENTIDADES PARA FUNCIONES DE UTILIDAD HOMOGÉNEAS*

Miguel A. Espinosa† y Juan David Prada-Sarmiento‡

Resumen

Haciendo uso de una función de utilidad homogénea y continua, este documento prueba identidades explícitas entre la mayoría de funciones resultantes del problema del hogar. A su vez, también se propone la caracterización de la homogeneidad para cada una de las funciones. Finalmente, se muestran métodos algebraicos explícitos para ir de la función de gasto a la utilidad indirecta, y de las demandas Marshallianas a sus contrapartes Hicksianas y vice versa. La principal ganancia de esto último, es que al no ser necesaria una tercera función, el problema de integrabilidad se ve enormemente simplificado al omitir la solución de un sistema de ecuaciones diferenciales.

Palabras Clave: Identidades, función de utilidad homogénea y teoría del hogar.

Clasificación JEL: D10, D11.

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1 Introduction

We can analyze the behaviour of a household with preferences over some consumption set through two standard optimization problems: maximization of a utility function representing the household's preferences subject to a budget constraint and minimization of the expenditure subject to a minimal level of satisfaction. Both problems have value functions (indirect utility and expenditure functions) and optimizers (Marshallian and Hicksian demand correspondences). Besides being interesting by themselves, under some assumptions all these objects represent or allow to recover the underlying preferences. Therefore it is of interest to be able to go from one object to another without delay. Although several books of intermediate and advanced microeconomic theory provide some identities between the objects derived from the household's problems, they do not provide explicit ways to go from one object to another one. For example, Jehle and Reny (2000) and Mas-Colell, Whinston and Green (1995) state four identities to go from the indirect utility function to the expenditure function and from Marshallian demands to Hicksian demands. These last two are not satisfactory enough because it is necessary to know either the indirect utility function or the expenditure function in order to use these identities. The need of at least two value functions in order to get a new one means that these identities are intensive in computational procedures. This paper proposes identities which allow to shift between the different objects associated with household optimization when the utility function is homogeneous, without the need for a third function.

To the best of our knowledge this intensive treatment of identities for homogeneous preferences have not been taken before. Among our main findings is the characterization about how to shift from the indirect utility function to the expenditure function and from the Marshallian demand to the Hicksian demand and vice versa, without the need of any other function.

2 Theoretical Framework

2.1 General Assumptions

Throughout this paper it is assumed that the consumer has a rational and continuous preference relation that can be represented by a continuous utility function. It is also assumed that prices are fixed and publicly announced and that consumer's wealth is an exogenous non-negative variable.

2.2 Definitions

2.2.1 Utility Function

We let the consumption set to be $\mathbb{R}^n_+$. The preferences of the consumer can be described by a utility function that expresses the maximum level of satisfaction that can be achieved with a given consumption bundle. It will be assumed that the utility function $u : \mathbb{R}^n_+ :\longrightarrow \mathbb{R}$ satisfies the following conditions:

C.1 $u(\cdot)$ is homogeneous of degree γ, i.e., for all $t > 0$ and all $\mathbf{x} \in \mathbb{R}^n_+$, $u(t\mathbf{x}) = t^\gamma u(\mathbf{x})$ where $\gamma \geq 0$ is the degree of homogeneity.

C.2 $u(\cdot)$ is strictly increasing in $\mathbf{x}$, i.e., if $\mathbf{x}_1 \gg \mathbf{x}_2$ then $u(\mathbf{x}_1) > u(\mathbf{x}_2)$ [1]

C.3 $u(\cdot)$ is a continuous and quasiconcave function over $\mathbb{R}^n_+$.

Homogeneity and continuity imply that $u(0) = 0$, and C.3 implies that $u(\mathbf{x}) > 0$ for $\mathbf{x} \gg \mathbf{0}$. It is important to note that C.1 and C.2 imply that $u(\cdot)$ is unbounded above. Finally, as shown by Prada (2010), if we assume C.1 with $0 < \gamma \leq 1$, quasiconcavity and that $u(\cdot)$ is increasing[2], then $u(\cdot)$ is concave.

2.2.2 Expenditure Function

Given a utility function $u(\cdot)$, the expenditure function expressing the minimum expenditure at which an agent can achieve a fixed level of utility $u \in \mathbb{R}_+$, taking the goods' price vector $\mathbf{p} \in \mathbb{R}^n_{++}$ as given, can be defined as:

$$e(u, \mathbf{p}) = \min_{\mathbf{x} \in \mathbb{R}^n_+} \{\mathbf{p} \cdot \mathbf{x} : u(\mathbf{x}) \geq u\} \tag{1}$$

The expenditure function is well defined, provided that the constraint set is nonempty. To see why, assume that there exists some $\mathbf{y} \in \mathbb{R}^n_+$ such that $u(\mathbf{y}) \geq u$. Then, the minimum expenditure has to be at most $\mathbf{p} \cdot \mathbf{y}$. Thus, we can optimize over the set $\left\{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{y}\right\}$. This set is compact if $\mathbf{p} \in \mathbb{R}^n_{++}$. If the utility function is unbounded above, the constraint set of the expenditure minimization problem is nonempty for all $u \in \mathbb{R}$. Then, the expenditure function exists for all $(\mathbf{p}, u) \in \mathbb{R}^n_{++} \times \mathbb{R}$.

[1] For $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$, $\mathbf{x}_1 \gg \mathbf{x}_2$ if and only if $\mathbf{x}_{1i} > \mathbf{x}_{2i}$ for all $i = 1, \ldots, n$.

[2] A function $f(\cdot)$ is increasing if and only if for $\mathbf{x}_1 \geq \mathbf{x}_2$ we have $f(\mathbf{x}_1) \geq f(\mathbf{x}_2)$, where $\mathbf{x}_1 \geq \mathbf{x}_2$ if and only if $x_{1i} \geq x_{2i}$ for all $i = 1, \ldots, n$.

2.2.3 Hicksian Demands

The set of consumption bundles that minimizes expenditure can be expressed as a function of u and $\mathbf{p}$. Such correspondence is known as the Hicksian demand correspondence and is defined as

$$\mathbf{x}^h(u, \mathbf{p}) = \arg\min_{\mathbf{x} \in \mathbb{R}^n_+} \{\mathbf{p} \cdot \mathbf{x} : u(\mathbf{x}) \geq u\} \tag{2}$$

By C.3 this is a convex set and by the maximum theorem the correspondence is upper-hemicontinuous. Note that if $u(\cdot)$ is strictly quasiconcave, then we have unique solution and the Hicksian demand is a continuous function.

It follows directly from the previous definitions that

$$e(u, \mathbf{p}) = \mathbf{p} \cdot \mathbf{x}^h(u, \mathbf{p}) \tag{3}$$

2.2.4 Indirect Utility Function

The indirect utility function expresses the maximum utility that the consumer can achieve as a function of wealth and goods' price vector (i.e $m \in \mathbb{R}_+$ and $\mathbf{p} \in \mathbb{R}^n_{++}$) . If the utility function $u(\cdot)$ is known, it can be defined as:

$$v(m, \mathbf{p}) = \max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x}) \;\; s.t \; \mathbf{p} \cdot \mathbf{x} \leq m \tag{4}$$

The indirect utility function is well defined if the utility function $u(\cdot)$ is continuous and if the budget set is compact (applying Weierstrass' extreme value theorem). Continuity holds by C.3 and compactness of the budget set holds if $\mathbf{p} \in \mathbb{R}^n_{++}$ and $m \in \mathbb{R}_+$.

