Long, Ngo Van; Siebert, Horst

Working Paper

Optimal foreign borrowing: the impact of the planning horizon on the half and full debt cycle

Kiel Working Papers, No. 348

Provided in cooperation with:
Institut für Weltwirtschaft (IfW)

Kieler Arbeitspapiere
Kiel Working Papers

Kiel Working Paper No. 348

OPTIMAL FOREIGN BORROWING: THE IMPACT OF THE PLANNING HORIZON ON THE HALF AND FULL DEBT CYCLE

by

Ngo van Long and Horst Siebert

Januar 1989

Institut für Weltwirtschaft an der Universität Kiel
The Kiel Institute of World Economics

ISSN 0342-0787

Institut für Weltwirtschaft
2300 Kiel, Düsternbrooker Weg 120

Kiel Working Paper No. 348

OPTIMAL FOREIGN BORROWING: THE IMPACT OF THE PLANNING HORIZON ON THE HALF AND FULL DEBT CYCLE

by

Ngo van Long and Horst Siebert

Januar 1989

Januar 1989

ISSN 0432-0787

OPTIMAL FOREIGN BORROWING: THE IMPACT OF THE PLANNING HORIZON ON THE HALF AND FULL DEBT CYCLE

NGO VAN LONG
AND HORST SIEBERT

1. Introduction

Foreign debt can be incurred for capital accumulation and for consumption. It is worth while to borrow for capital accumulation when the marginal productivity of capital is greater than the interest rate. And it is worthwhile to borrow for consumption, if the time preference rate exceeds the interest rate. If capital can be built instantaneously, the amount to be borrowed is completely determined by the condition for the optimal capital stock; a parametric change of a fixed planning horizon has no impact on the amount of debt incurred for capital accumulation.

The planning horizon has, however, an impact on the time profile of consumption and consequently on the time profile of debt incurred for consumption. It will be shown that if the rate of time preference exceeds the rate of interest, then, under some reasonable assumption about the relative magnitude of the prescribed terminal stock, a lengthening of the time horizon will lead to a higher consumption stream in some initial phase, and foreign debt will accumulate at a faster rate. In this case, debt incurred will be repaid in terminal time or not fully be repaid in infinite time, so that the country remains a debtor and never reaches a creditor position. We call this the half debt cycle.

Conversely, if the rate of interest is higher than the rate of time preference, then the effect of an increase in the time horizon is a fall in the initial consumption flow, provided the prescribed terminal stock is not too large. In addition, a debtor country may turn into a creditor country for some time interval. This is the case of a full debt cycle (World Bank 1985,p.47). This holds for finite time being sufficiently large and for infinite time. A second switch back to the borrower status is possible.

In a recent paper, Siebert (1987) considered the problem of foreign borrowing and capital accumulation, and focussed on the case of a fixed, finite time horizon T, with a zero terminal capital stock. We generalize Siebert's results in two directions: firstly, a prescribed terminal capital stock $\hat{K}$ is added as a terminal constraint, and secondly we investigate the effect of lengthening the time horizon.

2. The Model

The economy uses capital, K, and labour, L, to produce an output

$$Q = F(K,L).$$

We assume full employment and a constant population, so that we can write

$$Q = F(K). \qquad (1)$$

The function F(K) satisfies the usual properities:

$$F(0) = 0, \ F_K(K) > 0, \ F_{KK}(K) < 0.$$

The stock of capital depreciates at rate m. Let I denote gross investment. Then

$$K = I - mK. \qquad (2)$$

The stock of debt is denoted by B. If B is negative, it denotes a creditor position. Consumption equals gross output, minus gross investment minus interest payments plus additional borrowing:

$$C = Q - I - rB + \dot{B}. \tag{3}$$

Rearrange (3), making use of (1), to obtain

$$\dot{B} = C + I + rB - F(K). \tag{4}$$

The initial stock of debt is zero and the initial stock of capital is K_0 which is historically given. We allow instantaneous accumulation (or "jump" in K) financed by borrowing. This is possible because we place no bound on gross investment. The initial jump in capital from K_0 to say $K(0^+)$ is paid for by an equal jump in debt from 0 to $B(0^+)$. We allow $B(0^+)$ to exceed $K(0^+) - K_0$, so as to allow a jump in debt to finance borrowing for consumption. (However, it will be shown that $B(0^+) = K(0^+) - K_0$, because it is not optimal to finance additional consumption by a sudden jump in B). At this stage, for the sake of generality, we write

$$B(0^+) \geq K(0^+) - K_0.$$

Note that $K(0^+)$ is to be determined optimally.
At the terminal time T, the stock of net wealth is $K(T) - B(T)$. We require that terminal net wealth must be equal or exceed some specified level $\hat{K}$ [1].

$$K(T) - B(T) \geq \hat{K}.$$

The planning authority wishes to maximize the present value of the stream of utility W(C):

$$\text{Maximize} \int_0^T e^{-\delta T} W(C)\, dt$$

subject to (2), (4) and

i) $K(0^+)$, $B(0^+)$ to be chosen, subject to the constraint

$$B(0^+) = K(0^+) - K_0 \qquad (5)$$

ii) $K(T)$ and $B(T)$ to be determined, subject to

$$K(T) - B(T) \geq \hat{K} \qquad (6)$$

K_0, $\hat{K}$ and T are exogenously specified.

