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Paper by invitation

Decision Making by Hybrid Probabilistic - Possibilistic Utility Theory

Summary: It is presented an approach to decision theory based upon non-probabilistic uncertainty. There is an axiomatization of the hybrid probabilistic-possibilistic mixtures based on a pair of triangular conorm and triangular norm satisfying restricted distributivity law, and the corresponding non-additive S-measure. This is characterized by the families of operations involved in generalized mixtures, based upon a previous result on the characterization of the pair of continuous t-norm and t-conorm such that the former is restrictedly distributive over the latter. The obtained family of mixtures combines probabilistic and idempotent (possibilistic) mixtures via a threshold.

Key words: Decision making, Utility theory, Possibilistic mixture, Hybrid probabilistic-possibilistic mixture, Triangular norm, Triangular conorm, Pseudo-additive measure.

JEL: C44, L97.

The classical utility theory of John von Neumann and Oskar Morgenstern (1944) is based upon the notion of probabilistic mixtures. The more recently developed general theory of non-additive measures (see Dieter Denneberg 1994, Endre Pap 1995) provides a good mathematical base for modeling complex systems in economics through areas such as preference theory, utility theory, game theory, operational research and decision making theory, see Antoine Billot (1992). Early applications of non-additive measures in economics trace back to George Schackle and Lennox Sharman (1952), as he put forward an alternative concept to that of subjective probability, which he called potential surprise and what we today call necessity, as dual to possibility measure. In this same timeframe French mathematician Gustav Choquet (1953) introduced the concept of capacity as non-additive measure and the corresponding integral, which turns out to be useful in models of the theory of potential. Ellberg paradox (1969) shows the limits of subjective, and therefore additive, probability as a system for modeling beliefs. Glenn Shafer (1976) introduced belief functions, which previously had been called attribute of opinion. Bernoulli and Lambert further worked with non-additive probabilities when they modeled the cognitive behaviour of agents. David Schmeidler (1982) has introduced capacities into the heart of methodology of utility expectation. Following Schmeidler (1982) and Itzhak Gilboa (1987) introduced non-additive subjective probability. As special type of non-additive measures, called pseudo-additive measures (called also decomposable measures by Siegfrid Weber 1984), have been investigated by Michio Sugeno and

Toshiaki Murofushi (1987), V. P. Maslov (1987) and Pap (1990, 1995, 2002). Didier Dubois et al. (1996) have recently proven that the notion of mixtures can be extended to pseudo-additive measures. R. Cox's well-known theorem (1946) (see Jeff B. Paris 1994), which justifies the use of probability for treating uncertainty, has been discussed in many papers. Recently, the theory has come under criticism, and some relaxation on the conditions has been made, which thus implies that also some non-additive measures can satisfy the required conditions, see Joseph Y. Halpern (1999). Relaxing the condition on strict monotonicity, to only monotonicity on the function which occurs in the conditioning requirement, leads then to the pair of t-norm S and t-norm T which satisfies (RD) and the corresponding S-measure satisfy also all other required conditions. We present here an answer to the following question: what else remains possible beyond idempotent (possibilistic) and probabilistic mixtures?

The solution takes the advantage of a result obtained by Peter E. Klement, Radko Mesiar, and Pap (2000) (see for special case Carlo Bertoluzza 1993) on the relaxed distributivity of triangular norm over a triangular conorm (called restricted distributivity). This result has a drastic consequence on the notion of mixtures. Beyond possibilistic and probabilistic mixtures, only a form of hybridization is possible such that the mixture is possibilistic under a certain threshold, and probabilistic above. - See Dubois, Pap, and Henri Prade (2000, 2001). Of necessity to mention, the models in investigations are involved many aggregation functions (operators), see Michel Grabisch et al. (2009).

Before we present the hybrid axiomatic framework for utility theory, we recall both existing sets of utility axioms: classical, probabilistic (Von Neumann and Morgenstern), on one side, and possibilistic framework of Dubois and Prade for utility theory, on other. Comparing both axiomatization, we came to hybrid one, which generalize possibilistic and probabilistic mixtures.