2.2.5 Marshallian Demands

The correspondence that gives the set of consumption bundles that maximize the utility in terms of the goods' vector price $\mathbf{p} \in \mathbb{R}^n_{++}$ and wealth m is called the Marshallian demand correspondence. It is defined as

$$\mathbf{x}(m, \mathbf{p}) = \arg\max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x})$$

By Berge's maximum theorem this correspondence is upper-hemicontinuous and by C.3 it is convex valued. Note that if $u(\cdot)$ is strictly quasiconcave, then we have a unique solution to the utility maximization problem. In that case the Marshallian correspondence is single valued and it defines a continuous function, that will be called the Marshallian demand function.

2.2.6 Inverse Demands

The inverse demand correspondence gives the set of price vectors that, for fixed $\mathbf{x} \in \mathbb{R}^n_{++}$ and m, induce the minimum optimal level of utility. That is

$$\mathbf{p}(m, \mathbf{x}) = \arg \min_{\mathbf{p} \in \mathbb{R}^n_+} \{v(m, \mathbf{p}) : \mathbf{p} \cdot \mathbf{x} \leq m\} \tag{5}$$

This correspondence is nonempty. Note that by Berge's maximum theorem the indirect utility function $v(m, \mathbf{p})$ is continuous if the utility function is continuous. If $\mathbf{x} \in \mathbb{R}^n_{++}$, then the constraint set of the minimization problem that defines $\mathbf{p}(m, \mathbf{x})$ is compact and at least one solution exists. Also, since $v(m, \mathbf{p})$ is quasiconvex in prices and wealth, the correspondence is convex valued.

By the standard duality in consumption, under assumptions C.1, C.2 and C.3 we will have that $u(x) = v(m, \mathbf{p})$ when $\mathbf{p} \in \mathbf{p}(m, \mathbf{x})$.

2.3 Basic Relationships

The following Proposition summarizes some well known relationships among the objects previously defined.

Proposition 1 *Let $u(\cdot)$ be a continuous function satisfying C.2. For positive prices, wealth and utility $u > u(\mathbf{0})$ we have*

1. *Walras' law:*

   $\mathbf{p} \cdot x(m, \mathbf{p}) = m$. *This holds for any* $\mathbf{p}$.

2. *Demand identities:*

   $x(m, \mathbf{p}) = x^h(v(m, \mathbf{p}), \mathbf{p})$.

   $x^h(u, \mathbf{p}) = x(e(u, \mathbf{p}), \mathbf{p})$.

3. *Optimal value function identities:*

   $e(v(m, \mathbf{p}), \mathbf{p}) = m$.

   $v(e(u, \mathbf{p}), \mathbf{p}) = u$.

4. *Other identities:*

   $u\left(x^h(u, \mathbf{p})\right) = u$.

$\mathbf{p} \cdot x(m, \mathbf{p}) = e(v(m, \mathbf{p}), \mathbf{p})$.

Proof. This is a standard proposition found in any intermediate and advanced microeconomics test. See for example Mas-Colell, Whinston and Green (1995), Proposition 3.E.1, and Jehle and Reny (2000), Theorems 1.8 and 1.9. ■

Note that this Proposition does not require the utility to be homogeneous.

3 Identities Between Representations of a Homogeneous Utility Function

We follow Espinosa, Bonaldi and Vallejo (2009)[3]: identities are defined as "equations by means of which an explicit functional form of a representation of preferences is expressed in terms of an explicit functional form of other representation of those preferences".

The next result is similar to the one presented in Jehle and Reny (2000), Theorem 2.3.

Proposition 2 *If the utility function satisfies C.2 and C.3, then*

$$u(\mathbf{x}) = \min_{\mathbf{p} \in \mathbb{R}^n_+} \{v(m, \mathbf{p}) : \mathbf{p} \cdot \mathbf{x} \leq m\}$$

Proof. Fix $\mathbf{x} \in \mathbb{R}^n_{++}$. Define $\tilde{u}(\mathbf{x}) = \min_{\mathbf{p} \in \mathbb{R}^n_+} \{v(m, \mathbf{p}) : \mathbf{p} \cdot \mathbf{x} \leq m\}$ and take a minimizer $\mathbf{p} \in \mathbf{p}(m, \mathbf{x})$. Then, we have $\tilde{u}(\mathbf{x}) = v(m, \mathbf{p}) \geq u(\mathbf{x})$ because $\mathbf{p} \cdot \mathbf{x} \leq m$ and because of the definition of $v(m, \mathbf{p})$.

On the other hand, since $u(\mathbf{x})$ is quasiconcave, the upper contour level set is convex. Thus, by the separating hyperplane theorem, there exists a $\mathbf{q} \neq \mathbf{0}$ and a $r \in \mathbb{R}$ such that $\mathbf{q} \cdot \mathbf{x} \leq r \leq \mathbf{q} \cdot \mathbf{y}$ for all $\mathbf{y} \in \mathbb{R}^n_+$ such that $u(\mathbf{y}) \geq u(\mathbf{x})$.

We have $\mathbf{q} \in \mathbb{R}^n_+$. If not, there is some i such that $q_i < 0$. Then, making $\mathbf{y} = \mathbf{x} + \boldsymbol{\epsilon} + \alpha \mathbf{1}_i$ where $\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon & \ldots & \epsilon \end{bmatrix}'$ for some $\epsilon > 0$ and $\mathbf{1}_i = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix}'$ we have that for $\alpha > 0$ big enough $\mathbf{q} \cdot \mathbf{y} < r$ and $u(\mathbf{y}) > u(\mathbf{x})$. Then, it follows that $r \geq 0$ since $\mathbf{x} \in \mathbb{R}^n_+$. Note that $\mathbf{x}$ is a solution to the utility maximization problem with prices $\mathbf{q}$ and wealth r. Then, $\tilde{u}(\mathbf{x}) \leq v(r, \mathbf{q}) = u(\mathbf{x})$. ■

Fix any $\mathbf{x} \in \mathbb{R}^n_{++}$. Let $\mathbf{p} \in \mathbf{p}(m, \mathbf{x})$ as defined in Equation 5. It follows from Proposition 2 that $u(\mathbf{x}) = v(m, \mathbf{p})$. Since $\mathbf{p} \cdot \mathbf{x} \leq m$ we have that $\mathbf{x} \in \mathbf{x}(m, \mathbf{p})$. That is why the correspondence $\mathbf{p}(m, \mathbf{x})$ is called "inverse demand".

With this duality result we can easily prove the following result:

[3] From now on Espinosa, Bonaldi and Vallejo (2009) will be referred as EBV.

Corollary 1 *If the utility function $u(\cdot)$ satisfies C.2 and C.3, it is homogeneous of degree γ (satisfies C.1) if and only if the expenditure function $e(u,\mathbf{p})$ is homogeneous of degree $\frac{1}{\gamma}$ in u (and so it can be written as $e(u,p) = u^{\frac{1}{\gamma}} e(1,\mathbf{p})$).*

Proof.

