Implied Volatilities of Caps: a Gaussian approach

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Abstract

Implied volatilities of interest rate derivatives present some distinctive features, like the inverse relation with the underlying rates and the humped or decreasing shape of their term structure. The objective of this paper is to analyze and explain such features in a Gaussian framework. We will use an approximate relation which separates in a simple and natural way the effects on the implied volatility of the level and of the uncertainty of the interest rates. This is a useful tool for understanding the features of different models and to interpret some characteristics of the market.

1   Introduction

It is market practice to quote interest rate derivatives traded "Over the Counter" in terms of their implied volatility. For this reason, the term structure of at the money cap volatilities as well as the volatility surface of at the money swaptions are directly observed. This paper analyzes the case of caps. In particular we observe that, in the time series we analyzed, there are two, rather evident, facts. The first one is that the level of the volatility is inversely related to the level of the interest rates. The second one is that volatilities are either a decreasing or a humped function of maturity. These facts have been observed also by other authors on different time series (see e.g. Rebonato (2003) and Brigo and Mercurio (2002)), so that they can be addressed as "stylized facts". Rebonato (2003) suggests that the structure of implied volatility is humped in periods of normal market conditions and decreasing when markets are "excited". In a recent paper, Ho and Goodman (2003), compared the implied volatility of 5 year Caps and the 5 year swap yields from the US and the Japanese market from 1996 to 2003, reporting the same inverse relation and proposing an empirical formula. Interpreting and explaining such phenomena is indeed an interesting and important issue.

For our analysis, we model the evolution of the underlying interest rate term structure according to the HJM paradigm. Any model used for pricing and hedging interest rate derivatives has to prove its efficiency on the capacity of reproducing the observed term structure of implied volatilities. Heath, Jarrow and Morton (1994) introduced a modeling approach that models directly the dynamics of the entire term structure of interest rates: a particular model within the HJM class can be obtained by simply specifying the diffusion coefficient driving the stochastic evolution of the term structure. Ho and Lee (1986) and Hull and White (1990) proposed some parsimonious models that belong to the HJM class. Mercurio and Moraleda (1996) and Ritchen and Chuang (1999) introduced a gaussian HJM model with the specific intent to reproduce a humped volatility term structure. These models were compared by Angelini and Herzel (2004a) in terms of their capability of fitting the shape of the term structure of volatilities. Brace et al. (1997) determined how to specify a HJM model to exactly reproduce the observed volatility term structure.

In a HJM model the price of any traded security can (in principle) be determined by the the current term structure of the interest rates and the diffusion coefficient that specifies the model. Therefore, in a HJM context, the stylized facts mentioned above can be explained by the combined action of two factors: the level and the volatility of the interest rate term structure.

In Angelini and Herzel (2004b) it is proven a simple relation, valid up to terms of degree greater than three, between the implied volatilities, the standard deviations and the levels of forward rates. Such a relation allows to disentangle the effect on the implied volatility due to the level of the forward rate and to its stochastic dynamics. In this paper the formula will be tested on the above mentioned HJM models calibrated to market data using a cross-sectional approach. We will also study the type of structures that these models are able to produce and to what extent, giving concrete examples. This will lead to a better understanding of the characteristics of the models and provide with a useful tool to decide which model to use according to the current features of the market.

We will discuss the interpretation given by Rebonato (2003) about the shape of the term structure, showing that, in some periods, a decreasing shape of the implied volatility can be explained by a particular structure of the interest rate and a "normal" (i.e. not excited) volatility structure of forward rates. By calibrating the Hull-White model to market data we will find a remarkable correspondence between excited periods and the rate of mean reversion of the model.

It is also possible to apply the relation with a time series approach, by estimating (historically) the standard deviation of the underlying rate without specifying the parametric form of the model. In this way one can recover a possible way to estimate the market premium for volatility risk.

The rest of the paper is structured as follows: in Section 2 we analyze the dynamics of the US and Euro fixed income markets from 7/9/99 to 31/3/03, identifying some aspects relevant to our analysis. In Section 3 we will apply the approximate relation between implied volatility, current term structure and diffusion coefficient on some particular HJM model calibrated to the data using a cross-sectional approach. In Section 4 we will adopt a time series approach to recover market cap volatilities from the level and the volatility of interest rates.

2 Gaussian framework

We will work in the HJM framework, making the assumption that the dynamics of the instantaneous forward rate $f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}$, where $P(t,T)$ is the price, at the current time t, of the zero coupon bond with maturity T, are

$$df(u,T) = (\cdot)du + \beta(u,T)dW_u, \qquad (2.1)$$

with $\beta(u,T)$ a deterministic function. In other words, we will work with Gaussian models. In this context, Angelini and Herzel (2004b) proved an approximation formula for caplet implied volatilities, which we will briefly recall, for convenience of the reader.

The simple forward rate with reset $T - \tau$ and maturity T is defined as

$$F(t, T - \tau, T) = \frac{P(t, T - \tau) - P(t, T)}{\tau P(t, T)}.$$

Let us consider an at-the-money forward caplet with reset time $T - \tau$ and expiration T. The caplet is at-the-money forward when the strike rate is equal to $F(t, T - \tau, T)$. It is market standard to use Black Formula to price caps and floors. The market implied volatility $\sigma(t, T)$ is defined as the annualized version of the volatility that put into the Black Formula equals the market price of the caplet. The model implied volatility $\sigma^M(t, T)$ is defined as the annualized version of the volatility that put into the Black formula equals the model price of the caplet. The following proposition holds:

Proposition 2.1 *Let the instantaneous forward rates evolve according to (2.1). Then*
$$\sigma_1(t, T) = H(t, T) Z(t, T),$$

where
$$H(t, T) = \frac{\tau F(t, T - \tau, T) + 1}{\tau F(t, T - \tau, T)}. \tag{2.2}$$

and
$$Z^2(t, T) = \frac{1}{T - \tau - t} \int_t^{T-\tau} \left(\int_{T-\tau}^T \beta(u, x) dx \right)^2 du. \tag{2.3}$$

is a first order approximation for the model implied volatility $\sigma_M(t, T)$ of the at-the-money forward caplet with maturity T. The error of the approximation is

$$
\begin{aligned}
\sigma_M(t, T) - \sigma_1(t, T) &= \frac{1}{24\sqrt{T - \tau - t}} H(t, T) \left(H(t, T)^2 - 1 \right) \Sigma(t, T)^3 \\
&+ o(\Sigma(t, T)^5), \quad \Sigma(t, T) \to 0,
\end{aligned}
$$

where $\Sigma(t, T) = Z(t, T)\sqrt{T - \tau - t}$.

The approximating formula for the implied volatility can be written as

$$\sigma_M(t, T) \approx H(t, T) Z(t, T). \tag{2.4}$$

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maturity	1Y	2Y	3Y	4Y	5Y	7Y	8Y	9Y	10Y
std US	1.13	1.22	1.22	1.16	1.09	1.09	1.09	1.07	1.06
std Eu	0.62	0.68	0.67	0.68	0.69	0.63	0.60	0.61	0.60
skew US	-0.32	-0.07	-0.46	-0.73	-0.25	-0.27	-0.26	-0.19	-0.23
skew Eu	0.04	-0.07	0.24	0.25	0.18	0.30	0.32	0.16	0.45
kurt US	9.60	9.57	8.63	9.06	5.48	6.10	6.56	7.78	7.92
kurt Eu	5.68	6.80	4.16	4.12	5.32	4.78	4.15	4.27	4.52

Table 1: Sample statistics for US and Euro interest rates from 07/09/1999 to 31/03/2003. Standard deviations are annualized and in percent.

It separates, in a simple and natural way, the effects of the level of the forward rates (the term H) and of the volatility of bond prices (the term Z) on the implied volatility of caplets. In Angelini and Herzel (2004b) the error is also bounded on limited intervals as a function of $\Sigma(t, T)$.

3   Data analysis

In this section we analyze at-the-money forward cap implied volatility of the US and Euro market from 07/09/1999 to 31/03/2003. The maturities of the caps are $1, 2, 3, 4, 5, 7, 10$ years. From the bid and ask quotation we compute the mid volatility. For the same period, we consider spot interest rates of the US and Euro market for nine maturities (from 1 to 5 and from 7 to 10 years). All data were provided by Bloomberg.

We begin with a description of the underlying interest rates. We consider daily absolute differences and compute their sample statistics, standard deviation, skewness and kurtosis. The results are reported in Table 1.

The standard deviations are quite low, higher for the US market. In both cases, the structure is decreasing for long maturities, a typical fact usually explained by the mean reversion of interest rates and a little higher for middle maturities. The negative skewness in the US market is due to the fact that the period is mainly characterized by cuts of the official discount rate, passing from 6.5 points at the beginning of 2001 to 1.5 points at the end of the sample. In the Euro market the cuts began few months later and the official rate went from 3.75 to 2 points. The high kurtosis, especially for the US market, indicates the presence of extreme events, mostly due to the general economic situation during the period of observation and, more particularly, to the intense monetary policy of the Federal Reserve and the less pronounced one of the BCE.

