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Abstract

We establish the precise connections between *progressive taxation* and *inequality reduction*, in a setting where the level of tax revenue to be raised is endogenously fixed and tax schemes are balanced. We show that, in contrast with the traditional literature on taxation, the equivalence between *inequality reduction* and the combination of *progressivity* and *income order preservation* does not always hold in this setting. However, we show that, among rules satisfying *consistency* and, either *revenue continuity*, or *revenue monotonicity*, the equivalence remains intact.

Keywords: progressivity, inequality reduction, income order preservation, consistency, taxation

JEL classification: C70, D63, D70, H20

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1 Introduction

Progressivity is the requirement that a taxpayer with a higher income should pay at least as much rate of tax as a taxpayer with a lower income. *Inequality reduction* is the requirement that "income inequality" should be reduced after taxation.[1] It has long been perceived in the literature of taxation that the two principles are closely related (see, for instance, Musgrave and Thin, 1948; Fellman, 1976; and Kakwani, 1977). Jakobsson (1976) was the first to notice that this relation could be stated as an equivalence, provided tax functions preserve the order of incomes. The equivalence was proven later formally by Eichhorn et al. (1984) and Thon (1987).

In that literature, the two principles are defined as properties of a tax function, a real-valued function associating with any level of income a tax amount. We investigate the two principles in a different but related model of taxation introduced by O'Neill (1982) and Young (1987, 1988).[2] In this model, a taxation problem is identified by a profile of incomes and an amount of tax revenue. A (taxation) *rule* associates with each problem a profile of tax amounts of which the sum equals the desired tax revenue. We show that, in this model, the above logical equivalence no longer holds. In fact, *inequality reduction* implies neither *progressivity* nor *income order preservation*, as shown by our Examples 1 and 2. However, our main result shows that, among the rules satisfying two standard axioms known as *consistency* (the way any group of taxpayers split their total tax contribution depends only on their own taxable incomes) and *revenue continuity* (small changes in the tax revenue do not produce a jump in tax schedules), the equivalence remains intact. The role of revenue continuity in this result can also be played by the solidarity property known as *revenue monotonicity* (when the tax revenue increases, no one pays less).

2 Model and basic concepts

We study taxation problems in a variable-population model. The set of potential taxpayers, or *agents*, is identified by the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$, with generic element N. For each $i \in N$, let $y_i \in \mathbb{R}_+$ be i's (taxable) *income* and $y \equiv (y_i)_{i\in N}$ the income profile. A (taxation) *problem* is a triple consisting of a population $N \in \mathcal{N}$, an income profile $y \in \mathbb{R}^N_+$, and a tax revenue $T \in \mathbb{R}_+$ such that $\sum_{i\in N} y_i \geq T$. Let $Y \equiv \sum_{i\in N} y_i$. Let $\mathcal{D}^N$ be the set of taxation problems with population N and $\mathcal{D} \equiv \cup_{N\in\mathcal{N}}\mathcal{D}^N$.

Given a problem $(N, y, T) \in \mathcal{D}$, a *tax profile* is a vector $x \in \mathbb{R}^N$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq y_i$ and (ii) $\sum_{i\in N} x_i = T$.[3] We refer to (i) as *boundedness* and (ii) as *balancedness*.[4] A (taxation) *rule* on $\mathcal{D}$, $R\colon \mathcal{D} \to \cup_{N\in\mathcal{N}}\mathbb{R}^N$, associates with each problem $(N, y, T) \in \mathcal{D}$ a tax profile $R(N, y, T)$. We refer readers to Young (1987,

[1] This requirement is based on the inequality comparison known as the Lorenz dominance relation.

[2] See Moulin (2002) and Thomson (2003, 2006) for extensive treatments of this model and some other related allocation problems.

[3] Throughout the paper, for each $N \in \mathcal{N}$, each $M \subseteq N$, and each $z \in \mathbb{R}^N$, let $z_M \equiv (z_i)_{i\in M}$.