Assume C.1. Take $t > 0$. Note that

$$\begin{aligned}
e(u,\mathbf{p}) &= \min_{\mathbf{x}\in\mathbb{R}^n_+} \{\mathbf{p}\cdot\mathbf{x} : u(x) \geq u\} \\
&= \min_{\mathbf{x}\in\mathbb{R}^n_+} \{\mathbf{p}\cdot\mathbf{x} : tu(\mathbf{x}) \geq tu\} \\
&= \min_{\mathbf{x}\in\mathbb{R}^n_+} \left\{\mathbf{p}\cdot\mathbf{x} : u\left(t^{\frac{1}{\gamma}}\mathbf{x}\right) \geq tu\right\} \\
&= \min_{\tilde{\mathbf{x}}\in\mathbb{R}^n_+} \left\{t^{-\frac{1}{\gamma}}\mathbf{p}\cdot\tilde{\mathbf{x}} : u(\tilde{\mathbf{x}}) \geq tu\right\} \\
&= t^{-\frac{1}{\gamma}} \min_{\tilde{\mathbf{x}}\in\mathbb{R}^n_+} \{\mathbf{p}\cdot\tilde{\mathbf{x}} : u(\tilde{\mathbf{x}}) \geq tu\} \\
&= t^{-\frac{1}{\gamma}} e(tu,\mathbf{p})
\end{aligned}$$

and then $e(tu,\mathbf{p}) = t^{\frac{1}{\gamma}} e(u,\mathbf{p})$.

Now assume that the expenditure function is homogeneous of degree $\frac{1}{\gamma}$ in u. Then, $e(u,\mathbf{p}) = u^{\frac{1}{\gamma}} e(1,\mathbf{p}) = u^{\frac{1}{\gamma}} e(\mathbf{p})$ where we define $e(\mathbf{p}) \equiv e(1,\mathbf{p})$ as the normalized expenditure function. Since $m = e(v(m,\mathbf{p}),\mathbf{p}) = v(m,\mathbf{p})^{\frac{1}{\gamma}} e(\mathbf{p})$ we get

$$v(m,\mathbf{p}) = m^{\gamma} e(\mathbf{p})^{-\gamma}$$

From Proposition 2 we have

$$\begin{aligned}
u(t\mathbf{x}) &= \min_{\mathbf{p}\in\mathbb{R}_+} \left\{m^{\gamma} e(\mathbf{p})^{-\gamma} : \mathbf{p}\cdot t\mathbf{x} \leq m\right\} \\
&= t^{\gamma} \min_{\mathbf{p}\in\mathbb{R}_+} \left\{\left(\frac{m}{t}\right)^{\gamma} e(\mathbf{p})^{-\gamma} : \mathbf{p}\cdot\mathbf{x} \leq \frac{m}{t}\right\} \\
&= t^{\gamma} u(\mathbf{x})
\end{aligned}$$

Then, we get the homogeneity result on $u(\cdot)$ for $\mathbf{x} \in \mathbb{R}^n_{++}$. But by continuity we can extend this result to any $\mathbf{x} \in \mathbb{R}^n_+$. ■

From the proof of Corollary 1 we obtain another result that it is important on its own.

Corollary 2 *If the utility function $u(\cdot)$ satisfies C.2 and C.3, it is homogeneous of degree γ (satisfies C.1) if and only if the indirect utility function $v(m,\mathbf{p})$ is homogeneous of degree γ in m (and so it can be written* *as $v(m,p) = m^{\gamma} v(1,\mathbf{p})$).*

Proof. It follows from Corollary 1 and the identity $m = e(v(m,p),\mathbf{p})$. ■

Let $ae(u,\mathbf{p}) = \frac{e(u,\mathbf{p})}{u}$ be the average expenditure, and let $me(u,\mathbf{p}) = \frac{\partial e(u,\mathbf{p})}{\partial u}$ be the marginal average expenditure. If the utility function satisfies C.2 we have the following relation between ae and me:

Proposition 3 *The expenditure function is homogeneous of degree $\frac{1}{\gamma}$ in u if and only if the ratio of average to marginal expenditure equals γ.*

Proof. It is a consequence of Euler's theorem for homogeneous functions. See Proposition 2 in EBV. Alternatively, we could provide a direct proof given that $e(u,\mathbf{p}) = u^{\frac{1}{\gamma}} e(1,\mathbf{p})$. ■

We have an analogous result using the indirect utility function.

Corollary 3 *If the utility function is homogeneous of degree γ, then the ratio of the marginal utility of wealth (i.e $\frac{\partial v(m,\mathbf{p})}{\partial m}$) over the average utility of wealth $\frac{v(m,\mathbf{p})}{m}$ is γ.*

Proof. By Corollary 2 we have that $v(m,\mathbf{p}) = m^{\gamma} v(1,\mathbf{p})$. Define $v(\mathbf{p}) \equiv v(1,\mathbf{p})$. We have $\frac{\partial v(m,\mathbf{p})}{\partial m} = \gamma m^{\gamma-1} v(\mathbf{p})$. Then, the degree of homogeneity of the utility function is $\gamma = \frac{\frac{\partial v(m,\mathbf{p})}{\partial m}}{\frac{v(m,\mathbf{p})}{m}}$. ■

We now show a standard result about the inverse demand correspondence.

Proposition 4 *Let $u(\cdot)$ be a differentiable utility function that satisfies conditions C.1 to C.3. If the solution to the utility maximization problem is unique and interior, then the inverse demand correspondence is single valued. Also, the inverse demand function can be written as $p_i(m,\mathbf{x}) = \frac{m}{\gamma} \frac{\frac{\partial u(\mathbf{x})}{\partial x_i}}{u(\mathbf{x})}$ for $i = 1,\ldots,n$.*

Proof.

We need to prove that the Marshallian demand function is one-to-one. Fix price vectors $\mathbf{p}$ and $\mathbf{p}'$ and assume that $\mathbf{x} = \mathbf{x}(m,\mathbf{p}) = \mathbf{x}(m,\mathbf{p}')$. The first-order conditions for the utility maximization problem give us $\frac{\partial u(\mathbf{x})}{\partial x_i} = \lambda p_i$ and $\frac{\partial u(\mathbf{x})}{\partial x_i} = \lambda' p_i'$ for some constants $\lambda > 0$ and $\lambda' > 0$. Then, we have that for all $i = 1,\ldots,n$, $p_i = \frac{\lambda'}{\lambda} p_i'$. Finally, by monotonicity of the utility function we have $\mathbf{p} \cdot \mathbf{x} = \mathbf{p}' \cdot \mathbf{x} = m$. Then, $\lambda = \lambda'$ and therefore $\mathbf{p} = \mathbf{p}'$. This shows that the Marshallian demand function is invertible.