To get a first insight on the relation between the level of interest rates and the cap implied volatility we report in Figure 1 the time series of the 1-year cap implied volatility and of the 1-year spot rate for the US and the Euro markets. We focus on the US market (being the Euro market quite analogous). We identify a first period, from September 1999 until about the end of 2000, where the spot rate is high (between 6 and 8 percentage points) and the cap volatility stays below 15 %. Afterwards, during 2001, the Federal Reserve cuts the official rate, about once a month, bringing it from 6.5 to 1.75 percentage points. Consequently the 1-year spot rate presents a very decreasing trend along the whole year. The cap volatility steadily increases reaching a level of around 20 %, before September 11, 2001. After such date, the cap volatility jumps up to 35 % and the spot rate collapses of about 100 basis points in a few days. The Federal Reserve cuts the official rate four times until the end of the year. Most of 2002 is a period of very high cap volatility. After such period the Federal Reserve decides to cut again the official rate (November 6, 2002), and the spot rate goes, at least for a certain time, below 2%. Around the same time the cap volatility descended under 50%. Summarizing, we can distinguish roughly four time periods: from the beginning of the sample (September 1999) to the end of 2000, from the beginning of 2001 to September 10, 2001, from September 11, 2001 until half of 2002 and then until the end of the sample (March 2003).

Notice, in Figure 1, the relation of inverse proportion between the cap volatility and the level of the interest rate (respectively top and middle). This fact is reported by several authors, like Rebonato (2003) for the case of the implied volatilities of swaptions. The cap implied volatility is also rather obviously related to the uncertainty of the market. We show this fact by estimating the daily instantaneous volatility of the spot rate with maturity 1 year using the standard deviation computed on a moving window of 250 observations (Figure 1, bottom). Qualitatively, one sees that its trend is quite similar to that of the implied volatility, namely that there is a relation of direct proportion among the two. Therefore, it is rather evident that the implied volatility of caps is determined by at least two factors: namely the level of the interest rates and their volatility. Next sections will try to make this statement more clear from a quantitative point of view.

Now we move to analyze the term structure of cap implied volatilities. As often observed, also on other data sample, we found basically three types of shapes: humped at medium-long maturities, humped at maturity 2 years and decreasing. The humped shape is usually considered an indication of "normal" behaviour of the market. The decreasing shape could be an indi-
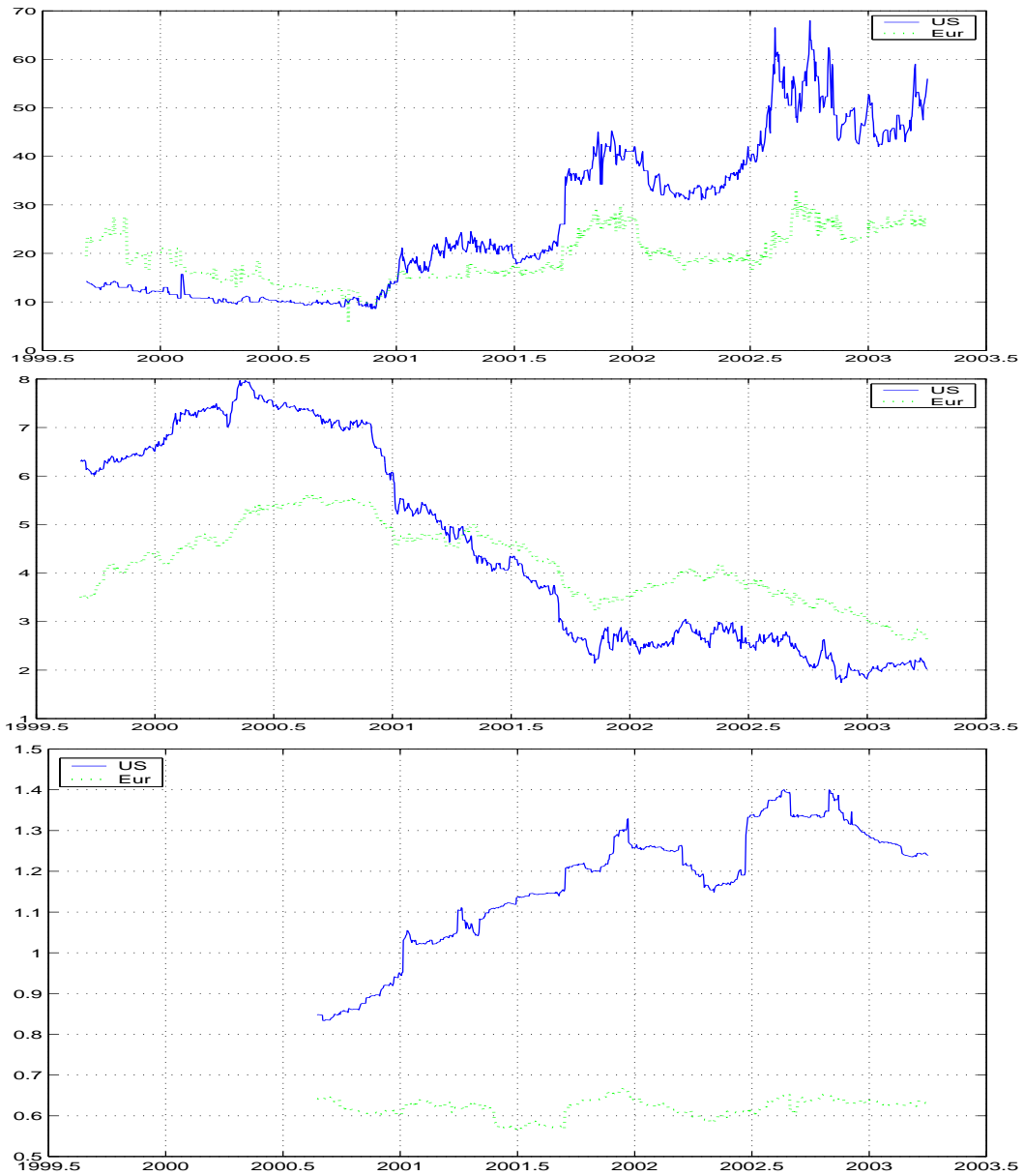
Figure 1 *1-year cap volatility (top), 1-year spot rate (middle) and daily estimates of annualized standard deviation of the spot rate with maturity 1 year on a moving window of 250 observations (bottom) of the US and Euro market from 07/09/1999 to 31/3/2003.*

cation that the market is uncertain about near future short rates and it is therefore in an "excited" status (see Rebonato (1999) §11.2). According to this interpretation, the first half of 2001 and the period from September 11 towards the end of 2003 may be considered "excited" periods (see Figure 2 and the following discussion).

A qualitative analysis of the daily shapes of the volatility term structures shows that, for the US market, there are 327 decreasing cases and 582 humped cases. The remaining 21 cases present essentially either a decreasing or humped shape, but they have some irregular behaviour, possibly due to some spurious data (for instance a high bid-ask spread), often on the cap with maturity two years. We found also an increasing structure, which however falls in the middle of two days with humped shape and it may be again due to spurious data. For the Euro market we have 296 decreasing cases, 554 humped cases. The remaining 80 cases present some irregularity, but they are mainly decreasing or humped.
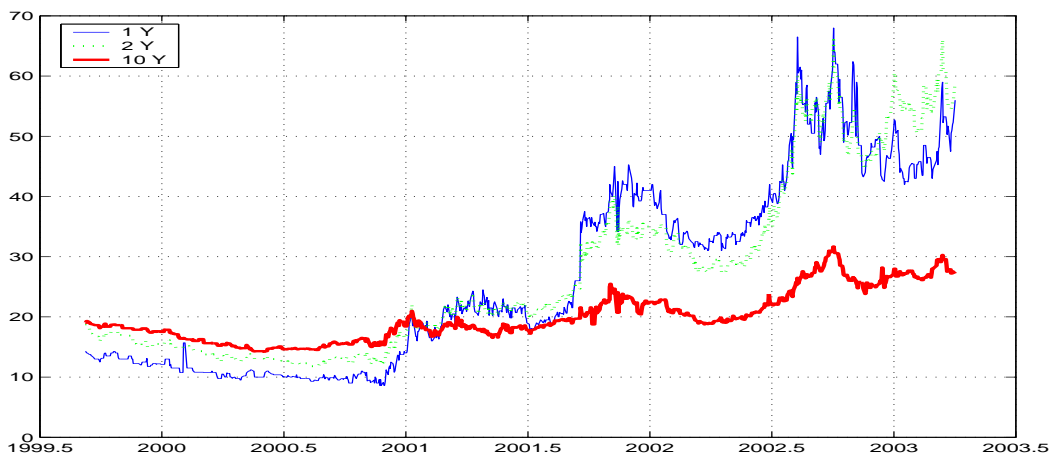

