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I. Introduction

In the course of economic development, nations may find it optimal to go through a cycle of debt. A recent theoretical paper by Long and Siebert (1989) gives sufficient conditions for this, under the assumption that the country is small and faces an exogenously given and time-independent world rate of interest. This raises an interesting question: would debt cycles be optimal (under certain conditions) if the country is large enough to influence the world rate of interest? In addition, is it possible that a debt cycle repeats itself indefinitely with the same frequency and magnitude? Can debt cycles exhibit the property of a converging spiral?

The present paper is a modest attempt to provide a partial answer to the above questions. For simplicity, we assume that there are only two countries, called the home country and the foreign country respectively. There are no impediments to international capital flows, and as a result, marginal products of capital in the two countries are equalized. In the foreign country, saving is a constant fraction of capital income. This is the standard "classical saving" assumption. The home country, however, chooses a consumption path that maximizes its intertemporal utility. This typically implies a non-constant saving ratio. In making its choice of an optimal consumption path, the home country perceives the effect of capital accumulation on the rate of interest. Thus, even though at each point of time no country exercises any influence on the current world rate of interest (itself being equal to the common marginal products of capital which are uniquely determined by the world stock of capital), the home country knows that the future time path of interest rate depends on the capital accumulation in both countries, and it takes this into account in its consumption-saving decision. The assumptions described above are admittedly strong, but they are not too far fetched.

The techniques we use in this paper to determine the optimality of debt cycles are drawn from Hopf bifurcation theory. This theory has been successfully applied to problems in economic theory; see for example Benhabib (1978), Benhabib and Nishimura (1978), Kemp and Long (1988) and (1989). That optimal policies may display cyclical behaviour is not an altogether surprising result. Ryder and Heal (1973) show that if preferences are intertemporally dependent, then consumption cycles may be optimal (depending on parameter values). It has been observed that many families spend a large amount of money for holiday every few years. The utility flow derived from the holiday is experienced not only during the holiday but also before it (anticipation) and after it (reminiscence). The intuition behind the optimality of debt cycles is different. Here we must recall that the home county cannot fully control the rate of capital accumulation in the foreign country; in particular the marginal product of capital in the steady state depends on the propensity to save of the capital owners in the foreign country. This lack of full control may make it optimal to orbit around the steady state.

II. The Model

We envision a world with two countries: country 1 (the home country) and country 2 (the foreign country). Each country produces an output using capital and labour. The two goods are perfect substitutes. There are no impediments to international capital flow. It follows that the marginal products of capital are equalized:

$$F_K(K_1,L_1) = F_K(K_2,L_2) \tag{1}$$

Where

K_i = capital stock employed in country i
L_i = country i's labour force.

For simplicity we assume that the production functions are identical and have all the usual neoclassical properties.

Labour is internationally immobile. We assume that labour grows at a constant rate:

$$dL_i/dt = nL_i \qquad n > 0 \tag{2}$$

In both countries the factor markets are perfectly competitive, so that the wage rate is equated to the marginal product of labour and the interest rate is the marginal product of capital.

Let A_i denote country i's stock of net wealth. Clearly the sum $A_1 + A_2$ must equal the world stock of capital, $K_1 + K_2$. To fix ideas, we shall interpret all capital movements as loans rather than direct investments (even though this distinction is of no consequence in our model). The difference $A_1 - K_1$ represents country 2's net indebtness to country 1. Note that

$$A_1 - K_1 = K_2 - A_2 \,. \tag{3}$$

In each country, national income is the value of output minus the interest on its debts:

$$Y_1 = F(K_1,L_1) + (A_1 - K_1)r \tag{4a}$$

$$Y_2 = F(K_2,L_2) - (A_1 - K_1)r \tag{4b}$$

where r is the interest rate.

Let R_2 denote the income of country 2's capital owners:

$$R_2 = rA_2 \tag{5}$$

(We assume that $A_2 \geq 0$ but we allow A_2 to be less than K_2, indicating that the foreign country is the net debtor in that case). We adopt the assumption that in the foreign country only capital owners save. Their saving ratio is a constant s. Workers in the foreign country consume all their income. Therefore the rate of change in country 2's net wealth is

$$\dot{A}_2 = srA_2 \tag{6}$$

Under the assumptions of constant returns to scale and equal population sizes, we have, from (1) and (3),

$$r = f'(k_1) = f'(k_2) = f'[(a_1 + a_2)/2] \tag{7}$$

where $k_i = K_i/L_i$, and f(.) is the per capita production function, and $a_i = A_i/L_i$. In other words, each country uses half of the world's capital stock. (If the assumption of equal population sizes is dropped, equation (7) will have to be modified, but there will be no essential changes to our result.)

