2009/51

Multi-assets real options

Joachim GAHUNGU and Yves SMEERS

CORE
Voie du Roman Pays 34
B-1348 Louvain-la-Neuve, Belgium.
Tel (32 10) 47 43 04
Fax (32 10) 47 43 01
E-mail: corestat-library@uclouvain.be
http://www.uclouvain.be/en-44508.html

CORE DISCUSSION PAPER
2009/51

Multi-assets real options

Joachim GAHUNGU[1] and Yves SMEERS[2]

September 2009

Abstract

Real options present a wide topic in investment litterature nowadays. However, despite big advances in the single asset investment pricing, the theory is miser of informations about problems involving more than one asset. We show in this paper that using dynamic programming, one can find an analytic trigger for a three assets simple exchange problem. Although we get a forward investment rule, one can not find the precise option value ex ante but only an average value. The precise option value depends on the first exit time from the continuation region which is stochastic.

This is a quite intuitive effect of the course of dimensionality of the problem. Valuating a single asset project gives a single condition for the optimal decision rule. The same holds for the simple exchange problem with two assets since the value of the project just depends on the price over cost ratio. In a three assets problem, as the project don't depend anymore of a single state variable, one can't run for a decision rule depending on the first exit time from the continuation region.

Keywords: real options, dynamic programming, price and cost uncertainty.

JEL Classification: C61, C63, G11

---

[1] Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium.
E-mail: joachim.gahungu@uclouvain.be
[2] Université catholique de Louvain, CORE and INMA, B-1348 Louvain-la-Neuve, Belgium.
E-mail: yves.smeers@uclouvain.be. This author is also member of ECORE, the association between CORE and ECARES.

Financial support from GDF-Suez is gratefully acknowledged.

This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

1 Introduction

A firm that observes a rise in demand of its output is induced to increase production to catch a higher immediate profit. To do so, it may have to invest in additionnal capacity. Sometimes, these investments have no salvage value. They are firm specific and hence cannot be recovered should the demand fall or the production cost rise. For instance, in power generation, a Combined Cycle Gas Turbine looses some of its value if the price of gas rises sharply and the utilisation of the plant decreases. There is little incentive for a competitor to buy the plant and hence no way to recover the entire initial expenditure. Such firm specific investment has a sunk cost : it is irreversible i.e. it cannot be recovered assuming downturn.

Irreversibility requires to act carefully. On one side, investing too early exposes to damages assuming downturn. On the other side, waiting implies lost immediate profits. One needs an investment rule that gives the perfect trade-off between the option to wait for a safer opportunity and the decision to go ahead with the investment as soon as one can capture an immediate profit.

The combination of the two features — irreversibility and time flexibility — implies an opportunity cost. Because of irreversibility, the opportunity to wait has a value that one must take into account in the investment decision. Investing entails not only the sunk cost of the project but also the cost of abandoning the option to invest later. The optimal behavior requires to wait until the value of launching the project just equals its sunk cost plus the opportunity cost of abandoning the right to wait. This is the well known analogy between investment problems and financial options : the right to delay an expenditure until an optimal time is similar to an american call option. The option to wait can accordingly be priced using either *contingent claim* or *dynamic programming* depending on whether the value of the project is or is not perfectly correlated with an asset traded on the market.[1]

Contingent claim analysis is the traditional approach for valuing options in finance. It requires that the value of the project — here the option — is at all moment replicated by a portfolio of traded assets. The value ot the option at any time is then the market value of that portfolio. This approach is quite demanding in term of assumptions. The interested reader can consult Wilmott[18], Shreve[16], Karatzas and Shreve[7] and Bingham and Kiesel[3]. Theses assumptions do not always fit the reality of investment in some physical assets. For instance, one cannot replicate the value of a nuclear plant by a portfolio of traded assets, let alone do this on a continuous basis by continuously trading the elements of that portfolio. We therefore resort to the alternative approach of dynamic programming in the rest of the paper.

Dynamic programming avoids problems of market completeness or perfect correlation with some traded asset. There is no strong financial assumption except the existence of an exogeneous discount rate. This approach is more akin to classical corporate finance. It does not require spanning assets but supposes a discount rate that embeds risk aversion, referred to as a "risk discount rate". The valuation of the option to wait by dynamic programming is based on Hamilton-Jacobi-Bellman equation. The essence of this method is to compare the immediate payoff coming from an immediate investment to the expected payoff coming from the same delayed investment. One has to wait until the value of the immediate action exceeds the expected value of the delayed project.

Finding the optimal timing of an investment in a real asset implies identifying

[1]A comparison of these two approaches is given in Dixit and Pindyck[4].

the external conditions that justify the investment. More specifically one is looking for special values of the demand and/or of the cost that will trigger an irreversible investment. The set of the values is the border between the continuation region in which the optimal decision is to wait and the exploitation region, in which we launch the project. Because this border is initially unknown, real option problems are free boundary problems and the determination of the unknown bound needs a double sets of conditions known as value matching and smooth pastings (see section 2).

The notion of real option was developed by Myers[12]. The idea (see Myers[12], Kester[8]) to link investment opportunities and American call options leads to the simple result than the option to wait is an advantage that allows one to invest more safely. McDonald and Siegel[11] show in their seminal paper that the incentive to wait rises with the uncertainty of the project leading to the famous paradigm stated in Dixit and Pindyck's book[4] : "(...) the simple NPV rule is not just wrong; it is often very wrong." Pindyck[14] notes the importance of delaying actions : for irreversible investment, one get one's money's worth waiting for new informations and higher value of the immediate post investment profit.

The theory of real option was extended from individual project valuation to the analysis of investment behaviour in market economics. Specifically, Leahy[9] shows that the optimal strategy of a firm acting in a competitive industry follows a myopic behavior. Oligopolistic situations were discussed by Slade[17], Baldursson[1], Baldursson and Karatzas[6]. Grenadier[5] finally proves than the Leahy's myopic arguments still holds in symmetric oligopolies for Cournot competition.

The real options litterature usually treats a single uncertainty factor. Directly related to our paper, the first two assets analysis was conducted by McDonald and Siegel[11] with the problem of "price and cost uncertainty".[2] Additionnal work on two assets uncertainty in irreversible investments includes Pindyck[15] and Bertola[2].

This paper extends the work of McDonald and Siegel[11]. It provides analytic solutions for more than two assets real options problems. McDonald and Siegel show that the two asset problem can be cast in the single state variable case and hence has a simple solution. With more than 2 assets one cannot reduce the problem to a single state variable and hence obtain an investment threshold that is given by a single number. The boundary is a surface and hence the option value is not fully known before the first exit time of the continuation region. It is this problem with more than 2 assets that we examine in this paper.

We structure the work as follows : the second section presents the price and cost uncertainty problem. We prove that the solution found by McDonald and Siegel[11] is the only form that can solve the equation along with the value matching and the smooth pasting conditions. The third section is the presentation of the solution of the 3 assets exchange problem. Section 4 shows simulations and gives useful comments. Section 5 extends the discussion to more than 3 assets. Section 6 gives solutions to additional problems. Section 7 concludes.

2 The (1,1) exchange problem

The first two asset real option model has been developed by McDonald and Siegel[11], hereafter referred to as "the price and cost uncertainty" or the (1,1) exchange prob-

[2]McDonald and Siegel[11] treated the first (one uncertainty) real option model — known as the McDonald and Siegel example — and the first two uncertainties problem — the price and cost uncertainty problem — in the same paper.

lem.[3] Consider the American perpetual right to exchange one asset for an other. Every problem that we will define herafter concerns American and perpetual security. See Margrabe[10] for a similar non perpetual European right.

Definition 1 (The (1,1) exchange problem)**.** *Consider the perpetual American option to pay the stochastic cost* $K(t)$ *against a project of stochastic value* $S(t)$*. When is the right time to exercise this option?*

We make the standard assumption of geometric Brownian motion processes and write

$$\begin{aligned} dS(t) &= \mu_S S(t)dt + \sigma_S S(t)dz_S(t, \omega_S) \\ dK(t) &= \mu_K K(t)dt + \sigma_K K(t)dz_K(t, \omega_K) \end{aligned}$$

with $\mathbb{E}[dz_S dz_K] = \rho_{SK} dt$ where dz_S and dz_K are respectively the S and K Wiener increments. We note :

- $\Omega = \Omega_S \times \Omega_K$ the set of all the events for the 2 processes.
- $\omega \in \Omega$ a special event for the set of the 2 processes $\{S(t), K(t)\}$ i.e.

  $$\omega = (\omega_S, \omega_K).$$

  The single term ω includes the randomness of the two processes.
- We define $\mathcal{F}_t$ to be the σ-algebra generated by the two variables $\{S(s), K(s)\}_{0 \le s \le t}$. Note that $\{\mathcal{F}_t\}$ is increasing i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \le t$. The random processes S and K are $\mathcal{F}_t$-adapted.

We use dynamic programming throughout the paper and note r the risk discount rate. It is of course not the risk-free interest rate : we work on the the real measure and adapt consequently the discount rate to risk aversion.

We define the continuation region as the region in which the exchange of asset K against asset S is not optimal. One can write the Bellman equation on the continuation region. The value of the project F is obviously a function of the two economic variables S and K.

$$F(S, K) = \max\left\{S - K, \frac{1}{1 + r dt}\mathbb{E}\Big[F(S, K) + dF\Big|S, K\Big]\right\}$$

In the continuation region

$$F(S, K) = \frac{1}{1 + r dt}\mathbb{E}\Big[F(S, K) + dF\Big|S, K\Big]$$

leading to the Bellman partial differential equation

$$\mu_S S F_S + \mu_K K F_K + \frac{1}{2}\Big(\sigma_S^2 S^2 F_{SS} + \sigma_K^2 K^2 F_{KK} + 2\rho_{SK}\sigma_S\sigma_K KSF_{SK}\Big) - rF = 0 \qquad (1)$$

that directly follows by the two dimensions Ito lemma.[4] Note that the differential equation (1) only holds on the continuation region. We refer to a solution of (1) as a Bellman function.

[3] We use dynamic programming instead of contingent claim in order to lay the ground of the methodology used in the following sections.

[4] For a formal derivation see Dixit and Pindyck[4], chapter 6, section 5, *Price and cost uncertainty*. Throughout this paper we use standard notation for the derivative e.g. $\partial_{SS} F = F_{SS}$ and so on.

2.1 The homogeneity argument

McDonald and Siegel[11] state the solution of (1) as the following homogeneous function.

$$F(S,K) = AK\left(\frac{S}{K}\right)^{\beta} \tag{2}$$

Introducing $F(S,K)$ in the differential equation (1), one finds that β satisfies the following quadratic equation

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\beta(\beta-1)\left(\sigma_S^2 + \sigma_K^2 - 2\rho_{SK}\sigma_S\sigma_K\right) + \beta(\mu_S - \mu_K) - (r - \mu_K) = 0 \tag{3}$$

hereafter referred to as the "fundamental quadratic" of the (1,1) exchange. This equation is known (see e.g. Dixit and Pindyck[4]) to have both a positive and a negative root, the positive root being greater than one. Because the value of the project increases with S, one must insert the positive root in the Bellman function.

Dixit and Pindyck[4] discuss various features of the solution of the characteristic equation. Specifically, the positive and negative roots respectively tend toward 1 and 0 as uncertainty increases.

In order to find the exchange trigger surface, one applies the two following optimal investment conditions.

1. The value matching condition : at the optimal exercise time τ, the value of the investment option matches the value of lauching the project i.e.

$$F(S(\tau), K(\tau)) = S(\tau) - K(\tau).$$

2. The smooth pasting conditions : at the optimal exercise point, the value of the project must be smooth with respect to the S and K variables. See Dixit and Pindyck[4] in chapter 4, Appendix C for a formal proof.

$$\begin{aligned} [\partial_S F]\ (S(\tau), K(\tau)) &= 1 \\ [\partial_K F]\ (S(\tau), K(\tau)) &= -1 \end{aligned}$$

The derivation of the final result follows McDonald and Siegel[11] and is now standard.

Proposition 1 (Optimal trigger of the (1,1) exchange problem)**.** *The trigger of the (1,1) exchange problem is given by*

$$\mathbb{T}^1 \equiv \left\{(S,K) \quad | \quad S = \frac{\beta_1}{\beta_1 - 1}K\right\} \tag{4}$$

where β_1 is the positive root of the fundamental quadratic $\mathcal{Q}(\beta)$.

Proof. See McDonald and Siegel[11] or Appendix A. □

We note this trigger $\mathbb{T}^1$ to indicate that this relation define a 1-dimensional manifold in the $S \times K$ space. Note that this result is strikingly similar to the investment trigger of the single uncertain factor problem. The only difference is that K now represents a random variable. One can find an expression for the volatility and the drift of the exchange and thus fall back on the single asset rule. Such simplification also appears in Margrabe[10].

The right time to exercise the option is the first exit time from the continuation region. It is given by a random time $\tau(\omega)$ defined by

$$\tau(\omega) = \min\left\{t \quad | \quad S(t,\omega) = \frac{\beta_1}{\beta_1 - 1}K(t,\omega)\right\}$$

where the vertical bar | means "such that". If the event $S(t,\omega) = \frac{\beta_1}{\beta_1-1}K(t,\omega)$ has zero measure, we set $\tau(\omega) = +\infty$.

The continuation region is given by

$$\mathcal{CR} \equiv \left\{(S,K) \quad | \quad S \leq \frac{\beta_1}{\beta_1 - 1}K\right\}$$

and because the random time $\tau : \omega \to [0,\infty]$ is the first exit time from a given subset $\mathcal{CR} \in \mathbb{R}^2$, τ is a stopping time (see Oksendal[13]).

