A Hull and White formula for a general stochastic volatility jump-diffusion model with applications to the study of the short-time behavior of the implied volatility

Elisa Alòs*
Dpt. d'Economia i Empresa, Universitat Pompeu Fabra,
Ramon Trias Fargas 25-27, 08005 Barcelona, Spain
elisa.alos@upf.edu

Jorge A. León†
Depto. de Control Automático, CINVESTAV-IPN,
Apartado Postal 14-740, 07000 México D.F., Mexico
jleon@ctrl.cinvestav.mx

Monique Pontier‡
Institut mathématique de Toulouse, Université de Toulouse,
31 062 TOULOUSE cedex 09, France
monique.pontier@math.univ-toulouse.fr

Josep Vives§
Dpt. Probabilitat, Lògica i Estadística, Universitat de Barcelona,
Gran Via 585, 08007 Barcelona (Catalunya), Spain
josep.vives@ub.edu

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Abstract

In this paper, generalizing results in Alòs, León and Vives (2007b), we see that the dependence of jumps in the volatility under a jump-diffusion stochastic volatility model, has no effect on the short-time behaviour of the at-the-money implied volatility skew, although the corresponding Hull and White formula depends on the jumps. Towards this end, we use Malliavin calculus techniques for Lévy processes based on Løkka (2004), Petrou (2006), and Solé, Utzet and Vives (2007).

Keywords: Hull and White formula, Malliavin calculus, Itô's formula for the Skorohod integral, jump-diffusion stochastic volatility models.

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1 Introduction and statement of the model

Stochastic volatility models are a well-known cornerstone in order to replicate some important features of the implied volatility, like its dependence with respect to the strike price. Unfortunately, in the continuous case, these models are insufficient to capture other crucial properties observed in financial markets, like the dependence of the implied volatility on time to maturity as it is shown in Lewis (2000). More precisely, empirical observations show that the at-the-money implied volatility skew slope explodes as time to maturity tends to zero, but this slope tends to a constant when we consider a stochastic volatility diffusion model (see for example Medvedev and Scaillet (2004)). This problem has motivated to consider jumps in the asset price dynamic models, between which we can mention the well-known model of Bates (see Bates (1996)), among others. This models allow flexible modelling and generate skews and smiles similar to those observed in market data.

In Alòs, León and Vives (2007b) a generalized Bates model, in the sense that the stochastic volatility does not follows a concrete equation, is studied. They prove, in particular, that for a volatility process independent of price jumps (as in the Bates case) the at-the-money skew slope behaviour at the expiry date is closely related to the derivative of the volatility process with respect to the Brownian motion driving the stock prices. They also show that the Malliavin calculus is a powerful tool to deal with volatility models.

The purpose of this paper is to extend the results of Alòs, León and Vives (2007b) to the case that the volatility can be correlated not only with the Brownian motion driving the stock prices, but also with the price jump process. Namely, we consider a log-price process, under the market chosen risk-neutral probability measure, given by

$$X_t = x + (r - \lambda k)t - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s(\rho dW_s + \sqrt{1-\rho^2}dB_s) + Z_t, \tag{1}$$

where, $t \in [0, T]$, x is the current log-price, r is the instantaneous interest rate, W and B are independent standard Brownian motions, $\rho \in [-1, 1]$, and Z is a compound Poisson process, independent of W and B, with intensity λ, Lévy measure ν, and with $k := \frac{1}{\lambda}\int_{\mathbb{R}}(e^y - 1)\nu(dy)$.

In Alòs, León and Vives (2007b), the volatility process σ is assumed to be a square-integrable stochastic process with right-continuous trajectories, bounded below by a positive constant and adapted to the filtration generated by W. Here we will assume the same hypothesis, but less restrictively, only that σ is adapted to the bigger filtration generated by W and Z. So, in this paper, we allow the volatility to have non-predictable jump times as advocated by Bakshi, Cao and Chen (1997) and Duffie, Pan and Singleton (2000), among others.