[4] Note that *boundedness* implies that each agent with zero income pays zero tax.

1988) for definitions of various taxation rules. A well-known example is the so-called *leveling tax* $L\colon \mathcal{D} \to \cup_{N\in\mathcal{N}}\mathbb{R}^N$ that makes post-tax incomes as equal as possible, provided no agent ends up being subsidized (i.e., paying a negative tax). Formally, for each $(N, y, T) \in \mathcal{D}$ and each $i \in N$, $L_i(N, y, T) \equiv \max\{y_i - 1/\lambda, 0\}$, where λ is a non-negative real number satisfying $\sum_{i\in N} \max\{y_i - 1/\lambda, 0\} = T$.

We now define our two main axioms of taxation.

Progressivity postulates that, for any pair of agents, the one with higher income should face a tax rate at least as high as the rate the other faces.

Axiom 1 ***Progressivity.*** *For each* $(N, y, T) \in \mathcal{D}$ *and each* $i, j \in N$, *if* $0 < y_i \leq y_j$, $R_i(N, y, T)/y_i \leq R_j(N, y, T)/y_j$.

Our second axiom says that "income inequality" should be reduced after taxation. This axiom is based on the following basic income inequality comparison. For each population $N \equiv \{1, \ldots, n\}$ and each pair of income profiles $y, y' \in \mathbb{R}^N_+$, y *Lorenz dominates* y' if, for each $k = 1, \ldots, n-1$, the proportion of the sum of the k lowest incomes to the total income at y is greater than (or equal to) the same proportion at y': that is, when $y_1 \leq y_2 \leq .... \leq y_n$ and $y'_1 \leq y'_2 \leq ... \leq y'_n$, for each $k = 1, ..., n-1$, $\sum_{i=1}^{k} y_i / \sum_{i=1}^{n} y_i \geq \sum_{i=1}^{k} y'_i / \sum_{i=1}^{n} y'_i$.

Axiom 2 ***Inequality reduction.*** *For each* $(N, y, T) \in \mathcal{D}$, *the post-tax income profile* $y - R(N, y, T)$ *Lorenz dominates the pre-tax income profile* y.

We will investigate logical relations between the two axioms, invoking in the process some of the following standard axioms.[5]

The first axiom requires that post-tax incomes be in the order of pre-tax incomes.

Axiom 3 ***Income order preservation.*** *For each* $(N, y, T) \in \mathcal{D}$ *and each pair* $i, j \in N$, *if* $y_i \geq y_j$, $y_i - R_i(N, y, T) \geq y_j - R_j(N, y, T)$.

The next axiom requires that the way any group of taxpayers split their total tax contribution depends only on their own taxable incomes.

Axiom 4 ***Consistency.*** *For each* $(N, y, T) \in \mathcal{D}$, *each* $M \subset N$, *and each* $i \in M$, $R_i(M, y_M, \sum_{i\in M} x_i) = x_i$, *where* $(x_i)_{i\in N} \equiv R(N, y, T)$ *and* $y_M \equiv (y_i)_{i\in M}$.

The next axiom says that small changes in revenue should not produce a jump in tax schedules.

Axiom 5 ***Revenue continuity.*** *For each* $N \in \mathcal{N}$, *each* $y \in \mathbb{R}^N_+$, *each sequence* $\{T^n : n \in \mathbb{N}\}$ *in* $\mathbb{R}_+$ *and each* $T \in \mathbb{R}_+$, *if* T^n *converges to* T, *then* $R(N, y, T^n)$ *converges to* $R(N, y, T)$.

Our final axiom says that no one pays less when the tax revenue increases.

Axiom 6 ***Revenue monotonicity.*** *For each* $(N, y, T) \in \mathcal{D}$ *and each* $T' \geq T$, $R(N, y, T') \geqq R(N, y, T)$.

[5] We refer readers to Thomson (2003, 2006) for detailed discussions on these axioms.