2.2 Elaboration of the solution : the uncorrelated case

It is worth elaborating on the form of equation (2), starting from the uncorrelated exchange problem and seeing what variable separation brings us. Consider the case where there is no correlation. The partial differential equation simplifies to

$$\mu_S SF_S + \mu_K KF_K + \frac{1}{2}\left(\sigma_S^2 S^2 F_{SS} + \sigma_K^2 K^2 F_{KK}\right) - rF = 0. \tag{5}$$

Assume a multiplicative form for the solution.

$$F(S,K) = Af(S)g(K)$$

with $A > 0$. The problem becomes separable as one can rewrite this differential equation as an equality between a function of S and a function of K. One can find the solution of this two dimensional real option problem by solving two separate one dimensional real option problems. Mathematically, one can prove that the multiplicative solution has to be of the form $AS^{\beta_1}K^{\lambda_2}$.

Proposition 2 (The power form of the (1,1) exchange). *Assuming a separable form for the Bellman function of the (1,1) exchange, it has the power form*

$$F(S,K) = AS^{\beta_1}K^{\lambda_2} \tag{6}$$

where β_1 and λ_2 are the roots of two fundamental quadratics.

Proof. See Appendix A. □

Assuming this particular solution, one can then prove that this function has to be homogeneous of degree one to satisfy the value matching and the smooth pasting conditions.

Applying the smooth pasting conditions one obtains

Proposition 3 (Condition at trigger). *At an optimal exercise time τ, the point $(S(\tau), K(\tau)) \in \mathbb{T}^1$ verifies*

$$S(\tau) = \left(-\frac{\beta_1}{\lambda_2}\right)K(\tau) \tag{7}$$

where β_1 and λ_2 are the exponents of the separable Bellman function (6).

Proof. Using the solution $F(S, K) = AS^{\beta_1}K^{\lambda_2}$ in the two smooth pasting conditions leads to the two relations

$$\frac{\beta_1}{S(\tau)}F(S(\tau), K(\tau)) = 1 \tag{8}$$

$$\frac{\lambda_2}{K(\tau)}F(S(\tau), K(\tau)) = -1. \tag{9}$$

The two processes $S(t, \omega)$ and $K(t, \omega)$ will always be positive because zero is an absorbant barrier for the geometric Brownian motion. Moreover $F(S, K)$ is the value of an option, so it has to be greater or equal to zero. Thus these two equations implies that $\beta_1 > 0$ and that $\lambda_2 < 0$. Now, we note that $(8) = -(9)$ i.e.

$$\frac{\beta_1}{S(\tau)}F(S(\tau), K(\tau)) = -\frac{\lambda_2}{K(\tau)}F(S(\tau), K(\tau))$$

and we simplify by $F(S(\tau), K(\tau))$ leading to

$$S(\tau) = \left(-\frac{\beta_1}{\lambda_2}\right)K(\tau) \tag{10}$$

which proves the statement. □

One observes that the combination of the two smooth pastings leads to one trigger condition. Simplifying by the function $F(S(\tau), K(\tau))$ eliminates the unknown coefficient A so the set of the two smooth pastings (8) and (9) is equivalently reformulated as the trigger (7) plus one of these two smooth pastings.

Solving for the value matching condition leads to homogeneity result.

Proposition 4 (Homogeneity in the (1,1) exchange problem)**.** *The Bellman function (6) of the (1,1) exchange problem satisfies*

$$\beta_1 + \lambda_2 = 1 \tag{11}$$

with $\beta_1 > 1$ *and* $\lambda_2 < 0$.

Proof. See Appendix A. □

This proposition shows that homogeneity should not be seen as an assumption. It is imposed by the value matching and the smooth pasting conditions in the exchange of two assets.

Knowing that the sum of the positive root β_1 and the negative root λ_2 must be equal to one, one can rewrite the general solution with a single exponent.

$$F(S, K) = AK\left(\frac{S}{K}\right)^{\beta_1} \tag{12}$$

This is just the McDonald and Siegel[11] form. To solve the Bellman differential equation, β_1 has to be the positive root of the quadratic equation

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\beta(\beta - 1)\left(\sigma_S^2 + \sigma_K^2\right) + \beta(\mu_S - \mu_K) - (r - \mu_K) = 0.$$

One can similarly fall back on McDonald and Siegel trigger of Proposition 1 by using the homogeneity condition $\beta_1 + \lambda_2 = 1$ on the trigger condition (7).

This variable separation also shows that one can alternatively write

$$F(S,K) = AS\left(\frac{K}{S}\right)^{\lambda_2}$$

without any difference of results. In the first formulation, the exchange option is expressed as a dynamic portfolio containing $K(t,\omega_K)$ perpetual american calls on underlying asset $S(t,\omega_S)/K(t,\omega_K)$ and of strike one. In the second, it is expressed as a dynamic portfolio containing $S(t,\omega_S)$ perpetual american puts on underlying asset $K(t,\omega_K)/S(t,\omega_S)$ and of strike one. The two options are obviously the same. This dual view of the problem is known as changing numéraire in risk neutral valuation. See e.g. Shreve[16], Wilmott[18] or Bingham and Kiesel[3] for exhaustive informations on changing numéraire.

This discussion gave a motivation for the multiplicative form of the McDonald and Siegel[11] solution : We first noted that in the absence of correlation, the Bellman differential equation is separable so the multiplicative form comes naturally in mind. We then noticed that solving our differential equation with the value matching and smooth pastings conditions requires homogeneity for the Bellman function. We now extend these findings to the general correlated case.

2.3 Elaboration of the solution : the correlated case

Putting back correlation in the differential equation, one cannot infer directly a separable form for the Bellman function and hence Proposition 2 no longer holds. However one can still try its multiplicative form

$$F(S,K) = AS^{\beta_1}K^{\lambda_2}.$$

One observes that this function solves Bellman differential equation with correlation and has to be homogeneous to solve the value matching and the smooth pasting conditions. Propositions 3 and 4 obviously still hold, leading to

Proposition 5 (The general solution of the (1,1) exchange problem)**.** *The Bellman function of the (1,1) exchange problem is given by*

$$F(S,K) = AK\left(\frac{S}{K}\right)^{\beta_1} \tag{13}$$

with β_1 *the positive root of a fundamental quadratic* $\mathcal{Q}(\beta) : \mathbb{R} \to \mathbb{R}$.

Proof. One can check by direct calculus that the solution (13) solves the Bellman partial differential equation with correlated assets along with the value matching and smooth pasting conditions. This only requires β_1 to be the positive root of the more involved fundamental quadratic

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\beta(\beta-1)\left(\sigma_S^2 + \sigma_K^2 - 2\rho_{SK}\sigma_S\sigma_K\right) + \beta(\mu_S - \mu_K) - (r - \mu_K) = 0$$

to deal with the correlation between S and K.

□

3 The (2,1) exchange problem

The previous problem is a particular case of a wider class of problems defined as the (n, m) *exchange problem*.

Definition 2 ((n,m) exchange problem)**.** *Consider the perpetual american option to exchange a bundle of* n *stochastic assets against a bundle of* m *others. When is the right time to exercise this option?*

McDonald and Siegel[11]'s "price and cost uncertainty" is thus a (1,1) exchange that we extend to a (2,1) exchange problem in this section. Of course, if one can solve the (2,1) exchange problem, one can also solve the (1,2) exchange problem in a similar way.

Example 1. *[The (2,1) exchange problem] Consider the perpetual American option to pay the sum of two stochastic sunk costs* $K_1(t)$ *and* $K_2(t)$ *for a project of stochastic value* $S(t)$*. When is the right time to exercise this option?*

The three assets follow geometric Brownian motions i.e. :

$$\begin{aligned}
dS(t) &= \mu_S S(t)dt + \sigma_S S(t)dz_S(t, \omega_S) \\
dK_1(t) &= \mu_{K_1} K_1(t)dt + \sigma_{K_1} K_1(t)dz_{K_1}(t, \omega_{K_1}) \\
dK_2(t) &= \mu_{K_2} K_2(t)dt + \sigma_{K_2} K_2(t)dz_{K_2}(t, \omega_{K_2})
\end{aligned}$$

We allow for correlation between random processes. Assume $\mathbb{E}[dz_S dz_{K_1}] = \rho_{SK_1} dt$, $\mathbb{E}[dz_S dz_{K_2}] = \rho_{SK_2} dt$, $\mathbb{E}[dz_{K_1} dz_{K_2}] = \rho_{K_1 K_2} dt$.

We note :

- $\Omega = \Omega_S \times \Omega_{K_1} \times \Omega_{K_2}$ the set of events for the 3 processes.
- $\omega \in \Omega$ a special event for the set of the 3 processes $\{S(t), K_1(t), K_2(t)\}$ i.e.

  $$\omega = (\omega_S, \omega_{K_1}, \omega_{K_2}).$$

  The single term ω includes the randomness of the three processes.
- We define $\mathcal{F}_t$ to be the σ-algebra generated by the random variables $\{S(s), K_1(s), K_2(s)\}_{0 \le s \le t}$. Note that $\{\mathcal{F}_t\}$ is increasing i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \le t$. The random processes S, K_1 and K_2 are $\mathcal{F}_t$-adapted.

The following discussion essentially adapts the procedure of the $(1, 1)$ exchange problem to derive the Bellman function. We first solve the separable problem and show that the value matching and the smooth pasting conditions imply a homogeneous form for the Bellman function. We then analyse the specific behavior of this three assets exchange problems.

3.1 The free boundary problem

The investment trigger is the set of all the triplets (S, K_1, K_2) for which it is optimal to invest. We shall show that it is a surface — or a 2-dimensional manifold — in the 3 dimensional space $S \times K_1 \times K_2$. This 2-dimensional manifold splits the whole space in two regions : the continuation region and the exploitation region.

The right time to exercise the option is the first exit time from the continuation region. In other terms, this is the first time when one hits the trigger surface. This is a random time $\tau : \Omega \to [0, \infty]$. We note

$$\tau : \omega \to \tau(\omega).$$

In a two variables problem — the (1,1) exchange — one observed that the right time to invest was given by an equation linking the two variables of the problem : the trigger condition has the form $S^* = \alpha K^*$. The set of optimal couples (S^*, K^*) is therefore a line in the $S \times K$ plan. The right time to exercise the option is

$$\tau = \min\Big\{t \quad | \quad S(t) = \alpha K(t)\Big\}.$$

In a three variables problem — like the (2,1) exchange — it is thus natural to expect that the investment trigger is a surface. One should find a 2-dimensional set of triplets (S^*, K_1^*, K_2^*) satisfying the optimal investment criterion. We expect something of the form

$$\tau = \min\Big\{t \quad | \quad \big(S(t), K_1(t), K_2(t)\big) \in \mathbb{T}^2\Big\}$$

with $\mathbb{T}^2$ designating a 2-dimensional manifold.

This investment trigger $\mathbb{T}^2$ is the unknown bound of a free boundary problem. Specifically, one needs to find the Bellman function $F(S, K_1, K_2)$ and the bound surface $\mathbb{T}^2$ such that :

- The Bellman function $F(S, K_1, K_2)$ solves

$$\begin{aligned} \mu_S S F_S &+ \mu_{K_1} K_1 F_{K_1} + \mu_{K_2} K_2 F_{K_2} \\ &+ \rho_{SK_1} \sigma_S \sigma_{K_1} S K_1 F_{SK_1} \\ &+ \rho_{SK_2} \sigma_S \sigma_{K_2} S K_2 F_{SK_2} \\ &+ \rho_{K_1 K_2} \sigma_{K_1} \sigma_{K_2} K_1 K_2 F_{K_1 K_2} \\ &+ \frac{1}{2}\Big(\sigma_S^2 S^2 F_{SS} + \sigma_{K_1}^2 K_1^2 F_{K_1 K_1} + \sigma_{K_2}^2 K_2^2 F_{K_2 K_2}\Big) - rF = 0 \end{aligned} \tag{14}$$

  in the continuation region.

- At each point $\big(S(\tau), K_1(\tau), K_2(\tau)\big) \in \mathbb{T}^2$, the value of the project matches the investment cost plus the option to defer :

  1. In value (value matching)

  $$F\big(S(\tau), K_1(\tau), K_2(\tau)\big) = S(\tau) - K_1(\tau) - K_2(\tau).$$

  2. In slope (smooth pastings)

  $$\begin{aligned} [\partial_S F]\ \big(S(\tau), K_1(\tau), K_2(\tau)\big) &= 1 \\ [\partial_{K_1} F]\ \big(S(\tau), K_1(\tau), K_2(\tau)\big) &= -1 \\ [\partial_{K_2} F]\ \big(S(\tau), K_1(\tau), K_2(\tau)\big) &= -1. \end{aligned}$$

3.2 The uncorrelated case

Like in the two assets problem, we motivate the form of the Bellman function starting from the no correlation case. With this assumption, the differential equation in the continuation region becomes

$$\begin{aligned}\mu_S S F_S &+ \mu_{K_1} K_1 F_{K_1} + \mu_{K_2} K_2 F_{K_2} \\ &+ \frac{1}{2}\Big(\sigma_S^2 S^2 F_{SS} + \sigma_{K_1}^2 K_1^2 F_{K_1 K_1} + \sigma_{K_2}^2 K_2^2 F_{K_2 K_2}\Big) - rF = 0.\end{aligned}$$

We can assume the separable multiplicative form

$$F(S, K_1, K_2) = Af(S)g(K_1)h(K_2)$$

for the Bellman function, where A is a constant to determine.

The function $F(S, K_1, K_2)$ represents the value of the option to exchange the sum of $K_1(t)$ and $K_2(t)$ for $S(t)$. It must be increasing in S and decreasing in both K_1 and K_2. The problem is separable as we first assume no correlation between random processes. Solving using variable separation leads to the following result.

Proposition 6 (The power form of the (2,1) exchange)**.** *Assuming a separable form for the Bellman function of the (2,1) exchange, it has the power form*

$$F(S, K_1, K_2) = AS^{\beta_1} K_1^{\lambda_2} K_2^{\gamma_2} \tag{15}$$

where β_1, λ_2 and γ_2 are the roots of three fundamental quadratics.

Proof. See Appendix B. □

Assuming this particular solution, one will prove that this function has to be homogeneous of degree one to solve the value matching and the smooth pasting conditions.

