Game complete analysis for financial markets stabilization

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Game Complete Analysis for Financial Markets Stabilization

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Abstract. The aim of this paper is to propose a methodology to stabilize the financial markets using Game Theory and in particular the Complete Study of a Differentiable Game, introduced in the literature by David Carfì. Specifically, we will focus on two economic operators: a real economic subject and a financial institute (a bank, for example) with a big economic availability. For this purpose we will discuss about an interaction between the two above economic subjects: the Enterprise, our first player, and the Financial Institute, our second player. The only solution which allows both players to win something, and therefore the only one desirable, is represented by an agreement between the two subjects: the Enterprise artificially causes an inconsistency between spot and future markets, and the Financial Institute, who was unable to make arbitrages alone, because of the introduction by the normative authority of a tax on economic transactions (that we propose to stabilize the financial market, in order to protect it from speculations), takes the opportunity to win the maximum possible collective (social) sum, which later will be divided with the Enterprise by contract.

1. Introduction

The recent financial crisis has shown that it is not enough to stabilize markets to prohibit or to restrict short-selling: in fact big speculators can influence the market and take advantage of arbitrage opportunities, caused by themselves.

In this paper is proposed a method to limit, with the introduction of a tax on financial transactions, the speculations by medium and big businesses and then to make more stable the financial market, but without inhibiting the possibilities of profits. At this purpose we will present and study a game as an example. We will try different types of equilibria of the game depending on the friendly, selfless, selfish, fearful or aggressive behavior of the two players, with the introduction of new concepts in Game Theory developed and applied in the last five years by David Carfì, such as devote and offensive equilibria. Finally, we shall propose for the two players a collaborative solution that will be the only one that gives to both mutual economic advantages.

Methodologies. The game G we propose takes place on 3 times, we say time 0, time 1 and time 2.

At time 0 the **Enterprise** can choose if to buy futures contracts to hedge the market risk of the underlying commodity, which (the Enterprise knows) should be bought at time 1, in order to conduct its business activities.

The **Financial Institute**, on the other hand, acts with speculative purposes, on spot markets (buying or short-selling the goods at time 0) and future market (doing the action contrary to that one made on the spot market: if the Financial Institute short-sold goods on spot market, it purchases its on the futures market, and vice versa) of the same product, which is of interest for the Enterprise.

The **Financial Institute** may so take advantage of the temporary misalignment of the spot and future prices that would be created as a result of a hedging strategy by the Enterprise. At the time 2 the Financial Institute will cash or pay the sum determined by its behavior in the futures market at time 1.

2. Financial preliminaries

Here we recall the concepts that we shall use in the present article:

- Any positive real number defines a **purchasing strategy**, on the other hand a negative real number defines a **selling strategy**.
- The **spot market** is the market where it is possible to buy and sell at current prices.
- **Futures** are contracts through which you undertake to exchange, at a predetermined price a certain quantity of the underlying commodity at the expiry of the contract.
- A **hedging operation** through futures by a trader consists in the purchase of future contracts in order to reduce exposure to specific risks on market variables (in this case on the price).
- A hedging operation is defined to be a **perfect hedging operation** when it completely eliminates the risk of the case.
- The **future price** is linked to the underlying spot price. Assuming that: i) the underlying commodity does not offer dividends; ii) the underlying commodity hasn't storage costs and has not convenience yield to take physical possession of the goods rather than future contract. Then, the general relationship linking future price and spot price, with unique interest capitalization at the time t, is the following one:

  $$F_0 = S_0 (1 + i)^t.$$

  If not, the arbitrageurs would act on the market until future and spot prices return to levels indicated by the above relation.

3. The game and stabilizing proposal

3.1 The description of the game

We assume that our first player is an Enterprise that may choose if to buy futures contracts to hedge, in a way that we assume perfect, by an upwards change in the price of the underlying commodity that the Enterprise knows to buy at time 1, to the conduct of its business.

Therefore, the **Enterprise** has the possibility to choose a strategy

$$x \in [0,1]$$

which represents the percentage of the quantity of the underlying M_1 that the Enterprise itself will purchase through futures, depending on whether it intends:

1) to not hedge ($x = 0$),

2) to hedge partially ($0 < x < 1$),

3) to hedge totally ($x = 1$).

On the other hand, our second player is a **Financial Institute** operating on the spot market of the underlying asset that the Enterprise knows it should buy at time 1. The Financial Institute works in our game also on the futures market:

i) taking advantage of possible gain opportunities - given by misalignment between spot prices and futures prices of the commodity -;

ii) or accounting for the loss obtained, because it has to close the position of short sales opened on the spot market.

They are just these actions to determine the win or the loss of the Financial Institute.

The Financial Institute can therefore choose a strategy

$y \in [-1,1]$

which represents the percentage of the quantity of the underlying M_2 that it can buy with its financial resources, depending on whether it intends:

1) to purchase the underlying on the spot market (y> 0),

2) to short sell the underlying on the spot market (y <0),

3) to not intervene on the market of the underlying (y = 0).

Now we illustrate graphically the bi-strategy space E × F of the game:

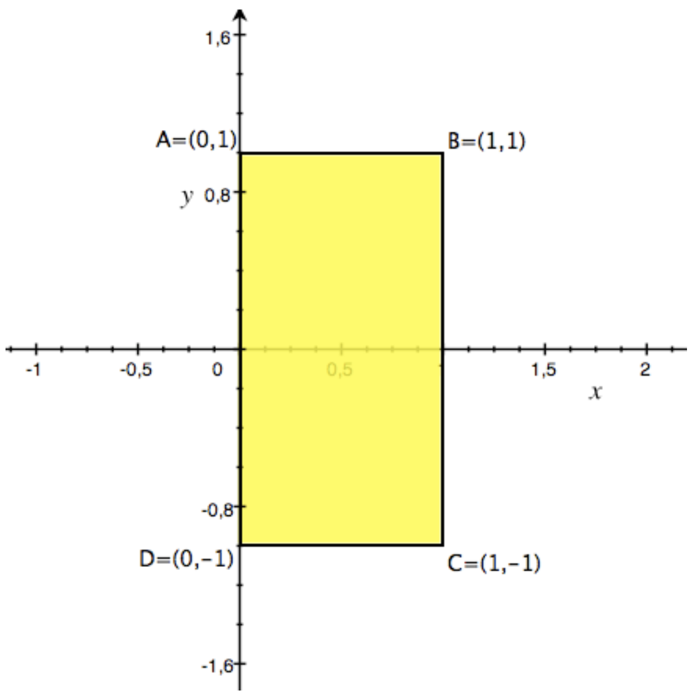


Figure 1. The bi-strategy space of the game

3.1 The payoff function of the Enterprise

The payoff function of the Enterprise, that is the function which represents quantitative loss or win of the Enterprise, referred to time 1, will be given by the win (or loss) obtained on goods not covered. The win relating with the not covered goods will be given by the quantity of the uncovered goods

$(1-x)M_1$,

multiplied by the difference

$F_0 - S_1(y)$,

between the future price at time 0 (the term F_0) - which the Enterprise should pay, if it decides to hedge - and the spot price $S_1(y)$ at time 1, when the Enterprise actually will buy the goods which it did not hedge.