1. Probabilistic Representation of Utilities (Von Neumann and Morgenstern Axioms of Preference)

Let X be a set of situations (consequences, outcomes). Let p be a simple probability measure on X, thus $p=(p(x_1),p(x_2),\ldots,p(x_n))$, where $p(x_i)$ are probabilities of outcome $x_i\in X$ occurring, i.e., $p(x_i)\geq 0$ for all $i=1,2,\ldots,n$, and $\Sigma_{i=1}^{n} p(x_i)=1$. Define $\mathbb{P}(X)$ as the set of simple probability measures on X. A particular lottery p is a point in $\mathbb{P}(X)$. A compound lottery is an operation defined on $\mathbb{P}(X)$ which combines two probability distributions p and p′ into a new one, denoted $V(p,p';\alpha,\beta)$, with $\alpha,\beta\in[0,1]$ and $\alpha+\beta=1$, and it is defined as: $V(p,p';\alpha,\beta) = \alpha\cdot p+\beta\cdot p'$. Notice that $V(p,p';\alpha,\beta)\in\mathbb{P}(X)$. Let $\precsim$ be a binary relation over $\mathbb{P}(X)$, i.e., $\precsim\ \subset \mathbb{P}(X)\times\mathbb{P}(X)$. Hence, we can write $(p,q)\in\precsim$, or $p\precsim q$ to indicate that lottery q is "preferred to or equivalent to" lottery p.

One of the possible axiom systems for the Von Neumann and Morgenstern type utility is given by

NM1. $\mathbb{P}(X)$ is equipped with a complete preordering structure $\precsim$.

NM2 (Continuity). For $p\prec q\prec r \Rightarrow \exists\alpha$: $q\sim V(p,r;\alpha,1-\alpha)$.

NM3 (Independence). $p\sim q \Rightarrow V(p,r;\alpha,1-\alpha) \sim V(q,r;\alpha,1-\alpha)$, $\forall r\in\mathbb{P}(X)$, $\forall\alpha\in[0,1]$.

NM4 (convexity). For $\forall p \prec q \Rightarrow p \prec V(p,q;\alpha,1-\alpha) \prec q$, $\forall \alpha \in]0,1[$.

The theorem below shows that the preference ordering on set of states which satisfies the proposed axioms can always be represented by a utility function.

Representation Theorem (Von Neumann and Morgenstern 1944). A preference ordering relation $\precsim$ on $\mathbb{P}(X)$ satisfies axioms **NM1**, **NM2**, **NM3** and **NM4** if and only if, there is a real-valued function U: $\mathbb{P}(X) \rightarrow \mathbb{R}$ such that

a) U represents $\precsim$, i.e. $\forall p,q \in \mathbb{P}(X)$, $p \precsim q \Leftrightarrow U(p) \leq U(q)$;

b) U is affine, i.e. $\forall p,q \in \mathbb{P}(X)$,

$U(\alpha \cdot p+(1-\alpha)\cdot q) = \alpha \cdot U(p)+(1-\alpha)\cdot U(q)$, for any $\alpha \in]0,1[$.

Moreover, U is unique up to a linear transformation.

2. Triangular Conorms and Pseudo-Additive Measures

A triangular conorm S (t-conorm for short) is a binary operation on the unit interval [0,1] such that for all x,y, z $\in$ [0,1] the following four axioms are satisfied:

(S1) Commutativity $S(x,y) = S(y,x)$;

(S2) Associativity $S(x,S(y, z)) = S(S(x,y), z)$;

(S3) Monotonicity $S(x,y) \leq S(x, z)$ whenever $y \leq z$;

(S4) Boundary Condition $S(x,0) = x$.

If S is a t-conorm, then its dual t-norm $T : [0,1]^2 \rightarrow [0,1]$ is given by

$$T(x,y) = 1 - S(1-x,1-y).$$

Example 2.1 The following are the three basic t-norms together with their dual t-conorms

(i) Minimum $T_{\mathbf{M}}$ and maximum $S_{\mathbf{M}}$ given by

$$T_{\mathbf{M}}(x,y) = \min(x,y), \qquad S_{\mathbf{M}}(x,y) = \max(x,y);$$

(ii) Product $T_{\mathbf{P}}$ and probabilistic sum $S_{\mathbf{P}}$ given by

$$T_{\mathbf{P}}(x,y) = x\ y, \qquad S_{\mathbf{P}}(x;y) = x+y-x\ y;$$

(iii) Lukasiewicz t-norm $T_{\mathbf{L}}$ and Lukasiewicz t-conorm $S_{\mathbf{L}}$ given by

$$T_{\mathbf{L}}(x,y) = \max(x+y-1,0), \quad S_{\mathbf{L}}(x,y) = \min(x+y,1).$$

We shall use results on t-conorms and t-norms from books of Bertold Schweizer and Abe Sklar (1989) and Klement, Mesiar, and Pap (2000).