Now fix $\mathbf{x} \in \mathbb{R}^n_+$. By the envelope theorem we have $\frac{\partial u(\mathbf{x})}{\partial x_i} = \lambda p_i(m,\mathbf{x})$. Multiplying by x_i and summing over i we obtain

$$\sum_{i=1}^{n} x_i \frac{\partial u(\mathbf{x})}{\partial x_i} = \sum_{i=1}^{n} \lambda p_i(m,\mathbf{x}) x_i = \lambda m$$

where we used Walras' law. Then, $p_i = m \frac{\frac{\partial u(\mathbf{x})}{\partial x_i}}{\sum_{i=1}^{n} x_i \frac{\partial u(\mathbf{x})}{\partial x_i}}$. Finally, note that the utility function is homogeneous of degree γ. Then, $\sum_{i=1}^{n} x_i \frac{\partial u(\mathbf{x})}{\partial x_i} = \gamma u(\mathbf{x})$. ■

Now we present and prove some identities that reduce the computational burden of shifting from one representation of preferences to another. Let $v(\mathbf{p}) \equiv v(1, \mathbf{p})$ and $e(\mathbf{p}) \equiv e(1, \mathbf{p})$.

Remember that Shephard's lemma and Roy's identity are valid if the solutions to the household's optimization problems are unique. When we use these results we are implicitly assuming uniqueness. Note also that we used some homogeneity properties that will be proved in Section 4.

Theorem 1 *If the utility function satisfies conditions C.1 to C.3, and the optimal value functions are differentiable in their parameters, then the next identities hold for all* $u,\ m \in \mathbb{R}_+$*:*

I.1. $v(m, \mathbf{p}) = u\left[\frac{m}{e(u,\mathbf{p})}\right]^{\gamma}$. *I.1'.* $v(m, \mathbf{p}) = \left(\frac{m}{e(\mathbf{p})}\right)^{\gamma}$ *and* $v(m, \mathbf{p}) = u\left[\frac{m}{\mathbf{p}\cdot\mathbf{x}^h(u,\mathbf{p})}\right]^{\gamma}$.

I.2. $v(m, \mathbf{p}(m, \mathbf{x})) = u(\mathbf{x})$.

I.3. $u(\mathbf{x}) = u\left[\frac{m}{e(u,\mathbf{p}(m,\mathbf{x}))}\right]^{\gamma}$. *I.3'.* $u(\mathbf{x}) = \left[\frac{1}{e(\mathbf{p}(1,\mathbf{x}))}\right]^{\gamma}$.

I.4. $x_i(m, \mathbf{p}) = -\frac{\partial v(m,\mathbf{p})}{\partial p_i} / \frac{\gamma v(m,\mathbf{p})}{m}$.

I.5. $x_i(m, \mathbf{p}) = \frac{m x_i^h(u,\mathbf{p})}{\mathbf{p}\cdot\mathbf{x}^h(u,\mathbf{p})}$. *I.5'.* $x_i(m, \mathbf{p}) = \frac{m\frac{\partial e(u,\mathbf{p})}{\partial p_i}}{e(u,\mathbf{p})} = \frac{m\frac{\partial e(\mathbf{p})}{\partial p_i}}{e(\mathbf{p})}$.

I.6. $x_i^h(u, \mathbf{p}) = \frac{u^{\frac{1}{\gamma}} x_i(1,\mathbf{p})}{v(\mathbf{p})^{\frac{1}{\gamma}}} = \frac{u^{\frac{1}{\gamma}} x_i(m,\mathbf{p})}{v(m,\mathbf{p})^{\frac{1}{\gamma}}}$. *I.6'.* $v(m, \mathbf{p}) = u\left[\frac{x_i(m,\mathbf{p})}{x_i^h(u,\mathbf{p})}\right]^{\gamma} = \left[\frac{x_i(m,\mathbf{p})}{x_i^h(1,\mathbf{p})}\right]^{\gamma}$.

I.7. $x_i^h(u, \mathbf{p}) = x_i(m, \mathbf{p})\left(\frac{u}{u(\mathbf{x}(m,\mathbf{p}))}\right)^{\frac{1}{\gamma}} = x_i(1, \mathbf{p})\left(\frac{u}{u(\mathbf{x}(1,\mathbf{p}))}\right)^{\frac{1}{\gamma}}$.

I.8. $e(u, \mathbf{p}) = m\left(\frac{u}{u(\mathbf{x}(m,\mathbf{p}))}\right)^{\frac{1}{\gamma}} = \left(\frac{u}{u(\mathbf{x}(1,\mathbf{p}))}\right)^{\frac{1}{\gamma}}$.

I.9. $x_i^h(u, \mathbf{p}) = -\frac{m u^{\frac{1}{\gamma}}}{\gamma} \frac{\frac{\partial v(m,\mathbf{p})}{\partial p_i}}{v(m,\mathbf{p})^{\frac{1+\gamma}{\gamma}}}$.

I.10. $x_i(m, \mathbf{p}) = x_i^h\left(u\left[\frac{m}{\mathbf{p}\cdot\mathbf{x}^h(u,\mathbf{p})}\right]^{\gamma}, \mathbf{p}\right) = x_i^h\left(\left[\frac{m}{\mathbf{p}\cdot\mathbf{x}^h(1,\mathbf{p})}\right]^{\gamma}, \mathbf{p}\right)$.

Proof. From Corollary 1 follows immediately that $e(u, \mathbf{p}) = u^{\frac{1}{\gamma}} e(\mathbf{p})$. Since $m = e(v(m, \mathbf{p}), \mathbf{p})$, we have $m = v(m, \mathbf{p})^{\frac{1}{\gamma}} e(\mathbf{p}) = v(m, \mathbf{p})^{\frac{1}{\gamma}} u^{-\frac{1}{\gamma}} e(u, \mathbf{p})$ and we get I.1 and I.1'.

We obtain I.2 simply by evaluating the inverse demand function in the indirect utility function. This identity follows from Proposition 2.

Identity I.3 can be obtained using I.2 and I.1. Identity I.3' follows in the next way:

$$u(\mathbf{x}) = u\left[\frac{m}{e(u, \mathbf{p}(m, \mathbf{x}))}\right]^{\gamma} = u\left[\frac{m}{u^{\frac{1}{\gamma}} e(1, \mathbf{p}(m, \mathbf{x}))}\right]^{\gamma} = \left[\frac{m}{e(\mathbf{p}(m, \mathbf{x}))}\right]^{\gamma}$$

To prove I.4 note that by Corollary 3, $\gamma = \frac{\frac{\partial v(m,\mathbf{p})}{\partial m}}{\frac{v(m,\mathbf{p})}{m}}$ and by Roy's Identity, Marshallian demands can be expressed as $x_i(m,\mathbf{p}) = -\frac{\partial v(m,\mathbf{p})}{\partial p_i} / \frac{\gamma v(m,\mathbf{p})}{m}$.