The application of the smooth pasting conditions leads to

Proposition 7 (Conditions at trigger)**.** *At an optimal exercise time τ, the point $(S(\tau), K_1(\tau), K_2(\tau)) \in \mathbb{T}^2$ verifies*

$$K_2(\tau) = \Big(\frac{\gamma_2}{\lambda_2}\Big) K_1(\tau) \tag{16}$$

$$S(\tau) = \Big(-\frac{\beta_1}{\lambda_2}\Big) K_1(\tau) \tag{17}$$

where β_1, λ_2 and γ_2 are the exponents of the separable Bellman function (15).

Proof. Introducing the solution $F(S(\tau), K_1(\tau), K_2(\tau)) = AS^{\beta_1}(\tau)K_1^{\lambda_2}(\tau)K_2^{\gamma_2}(\tau)$ in the three smooth pasting conditions leads to the three relations

$$\frac{\beta_1}{S(\tau)} F(S(\tau), K_1(\tau), K_2(\tau)) = 1 \tag{18}$$

$$\frac{\lambda_2}{K_1(\tau)} F(S(\tau), K_1(\tau), K_2(\tau)) = -1 \tag{19}$$

$$\frac{\gamma_2}{K_2(\tau)} F(S(\tau), K_1(\tau), K_2(\tau)) = -1. \tag{20}$$

Because $F(S(\tau), K_1(\tau), K_2(\tau))$ is positive, β_1 has to be positive and λ_2 and γ_2 has to be negative.

Then we note that $(18) = -(19)$ and $(19) = -(20)$. Simplifying by $F(S(\tau), K_1(\tau), K_2(\tau))$ in these two equations we obtain the announced statement. □

The three smooth pastings lead to two trigger conditions. The simplification by the function $F(S(\tau), K_1(\tau), K_2(\tau))$ eliminates the unknown coefficient A then the set of the three smooth pastings (18), (19) and (20) is equivalent to the set of the conditions at trigger (16) and (17) plus one of these three smooth pastings.

From now on we focus on the two conditions at trigger (16) and (17).

$$\begin{aligned}\frac{K_2(\tau)}{K_1(\tau)} &= \frac{\gamma_2}{\lambda_2} \\ \frac{S(\tau)}{K_1(\tau)} &= -\frac{\beta_1}{\lambda_2}\end{aligned}$$

For a given triplet $(\beta_1, \gamma_2, \lambda_2)$, these two relations define a 1-dimensional manifold that gives a relation linking the price $S(\tau)$ and the costs $K_1(\tau)$ and $K_2(\tau)$ at each exercice time τ. To find the investment trigger, one has to find the set of allowed triplets $\{(\beta_1, \lambda_2, \gamma_2)\}$ which we shall show is itself a 1-dimensional manifold. Then one obtains the trigger surface as the cartesian product of these two 1-dimensional manifolds.

We begin by noting that the right time to exercise the option is the first exit time from the continuation region given by

$$\tau = \min\left\{t \quad | \quad \frac{K_2(t)}{K_1(t)} = \frac{\gamma_2}{\lambda_2}, \frac{S(t)}{K_1(t)} = -\frac{\beta_1}{\gamma_2}, (\beta_1, \lambda_2, \gamma_2) \in \Sigma\right\}.$$

We now characterise the set Σ.

From Proposition 7 and using the value matching

$$F(S(\tau), K_1(\tau), K_2(\tau)) = S(\tau) - K_1(\tau) - K_2(\tau)$$

one can express A in terms of value $S(\tau)$.

$$\frac{1}{A} = \beta_1 S^{\beta_1 - 1}(\tau) K_1^{\lambda_2}(\tau) K_2^{\gamma_2}(\tau) = \beta_1 S^{\beta_1 - 1}(\tau)\left(\frac{-\lambda_2}{\beta_1}\right)^{\lambda_2} S^{\lambda_2}(\tau)\left(\frac{-\gamma_2}{\beta_1}\right)^{\gamma_2} S^{\gamma_2}(\tau) \qquad (21)$$

Applying the value matching condition with the relations (21), (16) and (17) leads to homogeneity result.

Proposition 8 (Homogeneity in the (2,1) exchange problem)**.** *The Bellman function (15) of the (2,1) exchange problem satisfies*

$$\beta_1 + \lambda_2 + \gamma_2 = 1 \qquad (22)$$

with $\beta_1 > 1$ *and* $\lambda_2, \gamma_2 < 0$.

Proof. See Appendix B. □

Rewriting the Bellman function as

$$F(S, K_1, K_2) = AS\left(\frac{K_1}{S}\right)^{\lambda_2}\left(\frac{K_2}{S}\right)^{\gamma_2} \qquad (23)$$

and introducing the homogeneity condition in (21) one obtains the coefficient A in terms of λ_2 and γ_2.

$$A(\lambda_2, \gamma_2) = \frac{1}{(1 - \lambda_2 - \gamma_2)^{(1-\lambda_2-\gamma_2)}(-\lambda_2)^{\lambda_2}(-\gamma_2)^{\gamma_2}} \qquad (24)$$

Note that the homogeneity condition reduces the possible values for the triplets $(\beta_1, \gamma_2, \lambda_2)$: one has 3 unknowns values and one condition then the set of possible triplets become a 2-dimensional manifold.

One can rewrite our stopping time solution as :

$$\tau = \min \left\{ t \quad | \quad \frac{K_2(t)}{K_1(t)} = \frac{\gamma_2}{\lambda_2}, \frac{S(t)}{K_1(t)} = \frac{\lambda_2 + \gamma_2 - 1}{\lambda_2}, (\lambda_2, \gamma_2) \in \Sigma' \right\}$$

where the set Σ' remains to define. Substituting the solution (23) in the uncorrelated Bellman partial differential equation (14), one find that the couple (λ, γ) solves the following equation.

$$\begin{aligned}
\mathcal{Q}(\gamma, \lambda) \equiv\ & \frac{1}{2}\lambda(\lambda - 1)\left(\sigma_S^2 + \sigma_{K_1}^2 - 2\rho_{SK_1}\sigma_S\sigma_{K_1}\right) + \lambda(\mu_{K_1} - \mu_S) \\
+\ & \frac{1}{2}\gamma(\gamma - 1)\left(\sigma_S^2 + \sigma_{K_2}^2 - 2\rho_{SK_2}\sigma_S\sigma_{K_2}\right) + \gamma(\mu_{K_2} - \mu_S) \\
-\ & (r - \mu_S) = 0.
\end{aligned}$$

The right time to exercise the option is thus given by

$$\tau = \min \left\{ t \quad | \quad \frac{K_2(t)}{K_1(t)} = \frac{\gamma_2}{\lambda_2}, \frac{S(t)}{K_1(t)} = \frac{\lambda_2 + \gamma_2 - 1}{\lambda_2}, \ \mathcal{Q}(\gamma_2, \lambda_2) = 0 \right\}.$$

3.3 The correlated case

The problem is no longer separable in the correlated case and Proposition 6 no longer holds. One can still try the multiplicative form

$$F(S, K_1, K_2) = S^{\beta_1} K_1^{\lambda_2} K_2^{\gamma_2}$$

to check whether it verifies the differential equation (14) with correlation.

One observes that this function solves the Bellman differential equation with correlation and has to be homogeneous to solve the value matching and the smooth pasting conditions. Propositions 7 and 8 still hold and can be restated as follows :

Proposition 9 (The general solution of the (2,1) exchange problem)**.** *The Bellman function of the (2,1) exchange problem is given by*

$$F(S, K_1, K_2) = A(\lambda_2, \gamma_2) S \left(\frac{K_1}{S}\right)^{\lambda_2} \left(\frac{K_2}{S}\right)^{\gamma_2} \tag{25}$$

where (λ_2, γ_2) *belong to the 0-level curve of an interaction fundamental quadratic form* $\mathcal{Q}(\lambda, \gamma) : \mathbb{R}^2 \to \mathbb{R}$ *and* $A(\lambda_2, \gamma_2)$ *is defined by relation (24).*

Proof. One can check by direct calculus that the solution (25) solves the Bellman differential equation with correlated assets along with the value matching and the smooth pastings conditions.

Substituting the general solution (25) in the differential equation (14), one find that the couple (λ, γ) has to solve the following equation.

$$\begin{aligned}
\mathcal{Q}(\lambda, \gamma) \equiv\ & \frac{1}{2}\lambda(\lambda - 1)\left(\sigma_S^2 + \sigma_{K_1}^2 - 2\rho_{SK_1}\sigma_S\sigma_{K_1}\right) + \lambda(\mu_{K_1} - \mu_S) \\
+\ & \frac{1}{2}\gamma(\gamma - 1)\left(\sigma_S^2 + \sigma_{K_2}^2 - 2\rho_{SK_2}\sigma_S\sigma_{K_2}\right) + \gamma(\mu_{K_2} - \mu_S) \\
+\ & \lambda\gamma\left(\sigma_S^2 - \rho_{SK_1}\sigma_S\sigma_{K_1} - \rho_{SK_2}\sigma_S\sigma_{K_2} + \rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}\right) \\
-\ & (r - \mu_S) = 0.
\end{aligned} \tag{26}$$

□

Extending the previous notation, we refer to the binary quadratic form $\mathcal{Q}(\gamma,\lambda)$ as the fundamental quadratic form of the problem.[5] We refer to the values (λ,γ) for which this quadratic form vanishes as the 0-level curve[6] of $\mathcal{Q}(\gamma,\lambda)$.

This 0-level curve now reduces the possible set of triplets $(\beta_1,\lambda_2,\gamma_2)$ to a 1-dimensional manifold. One can then write the final expression of the investment trigger.

Proposition 10. *The right time to exercise the option is the first exit time defined by*

$$\tau = \min\left\{t \quad | \quad \frac{K_2(t)}{K_1(t)} = \frac{\gamma_2}{\lambda_2}, \frac{S(t)}{K_1(t)} = \frac{\lambda_2+\gamma_2-1}{\lambda_2}, \ \mathcal{Q}(\lambda_2,\gamma_2)=0\right\}.$$

Proof. The result was directly derived from the preceding propositions. □

We now summarise the steps leading to the solution.

- Using the smooth pastings conditions, one obtains a 1-dimensional manifold for the investment trigger. This manifold is parametrised by the triplet $(\beta_1,\gamma_2,\lambda_2)$.
- Using the value matching condition and the differential equation, one obtains two conditions on the triplets $(\beta_1,\gamma_2,\lambda_2)$: an homogeneity condition and a quadratic form. Both conditions reduce the allowed triplets $(\beta_1,\gamma_2,\lambda_2)$ to a 1-dimensional manifold.
- To find the trigger, one has to introduce the 1-dimensional manifold for $(\beta_1,\gamma_2,\lambda_2)$ in the 1-dimensional manifold for the trigger : this defines the 2-dimensional manifold for the general trigger.

3.4 Remarks

Note four points on Proposition 10 :

1. The right time to exercise the option allows one to determine the trigger surface.

$$\mathbb{T}^2 \equiv \left\{(S,K_1,K_2) \quad | \quad \frac{K_2}{K_1} = \frac{\gamma_2}{\lambda_2}, \quad \frac{S}{K_1} = \frac{\lambda_2+\gamma_2-1}{\lambda_2}, \quad \mathcal{Q}(\lambda_2,\gamma_2)=0\right\} \tag{27}$$

One can write two alternative descriptions of the trigger using the conditions (16) and (17). Direct calculus leads to the following two other sets.

$$\mathbb{T}^2 \equiv \left\{(S,K_1,K_2) \mid \frac{K_2}{K_1} = \frac{\gamma_2}{\lambda_2}, \quad \frac{S}{K_2} = \frac{\lambda_2+\gamma_2-1}{\gamma_2}, \quad \mathcal{Q}(\lambda_2,\gamma_2)=0\right\} \tag{28}$$

$$\mathbb{T}^2 \equiv \left\{(S,K_1,K_2) \mid \frac{K_2}{K_1} = \frac{\gamma_2}{\lambda_2}, \ S = \frac{\lambda_2+\gamma_2-1}{\lambda_2+\gamma_2}(K_1+K_2), \ \mathcal{Q}(\lambda_2,\gamma_2)=0\right\} \tag{29}$$

The formulation (29) of the trigger is meaningful : it is a direct link between the revenue and the total cost. Both λ_2 and γ_2 are negative, thus the real option factor $(\lambda_2+\gamma_2-1)/(\lambda_2+\gamma_2)$ in (29) is always higher than 1. We'll show in Section 4.1 that it is also strictly increasing with the volatility of the three assets.

[5] A quadratic form in 2 variables is called a binary quadratic form.

[6] One can also use the terminology of 0-contour line.

2. $\mathbb{T}^2$ is the cartesian product of the set of possible triplets (S, K_1, K_2) assuming known values for the triplet $(\beta_1, \gamma_2, \lambda_2)$ by the set of possible triplets $\{(\beta_1, \gamma_2, \lambda_2)\}$. Since this is the product of two 1-dimensional manifold, the resulting set is a two dimensional manifold, as expected.

We note

$$\underbrace{\left\{\Big(S(\tau), K_1(\tau), K_2(\tau), \beta_1, \gamma_2, \lambda_2\Big)\right\}}_{D=2} =$$

$$\underbrace{\left\{\Big(S(\tau), K_1(\tau), K_2(\tau)\Big)(\beta_1, \gamma_2, \lambda_2)\right\}}_{D=1} \otimes \underbrace{\left\{(\beta_1, \gamma_2, \lambda_2)\right\}}_{D=1}$$

3. The (2,1) problem differs from the (1,1) problem by a major feature : one does not know the value of the two exponents in the Bellman solution before reaching the trigger. The Bellman function is indeed only known when reaching the trigger; it can take different values because the trigger surface is a manifold of dimension greater than 1.