The t-conorm S (respectively t-norm T) is called *strict* if it is continuous and strictly monotone on the open square $]0,1[^2$. The continuous t-conorm S (respectively t-norm T) is called *nilpotent* if each $a \in]0,1[$ is a nilpotent element of S (respectively of T), i.e., for every $a \in]0,1[$ there exists $n \in \mathbb{N}$ (the set of natural numbers) such that $a_S^{(n)} = 1$ (respectively $a_T^{(n)}=0$), where $a_S^{(n)}$ is the n-th power of a given by S(a,...,a) (respectively T(a, ..., a)), i.e., repeating the value a n-times.

The following representations hold.

Theorem 2.2. A function $S:[0,1]^2 \rightarrow [0,1]$ is a continuous Archimedean triangular conorm,i.e., for all $x \in]0, 1[$ we have $S(x,x)>x$, if and only if there exists a continuous, strictly increasing function s:[0,1] $\rightarrow$ [0, +\infty] with s(0)=0 such that for all $x,y \in [0,1]$

$$S(x,y) = s^{-1}(\min(s(x) + s(y),s(1))).$$

The analogous theorem holds for continuous Archimedean triangular norms:

Theorem 2.3. A function $T:[0,1]^2 \to [0,1]$ is a continuous Archimedean triangular norm,i.e., for all $x \in]0, 1[$ we have $T(x,x)<x$, if and only if there exists a continuous, strictly decreasing function

$$t:[0,1] \to [0, +\infty] \text{ with } t(1)=0 \text{ such that for all } x,y \in [0,1]$$
$$T(x,y) = t^{-1}(\min(t(x) + t(y),t(0))).$$

The functions s and t from Theorems 2.2 and 2.3 are then called additive generators of S and T, respectively.

The following relation for a pair of t-conorm S and a t-norm T will be important for the extension of the utility theory.

Definition 2.4. A t-norm T is restricted distributive over a t-conorm S if for all $x,y, z \in [0,1]$ we have

$$\text{(RD)} \qquad T(x,S(y, z)) = S(T(x,y),T(x, z)),$$

whenever $S(y, z) < 1$.

The following theorem is needed from the monograph of Klement, Mesiar, Pap (2000) which gives the complete characterization of the family of continuous pairs (S,T) which satisfy the condition (RD).

Theorem 2.5. A continuous t-norm T is conditionally distributive over a continuous t-conorm S if and only if there exists a value $a \in [0,1]$, a strict t-norm T^* and a nilpotent t-conorm S^* such that the additive generator s^* of S^* satisfying $s^*(1) = 1$ is also a multiplicative generator of T^* such that T on the square $[0,a]^2$ is an arbitrary continuous t-norm T_1, on the square $[a,1]^2$ is t-norm T*, and on the remaining part of the unit square it is equal to min, i.e., in the ordinal sum notations

$$T = (< 0,a,T_1 >;< a,1,T^* >),$$

and S on the square $[0,a]^2$ is max, on the square $[a,1]^2$ is t-conorm S*, and on the remaining part of the unit square it is equal to max, i.e., in the ordinal sum notations $S = (< a,1,S^* >)$.

The representation of the pair (S,T) of continuous t-conorm and t-norm, respectively, which satisfy the condition (RD), based on Theorem 2.5, we denote by $(<S_{\mathbf{M}}, S^*>,<T_1,T^*>)_a$.

Example 2.6

(i) The extreme case a = 0 reduces on the pair $S_{\mathbf{L}}$ and $T_{\mathbf{P}}$.

(ii) The other extreme case a = 1 reduces on the pair $S_{\mathbf{M}}$ and an arbitrary continuous t-norm T_1.

(iii) For 0<a<1 the pair S and T gives us the hybrid idempotent probabilistic case.

We restrict ourselves to the situation $(<S_{\mathbf{M}}, S_{\mathbf{L}}>,<T_1,T_{\mathbf{P}}>)_a$, since this is the most important case and all other cases can be obtained by isomorphisms (see Klement, Mesiar, and Pap 2000).

Let X be a fixed non-empty finite set.

Definition 2.7. Let S be a t-conorm and let 2^X be the family of all subsets of X. A mapping $m: 2^X \to [0,1]$ is called an S-measure if $m(\emptyset)=0$, $m(X)=1$ and if for all $A,B \in 2^X$ with $A \cap B = \emptyset$ we have

$$m (A \cup B) = S(m(A),m(B)).$$

Remark 2.8.