The Lagrangean for this problem is

$$Z = W(C) + \lambda\,[I - mK] + \rho\,[C + I - F(K) + rB] \qquad (7)$$

The optimality conditions are

$$\frac{\partial Z}{\partial C} = 0 \quad \Rightarrow \quad W_C = -\rho \qquad (7a)$$

$$\frac{\partial Z}{\partial I} = 0 \quad \Rightarrow \quad \lambda = -\rho \qquad (7b)$$

$$\dot{\lambda} = \delta\lambda - \frac{\partial Z}{\partial K} \quad \Rightarrow \quad \dot{\lambda} = \lambda(\delta + m) + \rho F_K(K) \qquad (7c)$$

$$\dot{\rho} = \delta\rho - \frac{\partial Z}{\partial B} \quad \Rightarrow \quad \dot{\rho} = \rho(\delta - r) \qquad (7d)$$

Define the J function as in Long and Vousden (1977) in order to obtain the transversality conditions:

$$J = \mu_1 \left(B(0^+) - K(0^+) + K_0\right) + \mu_2 \left(K(T) - B(T) - \hat{K}\right) \qquad (8)$$

then

$$-\frac{\partial J}{\partial B(0^+)} - \rho(0) = 0 \Rightarrow -\mu_1 - \rho(0) = 0 \qquad (8a)$$

$$-\frac{\partial J}{\partial K(0^+)} - \lambda(0) = 0 \Rightarrow \mu_1 - \lambda(0) = 0 \qquad (8b)$$

$$-\frac{\partial J}{\partial B(T)} + \rho(T) = 0 \Rightarrow \mu_2 + \rho(T) = 0 \qquad (8c)$$

$$-\frac{\partial J}{\partial K(T)} + \lambda(T) = 0 \Rightarrow -\mu_2 + \lambda(T) = 0 \qquad (8d)$$

$$\mu_1 \geq 0, \qquad \mu_1[B(0^+) - K(0^+) + K_0] = 0 \qquad (8e)$$

$$\mu_2 \geq 0\ , \qquad \mu_2[K(T) - B(T) - \hat{K}] = 0 \qquad (8f)$$

Combining (7a), (7b) and (8a) - (8f), we have

$$\lambda(T) = -\rho(T) = W_C\ (C(T)) > 0 \qquad (9a)$$

$$\lambda(0) = -\rho(0) = W_C\ (C(0)) > 0 \qquad (9b)$$

Therefore at the optimum

$$B(0^+) = K(0^+) - K_0 \qquad (10a)$$

$$K(T) - B(T) = \hat{K} \qquad (10b)$$

From (7b), (7c) and (7d), we obtain

$$F_K(K) - m = r \qquad (11)$$

Therefore $K^*(t)$ is a constant. It is the stock size at which the net marginal product of capital equals the rate of interest. It follows that

$$K(0^+) = K^* = K(T) \tag{12}$$

If $K_0 < K^*$, conditions (10a) and (12) imply that the country immediately raises its capital stock to K^* by borrowing the amount $B(0^+) = K^* - K_0$.

If $\hat{K} < K^*$, then from (10b) and (12) we know that

$$B(T) = K^* - \hat{K} > 0$$

We may interpret $B(T^+) = 0$ and $K(T^+) = \hat{K}$ as implying that the economy repays its debt at the end of the planning horizon, by affecting a downward jump in its capital stock, from $K(T^-) = K^*$ to $K(T^+) = \hat{K}$.

3. The Consumption Path

We are now in a position to determine the consumption path: From (7a) and (7d)

$$-\frac{W_{cc}\,C}{W_c}\quad\frac{\dot{C}}{C} = -\dot{\rho}\,/\,\rho = -(\delta - r)$$

Defining the elasticity of marginal utility $\eta = -\frac{W_{cc}\,C}{W_c} > 0$,

then

$$\dot{C}/C = -(\delta - r)/\eta \tag{13}$$

So C increases exponentially if r exceeds δ and decreases exponentially if $r < \delta$.

To proceed further, we assume that η is a constant. In other words, the function $W(C)$ is assumed to take the form

$$W(C) = \frac{C^{1-\eta}}{1-\eta} + A \quad , \quad \eta > 0, \ \eta \neq 1, \quad A = \text{constant}$$

