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**CORES AND STABLE SETS FOR INTERVAL-VALUED GAMES**

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# Cores and Stable Sets for Interval-Valued Games

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## Abstract

In this paper, interval-type solution concepts for interval-valued cooperative games like the interval core, the interval dominance core and stable sets are introduced and studied. The notion of  $\mathcal{I}$ -balancedness is introduced, and it is proved that the interval core of an interval-valued cooperative game is nonempty if and only if the game is  $\mathcal{I}$ -balanced. Relations between the interval core, the dominance core and stable sets of an interval-valued game are established.

JEL Classification: C71

*Keywords:* cooperative games, interval games, the core, the dominance core, stable sets

## 1 Introduction

Interval cooperative games are introduced by Branzei, Dimitrov and Tijs (2003) and Branzei et al. (2004) in the context of bankruptcy situations. Inspiring was the work of Yager and Kreinovich (2000) where an algorithm for fair division is presented. Methods of interval arithmetic and analysis

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(cf. Moore, 1995) have played a key role for new models of games based on interval uncertainty. Two-person zero-sum non-cooperative games with interval strategies and interval-valued payoff function are studied by Shashikhin (2004). Interval matrix games arising from situations where the payoffs vary within closed intervals for fixed strategies are introduced in Collins and Hu (2005). Carpenete et al. (2005) propose a method to associate a coalitional interval game to each strategic game. Alparslan Gök, Miquel and Tijs (2007) introduce different notions of balancedness and cores for interval-valued cooperative games using selections of such games, and focus on two-person interval-valued cooperative games. Throughout the foregoing literature motivation from different points of view for studying interval-valued games is provided. This paper deals with interval-type solution concepts for cooperative interval-valued games such as the interval core, the interval dominance core and stable sets. The interval core and the interval dominance core are set-valued interval-type solution concepts on the class of interval-valued cooperative games, i.e. multi-functions that associate with each interval-valued game a set of payoff vectors whose components are (closed) intervals of real numbers. The interval core of an interval-valued game may be the empty set or may contain (infinitely) many elements. We introduce the notion of  $\mathcal{I}$ -balancedness and extend the Bondareva-Shapley theorem for traditional cooperative games to the interval setting. The interval dominance core associates with each interval-valued game the subset of undominated elements of the interval imputation set. A stable set of an interval-valued game is a subset of the interval imputation set of the game satisfying both internal and external stability. We study relations between the interval core, the interval dominance core and stable sets on the class of interval-valued games.

The paper is organized as follows. In Section 2 we recall basic notions and facts from the theory of interval-valued cooperative games. In Section 3 we introduce the interval core and the notion of  $\mathcal{I}$ -balancedness, study properties of the interval core and prove that an interval-valued cooperative game has a nonempty interval core if and only if the game is  $\mathcal{I}$ -balanced. We illustrate via examples the difference between the interval core of an interval-valued game and its core, and some relations between the  $\mathcal{I}$ -balancedness of such a game and other types of balancedness (cf. Alparslan Gök, Miquel and Tijs (2007)). Unanimity interval-valued games are introduced and the structure of the interval core of such a game is explicitly described. Section 4 deals with the interval dominance core and stable sets, and their relations with the interval core on the class of arbitrary interval-valued cooperative games

and on the subclass of unanimity interval-valued games. In Section 5 we introduce the notion of convex interval-valued game showing that convexity is a sufficient condition for the non-emptiness of the interval core but not a necessary one, and suggest some topics for further research regarding cores and stable sets for interval-valued cooperative games.

## 2 Preliminaries

A *cooperative  $n$ -person game in coalitional form* is an ordered pair  $\langle N, v \rangle$ , where  $N = \{1, 2, \dots, n\}$  (*the set of players*) and  $v : 2^N \rightarrow \mathbb{R}$  is a map, assigning to each coalition  $S \in 2^N$  a real number, such that  $v(\emptyset) = 0$ . This function  $v$  is called the characteristic function of the game,  $v(S)$  is called the *worth (or value)* of coalition  $S$ . Often we identify a game  $\langle N, v \rangle$  with its characteristic function  $v$ . The set of coalitional games with player set  $N$  is denoted by  $G^N$ . We refer the reader to Tijs (2003) and Part I in Branzei, Dimitrov and Tijs (2005, 2008) for an introduction in classical cooperative game theory.