(i) Each S-measure m: $2^X \to [0,1]$ is uniquely determined by the values m({x}) with $x \in X$.

(ii) In the general case when X is an arbitrary non-empty set (also infinite) there is an additional condition on m in Definition 2.6, namely that it is continuous from below.

Example 2.9. A set function m: $2^X \to [0,1]$ is S_M-measure if and only if for all $A,B \in 2^X$ we have $m(A \cup B) = S_M(m(A),m(B))$. Usually it is called idempotent (possibility) measure with the corresponding distribution by π. Namely, for an arbitrary function $\pi: X \to [0,1]$, the set function m: $2^X \to [0,1]$ defined by

$$m(A)=\sup\{\pi(x) \mid x \in A\}$$

is an S_M-measure. We remark that only for X finite the notions of S_M-measure and possibility measures coincide, see Pap (1995).

We characterize which triangular norms can be used for extending the notion of independence to pseudo-additive measures in the sense of a prescribed triangular conorm. Since the term independence has a precise meaning in probability theory, we shall speak of separability in the framework of S-measures.

Definition 2.10. Two events A and B are said to be T-separable if

$$m(A \cap B)= T(m(A), m(B))$$

for a triangular norm T.

Under natural constraints, the only reasonable pseudo-additive measures admitting of an independence-like concept, are based on restricted distributive pairs (S, T) of conorms and t-conorms, and we have by Dubois, Pap, and Prade (2001):

(i) probability measures (and T = product);

(ii) possibility measures (and T is any t-norm);

(iii) suitably normalized hybrid set-functions m such that there is $a \in]0,1[$ which gives for A and B disjoint

$$m(A \cup B) = \begin{cases} m(A) + m(B) - a \text{ if } m(A) > a, m(B) > a, \\ \max(m(A), m(B)) \text{ otherwise,} \end{cases}$$

and for separability:

$$m(A \cap B) = \begin{cases} a + \frac{(m(A)-a)(m(B)-a)}{(1-a)} \text{ if } m(A) > a, m(B) > a, \\ a\, T_1(m(A)\backslash a, m(B)\backslash a) \text{ if } m(A) \le a, m(B) \le a, \\ \min(m(A), m(B)) \text{ otherwise.} \end{cases}$$

It is well-known, see Shafer (1996), that any probability distribution on a finite set X can be represented as a sequence of binary lotteries. A binary lottery is 4-uple (A,α,x,y), where A is a subset of X and $\alpha \in [0,1]$ such that $P(A)=\alpha$, and it represents the random event that yields x if A occurs and y otherwise.

Now we generalize this result to S-measure. Suppose tha m is a S-measure on $X=\{x_1, x_2, x_3\}$ and $m_i=m(\{x_i\})$. Suppose we want to decompose the ternary tree into the binary tree of the right side so that they are equivalent. We follow the calculations given by Dubois, Pap, Prade (2000). Thus the reduction of lottery property enforces the following equations

$$S(v_1,v_2)=1, \quad T(\mu, v_1)=m_2, \quad T(\mu,v_2)=m_3,$$

where T is the triangular norm that expresses separability for S-measures. The first condition expresses that (v_1,v_2) is in the mixture set (with no truncation for t-conorm S allowed). If these equations have unique solutions, then by iterating this construction, any distribution of a S-measure can be decomposed into a sequence of binary lotteries.The problem of normalization takes us to the following system of

$$(*) \qquad \alpha_1 =T(\mu, v_1), \quad \alpha_2 =T(\mu, v_2), \quad S(v_1,v_2) =1,$$

for given α_1 and α_2. We know that there always exists an unique solution (μ,v_1,v_2). We are interested in the analytical forms of (μ,v_1,v_2). We suppose without loss of generality that $\alpha_1 > \alpha_2$. Then we have the following cases:

Case I. Let $\alpha_1 > a$, $\alpha_2 > a$. Then (*) reduces on

$$\alpha_1= a+ (\mu -a)(v_1-a)\backslash(1-a), \quad \alpha_2= a+ (\mu -a)(v_2-a)\backslash(1-a), \quad 1= v_1 + v_2 -a.$$