3 Results

As in the literature on tax functions mentioned in the introduction, the combination of *progressivity* and *income order preservation* implies *inequality reduction.* Essentially, the same proof of Eichhorn et al. (1984) works, which will be provided for completeness in the appendix. Our next examples show, however, that *inequality reduction* implies neither *progressivity* nor *income order preservation.*

Example 1 *We construct a tax profile that reduces inequality but that is not progressive. The idea is that when there is too high a gap between the richest agent and anyone else, we impose a very large tax burden on the richest agent and a low and equal burden on all others. Consider* $y \equiv (2, 3, 15)$ *and* $T \equiv 10$*. Let* ε *be a number such that* $0 < \varepsilon \leq 1$*. Let* $(\varepsilon, \varepsilon, 10 - 2\varepsilon)$ *be the tax profile for this problem. Then the post-tax income profile is given by* $(2 - \varepsilon, 3 - \varepsilon, 5 + 2\varepsilon)$ *Note that both* income-order preservation *and* tax-order preservation *(i.e., rules do not impose lower tax burdens for agents with higher incomes) are satisfied. Since* $(2/20, 5/20, 20/20) \leq ((2 - \varepsilon)/10, (5 - 2\varepsilon)/10, 10/10)$*, the post-tax income profile Lorenz dominates* y*. Thus, the tax profile satisfies* inequality reduction*. Note that, at the above problem, the tax rate of agent* 1*,* $\varepsilon/2$*, is higher than the tax rate of agent* 2*,* $\varepsilon/3$*, thus violating* progressivity*. Therefore, any rule that takes this tax profile as its value at the above problem and that continues to satisfy* inequality reduction*, at any other problem, will suffice to show that* inequality reduction *does not imply* progressivity.[6]

Example 2 *We define a rule that reduces inequality but violates income order preservation. The idea is similar to the previous example. We impose a very large tax burden on the richest agent and no burden at all on other agents, when tax revenue is within a fixed range. Let* $N \equiv \{1, 2, \ldots, n\}$*. For each* $(N, y, T) \in \mathcal{D}$*, let* $\bar{T} \equiv \min\left\{y_{\sigma(n)} - y_{\sigma(n-2)}, Y(y_{\sigma(n)} - y_{\sigma(n-1)})/y_{\sigma(n)}\right\}$*, where* $Y = \sum y_i$ *and* $\sigma\colon N \to N$ *is a permutation such that* $y_{\sigma(1)} \leq y_{\sigma(2)} \leq \cdots \leq y_{\sigma(n)}$*.. Let* $T^m \equiv \min\{T, \bar{T}\}$ *and*

$$R(N, y, T) \equiv T^m e_{\sigma(n)} + L(N, y - T^m e_{\sigma(n)}, T - T^m),$$

where L *denotes the leveling tax and* $e_{\sigma(n)}$ *denotes the unit vector with* 1 *in the* $\sigma(n)$*-th component.*[7] *We show in the appendix that* R *satisfies* inequality reduction *(as well as* revenue continuity *and* revenue monotonicity*) but violates* income order preservation*.*

To recover the equivalence between *inequality reduction* and *progressivity* in our model, it suffices to impose two additional but standard axioms: *consistency* and *revenue continuity* (or *revenue monotonicity*).

[6]For example, when there is a group of agents whose incomes are sufficiently lower than those of the other agents, we define $R(\cdot)$ as in the example by choosing ε sufficiently close to zero. For other problems we set the value of $R(\cdot)$ at the tax profile provided by the *leveling tax*, which satisfies *inequality reduction* as well as the two order preservation axioms.

[7]Note that, when $y_{\sigma(n)} = y_{\sigma(n-1)}$, $\bar{T} = 0$ and $R(N, y, T) = L(N, y, T)$. Thus, the definition does not depend on the choice of σ and therefore $R(\cdot)$ is well-defined.