In order to prove I.5 note that from I.1,

$$\frac{\partial v(m,\mathbf{p})}{\partial p_i} = m^{\gamma}\frac{\partial}{\partial p_i}e(\mathbf{p})^{-\gamma} = -\gamma m^{\gamma} e(\mathbf{p})^{-\gamma-1}\frac{\partial}{\partial p_i}e(\mathbf{p}) = \frac{-\gamma v(m,\mathbf{p})\, x_i^h(1,\mathbf{p})}{e(\mathbf{p})}$$

where $\frac{\partial}{\partial p_i}e(\mathbf{p}) = x_i^h(1,\mathbf{p})$ follows from Shephard's lemma. Then using I.4 we get

$$x_i(m,\mathbf{p}) = -\left(\frac{-\gamma v(m,\mathbf{p})\, x_i^h(1,\mathbf{p})}{e(\mathbf{p})}\right) / \frac{\gamma v(m,\mathbf{p})}{m} = \frac{m x_i^h(1,\mathbf{p})}{e(\mathbf{p})}$$

Note now that since for all price vector $\mathbf{p}$ we have $\mathbf{p}\cdot\mathbf{x}^h(u,\mathbf{p}) = e(u,\mathbf{p}) = u^{\frac{1}{\gamma}}e(1,\mathbf{p}) = u^{\frac{1}{\gamma}}\mathbf{p}\cdot\mathbf{x}^h(1,\mathbf{p})$ then we must have $\mathbf{x}^h(u,\mathbf{p}) = u^{\frac{1}{\gamma}}\mathbf{x}^h(1,\mathbf{p})$ and therefore the Hicksian demand is homogeneous of degree $\frac{1}{\gamma}$ in u. Using this homogeneity we will get $x_i(m,\mathbf{p}) = \frac{m x_i^h(u,\mathbf{p})}{e(u,\mathbf{p})}$.

Identity I.6 is obtained by a simple algebraic manipulation of I.5' and taking into account that $v(\mathbf{p}) = e(\mathbf{p})^{-\gamma}$ and the homogeneity of the Hicksian demand with respect to u. Identity I.6' follows from I.6.

Identity I.7 can be obtained by using I.6 and the definition of the indirect utility function as $v(m,\mathbf{p}) = u(\mathbf{x}(m,\mathbf{p}))$.

Identity I.8 follows from I.7 and Walras' law. We have

$$e(u,\mathbf{p}) = \sum_{i=1}^{n} p_i x_i^h(u,\mathbf{p}) = \sum_{i=1}^{n} p_i x_i^h(u,\mathbf{p}) = \sum_{i=1}^{n} p_i x_i(m,\mathbf{p})\left(\frac{u}{u(\mathbf{x}(m,\mathbf{p}))}\right)^{\frac{1}{\gamma}} = m\left(\frac{u}{u(\mathbf{x}(m,\mathbf{p}))}\right)^{\frac{1}{\gamma}}$$

Identity I.9 can be obtained by using I.1 and Shephard's lemma. First note that from I.1 $e(u,\mathbf{p}) = m\left[\frac{u}{v(m,\mathbf{p})}\right]^{\frac{1}{\gamma}}$. Then, taking derivative with respect to p_i we find

$$x_i^h(u,\mathbf{p}) = \frac{\partial e(u,\mathbf{p})}{\partial p_i} = -\frac{1}{\gamma} m u^{\frac{1}{\gamma}} v(m,\mathbf{p})^{-\frac{1+\gamma}{\gamma}}\frac{\partial v(m,\mathbf{p})}{\partial p_i}$$

which is the same as $x_i^h(u,\mathbf{p}) = -\frac{m u^{\frac{1}{\gamma}}}{\gamma}\frac{\frac{\partial v(m,\mathbf{p})}{\partial p_i}}{v(m,\mathbf{p})^{\frac{1+\gamma}{\gamma}}}$.

Finally, to get I.10, remember that $x_i(m,\mathbf{p}) = x_i^h(v(m,\mathbf{p}),\mathbf{p})$. Then, the identity follows by substituting I.1. ■

3.1 A Brief Discussion

The main findings of the last Section are summarized in the next table.

Summarized Matrix of Identities for Theorem 1[4]

Optimal Functions	$u(\mathbf{x})$	$e(u,\mathbf{p})$	$\mathbf{x}^h(u,\mathbf{p})$	$v(m,\mathbf{p})$	$\mathbf{x}(m,\mathbf{p})$
$u(\mathbf{x})$	X				
$e(u,\mathbf{p})$	P.1	X	ShL	I.1	I.5' or I.10 and ShL
$\mathbf{x}^h(u,\mathbf{p})$	P.1	Def	X	I.1'	I.10 or I.5
$v(m,\mathbf{p})$	P.1 or I.2	I.1	I.9	X	I.4
$\mathbf{x}(m,\mathbf{p})$	P.1	Sec. 3.1.1 and Def	Sec. 3.1.1	I.6' and Sec. 3.1.1	X

Table 1: Summary of identities.

The correct way to read this matrix is from row to column. For example, if we have $v(m,\mathbf{p})$ and we want to get the i-th Hicksian demand we should to use the identity number 9. *Def* means that the result follows from the definitions given in the section 2, *ShL* means Shephard's Lemma, *Sec. 3.1.1* means that we need to solve the system of equations shown in Sub-subsection 3.1.1, and *P.1* means Process 1 which will be developed next.

3.1.1 Recovering the Hicksian demands directly from Marshallian demands

From identity I.5 we have $x_i(1,\mathbf{p})\,e(\mathbf{p}) = x_i^h(1,\mathbf{p})$. But by definition we have $e(\mathbf{p}) = \mathbf{p}'\mathbf{x}^h(1,\mathbf{p})$. Therefore we get the system of equations $\mathbf{x}^h(1,\mathbf{p}) = \mathbf{x}(1,\mathbf{p})\,\mathbf{p}'\mathbf{x}^h(1,\mathbf{p})$ and the Hicksian demand satisfies the matrix identity

$$[I_n - \mathbf{x}(1,\mathbf{p})\,\mathbf{p}']\,\mathbf{x}^h(1,\mathbf{p}) = \mathbf{0}$$

Note that by Sylvester's determinant theorem we have

$$\det[I_n - \mathbf{x}(1,\mathbf{p})\,\mathbf{p}'] = \det[1 - \mathbf{p}'\mathbf{x}(1,\mathbf{p})] = 0$$

[4] For the identities I.7 and I.8 we assume that the functional form of the utility is known.

because $\mathbf{p}'\mathbf{x}(1,\mathbf{p}) = 1$ by Walras' law. Since we must have $\mathbf{x}^h(1,\mathbf{p}) \neq \mathbf{0}$, for homogeneous utility functions the Hicksian demand is one of the nonnegative eigenvectors of $\mathbf{x}(1,\mathbf{p})\,\mathbf{p}'$ associated with the eigenvalue of value one. The system of equations given by this eigenvector identity is of rank $n-1$ and we need another equation to pin down the Hicksian demand.

Because of the normalizations used, we can uniquely pick the eigenvalue that satisfies

$$\mathbf{p}'\mathbf{x}^h(1,\mathbf{p}) = 1$$

and this will give us the Hicksian demand solely from the Marshallian demand, without the need of any other function.

Thus, we can easily solve the integrability problem for continuous, monotone and quasiconcave homogeneous utility functions.

3.1.2 Recovering the Marshallian demands directly from Hicksian demands

From identity I.10 we have that $x_i(m,\mathbf{p}) = x_i^h\left(\left[\frac{m}{\mathbf{p}\cdot\mathbf{x}^h(1,\mathbf{p})}\right]^\gamma, \mathbf{p}\right)$. This allow us to recover the Marshallian demand directly from the vector of Hicksian demands without the use of a third function. Note that identity I.5 also gives us the Marshallian demands.

The usual way to recover Marshallian demands from Hicksian demands is to find the expenditure function using the definition $e(u,\mathbf{p}) = \mathbf{p}\cdot\mathbf{x}^h(u,\mathbf{p})$, then obtain an indirect utility function and use the identity $\mathbf{x}(m,\mathbf{p}) = \mathbf{x}^h(v(m,\mathbf{p}),\mathbf{p})$. Identities I.5 or I.10 summarize all this procedure in easy algebraic formulas.