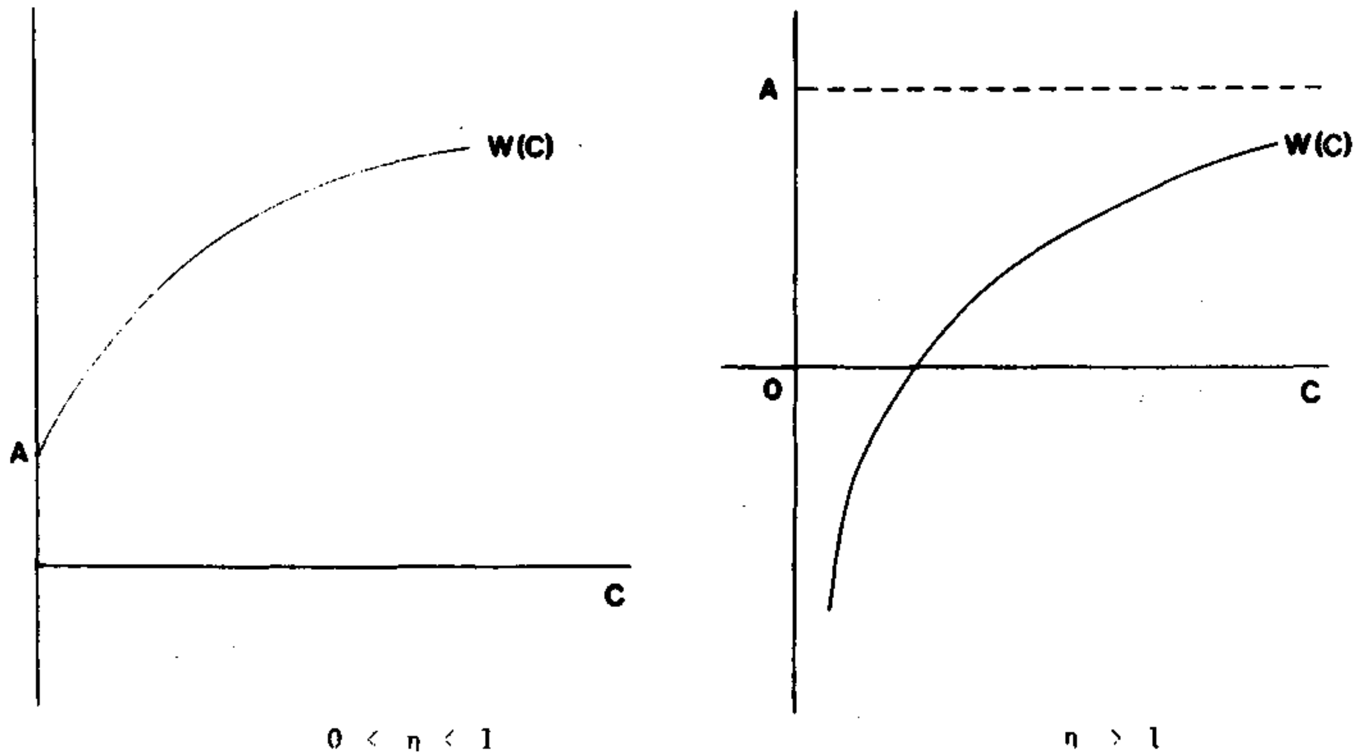

Figure 1

If $\eta = 1$, then $W(C) = \ln C + A$. Since η is a constant, (13) yields

$$C(t) = C(0)e^{(r-\delta)t/\eta} \qquad (0 \leq t \leq T) \qquad (14)$$

In order to emphasize the fact that both $C(0)$ and $C(t)$ depend on the time horizon T, we write (14) as

$$C(t, T) = C(0, T)\ e^{(r-\delta)t/\eta} \qquad (0 \leq t \leq T) \qquad (15)$$

To solve for $C(0, T)$, we use the equation

$$\dot{B} - rB = C - \tilde{Q}, \qquad (16)$$

where $\tilde{Q} = F(K^*) - mK^*$ is net output.

Multiply both sides of (16) by e^{-rt}

$$e^{-rt}(\dot{B} - rB) = e^{-rt}(C(t, T) - \tilde{Q}) \qquad (17)$$

Integrate both sides from 0^+ to T

$$\int_{0^+}^{T}\left[\frac{d}{dt}(Be^{-rt})\right]dt = \int_{0^+}^{T} C(t, T)\, e^{-rt}\, dt - \tilde{Q}\int_{0}^{T^+} e^{-rt} \qquad (17a)$$

From (15)

$$\int_{0^+}^{T} C(t,T)\, e^{-rt}\, dt = C(0,T)\int_{0}^{T} e^{\left(-r + \frac{r-\delta}{\eta}\right)t}\, dt$$

$$= C(0,T)\int_{0}^{T} e^{-Dt}\, dt \qquad (17b)$$

where, by definition[2)]

$$D = r - \frac{r-\delta}{\eta}\,. \qquad (18)$$

We assume $D \neq 0$. Therefore, (17a) becomes

$$e^{-rT}B(T) - B(0^+)$$

$$= C(0,T)\left[\frac{1 - e^{-DT}}{D}\right] - \tilde{Q}\left[\frac{1 - e^{-rT}}{r}\right] \qquad (19)$$

Now since $B(0^+) = K^* - K_0$

and $B(T) = K^* - \hat{K}$

equation (19) becomes

$$C(0,T)\left[\frac{1 - e^{-DT}}{D}\right]$$

$$= \left[\frac{1 - e^{-rT}}{r}\right] \left[\tilde{Q} - r K^*\right] + K_0 - e^{-rT} \hat{K} \qquad (20)$$

The left-hand side of (20) is the present value of the consumption stream. Note that the term $[1 - e^{-DT}]/D$ is positive for all $D \neq 0$, because if $D < 0$, then $e^{-DT} > 1$, and if $D > 0$ then $e^{-DT} < 1$. Equation (20) implies that for consumption to be positive, the prescribed terminal net wealth $\hat{K}$ must not be too large.

We now offer an intuitive interpretation of equation (20). Assume for the moment that $K_0 = \hat{K} = 0$. Then the left-hand side of (20) is the present value of the stream of wage income [$\tilde{Q} - rK^*$ is the net output minus the payment to capital, which goes to foreigners to service debts, so $\tilde{Q} - rK^*$ represents wage income]. Hence (20) says that the present value of the consumption stream (the left-hand-side of (20)) must be equal to the present value of wage income, if $K_0 = \hat{K} = 0$. If $K_0 > 0$ and $\hat{K} = 0$, then the country can afford to consume more than its present value of wage income. If $\hat{K} > 0$, the country has to consume less, because of its commitment to the bequest of $\hat{K}$.

3.1 Comparative Statics

Our result concerning the effect of an increase in T as the consumption profile is:

Proposition 1: If $\delta > r$, then a lengthening of the time horizon will raise the consumption flow provided that $e^{-rT} \hat{K} \geq K_0$.

Proof: Rewrite (20) as

$$C(0,T) = (K_0 - e^{-rT}\hat{K})\, D\, (1 - e^{-DT})^{-1}$$

$$+ (\tilde{Q} - rK^*)\,(D/r)\,(1 - e^{-DT})^{-1}\,(1 - e^{-rT}) \qquad (21)$$

Differentiate (21) with respect to T:

$$\frac{\partial C(0,T)}{\partial T} = \frac{re^{-rT}\hat{K}\,D(1 - e^{-DT}) - D^2 e^{-DT}(K_0 - e^{-rT}\hat{K})}{(1 - e^{-DT})^2}$$

$$+ (\tilde{Q} - rK^*)\,\frac{\partial}{\partial T}\left[\frac{D}{r}\;\frac{(1 - e^{-rT})}{(1 - e^{-DT})}\right] \qquad (22)$$

We only need to show that $\frac{\partial C(0,T)}{\partial T} > 0$.

The second term on the right-hand side of (22) is positive, if and only if $\delta > r$ (see Appendix 1). The first term is positive if $K_0 \leq e^{-rT}\hat{K}$. Recall that $D\,(1 - e^{-DT})$ is always positive for $D \neq 0$.