In mathematical language, ***the payoff function of the Enterprise is given by***

$f_1(x,y) = M_1((1-x)(F_0 - S_1(y))$,

where:

1) M_1 is the amount of goods that the Enterprise should buy at time 1;

2) (1-x) is the percentage of the underlying asset that the Enterprise will buy on the spot market at time 1 without any coverage (and therefore exposed to the fluctuations of the spot price of the goods);

3) F_0 is the future price at time 0. It represents the price established at time 0 that the Enterprise will have to pay at time 1 to buy the goods. By definition, assuming the absence of dividends, known income, storage costs and convenience yield to keep possession the underlying, the future price after (t - 0) time units is given by

   $F_0 = S_0(1+i)^t$,

   where $(1+i)^t$ is the capitalization factor with rate i at time t. By i we mean risk-free interest rate charged by banks on deposits of other banks, the so-called "LIBOR" rate. S_0 is, on the other hand, the spot price of the underlying asset at time 0. S_0 is a constant because it does not influence our strategies x and y.

4) $S_1(y)$ is the spot price of the underlying at time 1, after that the Financial Institute plays its strategy y. It is given by

   $S_1(y) = (S_0 + ny)(1+i)$

   where n is the marginal coefficient representing the effect of y on S_1. S_1 depends on y because, if the Financial Institute intervenes in the spot market by a strategy $y \neq 0$, then the price $S_1(y)$ changes because any demand change has an effect on the asset price.

   We are assuming the dependence $n \rightarrow ny$ in $S_1(y)$ as linear by assumption.

   The value $(S_0 + ny)$ then should be capitalized, because otherwise the two prices are referred on two different times (fibers). In fact, the price S_0 to which is added the incidence ny, should be "transferred" from time 0 to time 1.

The payoff function of the Enterprise. Therefore, knowing that

$S_1(y) = (S_0 + ny)(1+i)$

and that

$F_0 = S_0(1+i)$,

and replacing in

$$f_1(x,y) = M_1((1-x)(F_0 - S_1(y)),$$

we have:

$$f_1(x,y) = M_1((1-x)[S_0(1+i) - (S_0+ny)(1+i)].$$

After the appropriate simplifications, here is represented the payoff function of the Enterprise:

$$f_1(x,y) = M_1(1-x)(-ny(1+i)).$$

From now the value n(1+i) will be called ν_1 for simplicity of calculation.

3.2 The payoff function of the Financial Institute

The payoff function of the Financial Institute, that is the function representing the algebraic gain of the Financial Institute at time 1, is the multiplication of the quantity of goods bought on the spot market, that is

$$yM_2,$$

by the difference between the future price $F_1(x, y)$ (it is a price established at time 1 but cashed at time 2) transferred to time 1, that is

$$F_1(x, y)(1 + i)^{-1},$$

and the purchase price of goods at time 0, say S_0, capitalized at time 1 (in other words we are accounting for all balances at time 1).

Stabilizing strategy of normative authority. We therefore propose that - in order to avoid speculations on spot and future markets by the Financial Institute, which in this model is the only one able to determine the spot price (and consequently also the future price) of the underlying commodity - the normative authority imposes to the Financial Institute the payment of a tax on purchase of the goods.

So the Financial Institute can't take advantage of price swings caused by itself. This tax will be fairly equal to the incidence of the strategy of the Financial Institute on the spot price (instead we could propose a given proportion by the normative authority), so the price effectively paid for the goods by the Financial Institute is

$$S_0 + ny,$$

where ny is the tax paid by the Financial Institute.

Observation. We note that if the Financial Institute wins, it will act on the future market at time 2 to cash the win, but also in case of loss it must necessarily act in the future market and account for its loss because at time 2 (in the future market) it should close the short-sale position opened on the spot market.
In mathematical terms, the payoff function of the Financial Institute is:

$$f_2(x,y) = yM_2[F_1(x,y)(1+i)^{-1} - (S_0 + ny)(1+i)],$$

where:

1) y is the percentage of goods that the Financial Institute purchases or sells on the spot market of the underlying;

2) M_2 is the amount of goods that the Financial Institute can buy or sell on the spot market according to its disposable income;

3) S_0 is the price at which the Financial Institute bought the goods. S_0 is a constant because our strategies x and y does not have impact on it.

4) ny is normative tax on the price of the goods. We are assuming the tax is equal to the incidence of the strategy y of the Financial Institute on the price S_0.

5) $F_1(x,y)$ is the price of the future market (established) at time 1, after the Enterprise has played its strategy x.

The price $F_1(x,y)$ is given by

i. $F_1(x,y) = S_1(1+i) + mx$,

where (1 + i) that is the factor of capitalization of interests. By i we mean risk-free interest rate charged by banks on deposits of other banks, the so-called LIBOR rate. With m we intend the marginal coefficient that measures the impact of x on $F_1(x, y)$. $F_1(x, y)$ depends on x because, if the Enterprise buys futures with a strategy $x \neq 0$, the price F_1 changes because an increase of future demand influences the future price.

6) $(1+i)^{-1}$ is the discount factor. F_1 must be actualized at time t = 1 because the money for the sale of futures will be cashed in a time t = 2.

The payoff function of the Financial Institute. Therefore knowing that

$$F_1(x,y) = S_1(y)(1+i)$$

and replacing in

$$f_2(x,y) = yM_2[F_1(x,y)(1+i)^{-1} - (S_0 + ny)(1+i)],$$

we have:

$$f_2(x,y) = yM_2([(S_0 + ny)(1+i)(1+i) + mx](1+i)^{-1} - (S_0 + ny)(1+i)).$$

After the appropriate simplifications, below we represent the payoff function of the Financial Institute:

$$f_2(x,y) = yM_2mx(1+i)^{-1}$$

From now the value $m(1+i)^{-1}$ will be called μ_1 for easy of calculation.

Summarizing:

$$f(x,y) = (-\nu_1 yM_1(1-x), yM_2\mu_1 x).$$

4. Study of the game: the payoff space

4.1 Critical space of the game

Our game E×F is structured as follows:

Strategy set and payoff function of the first player:	$E = [0,1]$	$f_1(x,y) = -\nu_1 yM_1(1-x)$;
Strategy set and payoff function of the second player:	$F = [-1,1]$	$f_2(x,y) = yM_2\mu_1 x$.

Since we are dealing with a non-linear game it is necessary to study in the bi-win space also the points of the critical zone, which belong to the bi-strategy space. To find the critical area of the game we consider the Jacobian matrix and we put its determinant equal 0.

For what concern the gradients of f_1 and f_2, we have

$$\text{grad } f_1 = (M_1 y, -\nu_1 M_1 (1-x))$$
$$\text{grad } f_2 = (M_2 \mu_1 y, M_2 \mu_1 x).$$

The determinant of the Jacobian matrix is

$$\det J_f(x,y) = M_1 M_2 \nu_1 y \mu_1 x + M_1 M_2 \mu_1 (1-x)\, \nu_1 y.$$

Therefore the ***critical space of the game*** is

$$Z_f = \{(x,y) : M_1 M_2 \nu_1 y \mu_1 x + M_1 M_2 \mu_1 (1-x)\, \nu_1 y = 0\}.$$

Dividing by $M_1 M_2 \nu_1 \mu_1$, which are all positive numbers (strictly greater than 0), we have:

$$Z_f = \{(x,y) : yx + (1-x)\, y = 0\}.$$

Finally we have:

$$Z_f = \{(x,y): y = 0\}.$$

The critical area of our bi-strategy space is represented in the following figure by the dotted orange line, and it is the segment [H, K].