Figure 2 *Time series of the cap volatilities with maturity 1, 2 and 10 years from 07/09/1999 to 31/3/2003.*

Let us now concentrate the analysis on the US market. Figure 2 shows the time series of the cap volatility with maturities 1,2 and 10 years. The four periods previously identified by just looking at the 1-year volatility and spot rate, are also detectable from the analysis of the evolution of the term structure. In the first period until 23/02/2001, of 384 days, the structure is humped with maximum at a long maturity, namely four, five or seven years (apart from some irregularity). Here we can notice that the spread

between short term and long term interest rates is very small. Spot interest rate term structures will not be shown in the paper, but one can look at Figure 5 for a representation of it. Then there is period of about seventy days where a decreasing structure alternates with a hump at maturity two years, followed by other seventy days of hump at maturity two years; this lasts until 10/09/2001. The third period, until 26/08/2002, of two hundred fifty days, presents an essentially decreasing structure. Last, after about seventy days of alternating structures, decreasing and humped at maturity two years, the last eighty days there is a humped structure with maximum at maturity two years. The average of the term structures over each of these four periods (7/9/1999-23/2/2001, 26/2/2001-10/9/2001, 11/9/2001-26/8/2002, 27/8/2002-31/3/2003) is shown in Figure 3.

Approximation Formula (2.4) deals with caplet implied volatilities, which are not directly observed. Therefore a standard bootstrap procedure is applied to obtain them from caps. We denote by $\bar{\sigma}(t, T)$ the caplet bootstrapped volatilities at time t, with $T = t + 2\tau, t + 3\tau, \ldots, t + 20\tau$ and $\tau = 0.5$. We show the result of such a procedure for the decreasing and humped cases in Figure 4, where a day for each of the four period is shown. The type of structure of caplet volatilities is similar to that of caps, but it exhibits a more irregular behaviour. The cap volatility is essentially an average of the volatilities of the caplet involved in the cap. Since we are interested in understanding the differences, in the observed cap structures, between a hump at short maturity (2 years) and a decreasing behaviour (i.e. 1-year volatility higher than the 2-year one), we compared the initial part of the two structures. It turns out that the relation between one and two-year cap volatilities is preserved for caplets. In other words, the result of the bootstrap procedure, at least for short maturities, is always of the kind represented in Figure 4.

For every day in the sample, we look at the structure of the function $H(t, T)$, as defined in (2.2). To do so we computed, linearly interpolating the interest rates in the data set (adding the 6-month rate), the discount factors $P(t, T)$, for every t in the sample and for maturities $T = t + \tau, t + 2\tau, \ldots, t + 20\tau$ years and $\tau = 0.5$. We then have a sample of 930 curves of simple forward rates with reset $T - \tau$ and maturity T and of functions $H(t, T)$. According to the four periods separation, we computed the average curve over each period and the corresponding 95% confidence band. The results are shown in Figure 5. In the first period $H(t, T)$ is basically constant in T, while in the other three periods it is decreasing, but with a slope which is increasing over the time t. Comparing the behavior of the function $H(t, T)$ with the term structure of implied volatility we note a strong relation. In
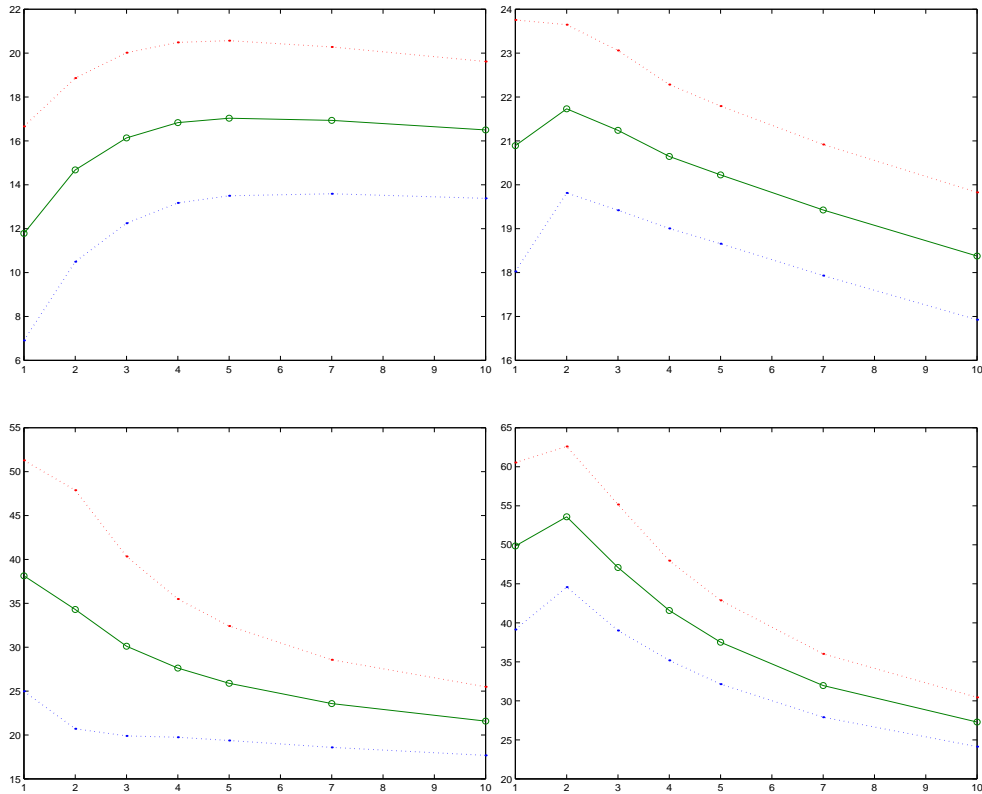
Figure 3 *Average of the US market cap implied volatility term structures with 95% confidence band. The sample is divided into four periods with similar characteristics: 7/9/1999-23/2/2001, 24/2/2001-10/9/2001, 11/9/2001-26/8/2002, 27/8/2002-31/3/2003.*

the first period, where $H(t,T)$ is constant, the implied volatility is humped at longer maturities, but quite flat. In the other three periods the shapes of the two are quite similar. Notice that in period 2 and 4 the average cap volatility presents a hump at maturity two years, smaller in period 2 and more pronounced in period 4. In those periods and especially in period 4, there are many days in which $H(t,T)$ has a hump at short maturities (namely two years). One can see this in the average of the function $H(t,T)$ by looking at the confidence bands. Some of these days were shown in the previous section.
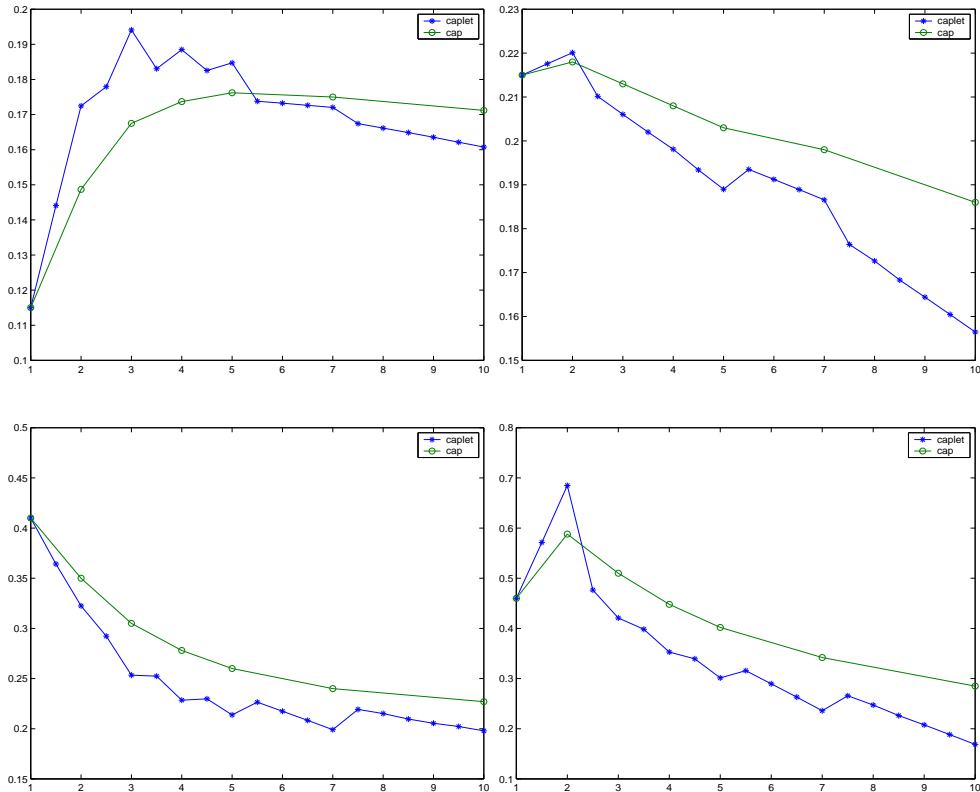
Figure 4 *Results of the bootstrap procedure from cap to caplet volatilities: the cap structure is humped at medium-long maturity (top left), humped (not pronounced) at short maturity (top right), decreasing (bottom left) and again humped (pronounced) at short maturity (bottom right).*