A useful tool to work with this model is the stochastic variation calculus for Lévy processes, also named Malliavin-Skorohod calculus. In this paper we link two different approaches of this calculus, one that comes from Solé, Utzet and Vives (2007) and another that comes from Løkka (2004) and Petrou (2006).

More concretely, following the ideas given by Alòs, León and Vives (2007b) we will obtain a generalized Hull and White formula for model (1). In comparison with the formula obtained in Alòs, León and Vives (2007b), our formula has an extra term because the volatility depends now on the jump price. This representation will allow us to show that the existence of correlation between the volatility process and the price jumps does not have any influence on the at the money skew of the implied volatility as time runs to expiry.

In the following, we denote by $\mathcal{F}^W, \mathcal{F}^B$ and $\mathcal{F}^Z$ the filtrations generated by the independent processes W, B and Z respectively. Moreover we define $\mathcal{F} := \mathcal{F}^W \vee \mathcal{F}^B \vee \mathcal{F}^Z$.

It is well-known that if we price an European call with strike price K by the formula

$$V_t = e^{-r(T-t)}E[(e^{X_T} - K)_+|\mathcal{F}_t], \tag{2}$$

where E is the expectation with respect to a risk neutral measure, there is no arbitrage opportunity. Thus V_t is a possible price for this derivative.

In the sequel we will use the following notation:

- The process $v_t := \left(\frac{Y_t}{T-t}\right)^{\frac{1}{2}}$, with $Y_t := \int_t^T \sigma_s^2 ds$, will denote the future average volatility.

- With $BS(t,x,\sigma)$ we will denote the classical Black-Scholes function with constant volatility σ, current log stock price x, time to maturity $T-t$, strike price K and interest rate r. Remember that this function can be written as

$$B(t,x,\sigma) = e^x \Phi\left(\frac{x - x_t^*}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}\right) - e^{x_t^*}\Phi\left(\frac{x - x_t^*}{\sigma\sqrt{T-t}} - \frac{\sigma}{2}\sqrt{T-t}\right)$$

  where $x_t^* = \log K - r(T-t)$ is the future log-price at t and Φ is the cumulative probability function of the standard normal law.

- With N we will denote the Poisson random measure on $[0,T]\times\mathbb{R}$ such that $Z_t = \int_{[0,t]\times\mathbb{R}} xN(ds,dx)$. Remember also that $\tilde{N}(ds,dx) := N(ds,dx) - ds\nu(dx)$ is the compensated Poisson random measure.

- We will consider the operator $\mathcal{L}_{BS}(\sigma) := \partial_t + \frac{1}{2}\sigma^2\partial_{xx}^2 + (r - \frac{1}{2}\sigma^2)\partial_x - r$ which satisfies $\mathcal{L}_{BS}(\sigma)BS(\cdot,\cdot,\sigma) = 0$.

The paper is organized as follows. In Section 2 we introduce the Malliavin calculus framework needed in the remaining of the paper. In section 3 we obtain the Hull and White formula. In Section 4, we apply it to the problem of describing the at the money short time skew of the implied volatility. Section 5 is devoted to the conclusions.

2 Required tools of Malliavin calculus for Lévy processes

2.1 Introduction

In this section we introduce the tools of Malliavin calculus for Lévy processes that we need in the rest of the paper.

Consider a complete probability space $(\Omega,\mathcal{F},P)$ and let $L = \{L_t, t\in[0,T]\}$ be a càdlàg Lévy process with triplet (γ,σ,ν). See for example the book of Sato (1999) for a general theory of Lévy processes.

It is well-known that L can be represented as

$$L_t = \gamma t + \sigma W_t + \iint_{(0,t]\times\{|x|>1\}} xN(ds,dx) + \lim_{\epsilon\downarrow 0}\iint_{(0,t]\times\{\epsilon<|x|\le 1\}} x\tilde{N}(ds,dx),$$

where W is a Brownian motion and N is the Poisson random measure associated to ν. It is also known that $\mathcal{F}^L = \mathcal{F}^W \vee \mathcal{F}^N$. See for example, Solé, Utzet and Vives (2007).