In per capita form, equation (6) becomes

$$\dot{a}_2 = sf'[(a_1 + a_2)/2]a_2 - na_2 \tag{8}$$

In the home country, we have the identity that the increase in per capita wealth equals per capita savings minus na_1:

$$\dot{a}_1 = f[(a_1 + a_2)/2] + [(a_1 - a_2)/2]f' - c - na_1 \tag{9}$$

where c is the home country's per capita consumption, and $(a_1 - a_2)/2$ represents the amount of wealth that the home country invests in the foreign country - given the assumption that marginal products of capital are equalized across countries. The expression $(a_1 - a_2)/2$ can be negative, in that case the home country is the debtor.

Notice that in our model, the foreign country may borrow for investment, but never for consumption. This is because in the foreign country, workers consume all their income and capitalists always save a fraction of their income. In the home country, however, consumption may exceed income over some time interval. In fact, as can be seen from equation (9), a_1 may become negative if c is sufficiently large over a sufficiently long period of time. We must therefore impose the restriction that in the limit, if the home country's net wealth (a_1) is negative, its output must be at least sufficient to service its debt:

$$\limsup_{t \to \infty} \{f[(a_1 + a_2)/2] - [(a_2 - a_1)/2]f'[(a_1 + a_2)/2]\} \geq 0 \qquad (10)$$

This constraint, together with the condition that the world's stock of capital is non-negative $(k_1 + k_2 = a_1 + a_2 \geq 0)$ imply that the home country cannot increase its per capita debt indefinitely.

We now add the usual assumptions that the utility function is strictly concave and increasing and that the marginal utility of consumption tends to infinity as consumption tends to zero. These, together with our earlier assumptions on the production functions, enable us to concentrate on interior solutions.

To summarize, the home country's problem is to find a time path of consumption, c(t), that maximizes its discounted stream of utility:

$$\text{Maximize} \int_0^{\infty} L_1(t)u[c(t)]\exp[-\delta]dt, \tag{11}$$

Since $L_1(t) = L_1(0)\exp(nt)$, we can write (11) as

$$\text{Maximize} \int_0^{\infty} u(c)\exp(-\rho t)dt, \qquad \rho \equiv \delta - n \tag{11'}$$

subject to (8), (9), (10) and the initial conditions

$$a_i(0) = a_{i0} \qquad (i = 1,2). \tag{12}$$

In what follows, we assume that $\delta - n > 0$.

The Hamiltonian for this problem is

$$H = u(c) + p_1[f(0.5a_1 + 0.5a_2) + 0.5(a_1 - a_2)f' - c - na_1]$$

$$+ p_2[sf'(0.5a_1 + 0.5a_2) - n]a_2 \tag{13}$$

Where p_1 and p_2 are the co-state variables, or the shadow prices of the stocks of wealth of the two countries.

At each instant of time, for given values of $p_1(t)$, $p_2(t)$, $a_1(t)$ and $a_2(t)$, consumption must be chosen to maximize the Hamiltonian. Thus

$$\partial H/\partial c = u'(c) - p_1 = 0 \tag{14}$$

From (14), we have

$$u''(c)(dc/dp_1) = 1$$

Therefore c is a decreasing function of p_1:

$$c = \phi(p_1) \tag{15}$$

with

$$dc/dp_1 = \phi'(p_1) = 1/u''(c) < 0 \tag{16}$$

The other necessary conditions for problem (11) are

$$\dot{a}_1 = \partial H/\partial p_1 = f(0.5a_1 + 0.5a_2) + 0.5(a_1 - a_2)f' - c - na_1$$