End of Proof

3.2 Interpretation

To give an intuitive explanation of Proposition 1, let us first consider the simple case, where $K_0 = 0 = \hat{K}$. Then equation (20) says that the present value of the consumption stream must equal the present value of the stream of wage income. Since consumption falls over time (due to $\delta > r$), it follows that consumption exceeds wage income for some initial phase, and falls short of wage income for the remaining time. See Figure 2:

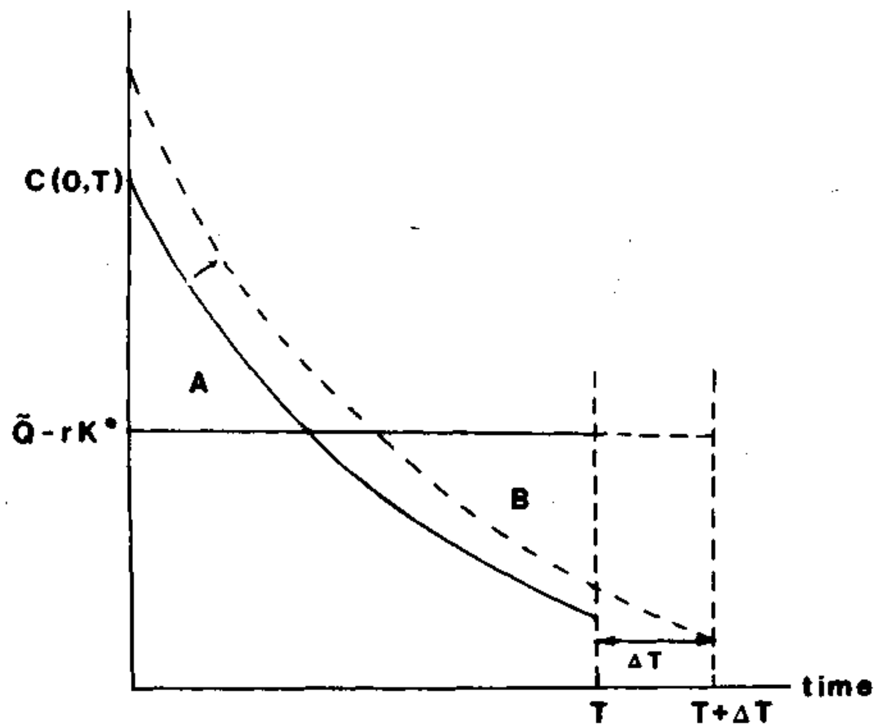

Figure 2

The present value of area B must equal the present value of area A. Now let the time horizon increase from T to (T + Δ T). We want to show that C(0, T + Δ T) is higher than C(0,T). The longer time horizon means that the present value of wage income stream (now extending from 0 to T + Δ T) will be higher, hence the country can afford to consume more. Therefore C(0,T + Δ T) > C(0,T). Since both paths follow the exponential rule, C(t,T + Δ) > C(t,T) for all t $\leq$ T. However, one can show that C(T + Δ T, T + Δ T) **is smaller than C(T,T).**

If $\hat{K} = 0$ and $K_0 > 0$, then a lengthening of T may not raise the consumption flow, because the initial wealth, K_0, has to be "used up" over a longer horizon. More generally, we interpret $e^{-rT}\hat{K} - K_0$ as the present value of net addition to wealth, to be given to future generations. A longer time horizon makes it possible to achieve this target less painfully.

Turning now to the case $\delta < r$, we have the corresponding result:

Proposition 2:
If $\delta < r$, then a lengthening of the time horizon will reduce the initial consumption flow, provided $e^{-rT}\hat{K}$ is sufficiently small relative to K_0.

The proof follows the same line of arguments used in the proof of Proposition 1. Figure 3 illustrates the consumption paths.

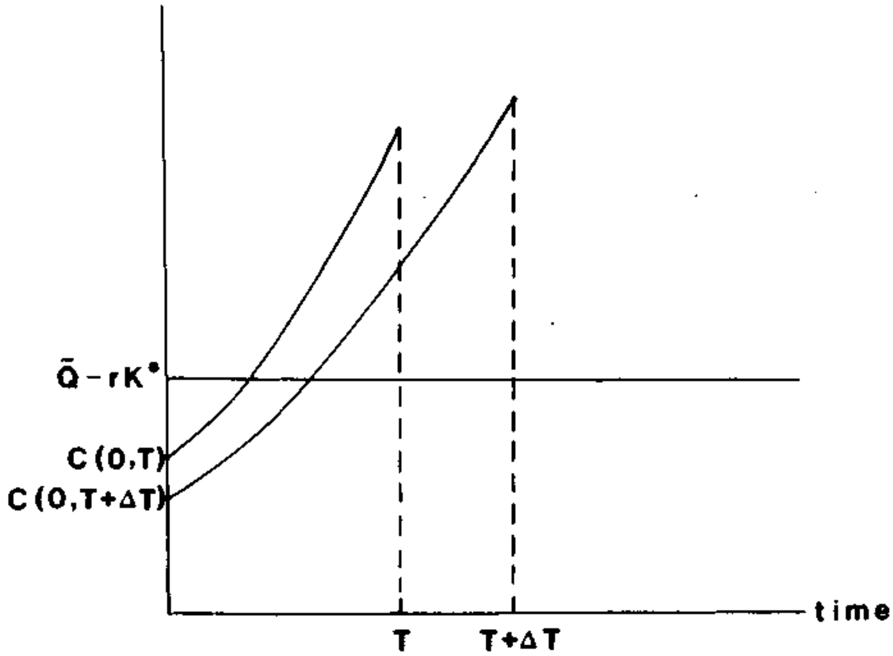

Figure 3

4. The Time Path of Debt

We now turn our attention to the time path of debt, and the effect of an increase in T of this path.