4. Finally, the preceding expression for τ is a stopping time. One can write :

$$\left\{\omega \quad | \quad \tau(\omega) \leq t\right\} =$$

$$\left\{\omega \quad | \quad \exists t_0 \leq 0 \ : \ \frac{K_2(t_0)}{K_1(t_0)} = \frac{\gamma_2}{\lambda_2}, \frac{S(t_0)}{K_1(t_0)} = \frac{\lambda_2 + \gamma_2 - 1}{\lambda_2}, \ \mathcal{Q}(\lambda_2, \gamma_2) = 0\right\}$$

Using $\gamma_2 = \lambda_2 \frac{K_2(t_0)}{K_1(t_0)}$ this can be rewritten

$$\left\{\omega \quad | \quad \tau(\omega) \leq t\right\} =$$

$$\left\{\omega \quad | \quad \exists t_0 \leq 0 \ : \ \frac{S(t_0)}{K_1(t_0)} = \frac{\lambda_2 + \lambda_2 \frac{K_2(t_0)}{K_1(t_0)} - 1}{\lambda_2}, \ \mathcal{Q}\Big(\lambda_2, \lambda_2 \frac{K_2(t_0)}{K_1(t_0)}\Big) = 0\right\} \quad \in \quad \mathcal{F}_t$$

since the process $\frac{K_2(t)}{K_1(t)}$ is $\mathcal{F}_t$-mesurable. Then $\tau(\omega)$ is a stopping time.So one can determine whether or not the trigger has been reached at time t based on informations available up to time t. This investment rule is therefore appropriate to forward numerical simulation as discussed in the following section.

A convenient reformulation of the trigger obtains after introducing the new stochastic process

$$\eta(t) = \frac{K_2(t)}{K_1(t)} \quad \forall t, \, \eta(t) \in \mathcal{F}_t.$$

One can write

$$\tau(\omega) = \min\left\{t \quad | \quad \frac{S(t)}{K_1(t)} = \frac{\lambda_2 + \lambda_2 \eta(t) - 1}{\lambda_2}, \ \mathcal{Q}(\lambda_2, \lambda_2 \eta(t)) = 0\right\}$$

and the trigger is defined by the collection of 1-dimensional manifold $\mathcal{C}^1(\eta)$

$$\mathbb{T}^2 = \{\mathcal{C}^1(\eta)\}_{\eta>0} = \left\{(S, K_1, K_2) \quad | \quad \frac{S}{K_1} = \frac{\lambda_2 + \lambda_2 \eta - 1}{\lambda_2}, \ \mathcal{Q}(\lambda_2, \eta\lambda_2) = 0\right\}_{\eta>0}$$

parametrised by

$$\eta = \frac{K_2}{K_1}.$$

In order to pave the way for numerical solution, the following section restate the optimal exercise procedure in algorithmic form.

3.5 Optimal exercise algorithm

1. Based on observations at time t, compute the ratio of the costs, noted $\eta(t)$.
$$\frac{K_2(t)}{K_1(t)} = \eta(t)$$
2. In the fundamental quadratic form, replace γ by $\eta\lambda$ and solve for λ. Set $\lambda_2 = \min\{\lambda\}$.
3. Using the last computed λ_2, compute $\gamma_2 = \eta\lambda_2$ to find the trigger value for the price by
$$S^*(t) = \frac{\lambda_2 + \gamma_2 - 1}{\lambda_2} K_1(t).$$
4. If $S = S^*$ then $t = \tau$ and
$$F(S, K_1, K_2) = A(\lambda_2, \gamma_2) S \left(\frac{K_1}{S}\right)^{\lambda_2} \left(\frac{K_2}{S}\right)^{\gamma_2}.$$
If $S \neq S^*$ then wait until time $t + dt$ and return to step 1.

A detailed explanation of the algorithm is given in Appendix B.

As indicated above the value of the Bellman function depends on the first exit time. As this cannot be predicted one cannot give the precise Bellman function before the first exit time. One can however define the expected value of the Bellman function where the expectation is taken over the possible exit times. This cannot be solved analytically but is amenable to simulation. This is what we turn to in the next section.

4 Simulations

We here illustrate the three assets exchange problem by a few simulations. We conduct our analysis in four steps.

We first give standard comparative statics. Starting from a basic set of parameters we illustrate the variation of the trigger with respect to the drift rates, the volatility rates and the correlations. We show that our solution has a good real option behavior. The trigger increases with high volatitily rates and high drift rates for the price. The trigger decreases with the two costs drifts rates since increasing costs imply that it is more valuable to invest now than wait and incur a bigger cost. As explained before, the trigger is a function of the K_2/K_1 ratio. Proper comparative must be conducted with a fixed cost ratio. We therefore assume a constant K_2/K_1 ratio for all the comparative statics simulations.

In a second step, we use Monte Carlo simulation to find the probability distribution of the costs ratio at the first exit time.

We thirdly use this distribution to compute an expected Bellman function of the exchange problem.

Finally, we check that the binary quadratic form behave well under reasonnable assumptions on the diffusion processes.

4.1 Comparative Statics

We illustrate the behavior of our solution. We start from the following set of parameters. The discount rate is 5% and all rates are annual.

$$\mu_S = 0.03 \qquad \mu_{K_1} = 0.015 \qquad \mu_{K_2} = 0.008$$
$$\sigma_S = 0.2 \qquad \sigma_{K_1} = 0.25 \qquad \sigma_{K_2} = 0.2$$

We split the year in 100 periods of time and scale variances accordingly. Correlations are $\rho_{SK_1} = 0.15$, $\rho_{SK_2} = 0.25$, $\rho_{K_1K_2} = 0.75$. We consider the cost ratio $K_2/K_1 = 0.8$ for the base case simulation and throughout the comparative statics.

For the base case simulation, we find a trigger

$$S^* = 3.4965K_1$$

and find a real option factor greater than 1.8 as expected.[7] Assume that we want to see the impact of a change in the μ_S parameter. We plot the trigger starting from the base case and changing μ_S over a broad range of values. We then do the same for μ_{K_1} and μ_{K_2} and obtain Figure 1, that depicts the impact of the drift rates on the trigger.

We do the same thing for the volatility rates and correlations to obtain Figure 2 and Figure 3.

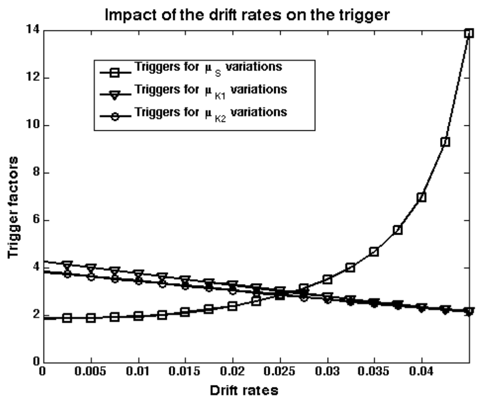

Fig. 1: Drift rates impact on trigger

[7] This factor encompasses the two costs then must be twice bigger than normal. One can check that putting all drift rates, all volatility rates and all correlations to zero lead to a trigger factor equal to 1.8 i.e. the deterministic rule since we assumed $K_2/K_1 = 0.8$.

$$S^* = 1.8K_1 = K_1 + 0.8K_1 = K_1 + K_2$$

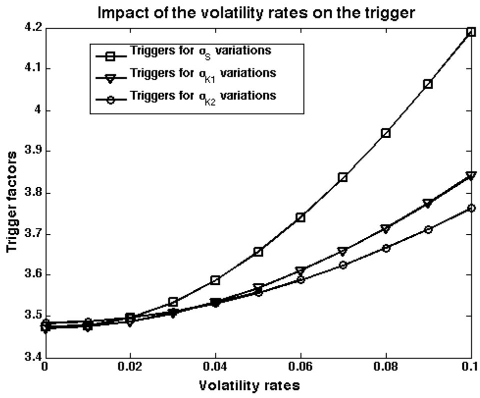

Fig. 2: Volatility rates impact on trigger

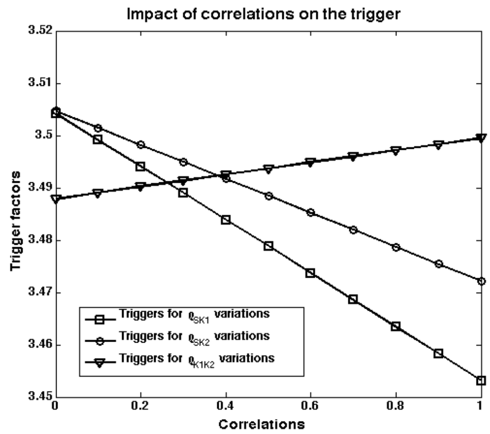

Fig. 3: Correlations impact on trigger

Figure 1 shows that the trigger is an increasing function of the drift rate of the reselling price i.e. the incentive to wait is increasing with the expected profit's growth. The value of the option to defer bursts when the drift rates rises up to the discount rate : it is then always more valuable to wait. However the trigger decreases with the cost's drift rates as they reduce the expected spread.

Figure 2 shows that the trigger always increases with the volatility rates. Increasing uncertainty on the reselling price makes the option to wait for better safety more valuable whereas an increasing uncertainty on a cost is an incentive to wait for a downturn in this cost.

Figure 3 shows that the trigger is a decreasing function of the correlation between the cost and the reselling price as this correlation reduces the expected spread. Conversely, the real option trigger rises with the inter-cost correlation as it increases the uncertainty over the spread.

We conclude that the three assets model complies with standard intuition on real options. As the uncertainty increases the option to postpone investment has a greater value and thus leads to wait. The drift is also a key parameter as a higher expected growth leads to a bigger incentive to wait for better conditions. The correlation factors have different impacts depending on their link with the reselling price. A correlation between the reselling price and a cost decreases the uncertainty over the spread and thus reduces the investment trigger. A correlation between costs increases the uncertainty over the spread and thus increases the investment trigger.

4.2 Probability of the first exit time

Section 3.5 summarises the investment rule. As explained earlier it depends on the first exit time of the continuation region through the stochastic parameter $\eta(\tau)$ defined as

$$\eta(t) = \frac{K_2(t)}{K_1(t)}, \quad \forall t.$$

At a exercise time τ, this parameter satisfies

$$\frac{K_2(\tau)}{K_1(\tau)} = \eta(\tau) = \frac{\gamma_2}{\lambda_2} \qquad\qquad S(\tau) = \frac{\lambda_2 + \gamma_2 - 1}{\lambda_2} K_1(\tau)$$

where (γ_2, λ_2) satisfies the 0-level curve of the fundamental quadratic (26). Suppose one would know the probability distribution of the random variable $\eta(\tau)$, that is $d\mathbb{P}[\eta(\tau) = \eta]$. It's direct computation can be algebraically tricky, but it can be estimated using direct Monte Carlo method by resorting on our forward investment rule. Suppose this is done, we note this probability density $\phi(\eta)$.[8]

$$d\mathbb{P}\Big[\eta(\tau) = \eta\Big] = \phi(\eta)d\eta$$

One can compute easily an expected value for the project.

Proposition 11 (The expected Bellman function of the project)**.** *Consider the (2,1) exchange problem. Given the probability density $\phi(\eta)$ of the costs ratio at the first exit time one can compute an expected Bellman function of the project $\bar{F}(S, K_1, K_2)$.*

$$\bar{F}(S, K_1, K_2) = \int_0^{+\infty} \phi(\eta) A(\eta) S \left(\frac{K_1}{S}\right)^{\lambda_2(\eta)} \left(\frac{K_2}{S}\right)^{\eta\lambda_2(\eta)} d\eta \tag{30}$$

with $A(\eta) = A(\lambda_2(\eta), \eta\lambda_2(\eta))$.

The expected value of the project is a function of S, K_1 and K_2. We noted $\lambda_2(\eta)$ because λ_2 is the negative root of the fundamental interaction quadratic form $\mathcal{Q}(\lambda, \eta\lambda)$ then it is unique and depends only on the value of the parameter η. For the same reason, the coefficient $A(\lambda_2, \eta\lambda_2)$ is just a function of η.

As we said earlier one can evaluate the probability density of the costs ratio at the first exit time using Monte Carlo method. This is done by simulating the evolution of the risk factors using the decision rule on a big number of scenarios to have a distribution of the η parameter associated with (the) exercise time.

We here give the graph of $\phi(\eta)$, the costs ratio distribution estimated by 15000 investment outcomes for η corresponding to (the) first exit time. This graph shows a bell-like curve for the costs ratio distribution. One can then have an approximation of the value of the project.

---

[8] To be more explicit

$$\mathbb{P}\Big[\eta - \frac{d\eta}{2} \leq \eta\big(\tau(\omega)\big) \leq \eta + \frac{d\eta}{2}]\Big] = \int_{\{\omega \in \Omega \,:\, \eta(\tau(\omega)) \in [\eta - d\eta/2, \eta + d\eta/2]\}} d\mathbb{P}(\omega) = \phi(\eta)d\eta.$$

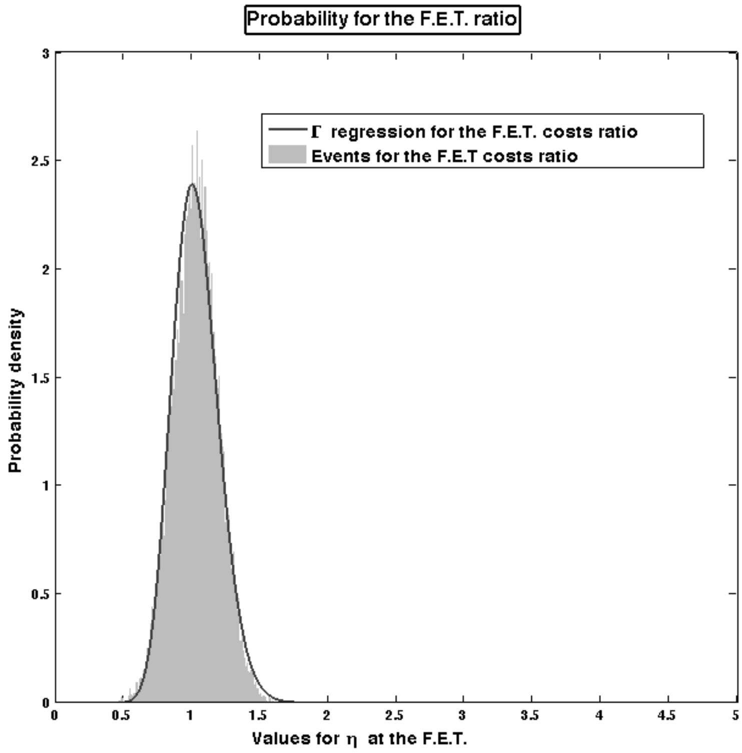

Fig. 4: Probabilitity distribution of the costs ratio at the F.E.T.