$$= f(0.5a_1 + 0.5a_2) + 0.5(a_1 - a_2)f' - \phi(p_1) - na_1 \tag{17}$$

$$\dot{a}_2 = \partial H/\partial p_2 = [sf'(0.5a_1 + 0.5a_2) - n]a_2 \tag{18}$$

$$\dot{p}_1 = \rho p_1 - (\partial H/\partial a_1)$$

$$= p_1[\rho + n - f' - 0.25(a_1 - a_2)f''] - 0.5p_2a_2sf'' \tag{19}$$

$$\dot{p}_2 = \rho p_2 - (\partial H/\partial a_2)$$

$$= p_2[\rho + n - 0.5sa_2f'' - sf'] - 0.25p_1(a_1 - a_2)f'' \tag{20}$$

From equation (18), it is clear that if the initial value of a_2 is positive then a_2 can never become negative. It is possible that in a steady state $a_2 = 0$. However, we shall ignore this singular case and concentrate on cases where the foreign country's net wealth is strictly positive (even though its debt may be positive or negative). We now seek to determine the necessary and sufficient condition for a steady state value $\bar{a}_2 > 0$. From (19), setting $\dot{p}_1 = 0$, we obtain:

$$p_2/p_1 = [n + \rho - f' - 0.25(\bar{a}_1 - \bar{a}_2)f'']/[0.5\bar{a}_2sf''] \tag{21}$$

Similaryly, from (20)

$$p_2/p_1 = [0.25(\bar{a}_1 - \bar{a}_2)f'']/[\rho - 0.5\bar{a}_2sf''] \tag{22}$$

Equations (21) and (22) yield

$$0.25\rho(\bar{a}_1 - \bar{a}_2) = (\rho + n - f')(\rho - 0.5\,\bar{a}_2sf'')/f'' \tag{23}$$

On the other hand, if $\bar{a}_2$ is positive in a steady state, then setting
$\dot{a}_2 = 0$ in (18), we have

$$f'(0.5\bar{a}_1 + 0.5\bar{a}_2) = n/s \tag{24}$$

This determines a unique steady state capital labour ratio $\bar{k}$, where $f'(\bar{k}) = n/s$, and $\bar{a}_1 + \bar{a}_2 = 2\bar{k}$ in the steady state. Noting that $\bar{a}_1 - \bar{a}_2 = (\bar{a}_1 + \bar{a}_2) - 2\bar{a}_2 = 2\bar{k} - \bar{a}_2$, equation (23) can be simplified to

$$\bar{a}_2 = \rho[\rho + n - (n/s) - 0.5\bar{k}f''(\bar{k})]/[(1 - s)(n + \rho)(-0.5f'')] \tag{25}$$

Hence $\bar{a}_2$ is in fact positive provided that the numerator of (25) is positive. Let us define the elasticity of marginal product of capital, evaluated at $\bar{k}$, as

$$\eta = -\bar{k}f''(\bar{k})/f'(\bar{k}) = -s\bar{k}f''(\bar{k})/n \tag{26}$$

Then, from (25), $\bar{a}_2$ is positive if and only if

$$\delta \equiv \rho + n > (1 - 0.5\eta)n/s \tag{27}$$

Having solved for $\bar{a}_2$, we can find $\bar{a}_1$, from the identity
$\bar{a}_1 + \bar{a}_2 = 2\bar{k}$. Steady-state consumption can then be solved using (17):

$$\bar{c} = f(\bar{k}) + (\bar{k} - \bar{a}_2)(n/s) - n(2\bar{k} - \bar{a}_2) \tag{28}$$

Therefore, provided that $f(\bar{k})$ is sufficiently large to ensure that $\bar{c} > 0$, we have identified a unique interior steady state. In addition, from (23) and (24), the home country is the net lender ($\bar{a}_1 - \bar{a}_2 > 0$) if and only if

$$\delta \equiv n + \rho < n/s \tag{29}$$

The results obtained in the preceeding analysis can be summarized in the following proposition.

<u>Proposition 1:</u> A unique steady state exists with $\bar{a}_2 > 0$ provided that (27) holds and $f(\bar{k})$ is sufficiently large to ensure that (28) is strictly positive. At the steady state the home country is the net lender if and only if rate of time preference is sufficiently small so that (29) holds.

The intuition behind Proposition 1 is easy to grasp. If the saving rate of foreign capital owners is low, then the marginal product of capital in the steady state will be high, and if the latter exceeds the home country's rate of time preference δ, then clearly it is optimal for the home country to be the net lender in the steady state.