In general, the construction of a Malliavin calculus for a certain process follows three main steps. First of all, to prove a chaotic representation property, secondly, to define formally the gradient and divergence operators and finally, to give their probabilistic interpretations. In this paper we use the approach given by Løkka (2004) and Petrou (2006) combined with the approach developed by Solé, Utzet and Vives (2007). As we will see, the point of view of Løkka (2004) and Petrou (2006) is more suitable for the purpose of our paper, because the form of the gradient operator in this approach simplifies our computations in a remarkable way.

2.2 The chaotic representation property

There are two ways to establish the chaotic representation property for a Lévy process. The first one was obtained by Itô (1956) and it holds for general Lévy processes. A Malliavin calculus based on this approach was developed in Solé, Utzet and Vives (2007). The second one, developed only for square integrable Lévy processes, was established by Løkka (2004) and Petrou (2006). In our case, we suppose $E[\int_0^T \sigma_s^2 ds] < \infty$ and $\int_{\mathbb{R}} x^2 \nu(dx) < \infty$ thus we have a square-integrable Lévy process. Moreover, in our case the Lévy measure ν is finite.

From Itô (1956), on the measurable space $([0,T]\times\mathbb{R}, B([0,T]\times\mathbb{R}))$, we can consider the centered independent random measure given by

$$M(E) := \sigma \int_{E(0)} dW_t + \iint_{E'} x\tilde{N}(dt,dx), \; E \in B([0,T]\times\mathbb{R})$$

where $E(0) := \{t \in [0,T] : (t,0) \in E\}$ and $E' := E - E(0)$. Its variance is given by

$$\mu(E) = \sigma^2 \int_{E(0)} dt + \iint_{E'} x^2 dt \nu(dx).$$

Remark that W can be seen as a centered independent Gaussian random measure on $[0,T]$ and $J(ds,dx) := x\tilde{N}(ds,dx)$ can be seen as a centered random measure on $[0,T]\times\mathbb{R}_0$ where $\mathbb{R}_0 = \mathbb{R} - \{0\}$. Thus we can write

$$M(ds,dx) = \sigma(W \otimes \delta_0)(ds,dx) + J(ds,dx).$$

where δ_0 is the Dirac's delta, that is, a unitary mass on the point $\{0\}$.

Also from Itô (1956) we can define stochastic multiple integrals I_n with respect to M with kernels in the Hilbert spaces

$$\mathbb{L}_n^2 := L^2(([0,T]\times\mathbb{R})^n, \mathcal{B}([0,T]\times\mathbb{R})^n, \mu^{\otimes n}),$$

in the usual way, and to prove that if $\{\mathcal{F}_t^X, t \in [0,T]\}$ is the completed natural filtration of X, for any random variable $F \in L^2(\Omega, \mathcal{F}_T^X, P)$ we have the chaotic representation

$$F = \sum_{n=1}^{\infty} I_n(f_n), \tag{3}$$

where the kernels are unique if we take them symmetric.

In the approach of Løkka (2004) and Petrou (2006) it is defined the centered independent random measure given by

$$\bar{M}(E) := \sigma \int_{E(0)} dW_t + \iint_{E'} d\tilde{N}(t,x).$$

with variance

$$\bar{\mu}(E) = \sigma^2 \int_{E(0)} dt + \iint_{E'} dt\nu(dx).$$

So, it can also be written by $\bar{M}(ds,dx) = \sigma(W \otimes \delta_0)(ds,dx) + \tilde{N}(ds,dx)$.