Consider first the case $\delta > r$. Given that the country has the target $\hat{K}$ to meet at time T, and that the present value of this target is $e^{-rT}\hat{K}$, it is clear that if this present value is not too large, then the country will want to borrow for consumption in some early periods, and that if $e^{-rT}\hat{K}$ is large, then the country may have to tighten its belt to meet the target, so that even though borrowing is in a sense attractive ($\delta > r$), the country cannot afford to do so. An increase in T, by postponing the date of meeting the target, would have a favourable effect on consumption, and allow debt to grow initially.

In order to verify the above intuitive arguments, we derive below the explicit time path of B(t) and examine its sensitivity with respect to the planning horizon.

For any fixed T, the time path of debt can be calculated using (17). Take the indefinite integral of (17):

$$Be^{-rT} = -\frac{C(0,T)\ e^{-Dt}}{D} + \frac{\tilde{Q}\ e^{-rt}}{r} + H \qquad (23)$$

where H is a constant of integration.

Multiply both sides by e^{rt}

$$B(t) = H\ e^{rt} + \frac{\tilde{Q}}{r} - \frac{C(0,T)}{D}\ e^{\left(\frac{r-\delta}{\eta}\right)t} \qquad (24)$$

At $t = 0^+$

$$B(0^+) = H + \frac{\tilde{Q}}{r} - \frac{C(0,T)}{D} \tag{25}$$

Hence $$H = B(0^+) - \frac{\tilde{Q}}{r} + \frac{C(0,T)}{D} . \tag{26}$$

Using (26) in (24) we obtain

$$B(t,T) = \left[B(0^+) - \frac{\tilde{Q}}{r} + \frac{C(0,T)}{D}\right]e^{rt}$$

$$+ \frac{\tilde{Q}}{r} - \frac{C(0,T)\, e^{\frac{(r-\delta)t}{\eta}}}{D} \tag{27}$$

But we have proved that $B(0^+) = K^* - K_0$. Hence

$$B(t,T) = \left[(K^* - K_0) - \frac{\tilde{Q}}{r} + \frac{C(0,T)}{D}\right]e^{rt}$$

$$+ \frac{\tilde{Q}}{r} - \frac{C(0,T)\, e^{\frac{(r-\delta)t}{\eta}}}{D} \tag{28}$$

We are now ready to prove the following result:

Proposition 3: If $\delta > r$, then $\dot{B} > 0$ for some initial phase, provided $e^{-rT}\hat{K}$ is not too large.

Proof: From (28),

$$\frac{dB}{dt} = \left[K^* - K_0 - \frac{\tilde{Q}}{r} + \frac{C(0,T)}{D}\right]re^{rt}$$

$$- \left(\frac{r - \delta}{\eta} \right) - \frac{C(0,T)\, e^{\frac{(r - \delta)t}{\eta}}}{D} \qquad (29)$$

At $t = 0$,

$$\left.\frac{dB}{dt}\right|_{t=0} = \left[(K^* - K_0) - \frac{\tilde{Q}}{r} + \frac{C(0,T)}{D} \right] r$$

$$- \left(\frac{r - \delta}{\eta} \right) \frac{C(0,T)}{D} \qquad (30a)$$

$$= \frac{C(0,T)}{D} \left[r - \frac{(r - \delta)}{\eta} \right] - \tilde{Q} + r(K^* - K_0) \qquad (30b)$$

$$= C(0,T) - \tilde{Q} + r(K^* - K_0) \qquad (30c)$$

Substitute (21) into (30c) and rearrange terms

$$\left.\frac{dB}{dt}\right|_{t=0} = (\tilde{Q} - rK^*) \left[\frac{D}{r} \frac{(1 - e^{-rT})}{(1 - e^{-DT})} - 1 \right]$$

$$+ \left[\frac{D}{1 - e^{-DT}} - r \right] K_0 - \left[\frac{D}{1 - e^{-DT}} \right] e^{-rT} \hat{K} \qquad (31)$$

Now, since $\delta > r$, we have $D > r$, and

$$\frac{D}{r} \frac{(1 - e^{-rT})}{(1 - e^{-DT})} > 1, \text{ i.e. } \frac{1 - e^{-rT}}{r} > \frac{1 - e^{-DT}}{D} \qquad (32)$$

In order to prove (32), one only needs to show that $(1 - e^{-yT})/y$ is a decreasing function of y. We leave this task to the reader. It suffices to note that the proof would be similar to the proof that $(e^{yT} - 1)/y$ is an increasing function of y, and the latter proof is in

Appendix 1. Also, since $D > r > 0$, $\frac{D}{1 - e^{-DT}} > D > r$. So if $e^{-rT}\hat{K}$ is not too large, the right-hand-side of (31) will be positive.

(End of Proof)

The result is illustrated in Figure 4, for a given $\hat{K}$. The longer the time horizon, the smaller is $e^{-rT}\hat{K}$, and the larger is $\dot{B}(0)$. In particular, if $\dot{B}(0)$ is positive and K_0 is smaller then $\hat{K}$ (as illustrated), then $\dot{B}$ must be negative eventually, and given T, there exists a unique t'(which depends on T) at which $\dot{B} = 0$, this t' can be solved explicitly by equating (29) to zero.