The loan the home country offers to the foreign country is

$$L \equiv \bar{a}_1 - \bar{k} = \bar{a}_1 - (\bar{a}_1 + \bar{a}_2)/2 = (\bar{a}_1 - \bar{a}_2)/2 \qquad (30a)$$

Therefore (23) can be written as

$$(f'')\rho L/[2(\rho - 0.5a_2 sf'')] = (\rho + n) - f' = \delta - n/s \qquad (30b)$$

Rearranging (30b) to obtain

$$n/s - [\rho L(-f'')]/[2\rho + \bar{a}_2 s(-f'')] = \delta \qquad (30c)$$

We may interpret the left side of (30c) as long run marginal revenue and the right side (= δ) as long run marginal opportunity cost. Compare this with the more familiar static expression

$$r(L) + r'(L)L = MC \qquad (30d)$$

where $r'(L) < 0$ (if we lend more, we depress the interest rate). Instead of $r'(L)$, in (30c) we have a more complicated expression,

$$- [\rho L(-f'')]/[2\rho + \bar{a}_2 s(-f'')] \ .$$

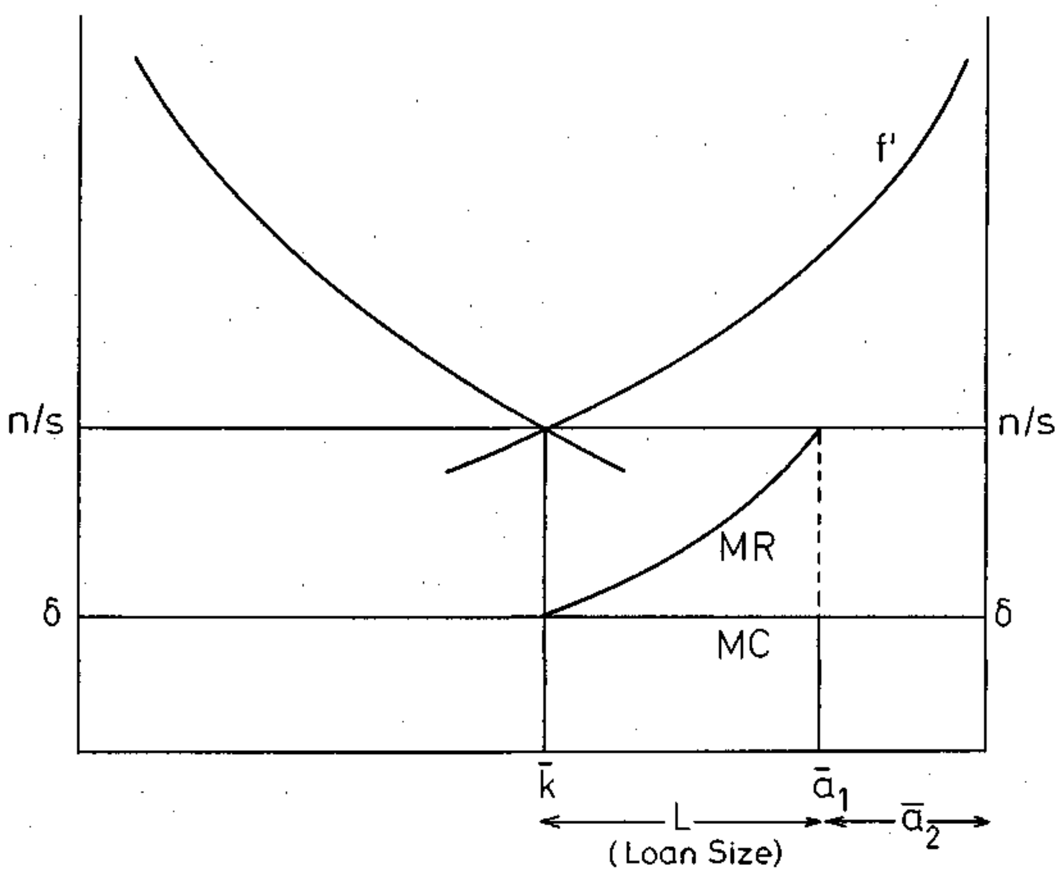
This is because we depress the interest rate only <u>over time</u>, as capitals are accumulated in both countries. In the foreign country, the key variable in accumulation is the saving rate (s) and in the home country, the key parameter is ρ.

The rule "long run marginal revenue must equal long run marginal opportunity cost", as stated by equation (30c), can be illustrated with the help of Figure 1. In this Figure, the two marginal product curves are drawn back to back. Their intersection determines $\bar{k}$, where both marginal products are n/s. The horizontal line δ represents long run marginal opportunity cost. The long run marginal revenue (MR) curve is drawn starting from $\bar{a}_1$, at the heights n/s, because at $\bar{a}_1$, L = 0 and the left side of (30c) collapses to n/s.