Proposition 1 *The following statements hold*:
(i) *Progressivity and income order preservation together imply inequality reduction.*
(ii) *Inequality reduction and consistency together imply progressivity.*
(iii) *Inequality reduction, together with consistency and revenue continuity* (*or revenue monotonicity*), *implies income order preservation.*

The proof of the proposition appears in the appendix.

Example 1 shows that *consistency* is essential for part (ii) of the proposition and also that adding *income order preservation* to *inequality reduction* is not sufficient to get *progressivity.* Example 2 shows that *consistency* is essential for part (iii) of the proposition.

The next result follows directly from Proposition 1.

Corollary 2 *For consistent and revenue-continuous* (*or revenue-monotonic*) *rules, the combination of progressivity and income order preservation is equivalent to inequality reduction.*

A Proofs

Proof. [**Proof of Proposition 1**] (i) Let R be a rule satisfying *progressivity* and *income order preservation.* Let $(N, y, T) \in \mathcal{D}$. Assume, without loss of generality, that $0 < y_1 \le y_2 \le \cdots \le y_n$. Let $x \equiv R(N, y, T)$. Then, by *progressivity,*

$$\frac{x_1}{y_1} \le \frac{x_2}{y_2} \le \cdots \le \frac{x_n}{y_n}. \tag{1}$$

Let $k \in \{1, ..., n-1\}$. By (1), $x_i y_j \le x_j y_i$, for all $i = 1, ..., k$ and $j = k+1, ..., n$. Thus, $\sum_{i=1}^k x_i \sum_{j=k+1}^n y_j \le \sum_{j=k+1}^n x_j \sum_{i=1}^k y_i$. Equivalently, $\sum_{i=1}^k x_i \sum_{j=1}^n y_j \le \sum_{j=1}^n x_j \sum_{i=1}^k y_i$, which says that

$$\sum_{i=1}^n y_i \sum_{i=1}^k (y_i - x_i) \ge \sum_{i=1}^k y_i \sum_{i=1}^n (y_i - x_i). \tag{2}$$

By *income order preservation*, the post-tax income profile $(y_i - x_i)_{i \in N}$ preserves the order of the pre-tax income profile y. Thus, (2) shows that the post-tax income profile Lorenz dominates the pre-tax income profile.

(ii) Let R be a rule satisfying *inequality reduction* and *consistency.* Suppose, by contradiction, that R is not *progressive.* Then, there exist $(N, y, T) \in \mathcal{D}$ and $i, j \in N$, such that $0 < y_i \le y_j$ and $R_i(N, y, T)/y_i > R_j(N, y, T)/y_j$. Let $a_i \equiv 1 - \frac{R_i(N,y,T)}{y_i}$ and $a_j \equiv 1 - \frac{R_j(N,y,T)}{y_j}$. Then, $a_i < a_j$, and therefore,

$$\frac{y_i}{y_i + y_j} > \frac{a_i y_i}{a_i y_i + a_j y_j} \ge \frac{\min\{a_i y_i, a_j y_j\}}{a_i y_i + a_j y_j}. \tag{3}$$

Let $T' \equiv R_i(N, y, T) + R_j(N, y, T)$. Consider $(\{i, j\}, (y_i, y_j), T') \in \mathcal{D}$. By *consistency,* $R_k(\{i, j\}, (y_i, y_j), T') = R_k(N, y, T)$ for each $k = i, j$, and therefore, $y_k - R_k(\{i, j\}, (y_i, y_j), T') = a_k y_k$ for each $k = i, j$. Thus, (3) contradicts *inequality reduction.*