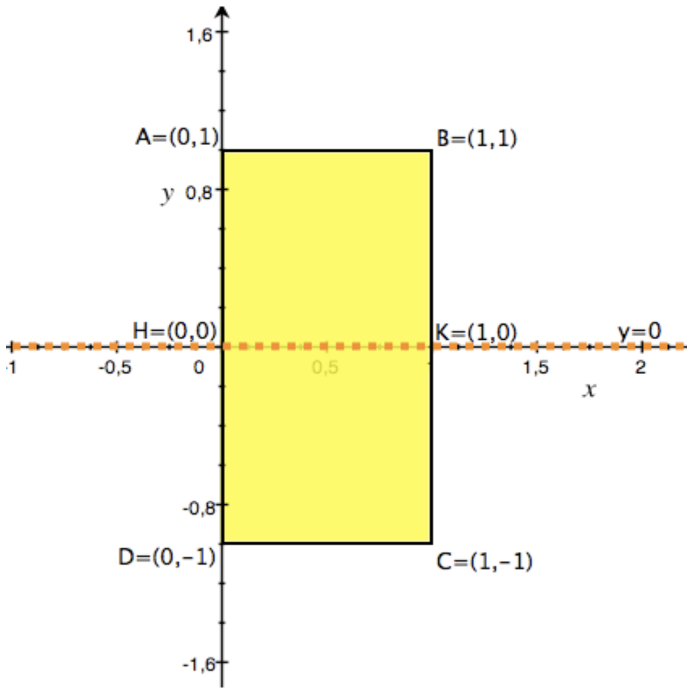


Figure 2. The critical space of the game

4.2 Payoff space

In order to represent graphically the payoff space f (E × F), we transform, by the function f, all the sides of bi-strategy rectangle E × F and the critical space Z of the game G.

1) The segment [A, B] is the set of all the bi-strategies (x,y) such that

$y = 1$ and $x \in [0,1]$.

Calculating the image of the generic point (x,1), we have:

$f(x,1) = (M_1[-\nu_1 (1-x)], M_2\mu_1 x)$.

Therefore setting

$X = M_1 [-\nu_1 (1-x)]$
$Y = M_2\mu_1 x$

and assuming $M_1 = 1$, $M_2 = 2$, and $\nu_1 = \mu_1 = 1/2$, we have

$X = -1/2 (1-x)$
$Y = x$.

Replacing Y instead of x, we obtain the image of the segment [A,B], defined as the set of the bi-wins (X,Y) such that

$X = -1/2 (1-Y) = -1/2+Y$
$Y \in [0,1/2]$.

It is a line segment with extremes A' = f (A) and B' = f (B).

Following the procedure described above for the other side of the bi-strategy rectangle and for the critical space, that are the segments [B, C], [C, D] [D, A] and [H, K], we get on the payoff space f(E × F) of our game G, the following figure:

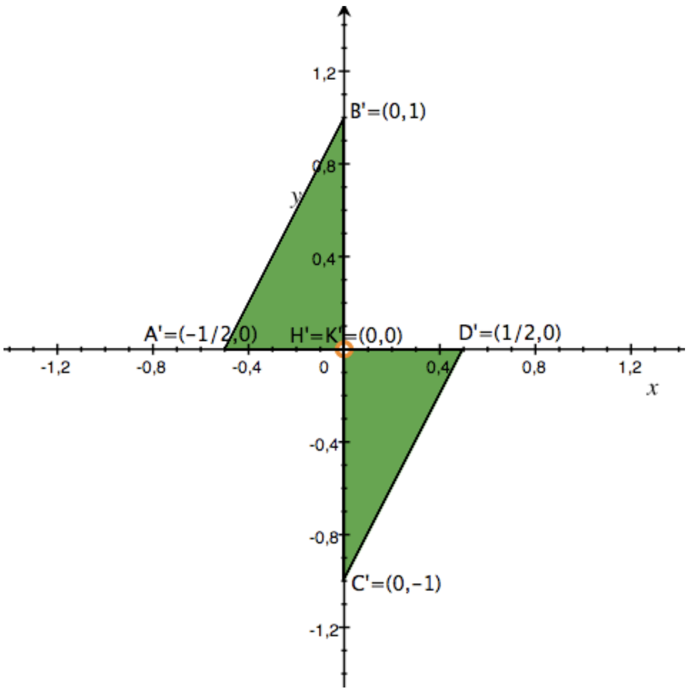


Figure 3. The payoff space of the game

We can see how the set of possible winning combinations of the two players took a curious butterfly shape that promises the game particularly interesting.

5. Study of the game and equilibria

5.1 Friendly phase

The superior extremum of the game, that is the bi-win $\alpha = (1/2,1)$, is a shadow maximum because it doesn't belong to the payoff space:

$$\alpha = (1/2,1) \notin f(E\times F).$$

The infimum of the game, that is the bi-win $\beta = (-1/2,-1)$, is a shadow minimum because it doesn't belong to the payoff space:

$$\beta = (-1/2,-1) \notin f(E\times F).$$

The weak maximal Pareto boundary of the payoff space is

$$[B'K'] \cup [H'D'].$$

The proper maximal Pareto boundary of the payoff space is represented by

$$\partial^{*} f (E\times F) = \{B', D'\}$$

The weak maximal Pareto boundary of the bi-strategic space is the retro-image of the weak maximal Pareto boundary of the payoff space, id

$$[BK] \cup [HD] \cup [HK].$$

The proper maximal Pareto boundary of the bi-strategic space is the reciprocal image of the proper maximal Pareto boundary of the payoff space, id

$$\partial^{*}_{f} (E\times F) = \{B , D\}.$$

The weak minimal Pareto boundary of the payoff space is

$$[A'H'] \cup [K'C'].$$

The proper minimal Pareto boundary of the payoff space is represented by

$$\underline{\partial} f (E\times F) = \{A', C'\}$$

The weak minimal Pareto boundary of the bi-strategy space is the reciprocal image of the weak minimal Pareto boundary of the payoff space, is

$$[AH] \cup [KC] \cup [HK]$$

The proper minimal Pareto boundary of the bi-strategy space is the reciprocal image of the proper minimal Pareto boundary of the payoff space, id

$$\underline{\partial}_{f} (E\times F) = \{A , C\}.$$

Now we show graphically the previous considerations:

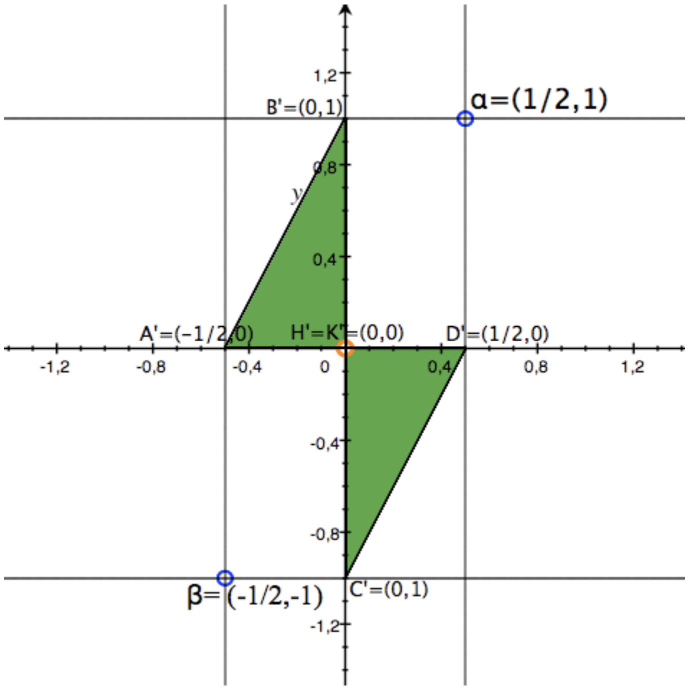


Figure 4. Pareto boundaries and extrema of the game

Control and accessibility of non-cooperative Pareto boundaries.