Let  $I(\mathbb{R})$  be the set of all closed intervals in  $\mathbb{R}$ . A *cooperative  $n$ -person interval game in coalitional form* is an ordered pair  $\langle N, w \rangle$  where  $N := \{1, 2, \dots, n\}$  is the set of players, and  $w : 2^N \rightarrow I(\mathbb{R})$  is the characteristic function which assigns to each coalition  $S \in 2^N$  a closed interval  $w(S) \in I(\mathbb{R})$ , such that  $w(\emptyset) = [0, 0]$ . For each  $S \in 2^N$ , the *worth set (or worth interval)*  $w(S)$  of the coalition  $S$  in the interval game  $w$  is a closed interval which will be denoted by  $[\underline{w}(S), \overline{w}(S)]$ , where  $\underline{w}(S)$  is the lower bound and  $\overline{w}(S)$  is the upper bound of  $w(S)$ . The family of all interval games with player set  $N$  is denoted by  $IG^N$ . Note that if all the worth intervals are *degenerate intervals*, i.e.,  $\underline{w}(S) = \overline{w}(S)$ , then the interval game  $\langle N, w \rangle$  corresponds to the classical cooperative game  $\langle N, v \rangle$  where  $v(S) = \underline{w}(S)$ . This means that traditional cooperative games can be embedded in a natural way in the class of interval-valued cooperative games. In the sequel, we recall some definitions and results from Alparslan Gök, Miquel and Tijs (2007), where the focus is on balancedness and cores for two-person interval-valued cooperative games. Let  $\langle N, w \rangle$  be an interval game; then  $v : 2^N \rightarrow \mathbb{R}$  is called a *selection* of  $w$  if  $v(S) \in w(S)$  for each  $S \in 2^N$ . The set  $Sel(w)$  of selections of  $w$  plays a key role in defining the imputation set and the core of an interval-valued cooperative game. Thus, the imputation set  $I(w)$  of  $\langle N, w \rangle$  is defined by  $I(w) = \cup \{I(v) | v \in Sel(w)\}$ , and the *core*  $C(w)$  of  $\langle N, w \rangle$  is defined

by  $C(w) = \cup \{C(v) | v \in Sel(w)\}$ . Clearly,  $C(w) \neq \emptyset$  if and only if there exists a  $v \in Sel(w)$  with  $C(v) \neq \emptyset$ . We notice that in an obvious way each degenerate interval-valued game, where each worth interval  $w(S)$  consists of one point, can be identified with a classical cooperative game and conversely, each classical cooperative game can be associated with a degenerate interval-valued game. To be more precise a classical cooperative game  $\langle N, v \rangle$  can be identified with  $\langle N, w \rangle$ , where  $w(S) = [v(S), v(S)]$  for each  $S \in 2^N$ . A map  $\lambda : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}_+$  is called a *balanced map* if  $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) e^S = e^N$ . An interval game  $\langle N, w \rangle$  is called *strongly balanced* if for each balanced map  $\lambda$  it holds that  $\sum \lambda(S) \bar{w}(S) \leq \underline{w}(N)$ . The family of all strongly balanced interval-valued games with player set  $N$  is denoted by  $BIG^N$ .

**PROPOSITION 2.1.** (*Alparslan Gök, Miquel and Tijs (2007)*) *Let  $\langle N, w \rangle$  be an interval game. Then, the following three assertions are equivalent:*

- (i) *For each  $v \in Sel(w)$  the game  $\langle N, v \rangle$  is balanced.*
- (ii) *For each  $v \in Sel(w)$ ,  $C(v) \neq \emptyset$ .*
- (iii) *The interval game  $\langle N, w \rangle$  is strongly balanced.*

From Proposition 2.1 it follows that  $C(w) \neq \emptyset$  for a strongly balanced game  $\langle N, w \rangle$ , since for all  $v \in Sel(w)$ ,  $C(v) \neq \emptyset$ .

We call an interval game  $\langle N, w \rangle$  *strongly unbalanced*, if there exists a balanced map  $\lambda$  such that  $\sum \lambda(S) \underline{w}(S) > \bar{w}(N)$ . Then,  $C(v) = \emptyset$  for all  $v \in Sel(w)$ , which implies that  $C(w) = \emptyset$ .

If all the worth intervals of an interval-valued game  $\langle N, w \rangle$  are degenerate intervals then strong balancedness corresponds to balancedness and strong unbalancedness corresponds to unbalancedness for the classical cooperative game  $\langle N, v \rangle$ .

### 3 The interval core

Let  $I = [\underline{I}, \bar{I}]$  and  $J = [\underline{J}, \bar{J}]$  be two intervals. We say that  $I$  is *weakly better* than  $J$ , which we denote by  $I \succcurlyeq J$ , iff  $\underline{I} \geq \underline{J}$  and  $\bar{I} \geq \bar{J}$ . Note that in case  $I \succcurlyeq J$ , the following conditions hold:

- (i) for each  $a \in J$  there is a  $b \in I$  such that  $a \leq b$ ;
- (ii) for each  $b \in I$  there is an  $a \in J$  such that  $a \leq b$ .