We obtain the unique solution

$$\mu=\alpha_1 + \alpha_2 -a, \qquad v_1= a+ (1-a)(\alpha_1-a)\backslash(\alpha_1+\alpha_2-2a),$$
$$v_2= a+ (1-a)(\alpha_2-a))\backslash(\alpha_1+\alpha_2-2a).$$

Case II. Assume $\alpha_1 > a \geq \alpha_2$. Then S=max, and $\mu \geq a$, $v_1 \geq a$ and $v_2 \leq \alpha_2 < a$. Hence assuming T_1= min (we shall only deal with this case) the equations (*) write:

$$\max (v_1,v_2) =1, \ \alpha_1= a+ (\mu -a)(1-a) \backslash (1-a) = \mu, \ \alpha_2= \min (\mu, v_2).$$

Assume v_1=1. Then $\mu = \alpha_1$ and $v_2 =\alpha_2$. We remark that this solution is unique, since assuming v_2=1 leads to $\mu=\alpha_2< a$, which is a contradiction.

Case III. Assume $\max(\alpha_1, \alpha_2) \leq a$. Then S=max. Assume again v_1=1. Then the first equation in (*) yields $\mu=v_1$. Assuming $T=T_1$=min the second equation of (*) leads to $v_2=\alpha_2$. Hence the same solution $(\alpha_1,1,\alpha_2)$ as in case II. Note that assuming v_2=1 again leads to a contradiction since then $\mu=\alpha_2$ and equation $\alpha_1=T(\alpha_2, v_1)$ has no solution.

For $\max(1, \alpha_2)=1$ the other two equations reduces on $\alpha_1= \min(\alpha_1,1)$, (or it can be considered the case with T_1 (we shall not examine this case) where we can take specially T_1= min) and so $\alpha_2=\min (\alpha_1, \alpha_2)$.

We have v_1=1 and $v_2= \alpha_2$.

3. Possibilistic Representation of Utilities (Dubois and Prade Axioms of Preferences)

The belief state about which situation in X is the actual one is supposed to be represented by a possibility distribution π. A possibility distribution π defined on X takes its values on a valuation scale V, where V is supposed to be linearly ordered. V is assumed to be bounded and we take sup(V)=1 and inf(V)=0. Define Pi(X) as set of consistent possibility distributions over X, i.e.,

$$Pi(X)=\{\pi:X\to V \mid \exists x\in X: \pi(x)=1\}.$$

The possibilistic mixture is an operation defined on Pi(X) which combines two possibility distributions π and π' into a new one, denoted $P(\pi,\pi';\alpha,\beta)$, with $\alpha,\beta\in V$ and $\max(\alpha,\beta)=1$, and it is defined as:

$$P(\pi,\pi';\alpha,\beta) = \max(\min(\alpha,\pi), \min(\beta,\pi')).$$

Let $\sqsubseteq$ be a binary relation over Pi(X), i.e., $\sqsubseteq\ \subset Pi(X)\times Pi(X)$. Hence, we can write $(\pi,\pi')\in\sqsubseteq$, or $\pi\sqsubseteq\pi'$ to indicate that possibilistic lottery π' is "preferred to or equivalent to" lottery π.

The proposed axiom systems for the Dubois and Prade type optimistic utility is given by

DP1 Pi(X) is equipped with a complete preordering structure $\sqsubseteq$.

DP2 (Continuity). For $\forall\pi\in Pi(X)$, $\exists\lambda$: $\pi\sim P(\underline{\pi}, \overline{\pi};\lambda,1)$, where $\overline{\pi}$ and $\underline{\pi}$ are a maximal and a minimal element of Pi(X) with respect to $\sqsubseteq$, respectively.

DP3 (Independence). $\pi\sim\pi' \Rightarrow P(\pi,\pi'';\lambda,\mu) \sim P(\pi',\pi'';\lambda,\mu)$, $\forall\pi''\in Pi(X)$, $\forall\lambda,\mu$.

DP4 (Uncertainty prone). $\pi\leq\pi' \Rightarrow \pi\sqsubseteq\pi'$.

The set of axioms **DP1**, **DP2**, **DP3** and **DP4** characterise the preference orderings induced by an optimistic utility.