Thus, we have purely algebraic ways to shift between the different kinds of demand. Note that both ways require the knowledge of the full vector of demands (i.e. $\mathbf{x}(m,\mathbf{p})$ or $\mathbf{x}^h(u,\mathbf{p})$).

3.1.3 Process 1: Integrability

The integrability process is simplified as follows:

1. Starting from the Marshallian demand function, recover the Hicksian demand function as shown above.

2. Given the Hicksian demand function, obtain the expenditure function using the identity $e(u,\mathbf{p}) = \mathbf{p}\cdot\mathbf{x}^h(u,\mathbf{p})$.

3. Given the expenditure function $e(u,\mathbf{p})$ obtain, the indirect utility function $v(m,\mathbf{p})$ using I.1 from Theorem 1.

4. Given the indirect utility function $v(m, \mathbf{p})$, recover the utility function $u(\mathbf{x})$ using Proposition 2. Alternatively, given the Marshallian demand we could find $\mathbf{p}(m, \mathbf{x})$ and then use the identity $u(\mathbf{x}) = v(m, \mathbf{p}(m, \mathbf{x}))$.

An important issue is the lack of uniqueness in the solution. If an specific $u(\mathbf{x})$ gives as solution of the utility maximization problem $\mathbf{x}(m, \mathbf{p})$, whatever monotonic and positive transformation to the utility function will give the same set of Marshallian demands $\mathbf{x}(m, \mathbf{p})$. However with the last process we will be able to pin down a specific utility function that is homogeneous of degree γ.

4 Homogeneity Properties of Representations of a Homogeneous Utility Function

Theorem 2 *If the utility function $u : \mathbb{R}_+^n \longrightarrow \mathbb{R}$ is a homogeneous function of degree $\gamma \in \mathbb{R}_+$, then:*

Homogeneity Properties for Homogeneous Utility Function

Optimal Functions		$\mathbf{p}$ *or* $\mathbf{x}$	m	u	$(m, \mathbf{p})$, $(u, \mathbf{p})$ *or* $(m, \mathbf{x})$
Indirect utility function	$v(m, \mathbf{p})$	$-\gamma$	γ		0
Marshallian demand	$x_i(m, \mathbf{p})$	-1	1		0
Expenditure function	$e(u, \mathbf{p})$	1		$\frac{1}{\gamma}$	$\frac{1+\gamma}{\gamma}$
Hicksian demand	$x_i^h(u, \mathbf{p})$	0		$\frac{1}{\gamma}$	$\frac{1}{\gamma}$
Inverse demand	$p_i(m, \mathbf{x})$	-1	1		0

Table 2: Characterization of homogeneity for the objects derived from household's optimization problems.

Proof. Note that if $f(\mathbf{x}, \mathbf{y}) : \mathbb{R}_+^{n+m} \longrightarrow \mathbb{R}_+$, $f(t\mathbf{x}, \mathbf{y}) = t^\alpha f(\mathbf{x}, \mathbf{y})$ and $f(\mathbf{x}, t\mathbf{y}) = t^\beta f(\mathbf{x}, \mathbf{y})$ then $f(t\mathbf{x}, t\mathbf{y}) = t^{\alpha+\beta} f(\mathbf{x}, \mathbf{y})$. For this reason, the last column of Table 2 is the sum of the three previous columns.

It is known that the indirect utility function is homogeneous of degree zero in the price vector $\mathbf{p}$ and wealth m. From Corollary 2 the indirect utility function is homogeneous of degree γ in the wealth m. Then, it is easily verified that the indirect utility function is homogeneous of degree $-\gamma$ in the price vector $\mathbf{p}$.

We now show that $x_i(m, \mathbf{p})$ is homogeneous of degree one in wealth. We know that $\mathbf{x} \in \mathbf{x}(m, \mathbf{p})$ maximizes utility subject to $\mathbf{p} \cdot \mathbf{x} \leq m$. Consider $\tilde{\mathbf{x}} = t\mathbf{x}$. We have $\mathbf{p} \cdot \tilde{\mathbf{x}} \leq tm$ and $u(\tilde{\mathbf{x}}) = t^\gamma u(\mathbf{x}) = t^\gamma v(m, \mathbf{p}) = v(tm, \mathbf{p})$ where we used the fact that the indirect utility function is homogeneous of degree γ in m. Then, $t\mathbf{x} \in \mathbf{x}(tm, \mathbf{p})$ and the Marshallian demand correspondence is homogeneous of degree one in

m. Since we know from microeconomic theory that the Marshallian demand is homogeneous of degree zero in $(m, \mathbf{p})$, then we must have that it is homogeneous of degree -1 in $\mathbf{p}$.

In Corollary 1 we showed that the expenditure function is homogeneous of degree $\frac{1}{\gamma}$ in u, and it is a well known fact that it is homogeneous of degree one in $\mathbf{p}$ (see Mas-Colell, Whinston and Green (1995), Proposition 3.E.2). It follows that this function is homogeneous of degree $\frac{1+\gamma}{\gamma}$ in $(u, \mathbf{p})$.

Since $e(u, \mathbf{p}) = \mathbf{p} \cdot \mathbf{x}^h(u, \mathbf{p})$ then for any $\mathbf{p}$ and $t > 0$ we have $\mathbf{p} \cdot t^\gamma \mathbf{x}^h(u, \mathbf{p}) = t^{\frac{1}{\gamma}} e(u, \mathbf{p}) = e(tu, \mathbf{p}) = \mathbf{p} \cdot \mathbf{x}^h(tu, \mathbf{p})$ and it follows that the Hicksian demand is homogeneous of degree $\frac{1}{\gamma}$ in u. From standard microeconomic theory we know that this demand is homogeneous of degree zero in $\mathbf{p}$. Then, it is homogeneous of degree $\frac{1}{\gamma}$ in $(u, \mathbf{p})$.

Finally, we know by 2 that $\overline{\mathbf{x}} \in \mathbf{x}(m, \mathbf{p}(m, \overline{\mathbf{x}}))$. Then, assuming that we are dealing with functions,

$$\mathbf{x}(m, \mathbf{p}(m, \overline{\mathbf{x}})) = \mathbf{x}(tm, \mathbf{p}(tm, \overline{\mathbf{x}})) = t\mathbf{x}(m, \mathbf{p}(tm, \overline{\mathbf{x}})) = \mathbf{x}\left(m, \frac{1}{t}\mathbf{p}(tm, \overline{\mathbf{x}})\right)$$

by the homogeneity properties of the Marshallian demand. Since this is true for any $\overline{\mathbf{x}}$, t and m we conclude $\mathbf{p}(tm, \overline{\mathbf{x}}) = t\mathbf{p}(m, \overline{\mathbf{x}})$. In a similar fashion we can prove that $\mathbf{p}(m, t\overline{\mathbf{x}}) = \frac{1}{t}\mathbf{p}(m, \overline{\mathbf{x}})$. Then, the inverse demand function is homogeneous of degree one in m and homogeneous of degree -1 in $\mathbf{x}$. ■

5 Example: The CES case

In order to show how the results of this paper work, we will find all relevant objects starting from the standard n-good CES utility function

$$u(\mathbf{x}) = \left(\sum_{i=1}^{n} \alpha_i x_i^\rho\right)^{\frac{\gamma}{\rho}}$$

where $\alpha_i > 0$, $\gamma > 0$, $\rho \neq 0$ and $\rho < 1$.