In this paper,  $n$ -tuples of intervals,  $I = (I_1, I_2, \dots, I_n)$ , where  $I_i \in I(\mathbb{R})$  for each  $i \in N$ , will play a key role. For further use we denote by  $I(\mathbb{R})^N$  the set of all  $n$ -dimensional vectors whose components are elements in  $I(\mathbb{R})$ . Let  $I_i$  be the payoff interval of player  $i$ , and  $I = (I_1, I_2, \dots, I_n)$  be an  $n$ -person interval-payoff vector. Then, according to Moore (1995), we have  $\sum_{i \in S} I_i = [\sum_{i \in S} \underline{I}_i, \sum_{i \in S} \bar{I}_i] \in I(\mathbb{R})$  for each  $S \in 2^N \setminus \{\emptyset\}$ . Next, we define interval-type solution concepts for interval-valued cooperative games  $w \in IG^N$ . Instead of  $w(\{i\})$ ,  $w(\{i, j\})$ , etc., we often write  $w(i)$ ,  $w(i, j)$ , etc. The *interval imputation set*  $\mathcal{I}(w)$  of the interval-valued game  $w$ , is defined by

$$\mathcal{I}(w) := \left\{ (I_1, I_2, \dots, I_n) \in I(\mathbb{R})^N \mid \sum_{i \in N} I_i = w(N), w(i) \preceq I_i, \text{ for all } i \in N \right\}.$$

We note that  $\sum_{i \in N} I_i = w(N)$  is equivalent with  $\sum_{i \in N} \bar{I}_i = \bar{w}(N)$  and  $\sum_{i \in N} \underline{I}_i = \underline{w}(N)$ , and  $w(i) \preceq I_i$  is equivalent with  $\underline{w}(i) \leq \underline{I}_i$  and  $\bar{w}(i) \leq \bar{I}_i$ , for each  $i \in N$ .

Furthermore,  $\sum_{i \in N} I_i = w(N)$  implies for all  $i \in N$  and for all  $t \in w(N)$  there exists  $x_i \in I_i$  such that  $\sum_{i \in N} x_i = t$ .

The *interval core*  $\mathcal{C}(w)$  of the interval-valued game  $w$ , is defined by

$$\mathcal{C}(w) := \left\{ (I_1, I_2, \dots, I_n) \in I(\mathbb{R})^N \mid \sum_{i \in N} I_i = w(N), \sum_{i \in S} I_i \succcurlyeq w(S), \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\}.$$

Here,  $\sum_{i \in N} I_i = w(N)$  is the efficiency condition and  $\sum_{i \in S} I_i \succcurlyeq w(S)$ ,  $S \in 2^N \setminus \{\emptyset\}$ , are the stability conditions of the payoff interval-valued vectors. Clearly,  $\mathcal{C}(w) \subset \mathcal{I}(w)$  for each  $w \in IG^N$ .

**EXAMPLE 3.1.** (*LLR-game*) Let  $\langle N, w \rangle$  be a three-person interval-valued glove game with  $w(1, 3) = w(2, 3) = w(1, 2, 3) = J \succcurlyeq [0, 0]$  and  $w(S) = [0, 0]$  otherwise. The interval core is  $\mathcal{C}(w) = \{([0, 0], [0, 0], J)\}$ .

**PROPOSITION 3.1.** Let  $w \in IG^N$ . The interval imputation set  $\mathcal{I}(w)$  of  $w$  is nonempty if and only if  $w(N) \succcurlyeq \sum_{i \in N} w(i)$ .

*Proof.* First, suppose that  $\mathcal{I}(w) \neq \emptyset$ . Take  $I = (I_1, I_2, \dots, I_n) \in \mathcal{I}(w)$ . Then  $I_i \succcurlyeq w(i)$ , for each  $i \in N$ . So,  $\sum_{i \in N} I_i \succcurlyeq \sum_{i \in N} w(i)$  by interval calculus and  $\sum_{i \in N} I_i = w(N)$  by the efficiency condition of  $\mathcal{I}(w)$ .

Next suppose that  $w(N) \succcurlyeq \sum_{i \in N} w(i)$ .