4   Cross-sectional approach

We consider three one-factor models: the continuous time version of the Ho-Lee model (Ho-Lee (1986)), denoted by HL, the extension of the Vasicek model (Vasicek (1977)) introduced by Hull and White (1990), denoted by HW, and the model proposed by Mercurio and Moraleda (1996) and Ritchen and Chuang (1999), henceforth MM. All of them belong to the class of Gaussian models proposed by Heath, Jarrow and Morton (1992). We chose these models because they are quite popular, analytically treatable and they have volatility term structures of the instantaneous forward rates with different characteristics. In particular they provide with an an-
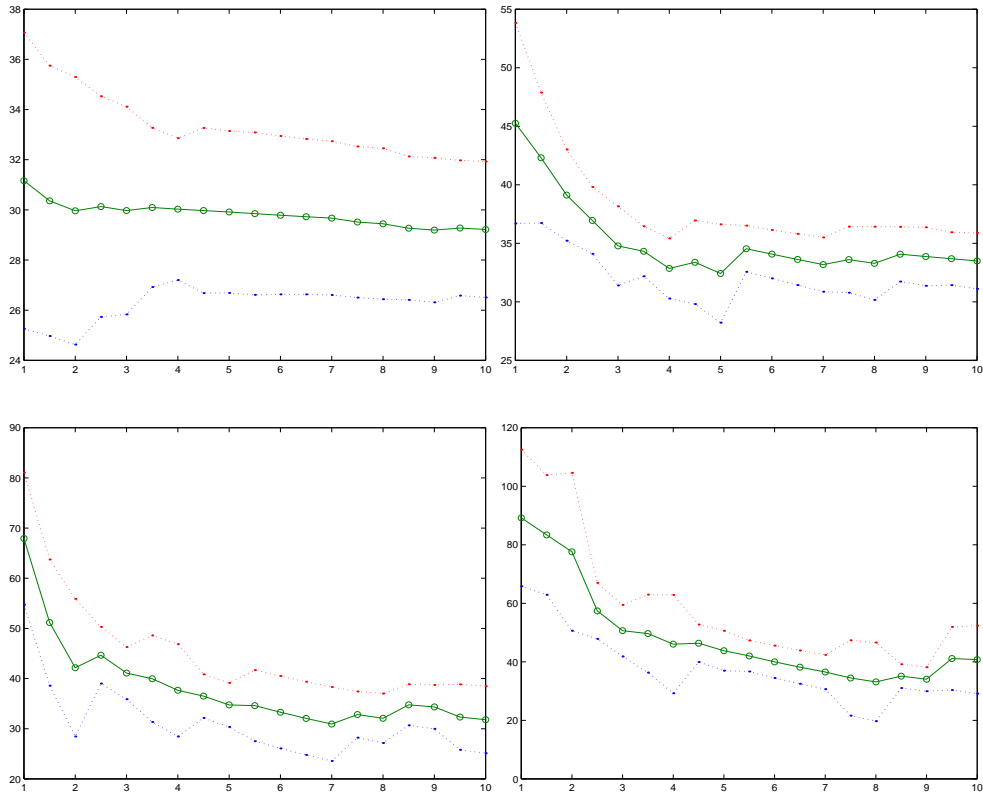
Figure 5 *Average of the function $H(t,T)$ of the US market with 95% confidence band. The sample is divided into four periods.*

alytical formula for caps. We will test the approximation Formula (2.4) on these models showing that it provides a good approximation of the model implied volatilities. Then we will use it to understand what kind of structures the models are able to produce and to what extent. Angelini and Herzel (2004a) analyzed this type of problem in calibrating different one and two factor gaussian models on the time series of discount factors and of implied volatilities of at-the-money interest rate caps of the euro market from 15/2/2001 to 4/7/2001. On this data set, they realized that the HL and HW model are not able to produce humped volatility structures, while the MM model is. In this section this issues is studied in a deeper way.

The three models are specified by the volatilities of the instantaneous forward rates which can be nested as

$$\beta(u,T) = (\sigma + \gamma(T-u))\exp(-a(T-u)).$$

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	1Y	2Y	5Y	7Y	10Y
HL	0.0025	0.0049	0.0086	0.0092	0.0147
HW	0.0024	0.0047	0.0098	0.0111	0.0199
MM	0.0021	0.0054	0.0093	0.0093	0.0120

Table 2: Mean of the relative errors of approximation Formula (2.4).

The HL model is obtained by setting $\gamma = a = 0$, the HW model by $\gamma = 0$. Therefore the HL model has constant volatility, the HW model a monotone volatility function, increasing (decreasing) when a is negative (positive), the MM model a volatility function which can be (when $T > u$) humped, decreasing or increasing.

For each of the models it is straightforward to compute the function $Z(t, T)$ and its shapes turn out to be similar to those of the volatility function $\beta(t, T)$ (see Angelini and Herzel (2004b) for details). Some of the possible shapes, determined by calibration to market data, will be shown below. The three Gaussian models were calibrated to the data set relative to the US market presented in Section 3 using a cross-sectional approach, namely by minimizing evry day the relative errors between model and market prices of caps. This turns out to be similar to minimizing the relative errors in cap volatilities (see Angelini and Herzel (2004a) for details). In this way we have 930 determinations of the three models selected.

For each day and for each model we can compute:

1. cap and caplet model implied volatilities (7 caps and 19 caplets);

2. the standard deviation $Z(t, T)$ as defined by (2.3);

Using Formula (2.4), we analyze the caplet volatilities produced by the different models and then look at the corresponding cap volatilities. Most of the times the two term structures are quite similar to each other, however there are some instances where a small hump for short maturities in the caplet volatilities may not result in a similar hump in the cap volatilities.

First of all we calculate the relative errors of the approximation Formula (2.4) each day of the sample and then their mean values. Results for each model are reported in Table 2.

Next we look at the capability of the models to reproduce the market implied volatility structure, when the latter is decreasing or humped. As a first description, we count the number of times when the model reproduces

the exact type of structure and then divide it by the total number of occurrences. The HL model reproduces the decreasing shape with a frequence of 0.8165 and of 0.2131 for the humped shape. In fact the HL model has a decreasing implied volatility function more than half of the times. The HW model has 0.6667 of probability of reproducing a decreasing structure and 0.0481 the humped structure. The MM model has 0.6789 in the first case and 0.6271 in the second case. The MM model is then the model that seems more capable of reproducing the hump and also the most flexible. Moreover, the cases where MM does not reproduce the hump are most often structures that look very close to be increasing, with a change only at the last maturity. In this cases MM would give an increasing structure very close to the market one.

Now we turn to an analysis of the structures using the approximation Formula (2.4). The HL case is particularly simple because the model function $Z(t,T)$ (2.3) is constant and therefore the shape of the caplet implied volatility is the same as the shape of $H(t,T)$. When this is humped, as sometimes observed, the HL model gives a humped volatility structure.

The HW model requires a deeper analysis: let us focus on the estimates of the parameter a. Only in 162 cases we find a positive number, while this is the condition required for this model to be mean-reverting. The mean value of the positive estimates is 0.0548, the maximum value is 0.2296 (day 763) and the minimum value is $4.31 \cdot 10^{-4}$ (day 743). In these cases the model function $Z(t,T)$ is decreasing and indeed, the cap volatility structure is always decreasing with only one single case of humped structure, with a small hump at maturity two years. In this last case, corresponding to the 10/10/2002, the value of the parameter a is 0.0095, quite small. We compare the average of the function $Z(t,T)$ over the days where the estimate for a is positive and compare it with that at the 10/10/2002. The result is reported in Figure 6 where one sees that the function relative to day 10/10/2002 decreases much slower than the average. Moreover, and most importantly, the function $H(t,T)$ is particularly humped in that day with respect to its average over those days where the parameter a is positive. Small and positive estimates of parameter a lead to slowly decreasing function $Z(t,T)$. In such cases a pronounced hump in the function $H(t,T)$ may produce a hump in the caplet volatility structure, as in the case of the 10/10/2002 (but also of other days). If the hump is sufficiently high and lasts for a couple of maturities, this may lead to a hump in the cap volatility structure. Higher values of the parameter a give more decreasing function $Z(t,T)$. In these cases it is quite hard for the caplet implied volatilities to present a hump at short maturities. In conclusion we can say that the Hull-White model,

imposing a positive a, as requested to have an economic interpretation as the velocity of mean reversion, is insufficient to fit humped structures of volatility. Moreover, if a is very small and positive, the model function $Z(t, T)$ (or the instantaneous volatility of instantaneous forward rates) is very close to be constant. Therefore a more parsimonious model like the Ho-Lee model would have a similar behaviour in terms of fitting.