In figure 4, the grey figure was obtained by generating 15000 paths on Matlab[9] and counting the number of η corresponding to the first exit time (in the graphic F.E.T.) in each elements of a partition 0.01-fine of the interval $[0, 10]$. We normalised and obtained a density distribution. The density distribution is bell shaped, but one can not infer that it is a normal as the parameter η can never be negative.[10]

The continuous black curve gives the same distribution filled by a general $\Gamma(\alpha, \beta)$ distribution. We used the Matlab function *gamfit* and obtained parameter $\alpha = 37.48$ and $\beta = 0.0275$ for the distribution of η at the first exit time.

The two alternatives are pretty concordant. We illustrate the fact that one can either use numerical integration to evaluate the expected Bellman function (using δ-fine partition of possible outcomes for η at the F.E.T.) or fit the probability density in order to manipulate heavy special functions.

4.3 The value of the project

The density $\phi(\eta)$, allows one to evaluate numerically the expected Bellman function as a Riemann sum. We obtain the expected Bellman function $\bar{F}(S, K_1, K_2)$. Since, the states variables are $\mathcal{F}_t$-measurable, the Bellman function is $\mathcal{F}_t$-measurable as well.

Computing the value of the project for a broad range of values for S, K_1 and K_2, we get the following representations. Because $F : \mathbb{R}^3 \to \mathbb{R}$ and one can just have a clear representation of a function from $\mathbb{R}^2 \to \mathbb{R}$, we always fixed $S = S_i$ and plotted

[9]All simulations were made on a Mac Mini 2 Core 2 Duo 2.2 GHz using Matlab 7.4.0. Time needed to run the 15000 events was around 5 minutes. Matlab files are available on demand. Please contact Joachim Gahungu at joachim.gahungu@uclouvain.be or +32 10 47 94 27.

[10]0 is an absorbant barrier for the geometric brownian motion.

$\bar{F}(S_i, K_1, K_2)$ as a function of K_1 and K_2. We did it for $S = (0, 5, 10, 15, 20)$ and drawn 5 surfaces on each graph.

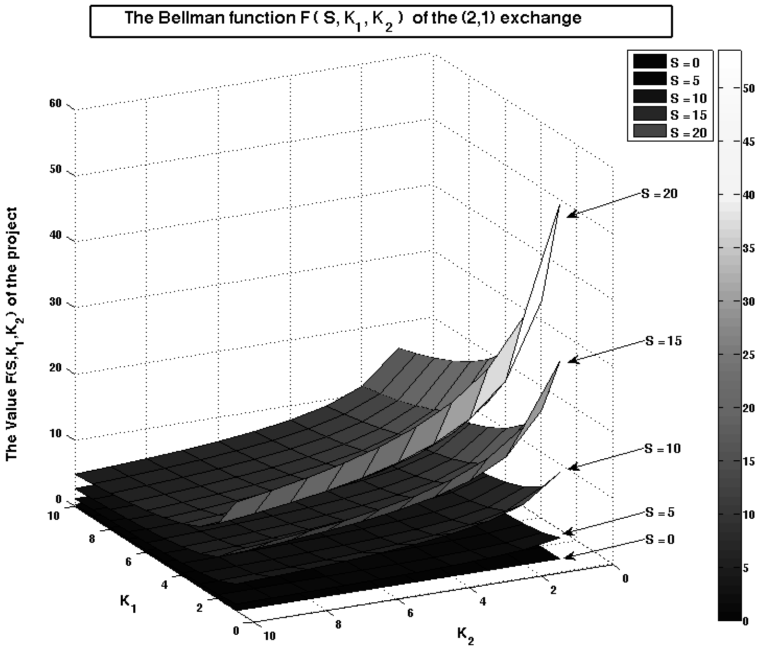

Fig. 5: The value of the (2,1) exchange

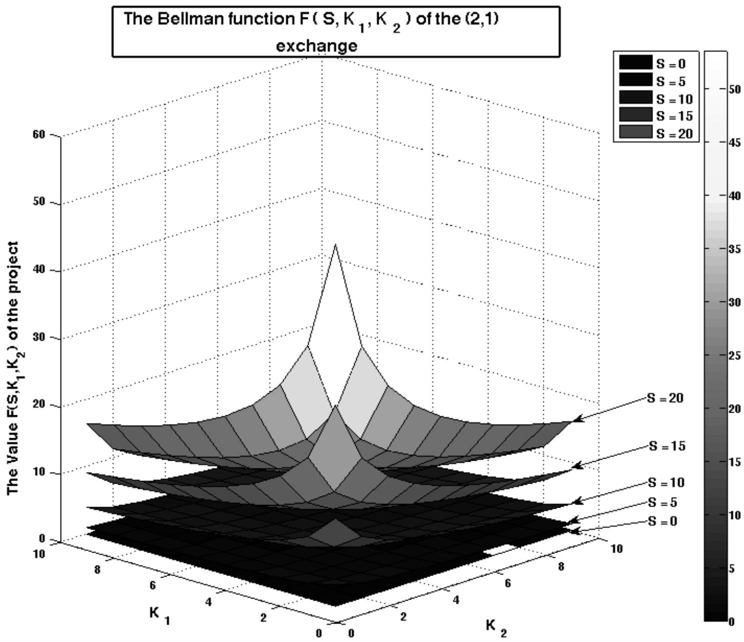

Fig. 6: The value of the (2,1) exchange

We see in these pictures that the estimated Bellman function has a good behavior.

It is decreasing in both K_1 and K_2 as one can see following each surface along the two axes. It is increasing in S as a higher value of S leads to a higher position of the surface.

4.4 Conditions on the fundamental quadratic form

Consider now the fundamental quadratic form $\mathcal{Q}(\lambda, \gamma)$. As a binary quadratic form, its 0-level curve contains an infinite number of points. Including the condition $\gamma = \eta\lambda$, the resulting one variable quadratic in λ has two roots.

$$\begin{aligned}
\mathcal{Q}(\lambda, \eta\lambda) = \; & \frac{1}{2}\lambda(\lambda-1)\left(\sigma_S^2 + \sigma_{K_1}^2 - 2\rho_{SK_1}\sigma_S\sigma_{K_1}\right) + \lambda(\mu_{K_1} - \mu_S) \\
& + \frac{1}{2}\eta\lambda(\eta\lambda-1)\left(\sigma_S^2 + \sigma_{K_2}^2 - 2\rho_{SK_2}\sigma_S\sigma_{K_2}\right) + \eta\lambda(\mu_{K_2} - \mu_S) \\
& + \eta\lambda^2\left(\sigma_S^2 - \rho_{SK_1}\sigma_S\sigma_{K_1} - \rho_{SK_2}\sigma_S\sigma_{K_2} + \rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}\right) \\
& - (r - \mu_S)
\end{aligned} \tag{31}$$

We need to show whether one of this roots has the necessary sign for the Bellman function to make economic sense. We consider the two following problems.

4.4.1 The exchange of $K_1 + K_2$ for S

The economics of the problem imposes that Bellman function of the (2,1) exchange is decreasing in both K_1 and K_2 i.e.

$$F(S, K_1, K_2) = AS\left(\frac{K_1}{S}\right)^{\lambda_2}\left(\frac{K_2}{S}\right)^{\gamma_2} \tag{32}$$

with both λ_2 and γ_2 negative. But $\lambda_2 = \eta\gamma_2$ with $\eta > 0$. To make sense, one thus should determine under what conditions

$$\textit{for all } \eta > 0, \;\; \mathcal{Q}(\lambda, \eta\lambda) \textit{ has a negative root } \lambda_2. \tag{33}$$

Note that these conditions also ensure that the trigger factor in (29) is greater than one.

4.4.2 The exchange of S for $K_1 + K_2$

From the preceding sections, it is clear that the Bellman function of the problem of exchanging S for $K_1 + K_2$ is

$$F(S, K_1, K_2) = AS\left(\frac{K_1}{S}\right)^{\lambda_1}\left(\frac{K_2}{S}\right)^{\gamma_1} \tag{34}$$

with both λ_1 and γ_1 positive. It leads to the same quadratic form (26) and to (31).

The economics of this problem requires that the Bellman function (34) decreases with S i.e. $\lambda_1 + \gamma_1 > 1$. One should thus determine under what conditions

$$\textit{for all } \eta > 0, \;\; \mathcal{Q}(\lambda, \eta\lambda) \textit{ has a positive root } \lambda_1 \textit{ s.t. } \lambda_1(1+\eta) > 1. \tag{35}$$

As we will see in Section 6, these conditions also warrants that the trigger factor of the (1,2) exchange problem is greater than one.

4.4.3 Conditions on the drift rates

One can not determine analytically under what conditions (33) and (35) hold because the only relation linking the correlation factors is the covariance matrix which is supposed to be positive definite. This condition is not easily algebraically handleable.

However, one can show using Monte Carlo simulation that (33) and (35) simultaneously hold providing the growth parameters μ_S, μ_{K_1} and μ_{K_2} are lower than the discount rate r.[11]

To summarize, assuming $\mu_S, \mu_{K_1}, \mu_{K_2} < r$, the quadratic

$$\mathcal{Q}(\lambda, \eta\lambda), \ \eta > 0$$

has a positive and a negative root, respectively λ_1 and λ_2 such that :

1. $\lambda_1(\eta) + \gamma_1(\eta) = (1+\eta)\lambda_1(\eta) > 1$
2. $\lambda_2(\eta) + \gamma_2(\eta) = (1+\eta)\lambda_2(\eta) < 0$.

5 The (n,m) exchange problem

The dynamic programming approach gives a solution to the general problem of exchanging a bundle of n assets for m others. Note $d = n + m$ as the total dimension of the problem. We extend the above discussion to obtain a forward investment rule and an expected Bellman function.

Example 2 (The (n,m) exchange problem)**.** *Assume one is considering the perpetual american option to pay a sum of* n *stochastic sunk costs i.e. a total cost* $a_1K_1(t) + a_2K_2(t) + \cdots + a_nK_n(t)$ *for a project whose value is a sum of* m *stochastic assets* $b_1S_1(t) + b_2S_2(t) + \cdots + b_mS_m(t)$. *When is the right time to exercise this option?*

We extend the above reasoning and proceeds.

1. We first assume no correlation
   (a) Solve by multiplicative variable separation and determine the power form.
   (b) Find the conditions at trigger using the smooth pasting conditions.
   (c) Find the homogeneity condition using the value matching condition.
   (d) Determine the fundamental quadratic form.
2. We then assume correlation
   (a) We take the multiplicative power form obtained in (1a) and check that it solves the Bellman differential equation.
   (b) On that basis we determine the fundamental quadratic form.

This formal work is left to the Appendix. We just show the general procedure to extract the trigger surface.

The investment trigger has to be a $d-1$ manifold.

For the d assets problem, one has:

[11] Matlab files available on demand. Please contact Joachim Gahungu at joachim.gahungu@uclouvain.be or +32 10 47 94 27.

- one value matching;
- d smooth pastings; these d smooth pastings lead to $d-1$ conditions at trigger and one condition for the A coefficient of the Bellman function.

For given exponents of the Bellman function, the application of the $d-1$ conditions at trigger reduces the trigger surface $(S_1(\tau), \dots, K_n(\tau))$ from a d to a 1-manifold.

It remains to find the admissible set of the d unknown exponents. The value matching condition implies an homogeneity equation, thus reduce the exponents admissible set from a d to a $d-1$ manifold.

Moreover the Bellman function has to be solution of the Bellman differential equation. This implies that the exponents satisfy an equation — the quadratic form — that reduces this admissible set to a $d-2$ manifold.

The investment trigger is the cartesian product of the 1-manifold for the trigger surface $(S_1(\tau), \dots, K_n(\tau))$ assuming known exponents by the $d-2$ manifold describing the allowed exponents. It is a $d-1$ manifold.

Given the trigger manifold, it remains to identify the first exit time. This is done by identifying whether the values of the processes allow one to determine univocally the d exponents. This requires $d-2$ additionnal conditions to identify a point of the $d-2$ admissible manifold. These $d-2$ additionnal conditions are given by the values of the processes at the first exit time. They are called the *filtration matching* relations.

Referring to the (2,1) exchange problem as an example. This problem has three state variables. One needs $3-2=1$ other relation coming from observations. This relation is precisely the value of the parameter $\eta(\tau) \in \mathcal{F}_\tau$.

Using *filtration matching* conditions, one can solve the (n, m) exchange problem.