The intersection of MR and MC determines the loan size L^*. Note that $\bar{a}_2$ is given by (25), and can be written as

$$\bar{a}_2 = 2\rho[\delta - (1 - 0.5\eta)(n/s)]s\bar{k}/[\delta(1 - s)n\eta]$$

Figure 1



3. The Existence of Debt Cycles

In order to study the question of existence of debt cycles around the steady state, we shall make use of Hopf Bifurcation Theorem, which we reproduce below for ease of reference. This theorem has been applied to economic problems by several authors, including Benhabib (1978), Benhabib and Nishimura (1979), Kemp and Long (1988 and 1989).

Theorem (Hopf): Let $\dot{x} = F(x,\mu)$ be a system of differential equations, where x is a vector of $2n$ variables, μ is a real parameter, and $F(x,\mu)$ analytic in a domain G for x and μ is bounded. Let there be a steady-state solution $\bar{x}(\mu)$ such that $F(\bar{x}(\mu),\mu) = 0$. For $\mu = 0$, let the Jacobian matrix of F have one pair of pure imaginary roots $a(\mu) \pm b(\mu)i$, $a(0) = 0$, $b(0) \neq 0$, and $da(\mu)/d\mu \neq 0$ at $\mu = 0$. Let μ be parameterized as $\mu = \mu(\epsilon)$, with $\mu(0) = 0$. Then there exists a family of periodic solutions $x = x(t,\epsilon)$ with the properties that $x(t,0) = \bar{x}(\mu(0))$ and that, for all sufficiently small ϵ, $x(t,\epsilon)$ is distinct from the steady-state solution $\bar{x}(\mu(\epsilon))$.

In our problem, the four differential equations (17), (18), (19), and (20) constitute the system $F(x,\mu)$ of the theorem. Our task therefore is to find parameter values such that the Jacobian matrix of the system (17) - (20) has a pair of pure imaginary roots and such that the real parts of those roots change sign at those values.

Linearizing the system (17) - (20) about the steady state ($\bar{a}_1$, $\bar{a}_2$, $\bar{p}_1$, $\bar{p}_2$), we obtain

$$\begin{bmatrix} da_1/dt \\ da_2/dt \\ dp_1/dt \\ dp_2/dt \end{bmatrix} = J \begin{bmatrix} a_1 - \bar{a}_1 \\ a_2 - \bar{a}_2 \\ p_1 - \bar{p}_1 \\ p_2 - \bar{p}_2 \end{bmatrix} \tag{31a}$$

where J is the Jacobian matrix with the following pattern

$$J = \begin{bmatrix} A & B & C & D \\ E & F & D & H \\ K & M & \rho-A & -E \\ M & N & -B & \rho-F \end{bmatrix} \tag{31b}$$

As shown by Kemp and Long (1988), the pattern given by (31b) is common to all Jacobian matrices of optimal control problems with two state variables and a positive discount rate. For our problem, the entries of matrix J are

$$A = \partial\dot{a}_1/\partial a_1 = (n/s) - n + 0.25(a_1 - a_2)f'' \tag{32a}$$

$$B = \partial\dot{a}_1/\partial a_2 = 0.25(a_1 - a_2)f'' \tag{32b}$$

$$C = \partial\dot{a}_1/\partial p_1 = -\varnothing'(p_1) = -1/u''(c) \tag{32c}$$

$$D = \partial\dot{a}_1/\partial p_2 = \partial\dot{a}_2/\partial p_1 = 0 \tag{32d}$$

$$E = \partial\dot{a}_2/\partial a_1 = 0.5a_2sf'' \tag{32e}$$

$$F = \partial\dot{a}_2/\partial a_2 = 0.5a_2sf'' \tag{32f}$$

$$H = \partial\dot{a}_2/\partial p_2 = 0 \tag{32g}$$

$$K = \partial\dot{p}_1/\partial a_1 = -p_1[0.75f'' + 0.5(f'''/f'')(n - n/s + \rho)] \tag{32h}$$

$$M = \partial\dot{p}_1/\partial a_2 = \partial\dot{p}_2/\partial a_1$$

$$= p_1[0.25(a_1 - a_2)(f''/a_2) - (n - n/s + \rho)(1/a_2)$$

$$-0.25f'' - 0.5(f'''/f'')(n - n/s + \rho)] \tag{32i}$$

$$N = \partial\dot{p}_2/\partial a_2 = p_1(f'')[0.25 + 0.5(a_1 - a_2)(1/a_2)]$$

$$-p_1(n - n/s + \rho)[0.5(f'''/f'') + (2/a_2)] \tag{32j}$$

As shown by Kemp and Long (1988, Appendix 1), the four roots of a matrix having the pattern (31b) must have the following form