The remaining estimates are negative, with a mean value of -0.0716, a maximum of $-3.05 \cdot 10^{-4}$ attained the 2/7/2002 and a minimum of -0.1484 attained the 29/11/2000. For instance, in the case of the minimum, there is a hump at short maturities in cap and caplet market volatilities (Figure 7 (top)). The HW model tries to reproduce this hump with an as increasing as possible function $Z(t, T)$, due also to the fact that the function $H(t, T)$ is slowly decreasing in T. The result is that, for long maturities, the model caplet volatilities is far from the market one. Therefore, forcing the HW model to fit an initially increasing structure, may lead to serious mispricing for long maturities derivatives. Instead the MM model, which is able to produce a humped $Z(t, T)$, follows more closely the market shape. A negative a could therefore be a sign of a humped market structure. In our sample, this happens 538 times. This is however not always true, especially if a is not very negative, and this is the case of 212 observations, occurring in days with an initially fast decreasing functions $H(t, T)$.

As a last application of the formula for the HW model we study two days, namely the 6/1/2003 and the 11/3/2003. This analysis is interesting because in these two days the estimates of the parameters of the HW model are very close, namely $\sigma = 0.012$ and 0.0119 and $a = 0.0523$ and 0.0521 respectively (the values of a are not far from the mean computed above), therefore the model function $Z(t, T)$ of the two days are very similar (Figure 8, top). However, the shapes of the function $H(t, T)$ are quite different: in the second day it has a hump at short maturities (Figure 8, middle). This corresponds to an analogous hump in the model caplet implied volatilities, which is not present the first day (Figure 8, (bottom)). The market caplet volatilities have similar structures, which the model is able to reproduce only in one of the two cases.

Last we look at the MM model. To confirm how the model is flexible and satisfactory at reproducing market volatility structures, we show the average of market and model cap volatilities (Figure 9, left) and caplet volatilities (Figure 9, right), first over the days when the market structure is decreasing (top) and then when is humped (bottom). The main difference from the other two models is relative to days of humped structure, as it was already observed: apart from some particular cases, like those analyzed above, on

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average the Ho-Lee and Hull-White models are quite far from reproducing a hump, basically due to their structures of volatility of the underlying interest rates.

Let us consider the approximation of the annualized standard deviation of the logarithm of model bond prices implied by the market volatility $\bar{\sigma}(t,T)$ given by

$$\bar{Z}(t,T) = \frac{\bar{\sigma}(t,T)}{H(t,T)}.$$

Looking at $\bar{Z}(t,T)$ may be seen as a short cut of the method proposed by Brace and Musiela (1994). Figure 13 represents the time series of $\bar{Z}(t,T)$, for maturities 1,2,10 years. It gives a different identification of periods of "normal" and "excited" status of the market that can be compared to that given in Section 3. The first period of uncertainty is the beginning of 2001, where the first cuts to the discount rate by the Federal Reserve were made. The analysis of $\bar{Z}(t,T)$ confirms the existence of such period, although it ends slightly earlier. It also appears that the long period after September 11, 2001 is not really characterized by "excitement". The function $\bar{Z}(t,T)$ is indeed quite humped in that period, becoming less humped and sometimes decreasing only in the second half of 2002. An interpretation of this is that, after September 11, the market was expecting a lowering of the interest rate level, as it was indeed the case. Then, especially in October and November 2002, the US economy was going through a time of difficulty and the market was uncertain. Only after the cut of November 6, 2002, the structure got back to a "normal" status.

Looking at Figure 13 we can guess that, at the beginning of 2001, being $\bar{Z}(t,T)$ approximately constant in time to maturity, the HL model would be able to fit sufficiently well the market implied volatility. In the period in the second half of 2002, characterized by a decreasing $\bar{Z}(t,T)$, one would expect the Hull-White model to give a satisfactory performance, and with a positive value of parameter a. This is indeed the case: in the first period under exam, the three models give similar fittings of the market volatility term structures, while in the second only the HW and MM models do. This fact may be read from Figure 10, where the estimates over the sample of the parameters of the three models are shown. First we notice the instability of the estimates, especially those of the MM model. During the first period the estimates of γ are close to zero and those of a of the HW and the MM model are close to each other and positive. The estimates of σ for the three models are basically the same (a similar situation can be spot around the month of May 2002). During the second period γ is again close to zero, while the

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parameter a of the HW model is positive and the σ of the HW and the MM model are very close to each other. Notice the strict correspondence between values of parameter a of the HW model with the period of "excitement" of the market. Decreasing functions $\bar{Z}(t,T)$ correspond to positive values and the higher the steep the greater the value assumed by the function. In conclusion, if one wants a model as parsimonious as possible to explain the volatility term structure, one should choose the HL model in the first period and the HW model in the second. The HW model is particularly fit during very "excited" periods.

Apart from these "excited" periods, the MM model outperforms the other two in terms of fitting of $\bar{Z}(t,T)$. Here we show this by comparing $\bar{Z}(t,T)$ with the corresponding functions of the models HL, HW and MM, calibrated on market data. All the functions are averaged over the four periods and the results are reported in Figure 11, clearly showing that the MM model has better fitting capabilities of market data.

5   Time series approach

The aim of this section is to analyze market implied volatilities of caps and caplets in terms of Formula (2.4), which gives an approximation of implied volatility of the cap(let) by means of the functions $H(t,T)$ and $Z(t,T)$. The first function can be readily extracted from the current term structure of interest rate, the second one, related to the standard deviation of the interest rates, has to be estimated. In the previous section we determined $Z(t,T)$ cross-sectionally, to fit the market volatility. Here it will be estimated on market data with a time series approach, using part of the information available to market participants, like past movements of interest rates.

To estimate $Z(t,T)$ we proceed as follows: let us observe that

$$
\begin{aligned}
Z(t,T)^2(T-\tau-t) &:= \int_t^{T-\tau} s(u, T-\tau, T)^2 du \\
&= E_t\left[\log^2(P(T-\tau, T))\right]
\end{aligned}
$$

Hence $Z(t,T)$ can be computed from the variance of the spot rate with time to maturity τ at time $T-\tau$. A standard way to estimate this is to calculate the daily standard deviation $v(t)$ of the spot rate with time to maturity τ and assume that it remains constant from t to $T-\tau$. So that the estimate for $Z(t,T)$ is $\hat{Z}(t,T) = v(t)$. Now we can compute $\hat{\sigma}(t,T) = H(t,T)\hat{Z}(t,T)$.

We consider the time series of US 1-year cap volatility, making the simplifying assumption of it being composed by a single, six month to one year,

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caplet. For each day t in the sample, we compute $\hat{Z}(t, T)$ by estimating $v(t)$ with the sample standard deviation on a moving window of length 250. Eight outliers were removed from the sample before proceeding with the estimate. The time series of $\bar{\sigma}(t, t+1Y)$ and $\hat{\sigma}(t, t+1Y)$ are represented in Figure 12. This seems to confirm once again that market implied volatility could be explained, at least as a first approximation, by the variability and the level of interest rates. This may justify the use of interest rate models where the implied volatility is determined by the process of the underlying, and not by an independent process as it is the case for stochastic volatility models. At a closer look to the figure, however, one can spot periods where the two series depart from each other: this may be interpreted either as a premium (positive or negative) for volatility risk or a gap between historical estimates and current market views.

References

[1] F. Angelini, S. Herzel, Consistent Calibration of HJM Models to Cap Implied Volatilities, forthcoming (Journal of Futures Markets), (2004a).

[2] F. Angelini, S. Herzel, An approximation of caplet implied volatilities in Gaussian models, forthcoming (Decisions in Economics and Finance), (2004b).

[3] T. Björk, *Arbitrage Theory in Continuous Time*, Oxford University Press, 1998.

[4] A. Brace, M. Musiela, A Multifactor Gauss Markov Implementation of Heath, Jarrow, and Morton, *Mathematical Finance*, 4, No. 1 (1994), 259-281.

[5] A. Brace, D. Gatarek, M. Musiela, The market model of interest rate dynamics *Mathematical Finance*, 7, No. 7 (1997), 127-147.

[6] D. Brigo, F. Mercurio, *Interest Rate Models: Theory and Practice*, Springer-Verlag, 2001.

[7] D. Heath, R. Jarrow, A. Morton, Bond Pricing and the Term Structure of Interest Rates *Econometrica*, 60, No. 1 (1992), 77-106.