(iii) Let R be a rule satisfying *consistency*, *revenue continuity* and *inequality reduction* (the same argument applies when *revenue continuity* is replaced by *revenue monotonicity*). Then, by the second statement, R satisfies *progressivity* and therefore *equal treatment of equals*, i.e., agents with the same income face the same tax burden. By Lemma 1 in Young (1987), R also satisfies *revenue monotonicity.* Suppose, by contradiction, that R violates *income order preservation.* Then, there exist $(N, y, T) \in \mathcal{D}$ and $i, j \in N$ such that $y_i < y_j$ and $y_i - x_i > y_j - x_j$, where $x \equiv R(N, y, T)$. By *consistency*, $R(\{i, j\}, (y_i, y_j), x_i + x_j) = (x_i, x_j)$. Let $n \in \mathbb{N}$ be such that

$$n - 1 > \frac{(y_j - x_j)(y_j - y_i)}{y_i(y_i - x_i - y_j + x_j)}. \tag{4}$$

Consider the problem $(N', y', T') \in \mathcal{D}$ with $N' = \{i, j\} \cup M$ such that $|M| = n - 1$, $M \cap N = \emptyset$, $y'_j = y_j$, $y'_k = y_i$ for each $k \in M \cup \{i\}$, and $T' = nx_i + x_j$. By *equal treatment of equals*, there exist $a, b \in \mathbb{R}_+$ such that for each $k \in M \cup \{i\}$, $R_k(N', y', T') = a$ and $R_j(N', y', T') = b$. If $a + b > x_i + x_j$, then by *consistency* and *revenue monotonicity*, $R(\{i, j\}, (y_i, y_j), a + b) = R(\{i, j\}, (y'_i, y'_j), a + b) = (a, b) \geq (x_i, x_j) = R(\{i, j\}, (y_i, y_j), x_i + x_j)$. Then $na + b > nx_i + x_j = T'$, contradicting *balancedness.* A similar contradiction occurs if $a + b < x_i + x_j$. Therefore, $a + b = x_i + x_j$. This, together with $na + b = nx_i + x_j$, implies $a = x_i$ and $b = x_j$. Therefore,

$$R_k(N', y', T') = \begin{cases} x_i & \text{if } k \in M \cup \{i\} \\ x_j & \text{if } k = j \end{cases}$$

Thus, by *inequality reduction*,

$$\frac{y_i}{y_j + ny_i} = \frac{\min_{k \in N'}\{y'_k\}}{\sum_{k \in N'} y'_k} \leq \frac{\min_{k \in N'}\{y'_k - R_k(N', y', T')\}}{\sum_{k \in N'}(y'_k - R_k(N', y', T')} = \frac{y_j - x_j}{(y_j - x_j) + n(y_i - x_i)},$$

which implies that

$$n \leq \frac{(y_j - x_j)(y_j - y_i)}{y_i(y_i - x_i - y_j + x_j)},$$

contradicting (4). ■

Proof. [**Proof of Example 2**] Let $N \equiv \{1, \ldots n\}$ and $(N, y, T) \in \mathcal{D}$. Without loss of generality, assume $y_1 \leq y_2 \leq \cdots \leq y_n$. Let $T^m \equiv \min\{T, \bar{T}\}$. Then $R(N, y, T) \equiv T^m e_n + L(N, y - T^m e_n, T - T^m)$.

To show that $R(\cdot)$ violates *income order preservation*, consider $(N, y, T) \equiv (\{1, 2, 3\}, (1, 3, 4), 2)$. Then $R(N, y, T) = (0, 0, 2)$ and the post-tax income profile is $(1, 3, 2)$, where the order of incomes of agents 2 and 3 is reversed.

We now show that $R(\cdot)$ satisfies *inequality reduction.* Let (N, y, T) be a problem and $y^* \equiv y - R(N, y, T)$ be the corresponding post-tax income profile. Let $\sigma \colon N \to N$ be such that $y^*_{\sigma(1)} \leq y^*_{\sigma(2)} \leq \cdots \leq y^*_{\sigma(n)}$. Let $y' \equiv y - T^m e_n$ and $T' = T - T^m$.