In order to use our identities we need first to find any optimal value function. Standard techniques of restricted optimization (see for example Simon and Blume (1994)) show that the indirect utility function is

$$v(m, \mathbf{p}) = m^\gamma \left(\sum_{i=1}^{n} \alpha_i \left(\frac{p_i}{\alpha_i}\right)^r\right)^{-\frac{\gamma}{r}}$$

where $r = \frac{\rho}{\rho - 1}$. Clearly,this function satisfies the conclusion of Corollary 1.

In order to see how Corollary 3 works, let's find $\frac{\partial v(m,\mathbf{p})}{\partial m} / \frac{v(m,\mathbf{p})}{m}$, which is

$$\gamma m^{\gamma}\left(\sum_{i=1}^{n}\alpha_i\left(\frac{p_i}{\alpha_i}\right)^r\right)^{-\frac{\gamma}{r}} / m^{\gamma}\left(\sum_{i=1}^{n}\alpha_i\left(\frac{p_i}{\alpha_i}\right)^r\right)^{-\frac{\gamma}{r}}$$

This clearly is the degree of homogeneity of the utility function.

It is also easily seen that the indirect utility function satisfies the properties shown in Table 2.

If we want to get expenditure function we can either use I.1 or I.9 and then use the expenditure function definition. To illustrate the power of our identities we will use I.1. This identity says that $v(m,\mathbf{p}) = u\left[\frac{m}{e(u,\mathbf{p})}\right]^{\gamma}$. Then, if we have $v(m,\mathbf{p})$ we can easily solve for $e(u,\mathbf{p})$, which is $\left[\frac{u}{v(m,\mathbf{p})}\right]^{\frac{1}{\gamma}} m$. Applying this identity we get

$$e(u,\mathbf{p}) = u^{\frac{1}{\gamma}}\left(\sum_{i=1}^{n}\alpha_i\left(\frac{p_i}{\alpha_i}\right)^r\right)^{\frac{1}{r}}$$

With this function in our hands it is easily verified that all conclusions from Corollary 1, Proposition 3 and the properties of Table 2 hold.

From $v(m,\mathbf{p})$ we can get the Marshallian demand using a simplified version of Roy's identity (I.4): it is a simplified version because it is not necessary to take derivative with respect to the wealth. Because this is a very traditional step, instead we get the Marshallian demand using the expenditure function and identity I.5'. This identity simply says $x_i(m,\mathbf{p}) = m\frac{\partial e(u,\mathbf{p})}{\partial p_i}/e(u,\mathbf{p})$.

Then, $\frac{mu^{\frac{1}{\gamma}}}{r}\left(\sum_{i=1}^{n}\alpha_i\left(\frac{p_i}{\alpha_i}\right)^r\right)^{\frac{1}{r}}\left(\frac{p_i}{\alpha_i}\right)^{r-1} / u^{\frac{1}{\gamma}}\left(\sum_{i=1}^{n}\alpha_i\left(\frac{p_i}{\alpha_i}\right)^r\right)^{\frac{1}{r}}$ and the Marshallian demand for the i-th good is:

$$m\left(\frac{p_i}{\alpha_i}\right)^{r-1} / \sum_{i=1}^{n}\alpha_i\left(\frac{p_i}{\alpha_i}\right)^r$$

The reader can easily verify that all stated properties of the Marshallian demand shown in Table 2 hold. A different way to verify that these are the Marshallian demands is to replace them into the utility function and obtain $v(m,\mathbf{p})$. Alternatively we can simply applying Roy's identity or the simplified version of Roy's identity.

Now, if we are interested in the Hicksian demand we can either use Shephard's Lemma or I.9. This identity says that the i-th Hicksian demand can be recovered as $-\left(mu^{\frac{1}{\gamma}}\right)\frac{\partial v(m,\mathbf{p})}{\partial p_i}/\gamma v(m,\mathbf{p})^{\frac{1+\gamma}{\gamma}}$. Substituting we get $-\left(mu^{\frac{1}{\gamma}}\right)m^{\gamma}\left(-\frac{\gamma}{r}\right)\left(\sum_{i=1}^{n}\alpha_i\left(\frac{p_i}{\alpha_i}\right)^r\right)^{-\frac{\gamma}{r}-1} r\left(\frac{p_i}{\alpha_i}\right)^{r-1} / \gamma m^{1+\gamma}\left(\sum_{i=1}^{n}\alpha_i\left(\frac{p_i}{\alpha_i}\right)^r\right)^{-\frac{1+\gamma}{r}}$ which simplifies to

$$x_i^h(u,\mathbf{p}) = u^{\frac{1}{\gamma}}\left(\frac{p_i}{\alpha_i}\right)^{r-1}\left(\sum_{i=1}^{n}\alpha_i\left(\frac{p_i}{\alpha_i}\right)^r\right)^{\frac{1-r}{r}}$$

The reader can verify the homogeneity properties of this function according Table 2.

Now we will find the inverse demands. We can do so using the identity proved in Proposition 4. From Process 1 we know that there is a duality relation between the direct and the indirect utility function, and we will exploit this relation. Let's start by differentiating $v(m, \mathbf{p})$ with respect to p_j and p_k.

$$p_j : \; m^{\gamma}\left(-\frac{\gamma}{r}\right)\left(\sum_{i=1}^{n}\alpha_i\left(\frac{p_i}{\alpha_i}\right)^{r}\right)^{-\frac{\gamma}{r}-1} r\alpha_j^{1-r}\left(p_j\right)^{r-1} - \lambda x_j = 0$$

$$p_k : \; m^{\gamma}\left(-\frac{\gamma}{r}\right)\left(\sum_{i=1}^{n}\alpha_i\left(\frac{p_i}{\alpha_i}\right)^{r}\right)^{-\frac{\gamma}{r}-1} r\alpha_k^{1-r}\left(p_k\right)^{r-1} - \lambda x_k = 0$$

Then, it follows that $\alpha_j^{1-r}\left(p_j\right)^{r-1}/\alpha_k^{1-r}\left(p_k\right)^{r-1} = \left(\frac{x_j}{x_k}\right)$ which is $p_j = \frac{p_k\alpha_j}{\alpha_k}\left(\frac{x_j}{x_k}\right)^{\frac{1}{r-1}}$. We know from Process 1 that we can replace this last equation in the budget restriction $\sum_{j=1}^{n} p_j x_j = m$. This results in $\sum_{j=1}^{n}\frac{p_k\alpha_j}{\alpha_k}\left(\frac{x_j}{x_k}\right)^{\frac{1}{r-1}} x_j = m$, which is $\frac{p_k}{\left(x_k\right)^{\frac{1}{r-1}}\alpha_k}\sum_{j=1}^{n}\alpha_j\left(x_j\right)^{\frac{r}{r-1}} = m$.

Then, the k-th inverse demand is

$$m\left(x_k\right)^{\frac{1}{r-1}}\alpha_k\left(\sum_{j=1}^{n}\alpha_j\left(x_j\right)^{\frac{r}{r-1}}\right)^{-1}$$

Note that if the utility function is known, then we can use Proposition 4 to find the inverse demand function.