[8] J. Ho and L. Goodman, Interest Rates - Normal or Lognormal?, *The Journal of Fixed Income*, 13, No. 2 (2003), 33-45.

[9] T. Ho, S. Lee, Term Structure Movements and Pricing Interest Rate Contingent Claims. *The Journal of Finance*, 41 (1986), 1011-1029.

[10] J. Hull, A. White, Pricing Interest Rate Derivative Securities. *The Review of Financial Studies*, 3 (1990), 573-592.

[11] Mercurio F. and J.M. Moraleda, An Analytically Tractable Interest Rate Model with Humped Volatility, discussion paper TI 96-116-2, (1996), Tinbergen Institute, Erasmus University Rotterdam, The Netherlands.

[12] R. Rebonato, *Interest Rate Option Models*, 2nd edition, Wiley, Chicherster, 1998.

[13] R. Rebonato, *Volatility and Correlation*, Wiley, Chicherster, 1999.

[14] R. Rebonato, *Term structure models: a review*, preprint (2003)

[15] P. Ritchken, I. Chuang, Interest Rate Option Pricing with Volatility Humps, *Review of Derivatives Research*, 3, (1999), 237-262.

[16] O. Vasicek, An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics*, 5 (1977), 177-188.
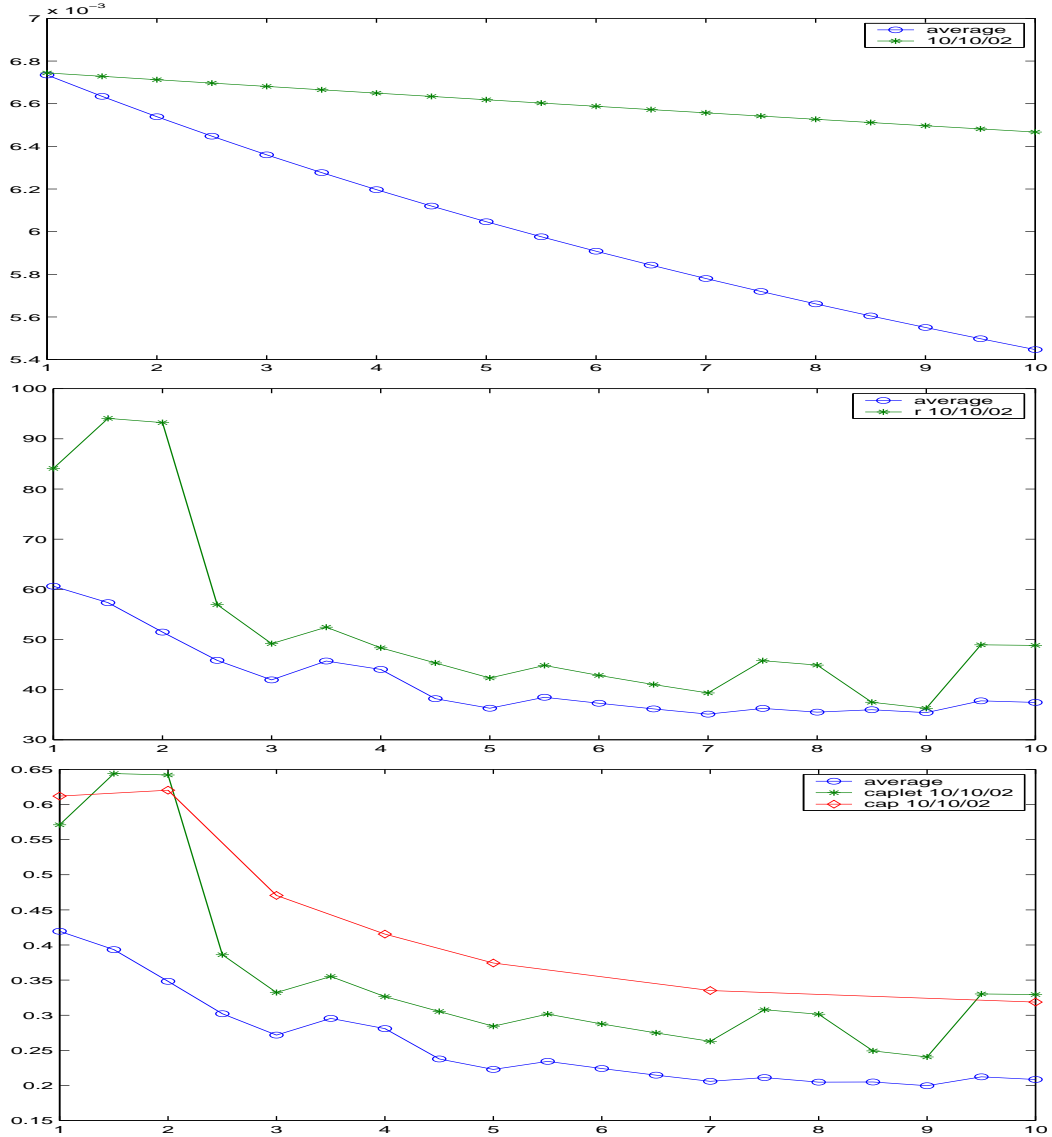
Figure 6 *Comparison of the 10/10/2002 with the other days where parameter a is positive: the function $Z(t,T)$ against its average (top), the function $H(t,T)$ against its average (middle), caplet model implied volatility against its average together with the corresponding cap volatility of the day (bottom).*
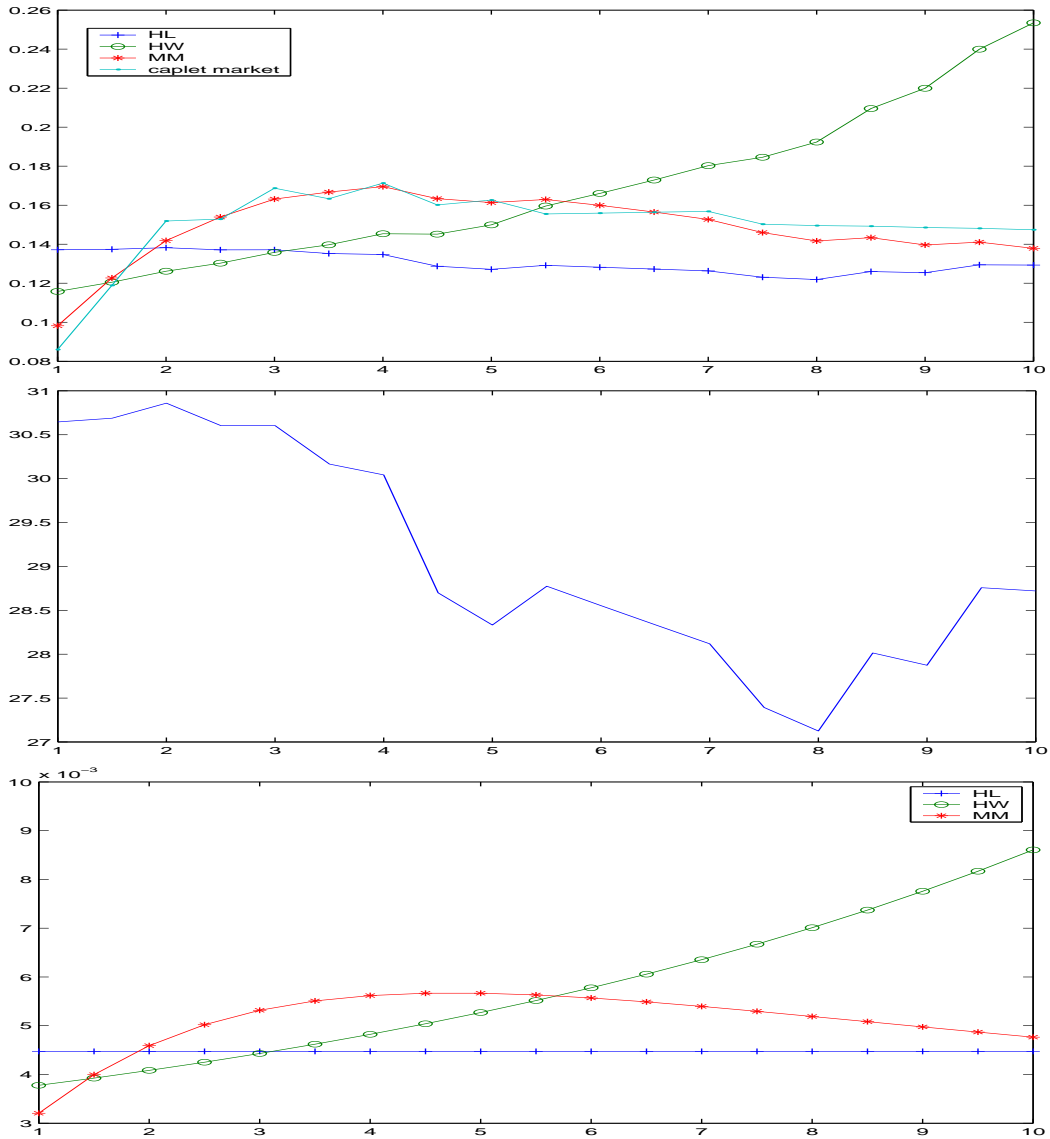
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Figure 7 *The 29/11/2000: market and model implied volatilities (top), the function $H(t,T)$ (middle), the function $Z(t,T)$ of the three models (bottom).*
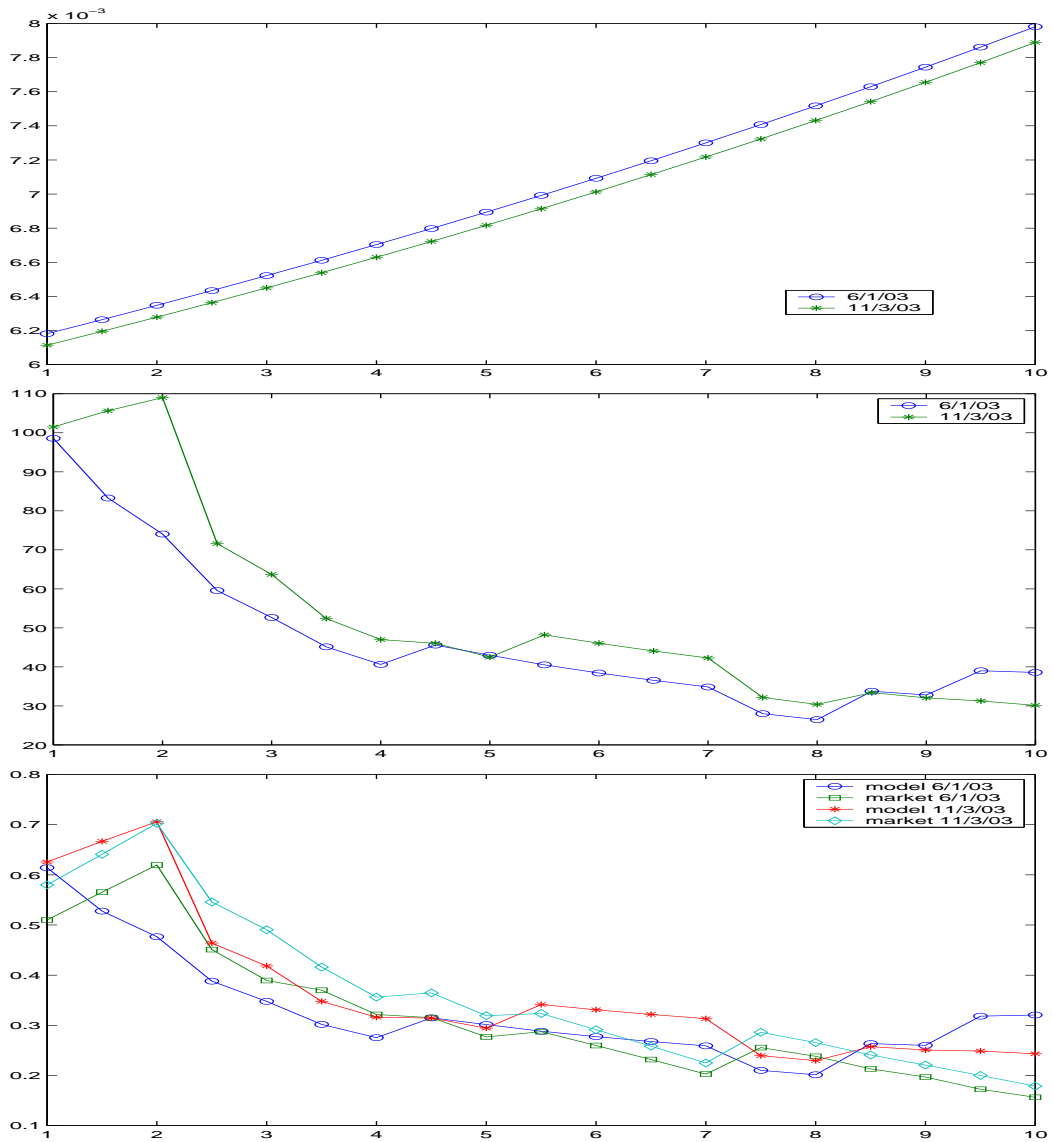
Figure 8 *The 6/1/2003 and 11/3/2003: the function $Z(t,T)$ of the HW model (top), the function $H(t,T)$ (middle), market and model caplet implied volatilities (bottom).*
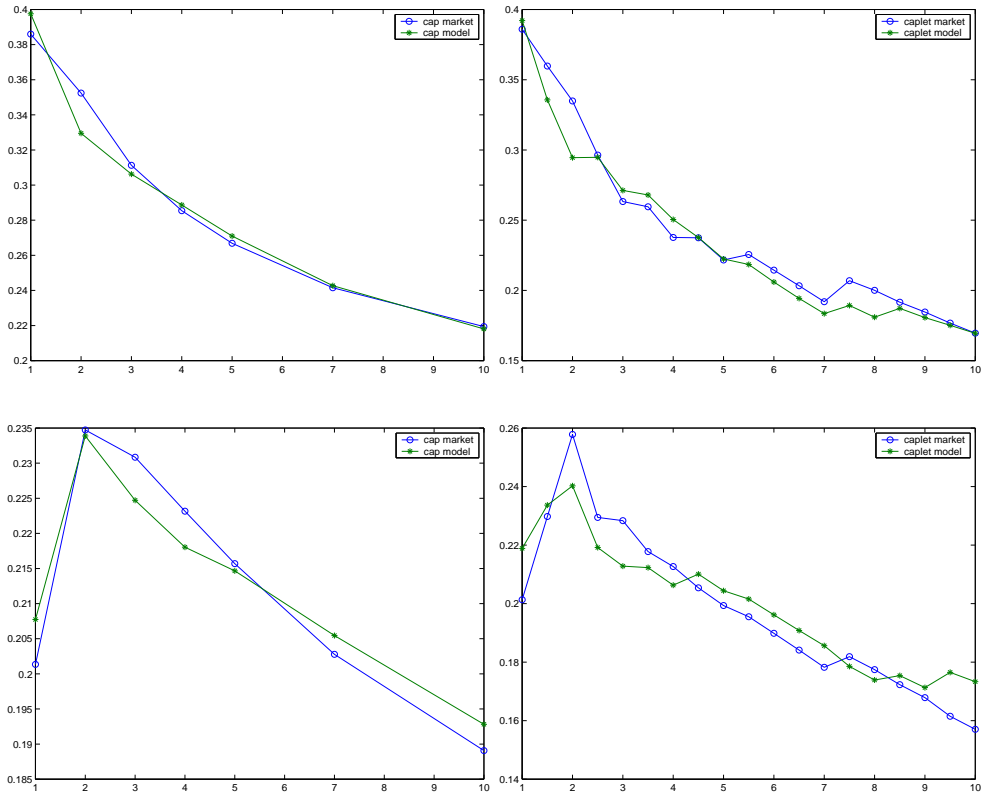
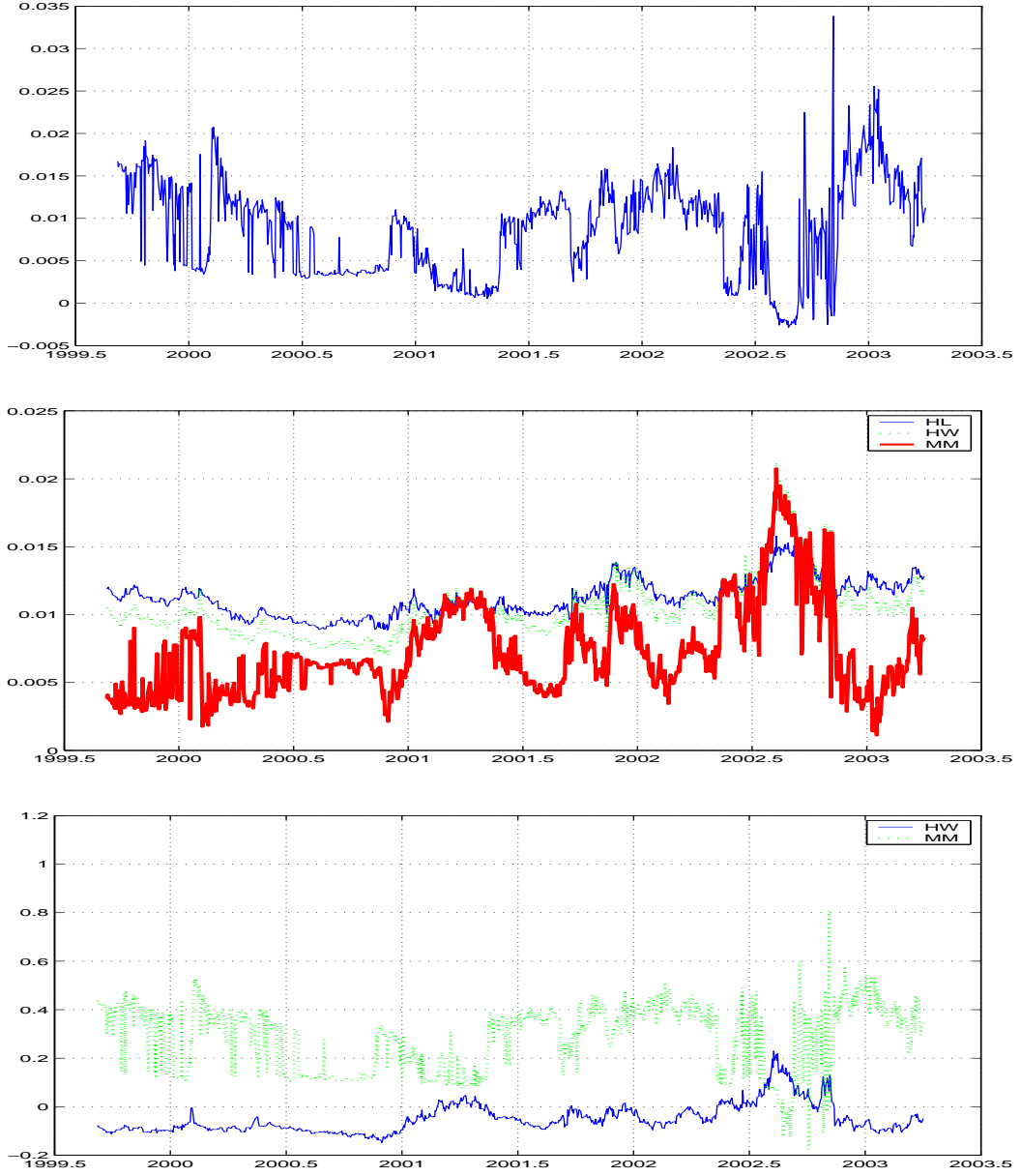**Figure 9** *MM model: average of market and model cap volatilities (left) and caplet volatilities (right) in the case of decreasing shapes (top) and of humped shapes (bottom).*