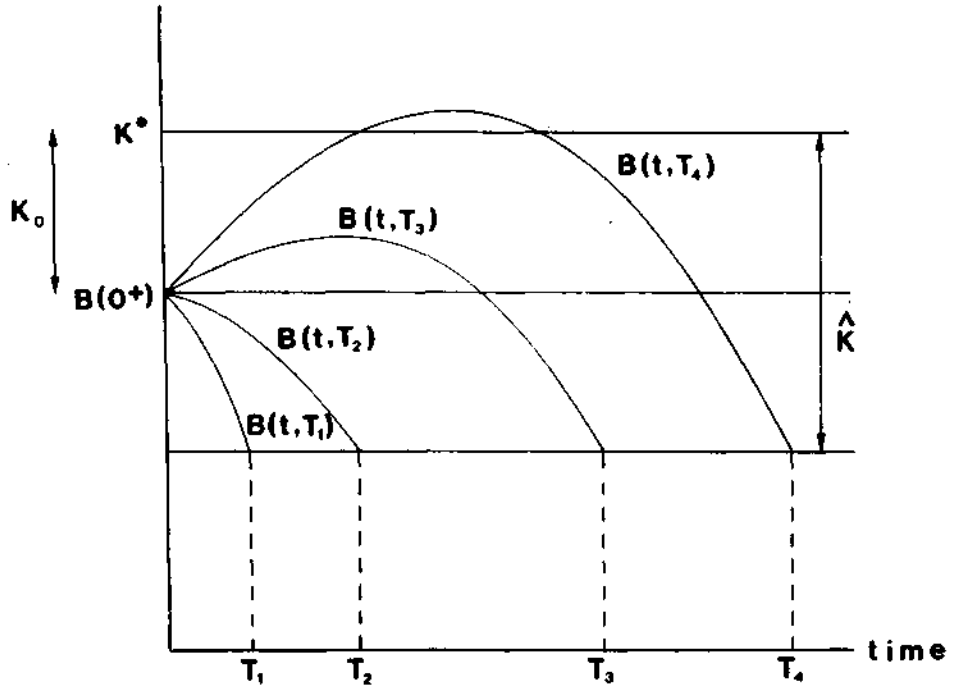

Figure 4 ($K > K_0$, $\delta > r$)

We note that when the stock of debt falls ($\dot{B} \leq 0$), it falls at an increasing rate ($\ddot{B} = r\dot{B} + \dot{C} < 0$, since $\dot{C} < 0$ for $\delta > r$). From this remark, it follows that in the case $\delta > r$, debt never increases after a period of decline: this would violate the concavity of B(t) when $\dot{B} \leq 0$. Another way of seeing this is to note that if $\dot{B}(0) < 0$, then from (30a)

$$\left[K^* - K_0 - \frac{\tilde{Q}}{r} + \frac{C(0,T)}{D} \right] r < 0$$

and

$$-\left(\frac{r - \delta}{\eta} \right) \frac{C(0,T)}{D} > 0 \qquad (30a)$$

For $t > 0$, the first term in (29) will be given weight $e^{rt} > 1$ while the second term will be given weight $e^{\frac{r - \delta}{\eta} t} < 1$, making $\dot{B}$ more negative still. Thus we have proved the following:

Proposition 4: If $\delta > r$ and $\hat{K} \leq K^*$, the country will not change its status from borrower to lender. This is the case of a half debt cycle.

For the opposite case where $r > \delta$, we have a richer story: it is possible that the optimal strategy involves switching of status twice (from borrower to lender and then back to borrower again).

Consider the simplest case with $r > \delta$, $\hat{K} = K_0 = 0$.
Then from (31) we have

$$\left. \frac{dB}{dt} \right|_{t=0} = (\tilde{Q} - rK^*) \left[\frac{D}{r} \frac{(1 - e^{-rT})}{(1 - e^{-DT})} - 1 \right] < 0 \qquad (33)$$

The negative sign follows from the fact that the sign of (32) is reversed when $r < \delta$; note that $r < \delta$ if and only if $r < D$.

Since $B(0^+) = K^* - K_0 = K^*$, and $B(T) = K(T) - \hat{K} = K(T) = K^*$, it follows that the time path of debt is U - shaped. This is illustrated in Figure 5. At time t_A, the country switches its status from a borrower to a lender. This switch is reversed at time t_B. This second switch hinges on the possibility to use part of the terminal capital stock to repay debt.

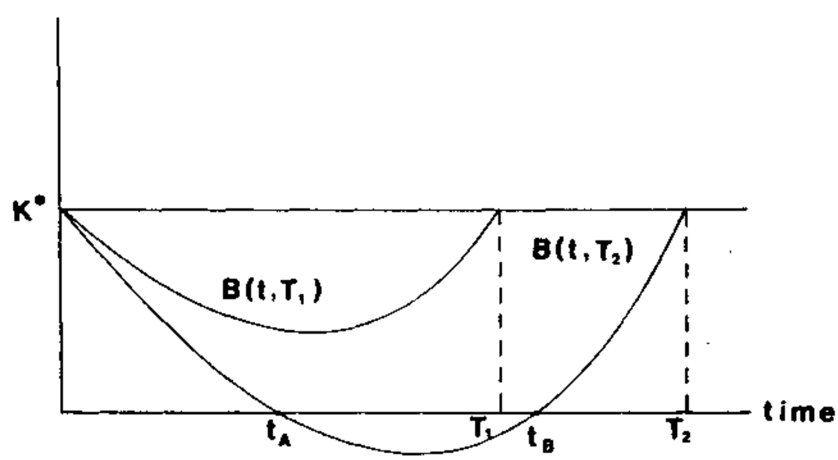

Figure 5 $(r > \delta,\ \hat{K} = K_0 = 0)$

Clearly if K_0 and $\hat{K}$ are positive but not too large, then the same story applies. Note also that the function $B(t)$ is convex when $\dot{B} \geq 0$, because $\ddot{B} = r\dot{B} + \dot{C} > 0$.

There are exceptions to the U - shaped case. For example, consider the case where $D < r$ but T is sufficiently short so that

$\frac{D}{1 - e^{-DT}} > r$. Then, from (31), $\dot{B}(0) > 0$ provided $\hat{K}$ is small. In this case, even though $\delta < r$, the country borrows for consumption and for capital accumulation. This is because it wants to maintain its capital stock at the efficient level K^* for all t, until time T, when the stock of capital is used to repay debt. Figure 6 illustrates this case.