Proposition 12 (The solution of the (n,m) exchange problem)**.** *Assuming a separable form for the (n,m) exchange problem, the Bellman function of the problem has the power form*

$$F(S_1, \cdots, S_m, K_1, \cdots, K_n) = A\left(\Pi_{i=1}^m S_i^{\beta_i}\right)\left(\Pi_{j=1}^n K_j^{\lambda_j}\right)$$

with

$$\begin{aligned} \beta_i &> 0, \ \forall i \\ \lambda_j &< 0, \ \forall j \\ \sum_{i=1}^m \beta_i + \sum_{j=1}^n \lambda_j &= 1. \end{aligned}$$

Moreover, the right time to exercise the (n,m) exchange option is given by

$$\tau = \min\left\{ t \quad | \quad \mathcal{Q}(\vec{\alpha}) = 0\,, \quad \frac{X_k(t)}{X_i(t)} = \frac{\alpha_k}{\alpha_i}\frac{c_i}{c_k} \quad \forall i, \forall k \right\}$$

with

$$\mathcal{Q}(\vec{\alpha}) \equiv \sum_{i=1}^{n+m} \mu_i \alpha_i + \frac{1}{2}\sum_{i=1}^{n+m}\sum_{j=1}^{n+m} \rho_{ij}\sigma_{ij}\left\{\delta_{ij}\alpha_i(\alpha_i - 1) + (1-\delta_{ij})\alpha_i\alpha_j\right\} - \rho.$$

Proof. See Appendix (C). □

6 Additional problems

This section gives solutions to additional problems. We first treat the (1,2) exchange problem and then take up the (0,2) and the (2,0) exchange problems. The methodology can be applied to multi-assets irreversible investment. Still, we leave these applications for future work.

Example 3 (The (1,2) exchange problem)**.** *Consider the perpetual American option to pay a stochastic sunk costs* $K(t)$ *for a project whose value is the sum of two random assets* $S_1(t) + S_2(t)$*. When is the right time to exercise this option?*

The Bellman function of this problem is

$$F(K, S_1, S_2) = AK\Big(\frac{S_1}{K}\Big)^{\gamma_1}\Big(\frac{S_2}{K}\Big)^{\lambda_1}. \tag{36}$$

For economic reasons, we expect $A > 0$. γ_1 and λ_1 are positive as the Bellman function must be increasing in both S_1 and S_2. The couple (λ_1, γ_1) belongs to the 0-level curve of a quadratic form $\mathcal{Q}'(\lambda, \gamma)$.

One then applies the value matching and smooth pasting conditions for this problem. The project is continuous at the border between the two regions i.e. $F(K(\tau), S_1(\tau), S_2(\tau)) = S_1(\tau) + S_2(\tau) - K(\tau)$. It must also be smooth at this border then $F_{S_1}(K(\tau), S_1(\tau), S_2(\tau)) = F_{S_2}(K(\tau), S_1(\tau), S_2(\tau)) = 1$ and $F_K(K(\tau), S_1(\tau), S_2(\tau)) = -1$. The three smooth pasting conditions lead to two conditions at trigger.

$$\frac{S_2(\tau)}{S_1(\tau)} = \frac{\lambda_1}{\gamma_1} \tag{37}$$

$$\frac{K(\tau)}{S_1(\tau)} = \frac{(\gamma_1 + \lambda_1 - 1)}{\gamma_1} \tag{38}$$

Note that we showed in Section 4.4 that $\gamma_1 + \lambda_1 > 1$ provided all the drift rates lower than the exogeneous discount rate. One pointed out that this condition is of the essence.

Equation (38) is a relation at trigger. At each moment t one measures $S_1(t)$, $S_2(t)$ and $K(t)$. One then uses the information at time t to check if the threshold has been reached using the standard algorithm. The investment trigger is defined by

$$\mathbb{T}'(2) \equiv \Big\{(S_1, S_2, K) \quad | \quad \frac{S_2}{S_1} = \frac{\lambda_1}{\gamma_1}, \quad \frac{K(\tau)}{S_1(\tau)} = \frac{(\gamma_1 + \lambda_1 - 1)}{\gamma_1}, \quad \mathcal{Q}(\gamma_1, \lambda_1) = 0\Big\}.$$

Using (37), one obtains the intuitive formulation

$$\mathbb{T}'(2) \equiv \Big\{(S_1, S_2, K) \quad | \quad \frac{S_2}{S_1} = \frac{\lambda_1}{\gamma_1}, \quad S_1 + S_2 = \Big(\frac{\lambda_1 + \gamma_1}{\lambda_1 + \gamma_1 - 1}\Big)K, \quad \mathcal{Q}(\gamma_1, \lambda_1) = 0\Big\}$$

and because $\gamma_1 + \lambda_1 > 1$ (see Section 4.4), the real options factor is greater than 1 as expected.

One alternatively uses the value matching or one smooth pasting to determine the coefficient A.

$$A = \frac{(\gamma_1 + \lambda_1 - 1)^{(\gamma_1 + \lambda_1 - 1)}}{\gamma_1^{\gamma_1}\lambda_1^{\lambda_1}} \tag{39}$$

Example 4 (The (0,2) exchange problem)**.** *Consider the perpetual American option to pay a deterministic sunk cost* $I = K(0)$ *for a project whose value is the sum of two random assets* $S_1(t) + S_2(t)$*. When is the right time to exercise this option?*

Intuition suggests to conjecture that the Bellman function of this project is

$$F(S_1, S_2) = BI\Big(\frac{S_1}{I}\Big)^{\gamma_1}\Big(\frac{S_2}{I}\Big)^{\lambda_1}$$

as this would relax the general form of the (1,2) exchange problem. However, such a motivation is not mathematically correct as the differential equation of the (1,2) exchange problem has one more state variable than the (0,2) exchange problem. We conjecture an alternative form of the Bellman function on the basis of variable separation on the (0,2) exchange problem.

The separable solution of the (0,2) exchange problem differential equation is

$$F(S_1, S_2) = AS_1^{\gamma_1} S_2^{\lambda_1}$$

with γ_1 and λ_1 positive and A expected to be positive after application of the boundary conditions. Solving the differential equation using this general function holds if the couple (γ_1, λ_1) is on the 0-level curve of an easily computed fundamental quadratic form $\mathcal{Q}''(\gamma, \lambda)$. One show easily that this special quadratic form is a particular case of the quadratic of the (1,2) exchange problem with $\mu_K = 0$ and $\sigma_K = 0$. Thus we know that $\lambda_1 + \gamma_1 > 1$.

The boundary conditions are the value matching $F(S_1(\tau), S_2(\tau)) = S_1(\tau) + S_2(\tau) - I$ and the two smooth pastings $F_{S_1}(S_1(\tau), S_2(\tau)) = F_{S_2}(S_1(\tau), S_2(\tau)) = 1$.

Solving with the two smooth pasting conditions gives the relation at trigger

$$\frac{\gamma_1}{\lambda_1} = \frac{S_1(\tau)}{S_2(\tau)} \tag{40}$$

and an expression for A in terms of the price processes at trigger $S_1(\tau)$.

$$A = \frac{1}{\gamma_1 S_1^{\gamma_1 - 1} S_2^{\lambda_1}} = \frac{1}{\gamma_1 (\frac{\lambda_1}{\gamma_1})^{\lambda_1} S_1^{\gamma_1 + \lambda_1 - 1}} \tag{41}$$

As one can not have the usual relation $\lambda_1 + \gamma_1 = 1$, we remain with two unknown powers.

Using the value matching condition, relation (41) for A and the relation at trigger (40), one obtains the investment rule

$$S_1(\tau) = \Big(\frac{\gamma_1}{\gamma_1 + \lambda_1 - 1}\Big) I. \tag{42}$$

One can then use the standard algorithm with the prices ratio at trigger as a source of information : given the price ratio at time t, one can find the two powers λ_1 and γ_1 and check if the investment criterion (42) holds.

The investment trigger has to verify the set of equations

$$\mathbb{T}''(2) \equiv \Big\{(S_1, S_2, K) \quad | \quad \frac{S_2}{S_1} = \frac{\lambda_1}{\gamma_1}, \quad S_1(\tau) = \Big(\frac{\gamma_1}{\gamma_1 + \lambda_1 - 1}\Big) I, \quad \mathcal{Q}''(\gamma_1, \lambda_1) = 0\Big\}.$$

Using (40), one obtains the intuitive formulation

$$\mathbb{T}''(2) \equiv \Big\{(S_1, S_2, K) \quad | \quad \frac{S_2}{S_1} = \frac{\lambda_1}{\gamma_1}, \quad S_1 + S_2 = \Big(\frac{\lambda_1 + \gamma_1}{\lambda_1 + \gamma_1 - 1}\Big) I, \quad \mathcal{Q}''(\gamma_1, \lambda_1) = 0\Big\}$$

and again note than the real options factor is greater than 1.

One can compute the coefficient A.

$$A = \frac{(\lambda_1 + \gamma_1 - 1)^{(\lambda_1+\gamma_1-1)}}{\lambda_1^{\lambda_1}\gamma_1^{\gamma_1} I^{(\lambda_1+\gamma_1-1)}} = \frac{(\lambda_1 + \gamma_1 - 1)^{(\lambda_1+\gamma_1-1)}}{\lambda_1^{\lambda_1}\gamma_1^{\gamma_1}} I^{(1-\lambda_1-\gamma_1)}$$

We thus fall back on our initial intuition as we can write the general solution of the (0,2) exchange problem as

$$F(S_1, S_2) = BI\left(\frac{S_1}{I}\right)^{\gamma_1}\left(\frac{S_2}{I}\right)^{\lambda_1}$$

with

$$B = \frac{(\lambda_1 + \gamma_1 - 1)^{(\lambda_1+\gamma_1-1)}}{\lambda_1^{\lambda_1}\gamma_1^{\gamma_1}}.$$

Example 5 (The (2,0) exchange problem)**.** *Consider the perpetual american option to pay the sum of two stochastic sunk costs* $K_1(t) + K_2(t)$ *for a project whose value* $S(0) = V$ *is deterministic. When is the right time to exercise this option?*

One can prove that the solution of the problem will take the form

$$F(K_1, K_2) = BV\left(\frac{K_1}{V}\right)^{\gamma_2}\left(\frac{K_2}{V}\right)^{\lambda_2}$$

with (γ_2, λ_2) a negative point (i.e. $\lambda_2 < 0$ and $\gamma_2 < 0$) of the 0-level curve of the fundamental quadratic form of the previous problem. The proof is immediate from the preceding example.

7 Conclusion

In this paper, we solve the real option problem of the exchange between n and m assets using dynamic programming. Our analysis assumes that each asset follows a geometric Brownian motion, and that the exchange option is American and perpetual. We are thus in the general framework of real option initiated by Mc Donald and Siegel[11].

We develop the approach on the simple two to one asset exchange that we later extend to the general case. We build up intuition by first considering the case of uncorrelated risk factors and find a product power form for the solution of the Bellman partial differential equation. We then note that this solution also solves the Bellman partial differential equation of the correlated case. In this process, we derive a quadratic relation that the exponents of the different factors of the product form need to satisfy. We refer to this relation as the fundamental quadratic form.

The fundamental quadratic form, together with the boundary optimality conditions define a stopping time that triggers the investment. This is obtained as follows. The value matching condition implies homogenity of degree one of the Bellman function. This is a second condition on the exponents of the Bellman function that reduces the set of possible exponents to a manifold of dimension 1. The smooth pasting conditions can be reduced to two relations between the three state variables that define a manifold of dimension 1.These relations involve the exponents of the Bellman function. All in all these define a trigger described by a manifold of dimension 2. A 2-dimensional trigger is quite intuitive for a three assets problem.

The Bellman function of the project depends on the value of the assets at the exercise point. In constrast with standard real option problems, the Bellman function cannot therefore be computed ex ante, that is, before the first exit time from the continuation region. We can however compute an ex ante value of the project as an expectation of the Bellman function using Monte Carlo simulation. We show that this expectation also satisfies Bellman partial differential equation.

We show in numerical treatment that the solution of this three assets problem has a good real option behavior. Among other things, the incentive to wait increases with the volatility of every asset.

We then extend this result to the n to m exchange problem. Using the same reasoning we find a trigger of dimension $m + n - 1$.

Then we define the key concept of this paper : the *filtration matching conditions*. With more than two assets, one has to use the filtration generated by the processes up to current time. In a $n + m$ exchange problem, one need $n + m - 2$ conditions coming from observations.

Additionnal problems are treated as application of the general method.

References

[1] BALDURSSON. Irreversible investment under uncertainty in oligopoly. *Journal of economic dynamics and control* (1997).

[2] BERTOLA, G. Irreversible investment. *Research in Economics* (1998).

[3] BINGHAM, N., AND KIESEL, R. *Risk-Neutral Valuation : Pricing and Hedging of Financial Derivatives*. Springer, 2001.

[4] DIXIT, A. K., AND PINDYCK, R. S. *Investment under Uncertainty*. Princeton University Press, January 1994.

[5] GRENADIER, S. Option exercice games : an application to the equilibrium investment strategies of firms. *Review of financial studies* (2002).

[6] KARATZAS, I., AND BALDURSSON, F. M. Irreversible investment and industry equilibrium. *Finance and stochastics 1* (1996), 69–89.

[7] KARATZAS, I., AND SHREVE, S. E. *Brownian Motion and Stochastic Calculus (Graduate Texts in Mathematics)*. Springer, August 2004.

[8] KESTER, C. Today's options for tomorrow's growth. *Harvard Business Review* (1984).

[9] LEAHY, J. Investment in competitive equilibrium : the optimality of myopic behavior. *Quaterly journal of economics* (1993).

[10] MARGRABE, W. The value of an option to exchange one asset for another. *The Journal of Finance 33*, 1 (1978), 177–186.

[11] MCDONALD, R., AND SIEGEL, D. The value of waiting to invest. *The Quarterly Journal of Economics 101*, 4 (1986), 707–728.

[12] MYERS, S. Determinants of corporate borrowing. *Journal of Financial Economics* (1977).

[13] OKSENDAL, B. *Stochastic Differential Equations*. Springer, 2007.

[14] PINDYCK, R. S. Irreversibility, uncertainty, and investment. *Journal of economic literature* (1991).

[15] PINDYCK, R. S. Investment of uncertain cost. *Journal of Financial Economics* (1993).

[16] SHREVE, S. *Stochastic Calculus for Finance*. Springer, 2005.

[17] SLADE, M. What does an oligopoly maximize? *Journal of Industrial Economics* (1994).

[18] WILMOTT, P. *Paul Wilmott on Quantitative Finance*. Wiley, March 2006.