Figure 10 *Time series of parameter estimates of γ (top), σ (middle) and a (bottom) of the HL, HW and MM models calibrated to the data set.*
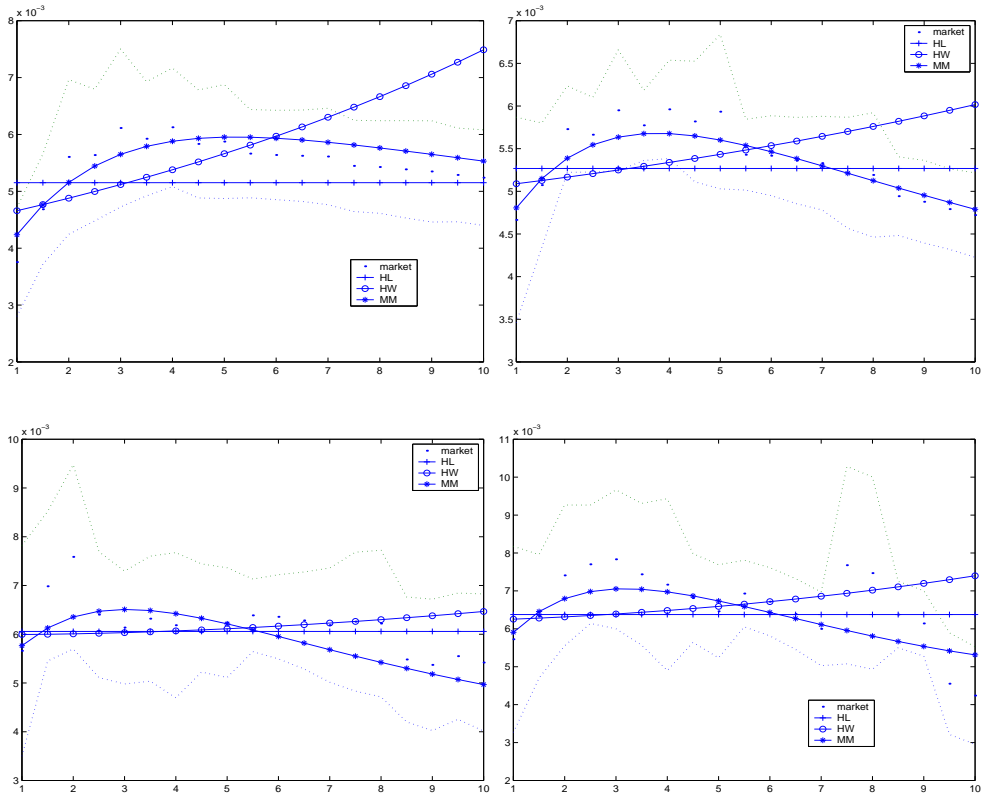
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Figure 11 *Average of the US market $\bar{Z}(t,T)$ over the four periods with 95% confidence band. This is compared with the corresponding functions of the HL, HW and MM models when calibrated to the data set.*
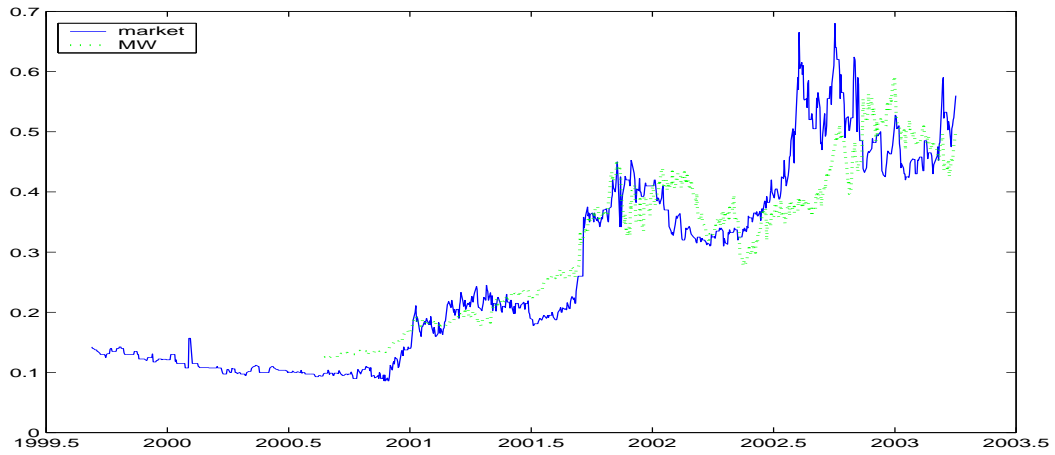
Figure 12 *Comparison between market and estimated 1-year cap volatility from 07/09/1999 to 31/3/2003.*
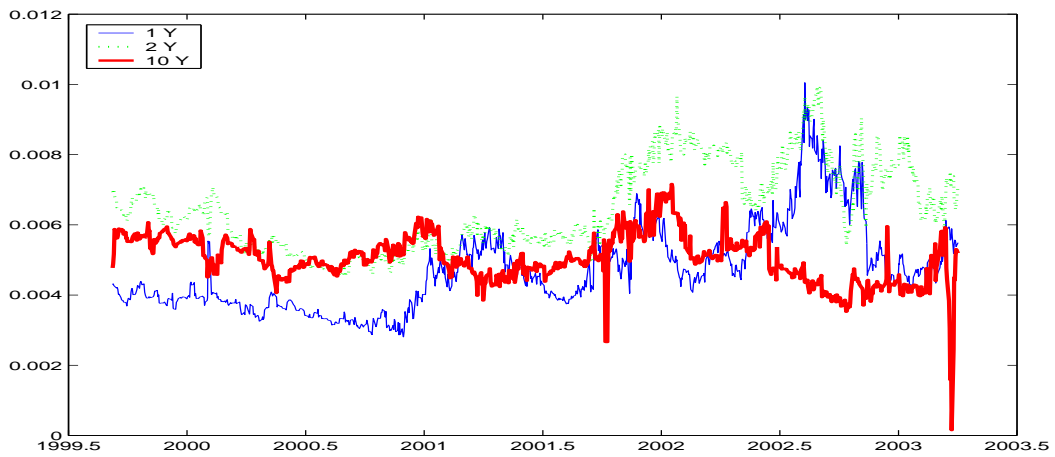

Figure 13 *The time series of $\bar{Z}(t,T)$ from 07/09/1999 to 31/3/2003, for maturities 1,2,10 years.*

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