Since $\bar{T} \leq y_n - y_{n-2}$, $y_{n-2} \leq y_n - \bar{T}$. Then, using the fact that $L(\cdot)$ satisfies *income order preservation*, we can show that $y^*_{n-2} \leq y^*_n$ and $y^*_1 \leq \cdots \leq y^*_{n-2} \leq y^*_n$. Hence for each

$i \leq n-2$, $\sigma(i) = i$. Note that if $T^m \leq y_n - y_{n-1}$, $\sigma(n-1) = n-1$ and $\sigma(n) = n$ and that if $T^m > y_n - y_{n-1}$, $\sigma(n-1) = n$ and $\sigma(n) = n-1$.[8]

Note that for each $i \leq n-1$, $R_i(N,y,T) = L_i(N,y',T')$ and $R_n(N,y,T) = T^m + L_n(N,y',T')$.

Case 1. $T^m \leq y_n - y_{n-1}$. Then for each $i \in N$, $\sigma(i) = i$. Then for each $k \leq n-1$,

$$\begin{aligned}
\frac{\sum_{i=1}^{k}(y_{\sigma(i)} - R_{\sigma(i)}(N,y,T))}{Y-T} &= \frac{\sum_{i=1}^{k}(y_i - R_i(N,y,T))}{Y-T} = \frac{\sum_{i=1}^{k}(y'_i - L_i(N,y',T'))}{Y'-T'} \\
&\geq \frac{\sum_{i=1}^{k} y'_i}{Y'} = \frac{\sum_{i=1}^{k} y_i}{Y-T^m} \geq \frac{\sum_{i=1}^{k} y_i}{Y},
\end{aligned}$$

where the first inequality holds by the inequality reduction property of $L(\cdot)$.

Case 2. $T^m > y_n - y_{n-1}$. Then for each $i \leq n-2$, $\sigma(i) = i$, $\sigma(n-1) = n$, and $\sigma(n) = n-1$.

For each $k \leq n-2$, by the same reasoning as above we show that the share of the sum of k lowest incomes after tax is greater than (or equal to) the sum of k lowest incomes before tax. For $k = n-1$,

$$\begin{aligned}
\frac{\sum_{i=1}^{n-1}(y_{\sigma(i)} - R_{\sigma(i)}(N,y,T))}{Y-T} &= \frac{\sum_{i=1}^{n-2}(y_i - R_i(N,y,T)) + y_n - R_n(N,y,T)}{Y-T} \\
&= \frac{\sum_{i=1}^{n-2}(y'_i - L_i(N,y',T')) + y_n - (T^m + L_n(N,y',T'))}{Y-T} \\
&= \frac{\sum_{i=1}^{n-2}(y'_i - L_i(N,y',T')) + y'_n - L_n(N,y',T')}{Y'-T'} \\
&\geq \frac{\sum_{i=1}^{n-2} y'_i + y'_n}{Y'} = \frac{\sum_{i=1}^{n-2} y_i + y_n - T^m}{Y-T^m} \\
&\geq \frac{\sum_{i=1}^{n-1} y_i}{Y},
\end{aligned}$$

where the first inequality holds for the inequality reduction property of $L(\cdot)$ and the second inequality holds because of $T^m \leq \bar{T} \leq \frac{Y(y_n - y_{n-1})}{y_n}$. ■

[8] If $T^m \leq y_n - y_{n-1}$, then $y'_{n-1} = y_{n-1} \leq y_n - T^m = y'_n$. Since $L(\cdot)$ preserves the order of incomes, $y'_{n-1} - L_{n-1}(N,y',T') \leq y'_n - L_n(N,y',T')$, that is, $y^*_{n-1} = y_{n-1} - L_{n-1}(N,y',T') \leq y_n - T^m - L_n(N,y',T') = y^*_n$. So $y^*_{n-1} \leq y^*_n$, which means $\sigma(n-1) = n-1$ and $\sigma(n) = n$. An analogous proof can be given for the case in which $T^m \geq y_n - y_{n-1}$.

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