A The (1,1) exchange problem

A.1 Proof of Proposition 1

Proof. We assumed a solution

$$F(S,K) = AK\Big(\frac{S}{K}\Big)^{\beta}.$$

Replacing F by this general solution in the differential equation (1) we find that β must be a root of the fundamental quadratic

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\beta(\beta-1)\Big(\sigma_S^2 + \sigma_K^2 - 2\rho_{SK}\sigma_P\sigma_I\Big) + \beta(\mu_S - \mu_K) - (r - \mu_K).$$

One still needs to apply the 3 boundary conditions. One notes β_1 the positive root of the fundamental quadratic in the following. At the point $(S(\tau), K(\tau))$ where investment is optimal, one has continuity in the value of the project and "high contact" conditions.

The first condition is the well known value matching

$$F(S(\tau), K(\tau)) = S(\tau) - K(\tau). \tag{43}$$

At the optimal point the value of the option to wait is just equal to the value of the running project. One gives up the option to wait for a project where its value is at least the sunk cost plus the option to defer. The second and third conditions are the "smooth pasting" conditions.

$$F_K(S(\tau), K(\tau)) = -1 \tag{44}$$

$$F_S(S(\tau), K(\tau)) = 1 \tag{45}$$

The interpretation of these relations is that if the Bellman function weren't smooth at the border between the two regions one could do better exercising at an other point.

The application of these conditions leads to

$$\frac{1}{A} = (\beta_1 - 1)\Big(\frac{S(\tau)}{K(\tau)}\Big)^{\beta_1} \qquad \frac{1}{A} = \beta_1\Big(\frac{S(\tau)}{K(\tau)}\Big)^{\beta_1 - 1}.$$

The trigger relation between $S(\tau)$ and $K(\tau)$ makes these two relations actually the same. One find the condition at trigger

$$S(\tau) = \frac{\beta_1}{\beta_1 - 1}K(\tau)$$

and the coefficient A.

$$A = \frac{(\beta_1 - 1)^{\beta_1 - 1}}{\beta_1^{\beta_1}}$$

Note that we obtained the complete solution without even using the value matching condition. This indicate that once the homogeneity stated, a third boundary condition is redundant. □

A.2 Proof of Proposition 2

Proof. Introducing the solution

$$F(S, K) = Af(S)g(K)$$

in the differential equation and dividing by $F(S, K)$, we obtain a separable expression in S and K. One can express this equation as an equality between a function of S and a function of K and write these two expressions as equal to a constant k.

$$\mu_S S \frac{f'(S)}{f(S)} + \frac{1}{2}\sigma_S^2 S^2 \frac{f''(S)}{f(S)} = -\mu_K K \frac{g'(K)}{g(K)} - \frac{1}{2}\sigma_K^2 K^2 \frac{g''(K)}{g(K)} + r = k$$

Each of these relation describe a — one dimensional — real option problem:

$$\begin{aligned} \mu_S S f'(S) &+ \frac{1}{2}\sigma_S^2 S^2 f''(S) - kf(S) = 0 \\ \mu_K K g'(K) &+ \frac{1}{2}\sigma_K^2 K^2 g''(K) - (r - k)g(K) = 0 \end{aligned}$$

Standard real options theory gives us the solutions.

$$f(S) = A\, S^{\beta_1} \qquad\qquad g(K) = K^{\lambda_2}$$

Of course β_1 and λ_2 are solutions of the two fundamental quadratics associated to these two equations. Namely we have respectly that β_1 is the positive root of $\mathcal{Q}_S(\beta)$ and λ_2 the negative root of $\mathcal{Q}_K(\lambda)$.

$$\begin{aligned} \mathcal{Q}_S(\beta) &= \frac{1}{2}\sigma_S^2 \beta(\beta - 1) + \mu_S \beta - k \\ \mathcal{Q}_K(\lambda) &= \frac{1}{2}\sigma_K^2 \lambda(\lambda - 1) + \mu_K \lambda - (r - k) \end{aligned}$$

The constants k, A and the trigger condition must be determined by the usual boundary conditions (43), (44), (45). □

A.3 Proof of Proposition 4

Proof. Use for instance the first smooth pasting condition

$$\frac{\beta_1}{S(\tau)} F(S(\tau), K(\tau)) = 1.$$

This is a condition on the coefficient A i.e.

$$A = \frac{S(\tau)}{\beta_1} \frac{1}{S^{\beta_1}(\tau) K^{\lambda_2}(\tau)} = \frac{1}{\beta_1 S^{\beta_1 - 1}(\tau) K^{\lambda_2}(\tau)}$$

and we will use it with the value matching condition and the condition at trigger to get the homogeneity.

Indeed the value matching yields

$$F(S(\tau), K(\tau)) = AS^{\beta_1}(\tau)K^{\lambda_2}(\tau) = S(\tau) - K(\tau)$$

then

$$AS^{\beta_1}(\tau)K^{\lambda_2}(\tau) = \frac{1}{\beta_1 S^{\beta_1 - 1}(\tau)K^{\lambda_2}(\tau)} S^{\beta_1}(\tau)K^{\lambda_2}(\tau) = S(\tau) - K(\tau) = S(\tau) + \frac{\lambda_2}{\beta_1} S(\tau)$$

and

$$\frac{S(\tau)}{\beta_1} = S(\tau) + \frac{\lambda_2}{\beta_1} S(\tau)$$

thus

$$1 = \beta_1 + \lambda_2.$$

Note that the homogeneous solution of the Proposition (1) allows one to determine β_1. One then find λ_2 using $\lambda_2 = 1 - \beta_1$. One can check that without correlation, the same constant k holds in the two quadratics $\mathcal{Q}_S(\beta)$ and $\mathcal{Q}_K(\lambda)$ defined in the proof of the Proposition 2 i.e. that

$$k = \frac{1}{2}\sigma_S^2 \beta_1(\beta_1 - 1) + \mu_S \beta_1 = r - \frac{1}{2}\sigma_K^2 \lambda_2(\lambda_2 - 1) - \mu_K \lambda_2.$$

□

B The (2,1) exchange problem

B.1 Proof of Proposition 6

Proof. Because the problem is separable, there exist two constants k_1 and k_2 such that the problem is split in three one dimensional real options differential equations:

$$\begin{aligned}
\mu_S S f'(S) &+ \frac{1}{2}\sigma_S^2 S^2 f''(S) - k_1 f(S) = 0 \\
\mu_{K_1} K_1 g'(K_1) &+ \frac{1}{2}\sigma_{K_1}^2 K_1^2 g''(K_1) - k_2 g(K_1) = 0 \\
\mu_{K_2} K_2 h'(K_2) &+ \frac{1}{2}\sigma_{K_2}^2 K_2^2 h''(K_2) - (r - k_1 - k_2) h(K_2) = 0
\end{aligned}$$

Their solution are easily found as

$$f(S) = S^{\beta_1} \qquad g(K_1) = K_1^{\lambda_2} \qquad h(K_2) = K_2^{\gamma_2}$$

Again β_1, λ_2 and γ_2 are solutions of the three fundamental quadratics associated to these three equations. The constants k_1, k_2 and A must be determined by the (2,1) exchange boundary conditions. □

B.2 Proof of Proposition 8

Using the value matching condition

$$F(S(\tau), K_1(\tau), K_2(\tau)) = S(\tau) - K_1(\tau) - K_2(\tau)$$

and using the Bellman function and the conditions at trigger given by Proposition 7, we obtain

$$F(S(\tau), K_1(\tau), K_2(\tau)) = \frac{1}{\beta_1 S^{\beta_1 - 1}(\tau) K_1^{\lambda_2}(\tau) K_2^{\gamma_2}(\tau)} S^{\beta_1}(\tau) K_1^{\lambda_2}(\tau) K_2^{\gamma_2}(\tau) = \frac{S(\tau)}{\beta_1}$$

$$= S(\tau) - K_1(\tau) - K_2(\tau) = S(\tau) + \frac{\lambda_2}{\beta_1} S(\tau) + \frac{\gamma_2}{\beta_1} S(\tau)$$

or

$$\frac{1}{\beta_1} S(\tau) = S(\tau) + \frac{\lambda_2}{\beta_1} S(\tau) + \frac{\gamma_2}{\beta_1} S(\tau).$$

Hence

$$\beta_1 + \lambda_2 + \gamma_2 = 1.$$

B.3 Optimal exercise algorithm (Section 3.5)

One does not know the time at which the investment trigger will be reached. This investment time is a random time τ described by a rule. We hope that τ is a stopping time.

Time τ is a stopping time if and only if one can determine whether the threshold is reached at time t or not from informations up to time t.[12] In other words, τ is a stopping time if one can say if $\tau \leq t$ according to informations up to time t i.e. to the σ-algebras generated by the random processes up to time t. If our decision rule is not a stopping time it's dramatic as one is not able to invest at the precise optimal moment.

To see that τ is indeed a stopping time, one has to prove that at each moment we can check if we are at the trigger or not. Because the problem is stationnary, it suffices to prove that we can check it at a single moment.

At time t one knows all the history of all processes for time before t. Precisely the sequence of σ-algebras generated by the three processes is the filtration $\mathcal{F}_t$ and each process is adapted to this filtration. One knows precisely at time t the value of the ratio of the two costs.

Relation (16) must hold at time t if it is optimal to exercise the option. In particular, one has the observations of the costs ratio. Define

$$\frac{K_2^{Obs}(t)}{K_1^{Obs}(t)} = \eta^{Obs}(t).$$

One does not know if the trigger has been reached. Maybe it did. Just in case state

$$\frac{K_2^{Obs}(t)}{K_1^{Obs}(t)} = \eta^{Obs}(t) = \frac{\bar{\gamma}_2}{\bar{\lambda}_2}$$

with for now $\bar{\gamma}_2$ and $\bar{\lambda}_2$ candidate powers for the Bellman function of the project. This is a relation between the two candidates and it is available now at time t i.e. it is

[12] A fluent and clear description of stopping time is given in Shreve[16]. See Oksendal[13] for mathematical aspects in Ito diffusions.

$\mathcal{F}_t$-adapted. One can put it in the quadratic form. It becomes a simple one variable second degree quadratic and one can find his unique negative roots. Thus using back this relation the values of the candidates $\bar{\lambda}_2$ and $\bar{\gamma}_2$ are univoquely determined.

Now we have to check if these candidates are good i.e. if one is at the trigger. If it is, the observation S^{Obs} matches the trigger condition on S. Use for instance relation (17). One must have[13] :

$$S^{Obs}(t) = \frac{\lambda_2 + \gamma_2 - 1}{\lambda_2} K_1^{Obs}(t)$$

If the last relation holds the trigger is reached. One can state $\lambda_2 = \bar{\lambda}_2$ and $\gamma_2 = \bar{\gamma}_2$. Then one is still at time t and one knows that $\tau = t$.

If the last don't holds one has to wait for more informations. One knows that $\tau > t$. One wait until time $t + dt$ for new values for the costs and price. One computes the new η^{Obs} and check if for this new η^{Obs} the trigger condition holds.

To sum up, as at time t one knows if $\tau = t$ or if $\tau > t$, τ is a stopping time. One now has a procedure to find the investment trigger.

C The (n,m) exchange problem

C.1 Proof of Proposition 12

Proof. We use the single variable X for the costs $\{K_k : k = 1, 2, \cdots, n\}$ and the prices $\{S_s : s = 1, 2, \cdots, m\}$. We note $X = \{X_i : i = 1, 2, \cdots, n+m\}$ with

$$\begin{aligned} X_< &= (X_1, \cdots, X_n) = (K_1, \cdots, K_n) \\ X_> &= (X_{n+1}, \cdots, X_{n+m}) = (S_1, \cdots, S_m). \end{aligned}$$

Similarly, we note $c = \{c_i : i = 1, 2, \cdots, n+m\}$ with

$$\begin{aligned} c_< &= (c_1, \cdots, c_n) = (a_1, \cdots, a_n) \\ c_> &= (c_{n+1}, \cdots, c_{n+m}) = (-b_1, \cdots, -b_m) \end{aligned}$$

such that the value of the option at the exercise point is simply

$$F(X(\tau)) = \sum_{i=1}^{n+m} c_i X_i(\tau).$$

The standard dynamic programming approach in real options leads us directly to the differential equation of the (n, m) problem exchange. We use the standard notation ρ_{ij} for the correlations between X_i and X_j.

$$\begin{aligned} \rho_{ii} &= 1 \qquad \forall i \\ \rho_{ij} &= \rho_{ji} \quad \forall i, \forall j \end{aligned}$$

[13] With a little algebra, such relation could have be stated in terms of K_2 or $K_1 + K_2$. One can directly check that

$$\begin{aligned} S^{Obs}(t) &= \frac{\lambda_2 + \gamma_2 - 1}{\lambda_2} K_1^{Obs}(t) = \frac{\lambda_2 + \gamma_2 - 1}{\lambda_2} \frac{\lambda_2}{\gamma_2} K_2^{Obs}(t) = \frac{\lambda_2 + \gamma_2 - 1}{\gamma_2} K_2^{Obs}(t). \\ S^{Obs}(t) &= \frac{\lambda_2 + \gamma_2 - 1}{\lambda_2} K_1^{Obs}(t) = \frac{\lambda_2 + \gamma_2 - 1}{\lambda_2 + \gamma_2} \frac{\lambda_2 + \gamma_2}{\gamma_2} K_1^{Obs}(t) = \frac{\lambda_2 + \gamma_2 - 1}{\lambda_2 + \gamma_2} (K_1^{Obs}(t) + K_2^{Obs}(t)). \end{aligned}$$

and δ_{ij} is the Cronecker delta i.e.

$$\delta_{ij} = \delta_{ji} = 0 \quad \text{if} \quad i \neq j$$
$$\delta_{ij} = 1 \quad \text{if} \quad i = j.$$

We note $X(\tau)$ the first exit time of the continuation region. The (n, m) exchange differential system is given by

$$\sum_{i=1}^{n+m} \mu_i X_i F_i(X) + \frac{1}{2} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \rho_{ij} \sigma_i \sigma_j X_i X_j F_{ij}(X) - \rho F(X) = 0 \tag{46}$$

$$F(X(\tau)) = \sum_{i=1}^{n+m} c_i X_i(\tau) \tag{47}$$

$$F_i(X(\tau)) = c_i \qquad \forall i \tag{48}$$

The first equation is just the Bellman differential equation. There is no immediate profit until the exchange occurs thus the simplest form for the differential equation. The second equation is the value matching condition while the final set of equations are smooth pastings.