Definition of Pareto control. The Enterprise can cause a Pareto bi-strategy x_0 if exists a strategy such that for every strategy y of the Financial Institute the pair (x_0, y) is a Pareto pair.

In this regard, in our game there are no maximal Pareto controls, nor minimal. So neither player can decide to go on the Pareto boundary without cooperation with the other one. The game promises to be quite complex to resolve in a satisfactory way for both players.

5.2 Nash equilibria

If the two players want to think only for themselves, they would choose the strategy that makes maximum their win regardless of the other player's strategy. In this case we talk about ***multifunction of best reply***.

It means to maximize for each player its payoff function considering every possible strategy of the other player.

In mathematical language the multifunction of best reply of the Enterprise is:

$$B_1: F \to E : y \to \max E_{f1(.,y)}$$

(i.e. the strategies of the Enterprise which maximize the section $f_{1(.,y)}$).

On the other hand the multifunction of best reply of the Financial Institute is:

$$B_2: E \to F: x \to \max F_{f2(x,.)}$$

(i.e. the strategies of the Financial Institute which maximize the section $f_{2(x,.)}$).

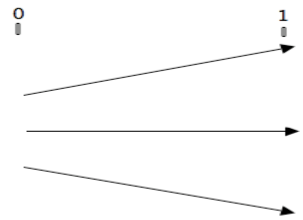
In practice, in order to find B_1 we use the partial derivative concerning x and we maximize it. On the other hand, in order to find B_2 we calculate the partial derivative concerning y and we maximize it.

Remembering that $M_1 = 1$, $\nu_1 = 0,5$, $M_2 = 2$ e $\mu_1 = 0,5$ are always positive numbers (strictly greater than 0), and that

$$f_1(x,y) = M_1[-\nu_1 y\,(1-x)],$$

we have

$$\partial_1 f_1 = \quad M_1\nu_1 y$$
$$M_1\nu_1 y > 0$$



$B_1(y) = 1$ se $y > 0$
$B_1(y) = E$ se $y = 0$
$B_1(y) = 0$ se $y < 0$

Remembering also that

$$f_2(x,y) = M_2\mu_1 xy$$

we have

$$\partial_2 f_2 = \quad M_2\mu_1 x$$
$$M_2\mu_1 x > 0$$

$B_2(x) = 1$ se $x > 0$
$B_2(x) = F$ se $x = 0$

Representing in red the graph of $B_1(y)$, and in blue that one of $B_2(x)$ we have:

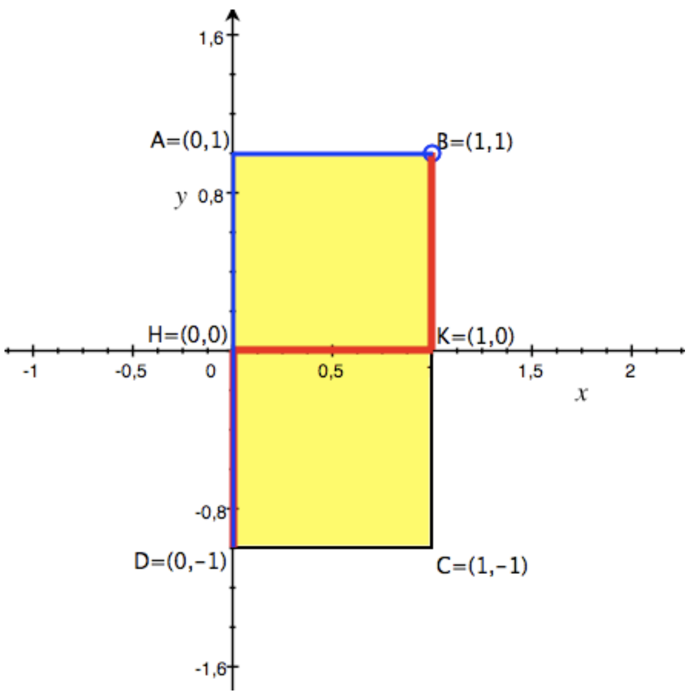


Figure 5. Nash equilibria.

The set of Nash equilibria, that are the intersections of two graphs of best reply is

$$Eq\ (B_1,B_2) = \{(1,1)\} \cup [HD].$$

The Nash equilibria can be considered good because they are on the weak maximal Pareto boundary (indeed the point K' is also part of the weak minimal boundary). Also among the Nash equilibria there are even the two the points that represent the proper maximal Pareto boundary, i.e. {B ', D'}. It is clear that if the two players pursue as aim the profit, and decide to choose their selfish strategy to obtain the maximum possible win, they will arrive on the boundary maximal weak. The selfishness, in this case, pays well. This purely mechanical examination, however, leaves us dissatisfied. The Enterprise has two Nash possible alternatives: not to hedge playing x = 0, or to hedge totally playing x = 1. Playing X = 0 it could both to win than lose, depending on the strategy played by the Financial Institute; opting instead for x = 1, the Enterprise guarantee to himself to leave the game without any loss and without any win. Analyzing the strategies of the Financial Institute relevant for Nash, we see that if the Enterprise adopts a strategy $x \neq 0$ the Financial Institute plays the strategy y = 1 winning something, or else if the Enterprise plays x = 0 the Financial Institute can play all its strategy set y = F indiscriminately without obtaining any win or loss. These considerations lead us to believe that the Financial Institute will play y = 1, in order to try to win at least "something", because if the Enterprise plays x = 0, its strategy y does not affect its win. The Enterprise, which knows the situation of the Financial Institute that very likely chooses the strategy y = 1, will hedge playing x = 1. So, despite the Nash equilibria are infinite, it is likely that the two players arrive in B = (1.1), which is part of the proper maximal Pareto boundary. Nash is a viable, feasible and satisfactory solution, at least for one of two players, presumably the Financial Institute.

5.3 Defensive phase

We suppose that the two players are aware of the will of the other one to destroy it economically, or are by their nature cautious, fearful, paranoid, pessimistic or risk averse, and then they choose the strategy that allows them to minimize their loss. In this case we talk about defensive strategies.