$$\lambda_{1,2,3,4} = (\rho/2) \pm [(\rho/2)^2 - (W/2) \pm 0.5(W^2 - 4 \det J)^{\frac{1}{2}}]^{\frac{1}{2}} \tag{33}$$

where W is defined as

$$W = -A^2 - F^2 - 2BE - 2DM + \rho A - \rho F - CK - NH \tag{34}$$

Kemp and Long (1988) also showed that if $W^2 - 4 \det J < 0$ then the roots are

$$\lambda_{1,2,3,4} = (\rho/2) \pm \alpha \pm \beta i \tag{35}$$

where α and β are real numbers (see Appendix for details). It follows that a pair of pure imaginary roots exist if parameter values can be found such that $\alpha = \rho/2$.
We now provide such a set of parameter values. Let x be any arbitrary positive number, and let

$$s = 0.2, \quad \rho = 1x, \quad n = 0.5x, \quad f(3.2) = 10x, \quad f(0) = 0,$$

$$f'(3.2) = 2.5x, \quad f''(3.2) = -x. \tag{36}$$

It is easy to check that there exists a unique steady state with

$$\bar{a}_1 = 5.4, \quad \bar{a}_2 = 1, \quad \bar{k} = (\bar{a}_1 + \bar{a}_2)/2 = 3.2 \;, \quad \bar{c} = 12.8x \tag{37}$$

In (36) and (37), the constant x reflects the choice of the unit of measurement of time, therefore x appears in all flow variables, but not in stock variables or in variables representing ratios. Thus if the rate of population growth is 5 per cent per year, then we set x = 0.10 so that n = 0.05.

The choice of unit of measurement of time has no effects on the equilibrium nor on its stability properties. The roots of the Jacobian matrix depends also on $\tau \equiv -U''/U'$ and on f'''. Let us set x = 1 for simplicity of computation, and choose τ and f''' such that

$$\det J = 1.3152 \tag{38a}$$

and

$$W = 1.50 \tag{38b}$$

From the Appendix, it is clear that to satisfy (38), we must have

$$\tau = 0.0458 \equiv \bar{\tau} \tag{39a}$$

$$f''' = 1.8400 \equiv \bar{f}''' \tag{39b}$$

It can be verified that (36), (37), and (39), with x = 1, yield a pair of pure imaginary roots and a pair of complex roots with positive real parts:

$$\lambda_{1,2} = 1 \pm i(0.5\sqrt{3}) \tag{40a}$$

$$\lambda_{3,4} = 0 \pm i(0.5\sqrt{3}) \tag{40b}$$

Furthermore, the real parts of (40b) is zero only at $\tau = \bar{\tau}$. In the neighborhood of $\bar{\tau}$, it is an increasing function of τ. For $\tau < \bar{\tau}$, we have two pair of complex roots, one pair with positive real part and one pair with negative real part. For $\tau > \bar{\tau}$, all four roots are complex with positive real parts. Applying Hopf Bifurcation Theorem, the existence of closed orbits (regular cycles) around the steady state is proven. To the left of $\bar{\tau}$, we have damped cycles, converging to the steady state. Figures 2 and 3 illustrate these two cases.