We skip the no-correlated case. We will directly check that the multiplicative power form solves the Bellman differential equation along with the value matching and the smooth pastings conditions.

Assume a multiplicative power form for the solution.

$$F(X) = A \prod_{i=1}^{n+m} X_i^{\alpha_i}$$

First we check that this function is solution of the general differential equation. To do so we compute :

$$F_i(X) = \alpha_i \frac{A}{X_i} \prod_{j=1}^{n+m} X_j^{\alpha_j} = \frac{\alpha_i}{X_i} F(X)$$

$$F_{ij}(X) = \begin{cases} \frac{\alpha_i \alpha_j}{X_i X_j} F(X) & \text{if} \quad i \neq j \\ \frac{\alpha_i(\alpha_i - 1)}{X_i^2} F(X) & \text{if} \quad i = j \end{cases}$$

Put the defined F in the Bellman differential equation. One find that

$$F(X) \left[\sum_{i=1}^{n+m} \mu_i \alpha_i + \frac{1}{2} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \rho_{ij} \sigma_{ij} \left\{ \delta_{ij} \alpha_i (\alpha_i - 1) + (1 - \delta_{ij}) \alpha_i \alpha_j \right\} - \rho \right] = 0$$

The function F is solution of the Bellman differential equation as soon as $\{\alpha_i \quad : \quad i = 1, 2, \cdots, n+m\}$ belong to the 0-level curve of an $n+m$ quadratic form.

$$\mathcal{Q}(\vec{\alpha}) \equiv \sum_{i=1}^{n+m} \mu_i \alpha_i + \frac{1}{2} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \rho_{ij} \sigma_{ij} \left\{ \delta_{ij} \alpha_i (\alpha_i - 1) + (1 - \delta_{ij}) \alpha_i \alpha_j \right\} - \rho$$

Here we showed that with special conditions between the roots, we have a solution for the Bellman differential equation. Note that we get a $n+m$ dimensions quadratic form.

In the (2,1) exchange problem, we found a 2-dimensions quadratic form, instead of a 3-dimensions quadratic form. That is because of the homogeneity propriety : one reduce the problem from 3 to 2 dimensions. Here we'll do the same.

If we show that the Bellman function must be homogeneous of degree one, we can reduce the fundamental quadratic from $n+m$ to $n+m-1$ dimensions. Homogeneity is the second and last part of the proof.

We want to proof that to solve the exchange problem, the Bellman function must be homogeneous of degree one. Let's start by the smooth pasting conditions.

$$\begin{aligned}
F_i(X(\tau)) &= \frac{\alpha_i}{X_i(\tau)} F(X(\tau)) = \frac{\alpha_i}{X_i(\tau)} A \prod_{k=1}^{n+m} X_k^{\alpha_k}(\tau) = c_i \qquad \forall i \\
&= A\alpha_i X_i^{\alpha_i - 1}(\tau) \prod_{\substack{k=1 \\ k \neq i}}^{n+m} X_k^{\alpha_k}(\tau) = c_i \qquad \forall i \\
\text{thus} \qquad A &= \frac{c_i}{\alpha_i X_i^{\alpha_i-1}(\tau) \prod_{\substack{k=1 \\ k\neq i}}^{n+m} X_k^{\alpha_k}(\tau)} \qquad \forall i.
\end{aligned}$$

One has to keep in mind that A is a constant. Then, with $i \neq k$, one has

$$A = \frac{c_i}{\alpha_i X_i^{\alpha_i-1}(\tau) \underbrace{\prod_{\substack{l=1 \\ l\neq i}}^{n+m} X_l^{\alpha_l}(\tau)}_{\left(\prod_{\substack{l\neq i \\ l\neq k}} X_l^{\alpha_l}(\tau)\right) X_k^{\alpha_k}(\tau)}} = \frac{c_k}{\alpha_k X_k^{\alpha_k-1}(\tau) \underbrace{\prod_{\substack{l=1 \\ l\neq k}}^{n+m} X_l^{\alpha_l}(\tau)}_{\left(\prod_{\substack{l\neq i \\ l\neq k}} X_l^{\alpha_l}(\tau)\right) X_i^{\alpha_i}(\tau)}}$$

Thus we find

$$\frac{X_k(\tau)c_k}{\alpha_k} = \frac{X_i(\tau)c_i}{\alpha_i}$$

Thus the implications of the smooth pastings are that, at the trigger

$$\frac{X_k(\tau)}{X_i(\tau)} = \frac{\alpha_k}{\alpha_i}\frac{c_i}{c_k} \qquad \forall i \neq k.$$

There is $n+m$ variables. There is C_{n+m}^2 such relations[14] although just $m+n-1$ of these equations are linearly independant. These relations come from the smooth pastings and only hold at trigger.

We finish the proof by the application of the value matching condition. This condition also holds only at the investment trigger.

$$A \prod_{l=1}^{n+m} X_l^{\alpha_l}(\tau) = \sum_{l=1}^{n+m} c_l X_l(\tau)$$

Now use the fact that at the trigger, A is given by

$$A = \frac{c_i}{\alpha_i X_i^{\alpha_i-1}(\tau) \prod_{\substack{k=1 \\ k\neq i}}^{n+m} X_k^{\alpha_k}(\tau)} \qquad \forall i.$$

[14]Here C_{n+m}^2 is the number of combinaisons of two elements picked in a set of $n+m$ elements.

Then for any given i, the value matching condition gives :

$$\frac{c_i}{\alpha_i X_i^{\alpha_i - 1}(\tau) \prod_{\substack{l=1 \\ l \neq i}}^{n+m} X_l^{\alpha_l}(\tau)} X_i^{\alpha_i}(\tau) \prod_{\substack{l=1 \\ l \neq i}}^{n+m} X_l^{\alpha_l}(\tau) = \sum_{l=1}^{n+m} c_l X_l(\tau)$$

Then

$$\frac{c_i X_i(\tau)}{\alpha_i} = \sum_{l=1}^{n+m} c_l X_l(\tau) = \sum_{l=1}^{n+m} c_l X_i(\tau) \frac{\alpha_l}{\alpha_i} \frac{c_i}{c_l}$$

thus

$$\sum_{l=1}^{n+m} \alpha_l = 1.$$

The Bellman function must be homogeneous of degree one.

$$F(X) = A \prod_{i=1}^{n+m} X_i^{\alpha_i} = A X_k \prod_{\substack{i=1 \\ i \neq k}}^{n+m} \left(\frac{X_i}{X_k}\right)^{\alpha_i}$$

$\square$

Recent titles

CORE Discussion Papers

2009/13. Jeroen V.K. ROMBOUTS and Lars STENTOFT. Bayesian option pricing using mixed normal heteroskedasticity models.
2009/14. Gauthier de MAERE d'AERTRYCKE and Yves SMEERS. The valuation of power futures based on optimal dispatch.
2009/15. Thierry BRECHET, Tsvetomir TSACHEV and Vladimir M. VELIOV. Prices versus quantities in a vintage capital model.
2009/16. François VANDERBECK and Laurence A. WOLSEY. Reformulation and decomposition of integer programs.
2009/17. Marc FLEURBAEY, Erik SCHOKKAERT and Koen DECANCQ. What good is happiness?
2009/18. David DE LA CROIX and Michel LUBRANO. The tradeoff between growth and redistribution: ELIE in an overlapping generations model.
2009/19. Thierry BRECHET and Fabien PRIEUR. Can education be good for both growth and the environment?
2009/20. Giacomo SBRANA and Andrea SILVESTRINI. What do we know about comparing aggregate and disaggregate forecasts?
2009/21. Marc GERMAIN, Henry TULKENS and Alphonse MAGNUS. Dynamic core-theoretic cooperation in a two-dimensional international environmental model.
2009/22. Claude D'ASPREMONT and Rodolphe DOS SANTOS FERREIRA. Household behavior and individual autonomy.
2009/23. Helmuth CREMER, Philippe DE DONDER and Pierre PESTIEAU. Education and social mobility.
2009/24. Maria Eugenia SANIN and Francesco VIOLANTE. Understanding volatility dynamics in the EU-ETS market: lessons *from* the future.
2009/25. Marco DI SUMMA and Laurence A. WOLSEY. Lot-sizing with stock upper bounds and fixed charges.
2009/26. Johanna M.M. GOERTZ and François MANIQUET. On the informational efficiency of simple scoring rules.
2009/27. Jean GABSZEWICZ, Ornella TAROLA and Skerdilajda ZANAJ. On uncertainty when it affects successive markets.
2009/28. Jerzy A. FILAR, Jacek B. KRAWCZYK and Manju AGRAWAL. On production and abatement time scales in sustainable development. Can we loosen the *sustainability screw* ?
2009/29. María Eugenia SANIN and Skerdilajda ZANAJ. Clean technology adoption and its influence on tradeable emission permit prices.
2009/30. Antoine BOMMIER, Marie-Louise LEROUX and Jean-Marie LOZACHMEUR. On the public economics of annuities with differential mortality.
2009/31. Gilles GRANDJEAN, Ana MAULEON and Vincent VANNETELBOSCH. Connections among farsighted agents.
2009/32. Axel GAUTIER and Xavier WAUTHY. On the nature of price competition under universal service obligations: a note.
2009/33. Santanu S. DEY and Laurence A. WOLSEY. Constrained infinite group relaxations of MIPs.
2009/34. Jean-François MAYSTADT and Philip VERWIMP. Winners and losers among a refugee-hosting population.
2009/35. Pierre DEHEZ. Allocation of fixed costs and the weighted Shapley value.
2009/36. Sabien DOBBELAERE, Roland Iwan LUTTENS and Bettina PETERS. Starting an R&D project under uncertainty.
2009/37. Carlotta BALESTRA and Davide DOTTORI. Aging society, health and the environment.
2009/38. Alain PHOLO BALA. Urban concentration and economic growth: checking for specific regional effects.
2009/39. Alain PHOLO BALA. Gates, hubs and urban primacy in Sub-Saharan Africa.
2009/40. Nicolas BOCCARD. On efficiency, concentration and welfare.

Recent titles

CORE Discussion Papers - continued

2009/41. Taoufik BOUEZMARNI, Jeroen V.K. ROMBOUTS and Abderrahim TAAMOUTI. A nonparametric copula based test for conditional independence with applications to Granger causality.
2009/42. Josez KONINGS and Hylke VANDENBUSSCHE. Antidumping protection hurts exporters: firm-level evidence from France.
2009/43. Pierre PESTIEAU and Uri M. POSSEN. Retirement as a hedge.
2009/44. Santanu S. DEY and Laurence A. WOLSEY. Lifting group inequalities and an application to mixing inequalities.
2009/45. Jean CAVAILHES, Pierre FRANKHAUSER, Dominique PEETERS and Isabelle THOMAS. Residential equilibrium in a multifractal metropolitan area.
2009/46. Daisuke OYAMA, Yasuhiro SATO, Takatoshi TABUCHI and Jacques-François THISSE. On the impact of trade on industrial structures: The role of entry cost heterogeneity.
2009/47. Ken-Ichi SHIMOMURA and Jacques-François THISSE. Competition among the big and the small.
2009/48. Frédéric BABONNEAU, Yurii NESTEROV AND Jean-Philippe VIAL. Design and operations of gas transmission networks.
2009/49. Olivier BOS. How lotteries outperform auctions for charity.
2009/50. Xavier WAUTHY and Nicolas BOCCARD. Entry accommodation under multiple commitment strategies: judo economics revisited.
2009/51. Joachim GAHUNGU and Yves SMEERS. Multi-assets real options.

Books

Public goods, environmental externalities and fiscal competition: 22 selected papers in public economics by Henry Tulkens, edited and introduced by Parkash Chander, Jacques Drèze, C. Knox Lovell and Jack Mintz, Springer, Boston 2006 (588 pp.).
V. GINSBURGH and D. THROSBY (eds.) (2006), *Handbook of the economics of art and culture.* Amsterdam, Elsevier.
J. GABSZEWICZ (ed.) (2006), La différenciation des produits. Paris, La découverte.
L. BAUWENS, W. POHLMEIER and D. VEREDAS (eds.) (2008), *High frequency financial econometrics: recent developments*. Heidelberg, Physica-Verlag.
P. VAN HENTENRYCKE and L. WOLSEY (eds.) (2007), *Integration of AI and OR techniques in constraint programming for combinatorial optimization problems*. Berlin, Springer.
P-P. COMBES, Th. MAYER and J-F. THISSE (eds.) (2008), *Economic geography: the integration of regions and nations*. Princeton, Princeton University Press.
J. HINDRIKS (ed.) (2008), *Au-delà de Copernic: de la confusion au consensus* ? Brussels, Academic and Scientific Publishers.
J-M. HURIOT and J-F. THISSE (eds) (2009), *Economics of cities.* Cambridge, Cambridge University Press.

CORE Lecture Series

C. GOURIÉROUX and A. MONFORT (1995), Simulation Based Econometric Methods.
A. RUBINSTEIN (1996), Lectures on Modeling Bounded Rationality.
J. RENEGAR (1999), A Mathematical View of Interior-Point Methods in Convex Optimization.
B.D. BERNHEIM and M.D. WHINSTON (1999), Anticompetitive Exclusion and Foreclosure Through Vertical Agreements.
D. BIENSTOCK (2001), Potential function methods for approximately solving linear programming problems: theory and practice.
R. AMIR (2002), Supermodularity and complementarity in economics.
R. WEISMANTEL (2006), Lectures on mixed nonlinear programming.