5.3.1 Conservative value and meetings

Conservative value of a player. It is defined as the maximization of its function of worst win.
Therefore the conservative value of the Enterprise is

$$v^{\#}_1 = \sup_{(x \in E)} f_1^{\#}$$

where $f_{1^{\#}}$ is the function of worst win of the Enterprise, and is given by

$$f_1^{\#} = \inf_{y \in F} f_1.$$

Remembering that

$$f_1\,(x,y) = M_1[-\nu_1 y\ (1-x)]$$

and that $M_1 = 1$, $\nu_1 = 0{,}5$, $M_2 = 2$ and $\mu_1 = 0{,}5$ are always positive numbers (strictly greater than 0), we have:

$$f_1^{\#} = \inf_{y \in F} M_1\,[-\nu_1 y\ (1-x)].$$

Therefore since the offensive strategies of the Financial Institute are

$$y^o = \begin{cases} 1 \text{ se } 0 \leq x < 1 \\ F \text{ se } x = 1, \end{cases}$$

we obtain:

$$f_1^{\#} = M_1\,[-\nu_1\ (1-x)] \text{ se } 0 \leq x < 1$$
$$f_1^{\#} = 0 \text{ se } x = 1$$

Graphically $f_1^{\#}$ appears as:

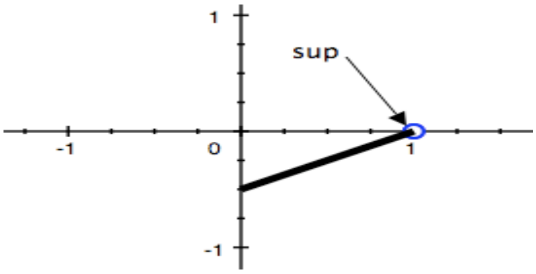


Figure 6. Graphical representation of $f_1^{\#}$, the function of worst win of the Enterprise.

So the defense (or conservative) strategy of the Enterprise is given by

$x^{\#} = 1$

and the conservative value of the Enterprise is

$v^{\#}_1 = \sup_{x \in E} \inf_{y \in F} M_1 [-\nu_1 y (1-x)] = 0.$

On the other hand, the conservative value of the Financial Institute is given by

$v^{\#}_2 = \sup_{y \in F} f_2^{\#}$

where $f_{2^{\#}}$ is the function of the worst win of the Financial Institute. It is given by

$f_2^{\#} = \inf_{x \in E} f_2.$

Remembering that

$f_2 (x,y) = M_2 \mu_1 xy$

and that $M_1 = 1$, $\nu_1 = 0,5$, $M_2 = 2$ and $\mu_1 = 0,5$ are always positive numbers (strictly greater than 0), we have:

$f_2^{\#} = \inf_{x \in E} M_2 \mu_1 xy.$

Therefore since the offensive strategies of the Enterprise are

$x^{o} =$ 0 se $y > 0$
E se $y = 0$
1 se $y < 0$

we obtain:

$f_2^{\#} = 0$ se $y \geq 0$
$f_2^{\#} = M_2 \mu_1 y$ se $y < 0$

Graphically $f_2^{\#}$ appears as:

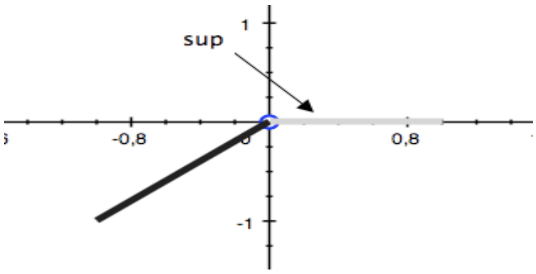


Figure 7. Graphical representation of $f_2^{\#}$, the function of worst win of the Financial Institute.

So the defense (or conservative) strategy of the Financial Institute is given by

$y^{\#} = [0,1]$

and the conservative value of the Financial Institute is

$v^{\#}_2 = \sup_{y \in F} \inf_{x \in E} M_2 \mu_1 xy = 0.$

Therefore the conservative bi-value is

$v^{\#}_f = (v^{\#}_1, v^{\#}_2) = (0,0)$.

Conservative meetings. They are represented by the bi-strategies $(x^{\#}, y^{\#})$, that are represented by the whole segment [B, K]. If the Enterprise and the Financial Institute decides to defend themselves against any opponent's offensive strategies they arrive on the payoffs subset [B', K'], which is part of the weak maximal Pareto boundary. B 'is even a point on the proper maximal boundary, while K' is also part of the weak minimal one. In this simplified model, although there is the possibility that the Financial Institute decides not to act on the market, obtaining in this way no profit and arriving in K ', the Financial Institute presumably will choose the defensive strategy $y^{\#} = 1$, because it's the only one that allows him to obtain the maximum possible profit (being able anyway not to incur losses). In this case the players arrive in B', the optimal solution for the Financial Institute. This happens because the Enterprise was unable with its strategies $x \in [0,1]$ to lead to a lowering of the futures price on the underlying.

Observation. In reality, however, in addiction to the Enterprise there are other traders, which could also cause a fall in futures prices and then, if the Financial Institute would choose a defensive strategy, presumably it would decide to not act on the market with $y^{\#} = 0$. In this case, the conservative meeting would be only one, i.e. K = (1,0).

5.3.2 Core and conservative parts of the game

Core of the payoff space. The core is the part of the maximal Pareto boundary contained in the upper cone of $v^{\#}_f = (v^{\#}_1, v^{\#}_2) = (0,0)$.

Therefore we have

core' (G) = [B'K'] ∨ [H'D'],

whose reciprocal image is

core (G) = [BK] ∨ [HD] ∨ [HK].

We can see graphically in red the part of the payoff space where the Enterprise would has a win greater than its conservative value $v^{\#}_1 = 0$ (x-axis in pink). On the other hand, in blue is shown the part of the payoff space where the Financial Institute obtains a win higher than its conservative value $v^{\#}_2 = 0$ (y-axis in light blue):

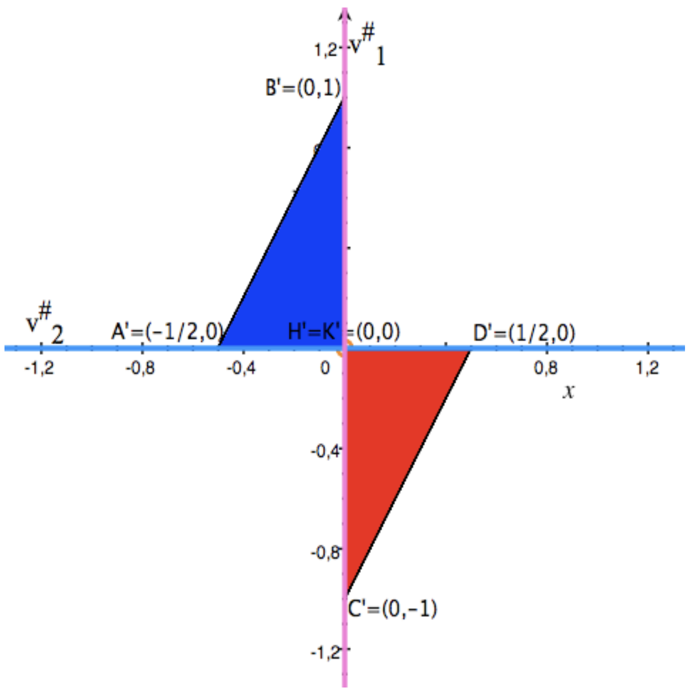


Figure 8. Core' and conservative parts on the payoff space.

We note that if both players choose their conservative strategies $x^{\#} = 1$ e $y^{\#} = [0,1]$, the Enterprise avoids to lose more of its conservative value $v^{\#}_1 = 0$ but is automatically unable to get also higher wins. The same discourse does not apply to the Financial Institute that may arrive on the segment [B'K']. The game is in substance blocked for the Enterprise, that is clearly disadvantaged in respect of the Financial Institute.