Figure 2 - Regular Cycles

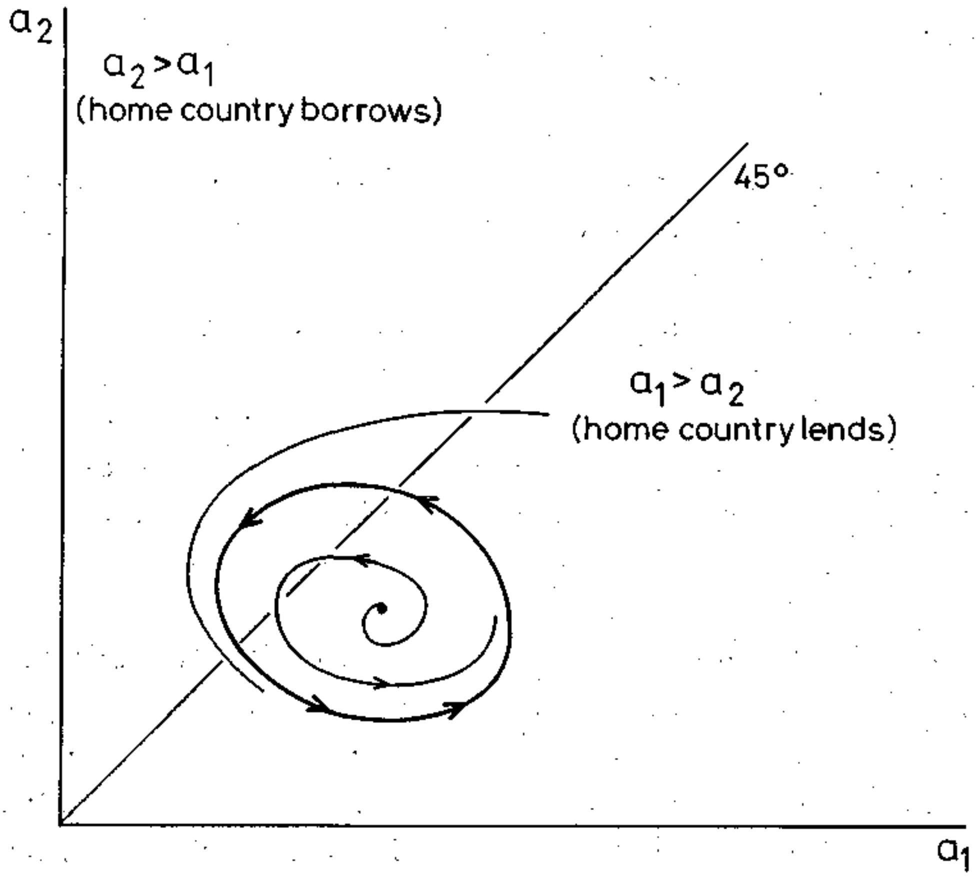
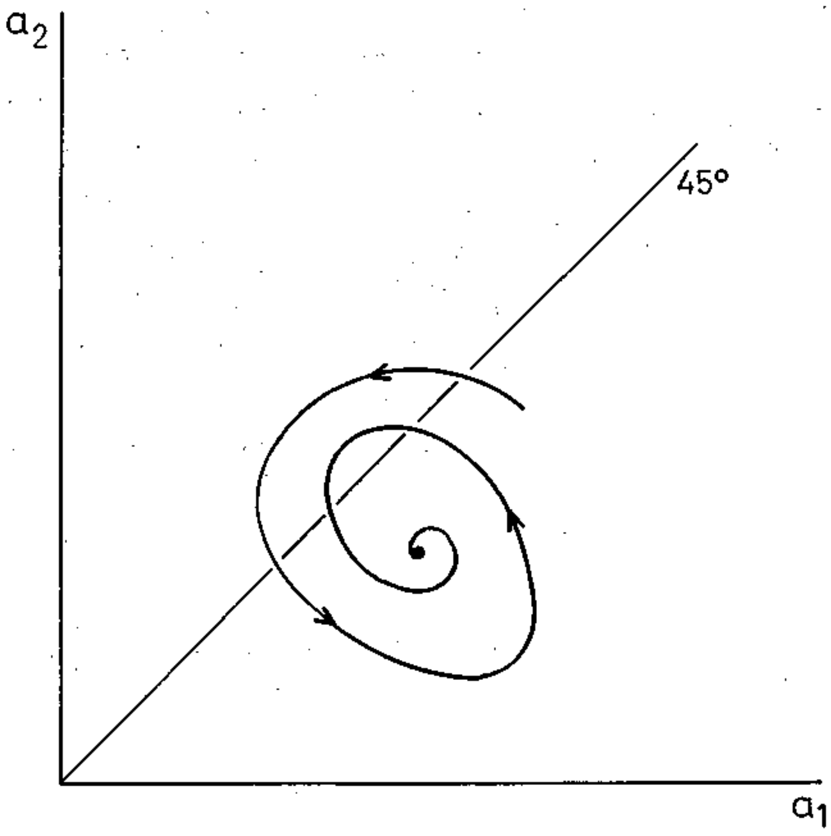


Figure 3 - Damped Cycles



4. Conclusion

We have shown that debt cycles may be optimal for the home country if the capital owners in the foreign country save a constant proportion of their income and workers do not save. This classical saving assumption was adopted because it simplifies computation. It seems likely that if we replace this saving rule by the assumption that in the foreign country a constant fraction of national income is saved, we would also obtain parameter values under which it is optimal for the home country to follow a cyclical policy with regards to foreign investment.