Then,  $I = (w(1), w(2), \dots, w(n-1), I_n)$ , where  $I_n = [\underline{I}_n, \bar{I}_n] = [\underline{w}(n) + \delta, \bar{w}(n) + \epsilon]$  with  $\epsilon = \bar{w}(N) - \sum_{i \in N} \bar{w}(i) \geq 0$  and  $\delta = \underline{w}(N) - \sum_{i \in N} \underline{w}(i) \geq 0$ , is an element of the interval imputation set.  $\square$

REMARK 3.1. *The interval core  $\mathcal{C}(w)$ , can be easily computed by solving a system of linear inequalities of the form:  $\sum_{i \in N} \underline{I}_i = \underline{w}(N)$ ;  $\sum_{i \in N} \bar{I}_i = \bar{w}(N)$  and  $\sum_{i \in S} \underline{I}_i \geq \underline{w}(S)$ ;  $\sum_{i \in S} \bar{I}_i \geq \bar{w}(S)$ , for each  $S \in 2^N \setminus \{\emptyset\}$ . We notice that the time complexity of the algorithm for computing the interval core  $\mathcal{C}(w)$  for  $w \in IG^N$  is the same as the time complexity of the algorithm for computing the core  $C(w)$  for  $v \in G^N$ .*

REMARK 3.2. *We notice that the elements of the sets  $C(w)$  and  $\mathcal{C}(w)$  are of different type, implying that we cannot compare the sets with respect to the inclusion relation. Specifically, the elements of  $C(w)$  are vectors  $x \in \mathbb{R}^N$ , whereas the elements of  $\mathcal{C}(w)$  are vectors  $I \in I(\mathbb{R})^N$ . But, if all the worth intervals of the interval-valued game  $\langle N, w \rangle$  are degenerate intervals then the interval core  $\mathcal{C}(w)$  corresponds in a natural way to the core  $C(w)$ , since  $([a_1, a_1], \dots, [a_n, a_n])$  is in the interval core  $\mathcal{C}(w)$  if and only if  $(a_1, \dots, a_n)$  is in the core  $C(w)$  for each  $a_i \in \mathbb{R}$  and  $i = 1, \dots, n$ . Furthermore, we could have situations in which  $\mathcal{C}(w) = \emptyset$  and  $C(w) \neq \emptyset$ , as Example 3.1 illustrates.*

EXAMPLE 3.2. *Let  $\langle N, w \rangle$  be a two-person interval-valued game with  $w(1, 2) = [6, 8]$ ,  $w(1) = [2, 4]$ ,  $w(2) = [5, 6]$  and  $w(\emptyset) = [0, 0]$ . For this game  $\mathcal{C}(w) = \emptyset$ . But,  $C(w) \neq \emptyset$  since  $C(v) \neq \emptyset$  for some selections  $v \in \text{Sel}(w)$ .*

PROPOSITION 3.2. *Let  $w \in IG^N$ . If the interval core  $\mathcal{C}(w)$  is nonempty then the core  $C(w)$  is nonempty.*

*Proof.* Take  $(I_1, I_2, \dots, I_n) \in \mathcal{C}(w)$ . Then,  $\sum_{i \in N} I_i = w(N)$  for each  $i \in N$ , implying that  $\sum_{i \in N} \underline{I}_i = \underline{w}(N)$  and  $\sum_{i \in N} \bar{I}_i = \bar{w}(N)$ , and  $\sum_{i \in S} I_i \succcurlyeq w(S)$ , implying that  $\sum_{i \in S} \underline{I}_i \geq \underline{w}(S)$  and  $\sum_{i \in S} \bar{I}_i \geq \bar{w}(S)$ . Let  $\langle N, v \rangle$  be the selection of  $w$  with  $v(S) = \underline{w}(S)$ ,  $v(N) = \underline{w}(N)$  and let  $x_i = \underline{I}_i$ . Then,  $\sum_{i \in S} x_i \geq \underline{w}(S)$  and  $\sum_{i \in N} x_i = \underline{w}(N)$  which shows that  $C(w)$  is nonempty.  $\square$

PROPOSITION 3.3. *Let  $w \in IG^N$ . Then, the interval core  $\mathcal{C}(w)$  of  $w$  is a convex set.*

PROPOSITION 3.4. *The interval core  $\mathcal{C} : IG^N \rightarrow I(\mathbb{R})^N$  is a superadditive map.*