Observation. But remembering the first observation, the game would be blocked for both, with the Financial Institute also unable to get higher wins to its conservative value $v^{\#}_2 = 0$ if it decides to play its defensive strategy $y^{\#} = 0$.

Conservative part of the game on the bi-strategy space. It is the set of the pairs (x,y) such that

$$f_1(x) \geq v^{\#}_1 \wedge f_2(y) \geq v^{\#}_2 .$$

Remembering that

$$f_1(x,y) = M_1[-\nu_1 y(1-x)]$$

and that

$$v^{\#}_1 = 0,$$

the conservative part of the Enterprise on the bi-strategy space is given by

$$(E\times F)^{\#}_1 = M_1[-\nu_1 y(1-x)] \geq 0,$$

which developed becomes

$$-\nu_1 M_1 y \leq 0 \vee x \leq 1$$

or

$$-\nu_1 M_1 y \geq 0 \vee x \geq 1.$$

Knowing that $M_1 = 1$ and $\nu_1 = 0{,}5$ are always positive numbers (strictly greater than 0), graphically we have the following result:

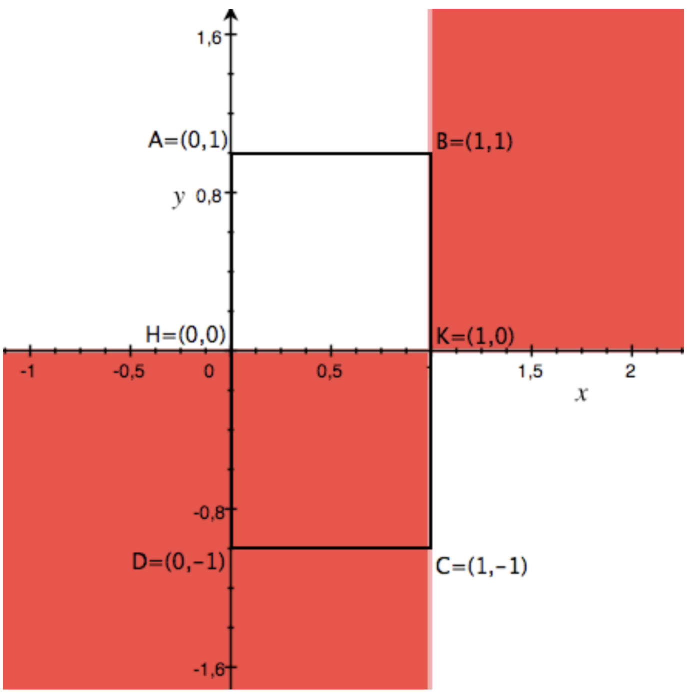


Figure 9. Conservative part of the Enterprise (in red) on the bi-strategy space**.**

Now talk about the Financial Institute. Remembering that

$$f_2(x,y) = M_2\mu_1 xy$$

and that

$$\nu''_2 = 0,$$

the conservative part of the Financial Institute on the bi-strategy space is given by

$$(E\times F)^{\#}_2 = M_2\mu_1 xy \geq 0.$$

Remembering that $M_2 = 2$ and $\mu_1 = 0,5$ are always positive numbers (strictly greater than 0), graphically we have the result shown in the following page:

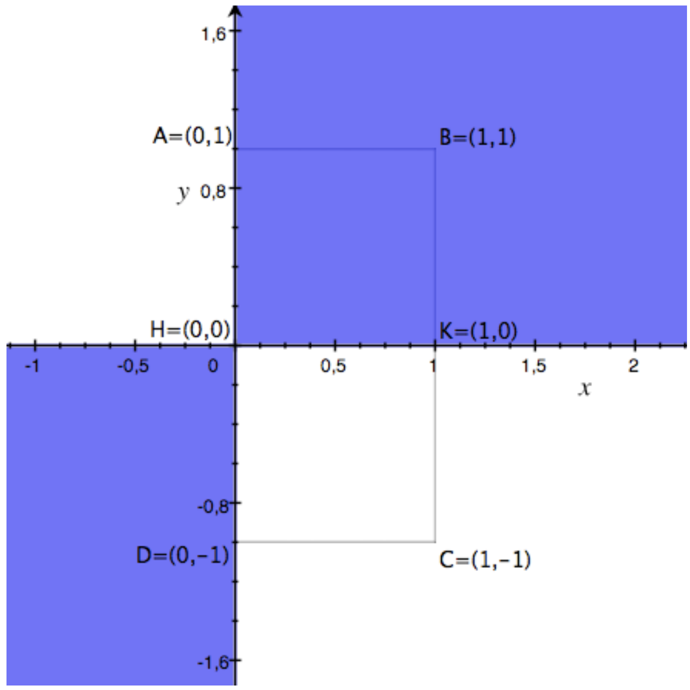


Figure 10. Conservative part of the Financial Institute (in light blue) on the bi-strategy space.

Then superimposing the graph of the conservative part (we talk about bi-strategy space) of the Enterprise and that one of the Financial Institute, we have the conservative part of the game on the bi-strategy space.

It is given by

$$(E\times F)^{\#} = (E\times F)^{\#}_1 \wedge (E\times F)^{\#}_2,$$

and i.e.

$$(E\times F)^{\#} = M_1[-\nu_1 y(1-x)] \geq 0 \wedge M_2\mu_1 xy \geq 0.$$

We observe the graphical result:

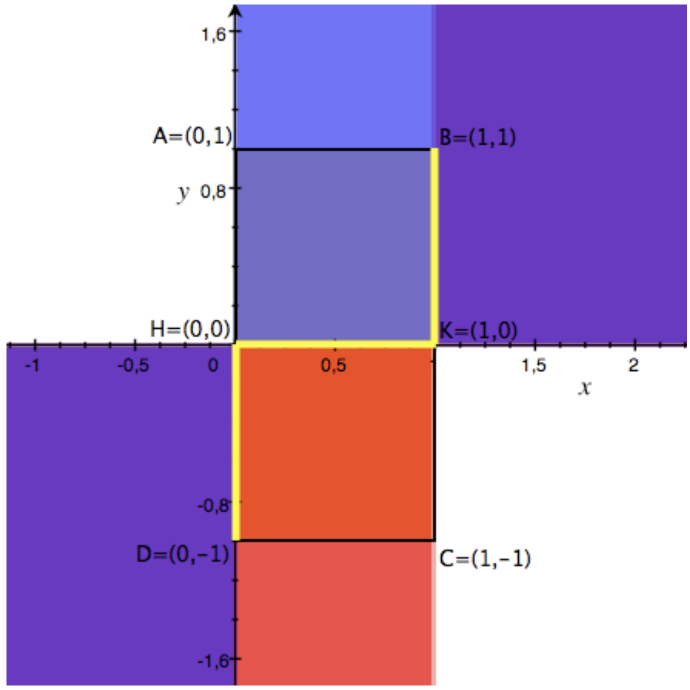


Figure 11. Conservative part of the game (in yellow) on the bi-strategy space.

We see easily that the conservative part of the game, on the bi-strategy space, is given by

$(E\times F)^{\#} = [BK] \vee [KH] \vee [HD]$.

5.3.3 Conservative nodes of the game

Conservative nodes. They are represented by the pairs (x,y) such that

$f_1(x,y) = v^{\#}_1$ e $f_2(x,y) = v^{\#}_2$.