Representation Theorem (Dubois and Prade 1998). A preference ordering relation $\precsim$ on $\mathbb{P}(X)$ satisfies axioms **DP1**, **DP2**, **DP3** and **DP4** if and only if, there exist:

a) a linearly ordered utility scale U, with inf(U)=0 and sup(U)=1;
b) a preference function $u:X\to U$ such that $u^{-1}(1)\neq\emptyset\neq u^{-1}(0)$, and
c) an onto order preserving function $h:V\to U$ such that h(0)=0, h(1)=1,

in such a way that it holds: $\pi\sqsubseteq\pi'$ iff $\pi\lhd_u\pi'$, where $\lhd_u$ is the ordering on Pi(X) induced by the qualitative utility

$$QU^+(\pi) = \max_{x\in X} \min(h(\pi(x)),u(x)).$$

4. Hybrid Probabilistic-Possibilistic Mixture

Let (S,T) be a pair of continuous t-conorm and t-norm, respectively, which satisfy the condition (RD). Then by Theorem 2.4 they are of the form $(\langle S_{\mathbf{M}}, S^*\rangle,\langle T_1,T^*\rangle)_a$, where S^* is a nilpotent t-conorm, T_1 an arbitrary t-norm and T^* a strict t-norm.

In order to generalize stated sets of axioms for utility theory, we denote $X=\{x_1,x_2,\ldots,x_n\}$ set of outcomes, $\Delta(X)$ set of S-measures defined on X.

Definition 4.1. A hybrid mixture operation which combines two S-measures m and m′ into a new one, denoted $M(m,m';\alpha,\beta)$, with pair (α,β) belonging to

$$\Phi_{S,a} = \{(\alpha,\beta)\mid \alpha,\beta\in]0,1[,\ \alpha+\beta=1+a \text{ or } \min(\alpha,\beta)\leq a,\ \max(\alpha,\beta)=1\},$$

where $a\in[0,1]$, is defined by

$$M(m,m';\alpha,\beta) = S(T(\alpha,m), T(\beta,m')).$$

As we alredy told, without loosing the generality, we shall restrict to the case $(\langle S_{\mathbf{M}}, S_{\mathbf{L}}\rangle,\langle T_1,T_{\mathbf{P}}\rangle)_a$.

Remark 4.2.

It is easy to verify that hybrid mixture M given by Definition 4.1 satisfies the axioms **M1-M5** given by Dubois et al. (1996) on $\Phi_{S,a}$. This kind of mixtures exhaust the possible solutions to **M1-M5**.

Let (S,T) be a pair of continuous t-conorm and t-norm, respectively, of the form $(\langle S_{\mathbf{M}}, S_{\mathbf{L}}\rangle,\langle T_1,T_{\mathbf{P}}\rangle)_a$. Let u_1,u_2 be two utilities taking values in the unit interval [0,1] and let μ_1, μ_2 be two degrees of plausibility from $\Phi_{S,a}$. Then we define *the optimistic hybrid utility function* by means of the hybrid mixture as

$$U(u_1, u_2; \mu_1, \mu_2)= S(T(u_1, \mu_1),T(u_2,\mu_2)).$$

We introduce *the pessimistic hybrid utility function* $\underline{U}$ using the utility function U in the following way

$$\underline{U}(u_1, u_2; \mu_1, \mu_2) = 1- U(u_1, u_2; \mu_1, \mu_2).$$

5. Characterization of the Optimistic Hybrid Utility Function

We shall examine in details the optimistic hybrid utility function utility function.

Case I. Let $\mu_1 > a$, $\mu_2 > a$, i.e., $\mu_1+ \mu_2= 1+a$. Then we have the following sub-cases:

(a) Let $\mu_1 > a$, $u_2 > a$.

Then we have

$$U(u_1, u_2; \mu_1, \mu_2)= S(a+(u_1-a)(\mu_1-a)\backslash(1-a),a+(u_2-a)(\mu_2 - a))\backslash(1-a)).$$

Then $a+(u_i-a)(\mu_\iota- a))\backslash(1-a) > a$ for all i=1,2. Hence by the preceding equality

$$U(u_1, u_2; \mu_1, \mu_2)= (u_1(\mu_1-a) + u_2(1- \mu_1))\backslash(1-a).$$

(b) Let $u_1 \le a$, $u_2 > a$.