It is interesting to note that cycles also exist in predator-prey models under certain assumptions on parameter values. If the predators were intertemporal maximizers, cycles perhaps would still exist. In fact, our model is a kind of predator-prey model, because the home country actually manipulates the intertemporal terms of trade by changing its supply of capital over time, and because the rate of growth of each country's wealth depends on the wealth of the other country. Finally, it should be noted that in our model we have deliberately assumed that at any instant of time the marginal products of capital are equalized (thus ruling out taxes on foreign investment[1]). This assumption was made in order to isolate long run considerations from short run policy instruments. Furthermore, taxes on foreign investment may provoke retaliation from the foreign country. Our result, that debt cycles may exist in the long run, would remain valid even when complications such as differential taxes on capital income are admitted.

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1 For a sample of literature on taxes on foreign investment, see Kemp (1962), Long (1973), Ruffin (1985), and Manning and Shea (1989).

Appendix

The roots of the Jacobian matrix J

Let

$$W = -A^2 - F^2 - 2BE - 2DM + \rho A + \rho F - CK - NH \tag{A.1}$$

The matrix J has a pair of pure imaginary roots only if

$$W^2 - 4 \det J < 0 \tag{A.2}$$

(see Kemp and Long (1988), Appendix 1). In our case, det J is given by

$$\det J = AF\rho^2 - AF^2\rho - A^2F\rho + A^2F^2 - 2ABEF - BE\rho^2 + BEF\rho + ABE\rho$$
$$+ B^2E^2 + CEM\rho - 2CEFM + CE^2N - CFK\rho + CF^2K \tag{A.3}$$

As shown in Kemp and Long (1988), the roots can be calculated as follows:

Let $V = (\rho/2)^2 - (W/2)$ (A.4)

$$Y = 0.5(4 \det J - W^2)^{\frac{1}{2}} \tag{A.5}$$

$$R = (X^2 + Y^2)^{\frac{1}{2}} \tag{A.6}$$

Note that Y > 0 because of (A.2)

Define Θ by

$$\cos \Theta = V/R, \quad \sin \Theta = Y/R \tag{A.7}$$

Then the roots of J are

$$\lambda_{1,2,3,4} = (\rho/2) \pm \alpha \pm \beta i \tag{A.8}$$

where

$$\alpha = R^{\frac{1}{2}}\cos(\Theta/2), \ \beta = R^{\frac{1}{2}}\sin(\Theta/2) \tag{A.9}$$

Referring to (36) and (37) and the paragraph that follows (37), it can be verified from (A.1) - (A.9) that the stability properties of the equilibrium are independent of the choice of the unit of measurement of time: if x is multiplied by a constant, then all the four roots are multiplied by the same constant. Therefore we may set x = 1 without loss of generality.

It is clear from (A.8) and (A.9) that a necessary and sufficient condition for a pair of pure imaginary roots is $\alpha = \rho/2$. Therefore we must find parameter values and a value for Θ such that the following equations are satisfied:

$$\cos(\Theta/2) = (\rho/2)R^{-\frac{1}{2}} \tag{A.10}$$

$$\cos\Theta = [(\rho/2)^2 - (W/2)]R^{-1} \tag{A.11}$$

Clearly if R = 1, $\Theta = 2\pi/3$, $\rho = 1$ and W = 1.50 then (A.10) and (A.11) are satisfied, with det J = 1.3125. Having specified $\rho = 1$, n = 0.5, f'= 0.5, s= 0.2, f''= -1, we can find values of f'''and τ ($\equiv -u''/u'$) that yield W = 1.50 and det J = 1.3125. These values are 1.8400 and 0.0458 respectively.

It remains to show that the real parts of the pure imaginary roots change sign as τ crosses the value 0.0458. We only need to show that $d\alpha/d\tau$ is not zero. From (A.9),

$$d\alpha/d\tau = 0.5R^{-\frac{1}{2}}\cos(\Theta/2)(dR/d\tau) - 0.5R\sin(\Theta/2)(d\Theta/d\tau) \tag{A.12}$$

Let

$$z = \cos\Theta = V/R$$

Then

$$d\Theta/d\tau = (d\Theta/dz)(dz/d\tau) = (-\sin\Theta)^{-1}[R(dV/d\tau) - V(dR/d\tau)]R^{-2}$$

Some simple calculations show that $d\alpha/d\tau > 0$ which is what we set out to prove.

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