And therefor remembering that

$f_1(x,y) = M_1[-\nu_1 y(1-x)]$

and that

$v^{\#}_1 = 0$,

we have:

$M_1[-\nu_1 y(1-x)] = 0$.

Solving the equation, we obtain

$-\nu_1 M_1 y = 0$

and

$(1-x) = 0$.

Knowing that M_1, e ν_1 are always positive numbers (strictly greater than 0), we have:

$y = 0$

and

$x = 1$.

Therefore the point K = (1,0) is a conservative node.

Remembering also that

$f_2(x,y)= M_2\mu_1xy$

and that

$v^{\#}_2 = 0$,

we have:

$M_2\mu_1xy = 0$.

Knowing that M_2 e μ_1 are always positive numbers (strictly greater than 0), we have:

x = 0

e

y = 0.

Therefore also the point H = (0,0) is a conservative node.

5.4 Cooperative solutions

It is clear at this stage of the study of our game that the best way for two players to get both a win is to find a cooperative solution.
One way would be to divide the maximum collective profit, determined by the maximum of the function of collective gain

$g(X,Y) = X+Y$

on the payoffs space of the game G

$V = \max_S g$,

that is represented in terms of bi-wins with evidence by the point B', which is the only bi-win belonging to the straight line

$X+Y = 1$

and to the payoff space.

So the Enterprise and the Financial Institute play x = 1 and y = 1 in order to arrive at the B 'and then split the wins obtained by contract. Practically the Enterprise buys futures to create artificially between future and spot prices a misalignment that is exploited by the Financial Institute, which get the maximum win V = 1.
For a possible quantitative division of this win V = 1 between the Financial Institute and the Enterprise, we use the transferable utility solution applying the transferable utility Pareto boundary of the payoff space the non-standard Kalai Smorodinsky Solution (non-standard because we do not consider the whole game, but only the maximal Pareto boundary).
We proceed finding the superior extremum of our maximal Pareto boundary, which is

$\sup \partial^* f(E\times F) = \alpha = (1/2,1)$

and join it with the inferior extremum of our maximal Pareto boundary, which is given by

$\inf \partial^* f(E\times F) = v^{\#} = (0,0)$.

We note that the inferior extremum of our maximal Pareto boundary is equal to $v^{\#} = (0,0)$.

The coordinates of the point of intersection P between the straight line of maximum collective win, i.e.

X+Y = 1,

and the straight line which joins the superior extremum of the maximal Pareto boundary with the infimum, i.e.

Y = 2X,

give us the desirable division of the maximum collective win V = 1 between the two players.

We can see the following figure in order to make us more aware of the situation:

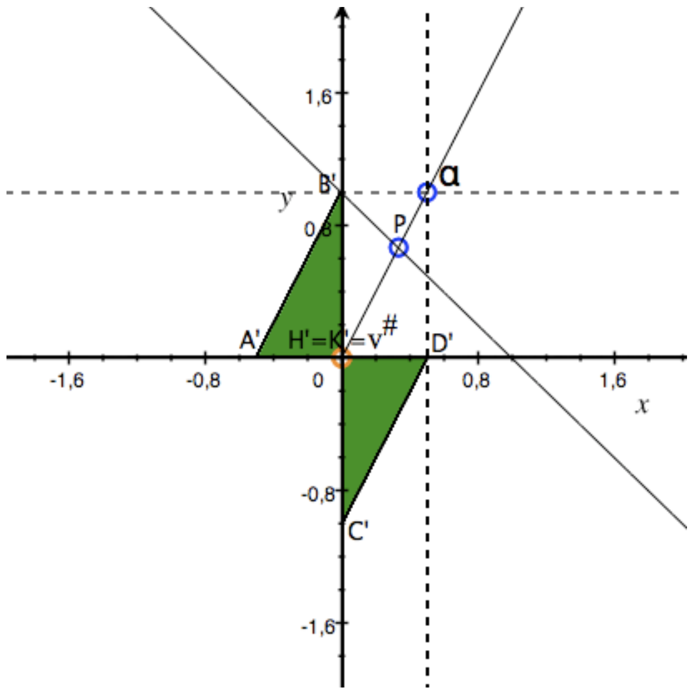


Figure 12. Transferable utility solution: cooperative solution.

In order to find the coordinates of the point P is enough to put in a system of equations

X+Y = 1

e

Y = 2X.

Substituting the Y in the first equation we have

X+2X = 1

and therefore

X = 1/3.

Substituting now the X in the second equation, we have

Y = 2/3.

Thus P = (1/3, 2/3) suggests as solution that the Enterprise receives 1/3 by contract by the Financial Institute, while at the Financial Institute remains the win 2/3.

6. Conclusions

The game just studied suggests a possible regulatory model that provides the stabilization of the financial market through the introduction of a tax on financial transactions. In fact, in this way it could be possible to avoid speculation by which our modern economy is constantly affected, and the Financial Institute could equally wins without burdening on the entire financial system with its unilateral manipulation of the asset price that it trades. In this game the only optimal solution is the cooperative one exposed in the previous section, otherwise the game appears like a sort of "your death, my life", as often happens in the economic competition, which leaves no escape if either player decides to work alone, without a mutual collaboration. In fact, all non-cooperative solutions lead dramatically to mediocre results for at least one of the parties. Now it is possible to provide an interesting key in order to understand the conclusions which we reached using the transferable utility solution. Since the point B = (1,1) is also the most likely Nash equilibrium, the number 1/3 (that the Financial Institute pays by contract to the Enterprise) can be seen as the fair price paid by the Financial Institute to be sure that the Enterprise chooses the strategy x = 1, so they arrive effectively to more likely Nash equilibrium B = (1,1), which is also the optimal solution for the Financial Institute.

7. Appendix

7.1 Offensive equilibria

If the two players want to think only to ruin the other one, would choose the strategy that makes maximum the loss of the other one. In this case it is necessary to talk about ***multifunction of worst offense.***
We have to minimize for each player the payoff function considering every possible strategy of the other player.
In mathematical language, the multifunction of worst offense of the Financial Institute to the Enterprise is:

$$O_2: E \to F : x \to \min E_{f1(x,\cdot)}$$

(i.e. the set of strategies of the Financial Institute that minimize the section $f_{1\,(x,\,\bullet)}$).
On the other hand, the multifunction of worst offense of the Enterprise to the Financial Institute is:

$$O_1 : F \to E : y \to \min F_{f2(\cdot,y)}$$

(i.e. the set of strategies of the Enterprise that minimize the section $f_{2\,(\bullet,y)}$).