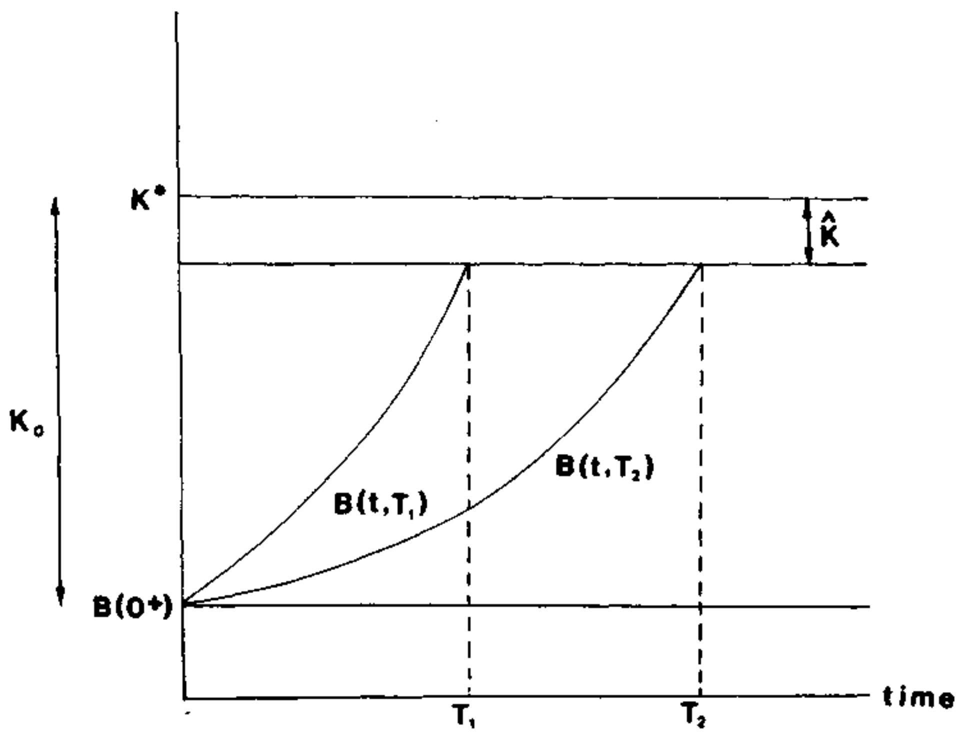

Figure 6 ($r > \delta$)

We summarize below our results for the case $r > \delta$.

Proposition 5: If $r > \delta$, and both Ko and $\hat{K}$ are sufficiently small, the country may at first borrow, then gradually switch to the lender status and finally accumulate debt again. However, if Ko is sufficiently large and the time horizon is very short, then the country may borrow both for consumption and capital accumulation, and debt increases over time.

Finally we note that the effect of an increase in the time horizon on the time path of debt has the same sign as its effect on the initial consumption. An increase in terminal time T reduces the level of initial debt. This is stated more precisely in the following proposition.

Proposition 6:

$$\text{SIGN} \ \frac{\partial B(t,T)}{\partial T} = \text{SIGN} \ \frac{\partial C(0,T)}{\partial T} .$$

Proof: Differentiate (28) with respect to T:

$$\frac{\partial B(t,T)}{\partial T} = \frac{1}{D} \left[e^{rt} - e^{\left(\frac{r-\delta}{\eta}\right)t} \right] \frac{\partial C(0,T)}{\partial T} \qquad (34)$$

The sign of the term inside the brackets [...] is exactly the same as the sign of $r - (\frac{r-\delta}{\eta})$, which is D. But this term is multiplied by $\frac{1}{D}$. This completes the proof.

5. The Case of Infinite Horizon

When the time horizon is infinite, there are no convincing reasons why one should impose conditions such as

$$\lim_{t\to\infty} B(t) \le 0,$$

or

$$\lim_{t\to\infty} [B(t) - K(t)] \le 0$$

It is more natural to specify that, in the limit, interest on the stock of debt must not exceed net national product:

$$\lim_{t\to\infty} [rB(t)] \le \lim_{t\to\infty} [F(K(t)) - mK(t)] \qquad (35)$$

This constraint ensures that the country will not keep on increasing its indebtness without bounds.
Notice that if $D \leq 0$ (i.e. $\delta \leq r(1 - \eta)$) then the integral (17b) does not converge when $T \to \infty$. This means that for infinite horizon we have to restrict our attention to the case $D > 0$.
Since the right-hand-side of (35) is finite, it is clear that if debt is positive, then

$$\lim_{T \to \infty} e^{-rT} B(T) = 0, \qquad (36)$$

and (18) gives

$$C(0,\infty) = [(\tilde{Q} - rK^*) + rK_0] (D/r). \qquad (37)$$

Substituting this into (28), we see that if $r - \delta < 0$ then in the limit, the stock of debt converges to $\tilde{Q}/r$, the capital value of net national output. The present value of the steady-state stream of wage income consumption however declines to zero asymtotially. Thus, in the half debt-cycle the country remains a debtor for eternity.

If $r - \delta > 0$ and $D > 0$, then $\left(\frac{r - \delta}{\eta}\right) < r$, and in this case (31) holds, but the stock of debt tends to minus infinity (the country is the lender in the long run and the country has gone through the full debt cycle), while its discounted value, $e^{-rt} B(t)$ converges to zero. In other words, the country's financial assets grow without bounds, but at a rate smaller than the interest rate. Consumption grows without bounds, at the rate $(r - \delta)/\eta < r$.

6. Conclusions

When capital can be accumulated instantaneously, external borrowing for capital accumulation will occur initially. The amount of debt is specified by the difference between the optimal capital stock and the initially given stock with the optimal stock being determined by the equality of the marginal productivity of capital and the interest rate. The amount of debt is independent of the time horizon.

Debt incurred for consumption, however, varies with the time horizon. It is worth while to borrow for consumption if the time preference rate exceeds the interest rate. In that case, the country never has an incentive to lend money; it therefore always remains a borrower, and we only have a half debt cycle. In the case of the half cycle, lengthening the time horizon will lead to a higher consumption stream in some initial phase; debt will accumulate at a faster rate. Thus, institutional arrangements that allow a longer time horizon like the establishment of efficient capital markets, will improve the present value of welfare for capital-poor countries. If the planning horizon approaches infinity and if interest on debt is limited by net national output, the stock of debt remains positive. It is limited by the capital value of net national output.