If we use this inverse demand in $v(m, \mathbf{p})$ we obtain the utility function. Replacing the i-th inverse demand we get $m^{\gamma}\left(\sum_{i=1}^{n}\alpha_i^{1-r}p_i^{r}\right)^{-\frac{\gamma}{r}} = m^{\gamma}\left(\sum_{i=1}^{n}\alpha_i^{1-r}\left(m\left(x_i\right)^{\frac{1}{r-1}}\alpha_i\left(\sum_{i=1}^{n}\alpha_j\left(x_j\right)^{\frac{r}{r-1}}\right)^{-1}\right)^{r}\right)^{-\frac{\gamma}{r}}$, which can be easily shown to be $\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)^{\frac{\gamma}{\rho}}$.

Next we show how to get the Hicksian demands directly from the Marshallian demands without the need of any other function. For purposes of illustration we now assume $n = 2$. The Marshallian demand vector is

$$\mathbf{x}(1, \mathbf{p}) = \begin{bmatrix} \frac{\left(\frac{p_1}{\alpha_1}\right)^{r-1}}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^{r}+\alpha_2\left(\frac{p_2}{\alpha_2}\right)^{r}} \\ \frac{\left(\frac{p_2}{\alpha_2}\right)^{r-1}}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^{r}+\alpha_2\left(\frac{p_2}{\alpha_2}\right)^{r}} \end{bmatrix}$$

so we have

$$
\begin{aligned}
I_n - \mathbf{x}(1,\mathbf{p})\,\mathbf{p}' &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{\left(\frac{p_1}{\alpha_1}\right)^{r-1}}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} \\ \frac{\left(\frac{p_2}{\alpha_2}\right)^{r-1}}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} \end{bmatrix} \begin{bmatrix} p_1 & p_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 - \frac{\left(\frac{1}{\alpha_1}\right)^{r-1} p_1^r}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} & -\frac{\left(\frac{1}{\alpha_1}\right)^{r-1} p_1^{r-1} p_2}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} \\ -\frac{\left(\frac{1}{\alpha_2}\right)^{r-1} p_2^{r-1} p_1}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} & 1 - \frac{\left(\frac{1}{\alpha_2}\right)^{r-1} p_2^r}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} \end{bmatrix}
\end{aligned}
$$

Note that $\det\left[I_n - \mathbf{x}(1,\mathbf{p})\,\mathbf{p}'\right] = 0$. Now

$$
\begin{bmatrix} 1 - \frac{\left(\frac{1}{\alpha_1}\right)^{r-1} p_1^r}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} & -\frac{\left(\frac{1}{\alpha_1}\right)^{r-1} p_1^{r-1} p_2}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} \\ -\frac{\left(\frac{1}{\alpha_2}\right)^{r-1} p_2^{r-1} p_1}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} & 1 - \frac{\left(\frac{1}{\alpha_2}\right)^{r-1} p_2^r}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} \end{bmatrix} \begin{bmatrix} x_1^h \\ x_2^h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

and thus we have the equation

$$
\begin{aligned}
\frac{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r - \left(\frac{1}{\alpha_1}\right)^{r-1} p_1^r}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} x_1^h &= x_2^h \frac{\left(\frac{1}{\alpha_1}\right)^{r-1} p_1^{r-1} p_2}{\alpha_1\left(\frac{p_1}{\alpha_1}\right)^r + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^r} \\
x_1^h &= x_2^h \frac{\left(\frac{1}{\alpha_1}\right)^{r-1} p_1^{r-1}}{\left(\frac{1}{\alpha_2}\right)^{r-1} p_2^{r-1}}
\end{aligned}
$$

From $p_1 x_1^h + p_2 x_2^h = 1$ (due to normalization) we get

$$
\begin{aligned}
p_1 x_1^h &= x_2^h \frac{\left(\frac{1}{\alpha_1}\right)^{r-1} p_1^{r-1} p_1}{\left(\frac{1}{\alpha_2}\right)^{r-1} p_2^{r-1}} = 1 - p_2 x_2^h \\
1 &= x_2^h \frac{\left(\frac{1}{\alpha_1}\right)^{r-1} p_1^{r-1} p_1 + \left(\frac{1}{\alpha_2}\right)^{r-1} p_2^{r-1} p_2}{\left(\frac{1}{\alpha_2}\right)^{r-1} p_2^{r-1}} \\
x_2^h &= \frac{\left(\frac{1}{\alpha_2}\right)^{r-1} p_2^{r-1}}{\left(\frac{1}{\alpha_1}\right)^{r-1} p_1^r + \left(\frac{1}{\alpha_2}\right)^{r-1} p_2^r}
\end{aligned}
$$

which of course is the Hicksian demand.

Finally, we will show how to recover the i-th Marshallian demand from the vector of Hicksian demands.

For this purpose we will use identity I.10: $x_i(m,\mathbf{p}) = x_i^h\left(\left[\frac{m}{\mathbf{p}\cdot\mathbf{x}^h(1,\mathbf{p})}\right]^\gamma, \mathbf{p}\right)$. Substituting the CES Hicksian demand we get

$$\begin{aligned}
x_i(m,\mathbf{p}) &= \frac{\frac{m}{\mathbf{p}\cdot\mathbf{x}^h(1,\mathbf{p})}\left(\frac{1}{\alpha_i}\right)^{r-1} p_i^{r-1}}{\sum_{j=1}^n \left(\frac{1}{\alpha_j}\right)^{r-1} p_j^r} \\
&= \frac{m}{\sum_{k=1}^n \left\{\frac{\left(\frac{1}{\alpha_k}\right)^{r-1} p_k^r}{\sum_{j=1}^n \left(\frac{1}{\alpha_j}\right)^{r-1} p_j^r}\right\}} \frac{\left(\frac{1}{\alpha_i}\right)^{r-1} p_i^{r-1}}{\sum_{j=1}^n \left(\frac{1}{\alpha_j}\right)^{r-1} p_j^r} \\
&= \frac{m\left(\frac{1}{\alpha_i}\right)^{r-1} p_i^{r-1}}{\sum_{k=1}^n \left(\frac{1}{\alpha_k}\right)^{r-1} p_k^r}
\end{aligned}$$

which of course is the Marshallian demand.

6 Conclusions

In this paper, identities that allow to shift between six different ways of representing a homogeneous utility function were derived. The homogeneity properties of those representations have also been outlined.

These results, which have been summarized using tables, are useful to simplify computational procedures when different representations of a utility function are required. For example, we got a simple algebraic formula to shift from the indirect utility function to the expenditure function and vice versa. As far as we know this useful identity has been has been ignored in the literature.

Finally, we proved an explicit algebraic way to get Hicksian demands from Marshallian demands and vice versa, without the need of any other function. This allow us to avoid solving differential equations to find the expenditure function, thus simplifying the integrability problem. Note however that these algebraic identities require the knowledge of the full vector of demands (i.e. $\mathbf{x}(m,\mathbf{p})$ or $\mathbf{x}^h(u,\mathbf{p})$). It remains to be analyzed under what conditions can the i-th Marshallian demand be recovered using only the i-th Hicksian demand and vice versa.

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