*Proof.* Let  $w_1, w_2 \in IG^N$ . Clearly,  $w_1 + w_2 \in IG^N$ . We will show that

$$\mathcal{C}(w_1) + \mathcal{C}(w_2) \subseteq \mathcal{C}(w_1 + w_2).$$

Take  $(I_1, I_2, \dots, I_n) \in \mathcal{C}(w_1)$  and  $(J_1, J_2, \dots, J_n) \in \mathcal{C}(w_2)$ . Then,

$$\sum_{k \in N} I_k + \sum_{k \in N} J_k = w_1(N) + w_2(N) \Rightarrow \sum_{k \in N} (I_k + J_k) = (w_1 + w_2)(N),$$

and, for each  $S \in 2^N \setminus \{\emptyset\}$ ,  $\sum_{k \in S} I_k \succcurlyeq w_1(S)$  and  $\sum_{k \in S} J_k \succcurlyeq w_2(S)$ , implying that  $\sum_{k \in S} \bar{I}_k \geq \bar{w}_1(S)$  and  $\sum_{k \in S} \bar{J}_k \geq \bar{w}_2(S)$ . Then, for each  $S \in 2^N \setminus \{\emptyset\}$ ,

$$\sum_{k \in S} \bar{I}_k + \sum_{k \in S} \bar{J}_k \geq \bar{w}_1(S) + \bar{w}_2(S) \Rightarrow \sum_{k \in S} (\bar{I}_k + \bar{J}_k) \geq (\bar{w}_1 + \bar{w}_2)(S).$$

Similarly,  $\sum_{k \in S} (\underline{I}_k + \underline{J}_k) \geq (\underline{w}_1 + \underline{w}_2)(S)$ . Hence, the interval core is a superadditive map.  $\square$

On the class of traditional cooperative games, the core (cf. Gillies (1953)) has the property of relative invariance with respect to strategic equivalence. This result can be easily extended to interval-valued games as we see in Proposition 3.5. For this extension we need the notion of additive interval-valued games. A game  $\langle N, a \rangle$  is called an *additive interval-valued game* if for each  $S \in 2^N$ ,  $a(S) = \sum_{i \in S} a(\{i\})$ . For such a game  $\mathcal{C}(a) = \{(a(\{1\}), a(\{2\}), \dots, a(\{n\}))\}$ .

**PROPOSITION 3.5.** *The interval core  $\mathcal{C} : IG^N \rightarrow I(\mathbb{R})^N$  is relative invariant with respect to strategic equivalence, i.e. for each  $w, a \in IG^N$  with  $a$  being an additive interval-valued game, and for each  $k > 0$  we have  $\mathcal{C}(kw + a) = k\mathcal{C}(w) + \mathcal{C}(a)$ .*

An interval-valued game  $w \in IG^N$  is called  *$\mathcal{I}$ -balanced* if for each balanced map  $\lambda : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}_+$  we have  $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)w(S) \preceq w(N)$ . The class of  $\mathcal{I}$ -balanced games is denoted by  $\mathcal{IBIG}^N$ . In the following proposition a relation between balancedness (in terms of selections) and  $\mathcal{I}$ -balancedness is given.

**PROPOSITION 3.6.** *Let  $\langle N, w \rangle$  be a strongly balanced interval-valued game; then  $\langle N, w \rangle$  is  $\mathcal{I}$ -balanced.*



*Proof.* Take a balanced map  $\lambda : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}_+$ . Then

$$\bar{w}(N) \geq \underline{w}(N) \geq \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) \bar{w}(S) \geq \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) \underline{w}(S).$$

So,  $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) w(S) \preceq w(N)$ . Hence,  $\langle N, w \rangle$  is  $\mathcal{I}$ -balanced.  $\square$

Note that the converse of the Proposition 3.6 is not true since there exists  $v \in \text{Sel}(w)$  with  $C(v) \neq \emptyset$ , implying that the core  $C(w)$  is nonempty, but the interval core may be empty as we learn from the Example 3.2.

In the classical theory of cooperative games it is proved (Bondareva (1963) and Shapley (1967)) that a game  $v \in G^N$  is balanced iff  $C(v)$  is nonempty. In the next theorem we extend this result to interval-valued cooperative games by using the duality theorem from linear programming theory (see Theorem 1.32 in Branzei, Dimitrov and Tijs, 2005).

**THEOREM 3.1.** *Let  $w \in IG^N$ . Then the following two assertions are equivalent:*

- (i)  $C(w) \neq \emptyset$ ;
- (ii) *The game  $w$  is  $\mathcal{I}$ -balanced.*

*Proof.* First, using Remark 3.1, we note that  $C(w) \neq \emptyset$  if and only if the following two equalities hold simultaneously:

$$\underline{w}(N) = \min \left\{ \sum_{i \in N} \underline{I}_i \mid \sum_{i \in S} \underline{I}_i \geq \underline{w}(S), \text{ for each } S \in 2^N \setminus \{\emptyset\} \right\}, \quad (4)$$

$$\bar{w}(N) = \min \left\{ \sum_{i \in N} \bar{I}_i \mid \sum_{i \in S} \bar{I}_i \geq \bar{w}(S), \text{ for each } S \right\}. \quad (5)$$