Remembering that

$$f_1(x,y) = M_1[-\nu_1 y\,(1-x)]$$

we have

$$O_2(x) = \begin{cases} 1 \text{ se } 0 \le x < 1 \\ F \text{ se } x = 1. \end{cases}$$

Remembering also that

$$f_2(x,y) = M_2\mu_1 xy,$$

we have

$$O_1(y) = \begin{cases} 0 \text{ se } y > 0 \\ E \text{ se } y = 0 \\ 1 \text{ se } y < 0. \end{cases}$$

We observe in the following figure the graphs of O_2 (in blue) and of O_1 (in red):

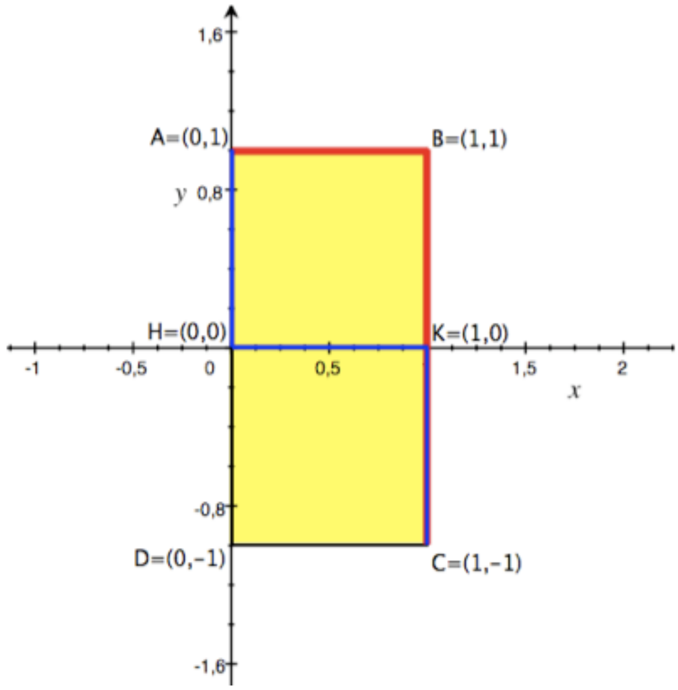


Figure 13. Offensive equilibria.

The set of offensive equilibria, i.e. the intersections of two graphs of worst offense is

$$Eq\ (O_1,O_2) = \{(0,1)\} \cup [KC].$$

The offensive equilibria may be considered bad because they are on the weak minimal Pareto boundary (indeed the point K 'is also part of the weak maximal boundary). In addition, among the offensive equilibria there are also the two points that represent the proper minimal Pareto boundary, i.e. {A', C'}. It is clear that if the two players want to attack the other one, and decide to choose their strategy just to spite the other player, they arrive on the weak minimal Pareto boundary. Analyzing the situation, probably the Financial Institute plays the strategy $y = 1$ because it is the only one able to maximize the damage of the Enterprise if it plays $x \neq 1$, while if the Enterprise chooses the strategy $x = 1$, the choice of strategy by the Financial Institute is indifferent about the damage (zero) procured to the Enterprise. On the other hand, knowing that the Financial Institute chooses the strategy $y = 1$ to try to hurt it, the Enterprise most likely chooses $x = 0$ to be sure that the Financial Institute gets the minimum possible win (which, in this case, is equal to 0). So, despite the offensive equilibria are infinite, the two players most likely arrive in A = (0,1), which is on the proper minimal Pareto boundary: the offensive strategies of both players can be considered a credible threat. We want to highlight as very likely even if the Enterprise plays its offensive strategies, in our game, however, the Financial Institute will not lose.

7.2 Equilibria of devotion

In the event that the two players wanted to "do good" to the other one, they would choose its strategy that maximizes the payoff of the other one. In this case is necessary to talk about ***multifunction of devotion***.
We have to maximize the payoff function of the other player considering every its possible strategy. In mathematical language for the Enterprise the multifunction of devotion is:

$$L_1 : F \to E : y \to \max_{f2(\cdot,y)} F,$$

(i.e. the set of strategies of the Enterprise that maximize the section $f_{2(\cdot,y)}$).

For the Financial Institute the multifunction of devotion is:

L_2: E → F : x → max $_{f1(x,\cdot)}$ E

(i.e. the set of strategies of the Financial Institute that maximize the section $f_{1(x,\cdot)}$).

In practice, in order to find L_1 we try the value of x that maximizes f_2; in order to find L_2 we try the value of y that maximize f_1 .

Remembering that $M_1 = 1$, $\nu_1 = 0,5$, $M_2 = 2$ and $\mu_1 = 0,5$ are always positive numbers (strictly greater than 0) and that the payoff function of the Enterprise is

$f_1(x,y) = M_1 [-\nu_1 y (1-x)]$,

and that the payoff function of the Financial Institute is

$f_2(x,y) = M_2 \mu_1 xy$,

we have:

$L_2(x) =$ -1 se $0 \leq x > 1$
F se $x = 1$

$L_1(y) =$ 1 se $y > 0$
E se $y = 0$
0 se $y < 0$

Now we illustrate in red the graph of $L_1(y)$, in blue that one of $L_2(x)$:

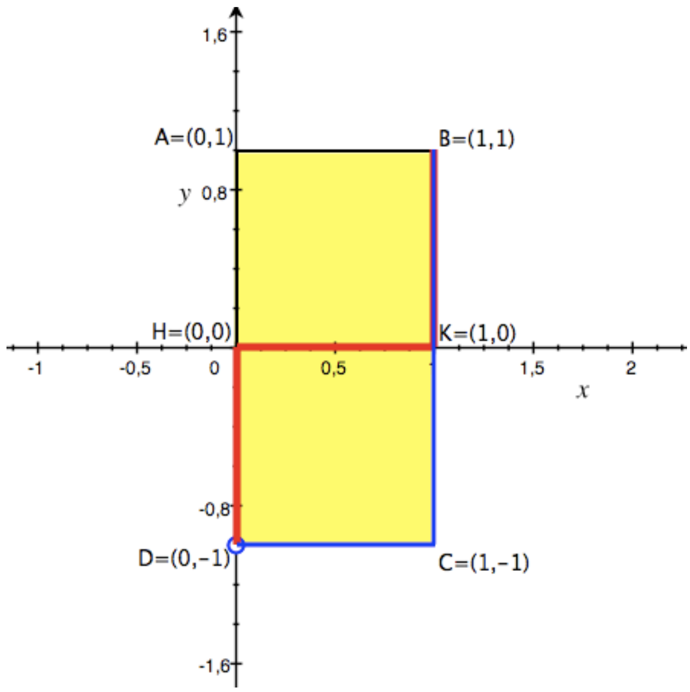


Figure 14. Equilibria of devotion.

The set of equilibria of devotion is

Eq $(L_1,L_2) = \{(0,-1)\} \cup [BK]$.

The equilibria of devotion can be considered good because they are on the weak maximal Pareto boundary (indeed the point K' is also part of the weak minimal boundary). Also among the devote equilibria there are even the two the points that represent the proper maximal Pareto boundary, i.e. {B',D'}. It is clear that if both players ignore their good and decide to choose their strategy selflessly so that the other one has the maximum possible win, they arrive on the weak maximal Pareto boundary. The Financial Institute probably plays the strategy $y = -1$ because it is the only one able to maximize the win of the Enterprise if it plays $x \neq 1$, while if the Enterprise chooses the strategy $x = 1$, the choice of strategy of the Financial Institute is indifferent about the win (equal to 0) of the Enterprise. On the other hand, the Enterprise, knowing that the Financial Institute chooses the strategy $y = -1$ in order to help it, most likely chooses $x = 0$. So the Financial Institute gets the highest possible win, which in this case is equal to 0. We can see that although the equilibria of devotion are infinite, the two players most likely arrive in $D = (0, -1)$, which is on the proper maximal Pareto boundary. In case of devote strategies adopted by the Financial Institute, most likely the Enterprise manages to win the maximum possible sum, while it is not the same for the Financial Institute.

8. References

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