Then we have

$$U(u_1, u_2; \mu_1, \mu_2) = S(u_1, a+((u_2-a)(\mu_2-a))\backslash(1-a))$$
$$= a+((u_2-a)(\mu_2-a))\backslash(1-a).$$

In a quite analogous way it follows for $u_1 > a$, $u_2 \le a$ that

$$U(u_1, u_2; \mu_1 , \mu_2)= a+ ((u_1-a)(\mu_1-a))\backslash(1-a).$$

(c) Let Let $u_1 \le a$, $u_2 \le a$. Then

$$U(u_1, u_2; \mu_1, \mu_2)= \max (u_1, u_2).$$

Case II. Let $\mu_1 \le a$, $\mu_1= 1$ (in a quite analogous way we can consider the case $\mu_2 \le a$, $\mu_1=1$). Then we have the following sub-cases, where S=max:

(a) Let $u_1>a$, $u_2 > a$. Then we have

$$U(u_1, u_2; \mu_1, \mu_1)= S(\mu_1, u_2)=u_2.$$

(b) Let $u_1 \le a$, $u_2 > a$. Then we have

$$U(u_1, u_2; \mu_1, \mu_1)=S(aT_1(u_1\backslash a, \mu_1\backslash a),u_2) =u_2.$$

(c) Let $u_1 > a$, $u_2 \le a$. Then we have

$$U(u_1, u_2; \mu_1, \mu_2) = S(\mu_1, u_2) = \max(\mu_1, u_2).$$

(d) Let $u_1 \le a$, $u_2 \le a$. Then we have

$$U(u_1, u_2; \mu_1, \mu_2) = \max(aT_1(u_1\backslash a, \ \mu_1\backslash a), u_2).$$

For T_1=min the case II and case Ic are exactly idempotent (possibilistic) utility.

6. Behaviour of the Decision Maker with respect to Hybrid Utility Function

Although the above description of optimistic hybrid utility is rather complex, it can be easily explained, including the name optimistic.

Case I is when the decision-maker is very uncertain about the state of nature: both μ_1 and μ_2 are high and the two involved states have high plausibility. Case Ia is when the reward is high in both states- then the behavior of utility is probabilistic. Case Ib is when the reward is low in state x_1 ($u_1 \le a$), but high on the other state. Then the decision-maker looks forward to the best outcome and the utility is a function of u_2 and μ_2 only. In case Ic when both rewards are low, the decision-maker is possibilistic and again focuses on the best outcome. Case II is when state x_1, is unlikely. In case IIa,b when the plausible reward is good, then the decision-maker looks forward to this reward. In case IIc where the most plausible reward is low then the decision maker still keeps some hope that state x_1, will prevail if u_2 is really bad, but weakens the utility of state x_1, because of it lack of plausibility. This phenomena subsides when the least plausible outcome is also bad, but the (bad) utility of x_1, participates in the calculation of the resulting utility, by discounting μ_1, even further. From the analysis, the optimistic attitude of an agent ranking decisions using the hybrid utility is patent.

In addition, we can provide the corresponding interpretations of the pessimistic hybrid utility function $\underline{U}$ as dual interpretations to the preceding cases of U. In order to do this, we must go through the above behavior analysis again, interpreting u_1 and u_2 and $\underline{U}$ as disutilities instead of utilities. For instance, in case IIa,b, the decision-maker is afraid that the worst outcome occurs (u_2>a is interpreted as penality).

7. Axioms for a Hybrid Probabilistic-Possibilistic Utility Theory

We propose the following set of axioms for a preference relation $\le_h$ defined over $\Delta(X)$ to represent optimistic utility (see Pap and Marija Roca 2006):

H1. *$\Delta(X)$ is equipped with a complete preordering structure $\lesssim_h$, i.e., $\lesssim_h$ is reflexive, transitive and complete.*

H2 (Continuity). If $m <_h m' <_h m''$ then

(i) $\exists\alpha \in]a,1[$: $m' \sim_h M(m,m'';1+a-\alpha,\alpha)$, if $m,m',m''>a$;

(ii) $\exists\alpha \in]0,a]$: $m' \sim_h M(m,m'';1,\alpha)$, otherwise.

H3 (Independence). For $\forall m,m',m''\in\Delta(X)$ and for $\forall\alpha,\beta\in\Phi_{S,a}$:

$$m' \leq_h m'' \Leftrightarrow M(m',m;\alpha,\beta) \leq_h M(m'',m;\alpha,\beta).$$

H4 (Uncertainty prone).
(i) $m \leq_h m' \Rightarrow m \leq_h M(m,m';\alpha,1+a-\alpha) \leq_h m'$, $\alpha\in]a,1[$, if $m,m'>a$;
(ii) $m < m' \Rightarrow m <_h m'$, otherwise.