We consider the matrix  $A$  whose columns are the characteristic vectors  $e^S$ ,  $S \in 2^N \setminus \{\emptyset\}$ , and apply the duality theorem of linear programming. Then, (4) holds true if and only if

$$\underline{w}(N) = \max \left\{ \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) \underline{w}(S) \mid \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) e^S = e^N, \lambda \geq 0 \right\}, \quad (6)$$

and (5) is satisfied if and only if

$$\bar{w}(N) = \max \left\{ \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) \bar{w}(S) \mid \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) e^S = e^N, \lambda \geq 0 \right\}. \quad (7)$$

Now, note that (6) holds if and only if

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) \underline{w}(S) \leq \underline{w}(N), \text{ for each } \lambda \geq 0 \text{ such that } \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) e^S = e^N, \quad (8)$$

whereas (7) is guaranteed if and only if

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) \bar{w}(S) \leq \bar{w}(N), \text{ for each } \lambda \geq 0 \text{ such that } \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) e^S = e^N. \quad (9)$$

Finally, we note that (8) and (9) express together the  $\mathcal{I}$ -balancedness of  $w$ .  $\square$

Let us note that the interval-valued game in Example 3.2 is not  $\mathcal{I}$ -balanced since  $w(1) + w(2) \not\geq w(12)$ . According to Theorem 3.1 we conclude that  $\mathcal{C}(w) = \emptyset$ .

**REMARK 3.3.** *If  $\mathcal{C}(w)$  is not empty, then  $\mathcal{C}(\underline{w})$  and  $\mathcal{C}(\bar{w})$  are both nonempty.*

We note that if all the worth intervals of the interval-valued game  $\langle N, w \rangle$  are degenerate intervals, then strongly balancedness and  $\mathcal{I}$ -balancedness of the game also correspond to the classical balancedness.

Next, we introduce the notion of unanimity interval-valued game and prove that such games are  $\mathcal{I}$ -balanced games. Let  $J \in I(\mathbb{R})^N$  and let  $T \in 2^N \setminus \{\emptyset\}$ . The unanimity interval-valued game based on  $J \succcurlyeq [0, 0]$  and  $T$  is defined by

$$u_{T,J}(S) = \begin{cases} J, & T \subset S \\ [0, 0], & \text{otherwise,} \end{cases}$$

for each  $S \in 2^N$ .

**PROPOSITION 3.7.** *For each unanimity interval-valued game  $u_{T,J}$  the corresponding interval core  $\mathcal{C}(u_{T,J})$  is equal to*

$$\mathcal{K} = \left\{ (I_1, \dots, I_n) \in I^N(\mathbb{R}) \mid \sum_{i \in N} I_i = J, I_i \geq 0 \text{ for all } i \in N, I_i = [0, 0] \text{ for } i \in N \setminus T \right\}.$$

*Proof.* In order to show that  $\mathcal{C}(u_{T,J}) \subset \mathcal{K}$ , let  $(I_1, \dots, I_n) \in \mathcal{C}(u_{T,J})$ . Clearly, for each  $i \in N$  we have  $I_i \geq u_{T,J}(\{i\})$  and  $u_{T,J}(\{i\}) = [0, 0]$ . So,  $\underline{I}_i \geq 0$  for all  $i \in N$ . Furthermore,  $\sum_{i \in N} I_i = u_{T,J}(N) = J$ . Since also  $\sum_{i \in T} I_i = J$ , we conclude that  $I_i = 0$  for  $i \in N \setminus T$ . So,  $(I_1, \dots, I_n) \in \mathcal{K}$ .

In order to show that  $\mathcal{K} \subset \mathcal{C}(u_{T,J})$ , let  $(I_1, \dots, I_n) \in \mathcal{K}$ . So,  $\underline{I}_i \geq 0$  for all  $i \in N$ ,  $I_i = [0, 0]$  if  $i \in N \setminus T$ ,  $\sum_{i \in N} I_i = J$ . Then  $(I_1, \dots, I_n) \in \mathcal{C}(u_{T,J})$ , because it also holds:

$$(i) \sum_{i \in S} I_i = [0, 0] = u_{T,J}(S) \text{ if } T \setminus S \neq \emptyset,$$

$$(ii) \sum_{i \in S} I_i = \sum_{i \in N} I_i = u_{T,J}(N) = J = u_{T,J}(S) \text{ if } T \subset S.$$