If, however, the time preference rate is lower than the interest rate, it is not worth while to borrow for consumption. Then, a lengthening of the time horizon will lower the consumption flow if the present value of the terminal capital stock to be handed over to future generations in terminal time is sufficiently small relative to the initial capital stock. A lengthening of the time horizon will also reduce debt in the initial periods. With the interest rate being greater than the time preference rate, the country may at first borrow and then change its status to a lender. Thus, the country may complete a full debt cycle (and it may eventually switch to a borrower again). This holds if both the initial and the terminal capital stock are sufficiently small. If the initial capital stock is sufficiently large relative to the terminal capital stock, debt increases over time.

We have made a number of simplifying assumptions, which must be relaxed in future studies. For example one may introduce adjustment costs which prevent instantaneous accumulation. Another step toward more realism would be to make the rate of interest dependent on the size of the debt. We hope to be able to address these issues in subsequent research.

APPENDIX 1

Proof that $\frac{\partial}{\partial T}\left[\frac{D}{r}\frac{(1-e^{-rT})}{(1-e^{-DT})}\right] > 0$ if and only if r.

$$\frac{\partial}{\partial T}\left[\frac{D}{r}\frac{(1-e^{-rT})}{(1-e^{-DT})}\right]$$

$$= \frac{D}{r}\left[\frac{re^{-rT}(1-e^{-DT}) - De^{-DT}(1-e^{-rT})}{(1-e^{-DT})^2}\right]$$

$$= \frac{1}{(1-e^{-DT})^2}\left[De^{-rT}(1-e^{-DT}) - \frac{D^2}{r}e^{-DT}(1-e^{-rT})\right]$$

$$= \frac{1}{(1-e^{-DT})^2\,[De^{-rT}(1-e^{-DT})]}$$

$$\left[1-\frac{D}{r}\left(\frac{e^{-DT}}{e^{-rT}}\right)\left(\frac{1-e^{-rT}}{1-e^{-DT}}\right)\right]$$

$$= \frac{1}{(1-e^{-DT})^2\,[De^{-rT}(1-e^{-DT})]}\left[1-\frac{\frac{1}{r}(e^{rT}-1)}{\frac{1}{D}(e^{DT}-1)}\right]$$

We now show that

$$\frac{\frac{1}{r}(e^{rT}-1)}{\frac{1}{D}(e^{DT}-1)} < 1 \quad \text{if and only if } D > r.$$

(Note: D > r if and only if δ > r).

Proof, that

$$\frac{\frac{1}{r}\left(e^{rT} - 1\right)}{\frac{1}{D}\left(e^{DT} - 1\right)} < 1 \qquad \text{if } D > r.$$

Define $f(y) = \frac{1}{y}\left(e^{yT} - 1\right)$. We need to show that $f(y)$ is an increasing function of y. Now

$$f'(y) = \frac{y\,T\,e^{yT} - \left(e^{yT} - 1\right)}{y^2}$$

So we must show $\quad 1 - e^{yT} + yT\,e^{yT} > 0$ for all $yT \neq 0$

i.e. show $\quad 1 - e^{z} + z e^{z} > 0$ for all $z \neq 0$

Now e^{z} is a convex function. Let $g(z) = e^{z}$. Then, from the properties of convex function, for all $z^* \neq z$,

$$g(z^*) - g(z) > g'(z)(z^* - z)$$

$$e^{z^*} - e^{z} > e^{z}(z^* - z)$$

Let $z^* \neq 0$. Then

$$1 - e^{z} > -z\,e^{z}$$

$$1 - e^{z} + z\,e^{z} > 0$$

End of Proof

Notes

* We appreciate comments from G. Franke and M. Rauscher.

1) In Siebert's model, $\hat{K} = 0$, so that $K(T) \geq B(T)$; it is also possible to require $B(T^+) = 0$, so that there is a downward jump in both K and B: the capital stock is used to repay debt.

2) D may be positive or negative. $D > r$ if and only if $\delta > r$. D is negative, if and only if $r(1-\eta) - \delta > 0$. It can be shown that if D is negative then the infinite horizon problem has no optimal solution; see Arrow and Kurz (1970), p. 156, on this last point. The expression

$$\left[\frac{1 - e^{-DT}}{D} \right]$$ is always positive,

regardless of the sign of D, because if $D < 0$ then $e^{-Dt} > 1$ for all $t > 0$, and if $D > 0$ then $e^{-Dt} < 1$, for all $t > 0$. By the same reason, $D(1 - e^{-Dt})$ is always positive.

References

Arrow, J. Kenneth, Mortecai Kurz (1970), Public Investment, The Rate of Return, and Optimal Fiscal Policy, 156, Baltimore.

Chipman, John S., *A Two-Period Model of International Trade and Payments*. Universität Konstanz, Working Paper, 194, Konstanz 1985.

Glick, Reuven, Homi J. Khavar (1986), The Costs and Benefits of Foreign Borrowing: A Survey of Multi-Period Models, *Journal of Development Studies*, 22, 279-299.

Van Long, Ngo, Neil J. Vousden (1977), Optimal Control Theorems. In: John D. Pitchford, Stephen J. Turnovsky (Eds.), *Applications of Control Theory to Economic Analysis*. Amsterdam, 11-34.

Sachs, Jeffrey (1984), *Theoretical Issues in International Borrowing*. Princeton Studies in International Finance, 54, Princeton, N.J.

Siebert, Horst (1985), *Economics of Resource-Exporting Country. Intertemporal Theory of Supply and Trade*. Greenwich.

Siebert, Horst (1987), Foreign Debt and Capital Accumulation. *Weltwirtschaftliches Archiv*, 83, 618-630.

World Bank (1985), World Development Report, Washington D.C.