Now, we define a function of optimistic utility for all $m\in\Delta(X)$ by

$$U^+(m) = S_{x_i\in X}\left(T\left(m(x_i),u(x_i)\right)\right),$$

where $u:X\to U$ is a preference function that assigns to each consequence of X a preference level of U, such that $u^{-1}(1)\neq\emptyset\neq u^{-1}(0)$. It is interesting to notice that U^+ preserves the hybrid mixture in the sense that:

$$U^+\left(M\left(m,m';\alpha,\beta\right)\right) = S\left(T\left(\alpha,U^+(m)\right),T\left(\beta,U^+(m')\right)\right) = M\left(U^+(m),U^+(m');\alpha,\beta\right).$$

In the proof of the next representation theorem the following lemma is crucial.
Lemma. Let $\precsim_u$ be the preference ordering on $\Delta(X)$ induced by utility function

$$U^+(m) = S_{x_i\in X}\left(T\left(m(x_i),u(x_i)\right)\right),$$

i.e., $m\precsim_u m'$ if and only if $U^+(m)\leq U^+(m')$. Then the binary relation $\precsim_u$ verifies set of axioms {**H1,H2,H3,H4**}.

Representation Theorem (Optimistic Utility).
Let $\Delta(X)$ be a set of S-measures defined on X, and $\leq_h$ a binary preference relation on $\Delta(X)$. Then the relation $\leq_h$ satisfies the set of axioms {**H1,H2,H3,H4**} if and only if there exist:

a) a linearly ordered utility scale U, with inf(U)=0 and sup(U)=1;
b) a preference function $u:X\to[0,1]$,

in such a way that $m\leq_h m'$ if and only if $m\precsim_u m'$, where $\precsim_u$ is the ordering in $\Delta(X)$ induced by the optimistic utility function defined as:

$$U^+(m) = S_{x_i\in X}(T(m(x_i), u(x_i))),$$

where (S,T) is a pair of continuous t-conorm and t-norm, respectively, which satisfy the condition (RD).

We will introduce, on the analogous way, the pessimistic criterion in the hybrid utility theory, but first, we have to modify the existing set of axioms. Namely, the axioms **H2.** and **H4.** have to be adapted to pessimistic preference criterion.

H2* (Continuity). If $m <_h m' <_h m''$ then:
(i) $\exists\alpha\in]a,1[$: $m' \sim_h M(m,m'';1+a-\alpha,\alpha)$, if $m,m',m''>a$;
(ii) $\exists\alpha\in]0,a]$: $m' \sim_h M(m,m'';\alpha,1)$, otherwise.

H4* (Uncertainty Aversion).
(i) $m \leq_h m' \Rightarrow m \leq_h M(m,m';\alpha,1+a-\alpha) \leq_h m'$, $\alpha\in]a,1[$, if $m,m'>a$;
(ii) $m < m' \Rightarrow m' <_h m$, otherwise.

Thus, the modified set of axioms, i.e. the set {**H1,H2*,H3,H4***} faithfully characterise the preference ordering induced by a pessimistic hybrid utility, which is dual to the optimistic one.

Representation Theorem (Pessimistic Utility).

Let $\Delta(X)$ be a set of S-measures defined on X, and $\leq_h$ a binary preference relation on $\Delta(X)$. Then the relation $\leq_h$ satisfies the set of axioms {**H1,H2*,H3,H4***} if and only if there exist

a) a linearly ordered utility scale U, with inf(U)=0 and sup(U)=1;

b) a preference function $u:X\to[0,1]$,

in such a way that $m\leq_h m'$ if and only if $m\lesssim_u m'$, where $\lesssim_u$ is the ordering in $\Delta(X)$ induced by the pessimistic utility function given by

$$U^{-}(m) = 1-S_{x_i\in X}(T(m(x_i), 1-u(x_i)),$$

where (S,T) is a pair of continuous t-conorm and t-norm, respectively, which satisfy the condition (RD).

8. Conclusions

A generalization is given of the Von Neumann and Morgenstern utility theory, which was based on the probability theory, now using special type of non-additive measures the so called pseudo-additive measures. An axiomatization of the hybrid probabilistic-possibilistic mixtures based upon a pair of triangular conorm and triangular norm satisfying restricted distributivity law, and the corresponding non-additive S-measure is presented. Conclusions illustrate that this is a maximal natural generalization in the sense that any further generalization would loose some natural requirement. Furthermore, an interpretation of the corresponding behavior of the decision maker is provided. The future work will be related to further interpretations in applications in economics.

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