□

REMARK 3.4. *On the class of traditional cooperative games, the unanimity games  $\langle N, u_T \rangle$  are defined by*

$$u_T(S) = \begin{cases} 1, & T \subset S \\ 0, & \text{otherwise,} \end{cases}$$

for each  $T \in 2^N \setminus \{\emptyset\}$ . The core  $C(u_T)$  of the unanimity game  $\langle N, u_T \rangle$  is given by

$$C(N, u_T) = \left\{ x \in \mathbb{R}^N \mid \sum_{i=1}^n x_i = 1, x_i = 0 \text{ if } i \in N \setminus T \right\}.$$

The core  $C(u_{T,[1,1]})$  of the unanimity interval-valued game  $u_{T,J}$  with  $J = [1, 1]$  is

$$C(u_{T,[1,1]}) = \left\{ I \in I(\mathbb{R})^N \mid \sum_{i \in N} I_i = [1, 1], \underline{I}_i \geq 0 \text{ for all } i \in N, I_i = [0, 0] \text{ for } i \in N \setminus T \right\}.$$

Hence, the interval core of the unanimity interval-valued game based on the degenerate interval  $J = [1, 1]$  corresponds to the core of the unanimity game in the traditional case. We recall that traditional unanimity games are convex games; in Section 5 we define convex interval-valued games and notice that unanimity interval-valued games are also convex.

## 4 The interval dominance core and stable sets

Let  $w \in IG^N$ ,  $I = (I_1, \dots, I_n), J = (J_1, \dots, J_n) \in \mathcal{I}(w)$  and  $S \in 2^N \setminus \{\emptyset\}$ . We say that  $I$  dominates  $J$  via coalition  $S$ , and denote it by  $I \text{ dom}_S J$ , if

- (i)  $I_i \succcurlyeq J_i$  for all  $i \in S$ ,
- (ii)  $\sum_{i \in S} I_i \preccurlyeq w(S)$ .

For  $S \in 2^N \setminus \{\emptyset\}$  we denote by  $D(S)$  the set of those elements of  $\mathcal{I}(w)$  which are dominated via  $S$ .

For  $I, J \in \mathcal{I}(w)$ , we say that  $I$  dominates  $J$  and denote it by  $I \text{ dom } J$  if there is an  $S \in 2^N \setminus \{\emptyset\}$  such that  $I \text{ dom}_S J$ .

$I$  is called *undominated* if there does not exist  $J$  and a coalition  $S$  such that  $J \text{ dom}_S I$ .

The *interval dominance core*  $\mathcal{DC}(w)$  of an interval-valued game  $w \in IG^N$  consists of all undominated elements in  $\mathcal{I}(w)$ , i.e. the complement in  $\mathcal{I}(w)$  of  $\cup \{D(S) | S \in 2^N \setminus \{\emptyset\}\}$ .

For  $w \in IG^N$  a subset  $A$  of  $\mathcal{I}(w)$  is a *stable set* if the following conditions hold:

- (i) (*Internal stability*) There does not exist  $I, J \in A$  such that  $I \text{ dom } J$  or  $J \text{ dom } I$ .
- (ii) (*External stability*) For each  $I \notin A$  there exist  $J \in A$  such that  $J \text{ dom } I$ .

Next, we study relations between the interval core, interval dominance core and stable sets for interval-valued cooperative games.

**THEOREM 4.1.** *Let  $w \in IG^N$  and let  $A$  be a stable set of  $w$ . Then,  $\mathcal{C}(w) \subseteq \mathcal{DC}(w) \subseteq A$ .*

*Proof.* In order to show that  $\mathcal{C}(w) \subseteq \mathcal{DC}(w)$  let us assume that there is  $I \in \mathcal{C}(w)$  such that  $I \notin \mathcal{DC}(w)$ . Then, there is a  $J \in \mathcal{I}(w)$  and a coalition  $S \in 2^N \setminus \{\emptyset\}$  such that  $J \text{ dom}_S I$ . Thus,  $J(S) \preccurlyeq w(S)$  and  $J_i \succcurlyeq I_i$  for all  $i \in S$  implying that  $I \notin \mathcal{C}(w)$ . To prove next that  $\mathcal{DC}(w) \subseteq A$  it is sufficient to show  $\mathcal{I}(w) \setminus A \subset \mathcal{I}(w) \setminus \mathcal{DC}(w)$ . Take  $I \in \mathcal{I}(w) \setminus A$ . By the external stability of  $A$  there is a  $J \in A$  with  $J \text{ dom } I$ . The elements in  $\mathcal{DC}(w)$  are not dominated. So,  $I \notin \mathcal{DC}(w)$ , i.e.,  $I \in \mathcal{I}(w) \setminus \mathcal{DC}(w)$ .  $\square$

The inclusions stated in the previous theorem may be strict. The following example, inspired by Tijs (2003), illustrates that the inclusion of  $\mathcal{C}(w)$  in  $\mathcal{DC}(w)$  might be strict.

**EXAMPLE 4.1.** *Let  $\langle N, w \rangle$  be the three-person interval-valued game with  $w(1, 2) = [2, 2]$ ,  $w(N) = [1, 1]$  and  $w(S) = [0, 0]$  if  $S \neq \{1, 2\}, N$ . Then  $D(S) = \emptyset$  if  $S \neq \{1, 2\}$  and  $D(\{1, 2\}) = \{I \in \mathcal{I}(w) \mid I_3 \succ [0, 0]\}$ . The elements  $I$  in  $\mathcal{I}(w)$  which are undominated satisfy  $I_3 = [0, 0]$ . Since the interval dominance core is the set of undominated elements in  $\mathcal{I}(w)$ , the interval dominance core of this game is nonempty but the interval core is empty.*

The next proposition shows that on the class of unanimity games the interval core and the interval dominance core coincide.

**PROPOSITION 4.1.** *Let  $\langle N, u_{T,J} \rangle$  be the unanimity interval-valued game based on coalition  $T$  and the payoff interval  $J \succ [0, 0]$ . Then,  $\mathcal{DC}(u_{T,J}) = \mathcal{C}(u_{T,J}) = \mathcal{K}$ .*

*Proof.* The second equality is already proved in Proposition 3.5. To prove the first equality note first that  $\mathcal{C}(u_{T,J}) \subset \mathcal{DC}(u_{T,J})$  by Theorem 4.1. We only have to prove that  $\mathcal{DC}(u_{T,J}) \subset \mathcal{C}(u_{T,J})$  or we need to show that for each  $I \notin \mathcal{C}(u_{T,J})$  we have  $I \notin \mathcal{DC}(u_{T,J})$ . Take  $I \notin \mathcal{C}(u_{T,J})$ . Then, there is a  $k \in N \setminus T$  with  $I_k \neq [0, 0]$ . Then,  $I' \text{ dom}_T I$ , where  $I'_i = [0, 0]$  for  $i \in N \setminus T$  and  $I'_i = I_i + \frac{1}{|T|}I_k$  for  $i \in T$ . So,  $I \notin \mathcal{DC}(u_{T,J})$ .  $\square$

The next example illustrates the fact that the interval core might coincide with the interval dominance core also for games which are not unanimity interval-valued games.

**EXAMPLE 4.2.** *Consider the game  $w$  in Example 3.1. We will show that  $\mathcal{DC}(w) = \mathcal{C}(w)$ . Take  $I = (I_1, I_2, I_3) \in \mathcal{I}(w)$ . Note that if  $I_1 \neq [0, 0]$  then  $([0, 0], I_2 + \frac{1}{2}I_1, I_3 + \frac{1}{2}I_2) \text{ dom}_{\{2,3\}} (I_1, I_2, I_3)$ . So,  $I \notin \mathcal{DC}(w)$ . Similarly, if  $I_2 \neq [0, 0]$  then  $I \notin \mathcal{DC}(w)$ . Hence,  $\mathcal{DC}(w) \subset \{([0, 0], [0, 0], J)\} = \mathcal{C}(w)$  by Example 3.1. On the other hand we know, in view of Theorem 4.1, that  $\mathcal{C}(w) \subset \mathcal{DC}(w)$ . So, we conclude that  $\mathcal{DC}(w) = \mathcal{C}(w)$ .*

## 5 Concluding remarks

An interesting subclass of interval-valued cooperative games is the class of convex interval-valued games.

An interval-valued cooperative game is *convex* if and only if  $w(S) + w(T) \preceq w(S \cup T) + w(S \cap T)$  for all  $S, T \in 2^N$ .

Convex interval-valued games turn out to have always a nonempty core. An example of a convex game is the unanimity interval-valued game  $u_{T,J}$ . Convexity of an interval-valued game is a sufficient condition for the nonemptiness of the interval core of the game. However, convexity of an interval-valued game is not a necessary condition for the nonemptiness of its interval core, as we see in Example 3.1. The *LLR*-game is not convex, since  $J + [0, 0] = w(1, 2, 3) + w(1) \not\preceq w(1, 3) + w(2, 3) = J + J$ , but its interval core is nonempty. For further results on convex interval-valued games we refer to Alparslan Gök, Branzei and Tijs (2008).

We end this section with some remarks on possible further research regarding cores and stable sets for interval-valued cooperative games. It is interesting to find sufficient conditions for the equality of the interval core and the interval dominance core. Trying to prove the convexity of the interval dominance core of any interval-valued game might be useful. Also studying the stable sets of an interval-valued game in terms of selections is another valuable topic for the extension of the theory of interval-valued games.

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