In this setting we can consider the multiple stochastic integrals $\bar{I}_n$ with respect to $\bar{M}$ with kernels in

$$\bar{\mathbb{L}}_n^2 := L^2(([0,T]\times\mathbb{R})^n, \mathcal{B}([0,T]\times\mathbb{R})^n, \bar{\mu}^{\otimes n}). \tag{4}$$

Notice that, for every $g_n \in \bar{\mathbb{L}}_n^2$, the function

$$f_n(t_1,x_1;t_2,x_2;....;t_n,x_n) := \frac{1}{h(x_1)\cdots h(x_n)} g_n(t_1,x_1;t_2,x_2;....;t_n,x_n)$$

belongs to $\mathbb{L}_n^2$, where $h(x) := x1\!\!1_{\{x \neq 0\}} + 1\!\!1_{\{x=0\}}$. Moreover, we have the relation

$$I_n(f_n) = \bar{I}_n(g_n). \tag{5}$$

This facts can be proved as usual. That is, first of all we see that it is true for step functions that are zero on the diagonals, and then we use limit arguments.

Consequently, for $F \in L^2(\Omega, \mathcal{F}_T^X, P)$ we have two chaotic representations

$$F = \sum_{n=1}^{\infty} \bar{I}_n\left(g_n\right) = \sum_{n=1}^{\infty} I_n\left(f_n\right) \tag{6}$$

where g_n are symmetric functions of $\bar{\mathbb{L}}_n^2$ and f_n are symmetric functions of $\mathbb{L}_n^2$.

Notice that, if $\nu = 0$, $\mu(E) = \bar{\mu}(E) = \sigma^2 \int_{E(0)} dt$ and $M = \bar{M} = \sigma(W \otimes \delta_0)$.

2.3 The Malliavin-type derivative

Let $Dom\ D$ and $Dom\ \bar{D}$ be the sets of random variables in $L^2(\Omega)$ such that

$$\sum_{n=1}^{\infty} nn! \left\| f_n \right\|_{L_n^2}^2 < \infty$$

and

$$\sum_{n=1}^{\infty} nn! \left\| g_n \right\|_{\bar{L}_n^2}^2 < \infty,$$

respectively.

By equality (5), it is easy to show that $Dom\ D = Dom\ \bar{D}$. In the following we will denote this subspace of $L^2(\Omega)$ by $\mathbb{D}^{1,2}$, that is, $\mathbb{D}^{1,2} := Dom\ D = Dom\ \bar{D}$.

The Malliavin derivative $D^M F$ of a random variable $F \in \mathbb{D}^{1,2}$ is the process $\left\{D_{t,x}^M F, (t,x) \in [0,T] \times \mathbb{R}\right\}$ defined by $D_{t,x}^M F := \sum_{n=1}^{\infty} n I_{n-1}(f_n((t,x),\cdot))$.

In a similar way, the Malliavin derivative $D^{\bar{M}} F$ of a random variable $F \in \mathbb{D}^{1,2}$ is defined as the process $\left\{D_{t,x}^{\bar{M}} F, (t,x) \in [0,T] \times \mathbb{R} \times \Omega\right\}$ given by $D_{t,x}^{\bar{M}} F := \sum_{n=1}^{\infty} n \bar{I}_{n-1}\left(g_n\left(t,x\right),\cdot\right)$.

We have

$$\begin{aligned} D_{t,x}^{\bar{M}} F &= \sum_{n=1}^{\infty} n \bar{I}_{n-1}\left(g_n(\left(t,x\right),\cdot)\right) = \sum_{n=1}^{\infty} n I_{n-1}\left(h(x) f_n(\left(t,x\right),\cdot)\right) \\ &= h(x) \sum_{n=1}^{\infty} n I_{n-1}\left(f_n(\left(t,x\right),\cdot)\right) = h(x) D_{t,x}^M F. \end{aligned} \tag{7}$$

Similarly, notice that $\mu(dt,dx) = h^2(x)\bar{\mu}(dt,dx)$.

Henceforth, in order to give the probabilistic interpretation of above operators, we assume that the underlying probability space is the canonical Lévy space $(\Omega^W \times \Omega^N, \mathcal{F}^W \otimes \mathcal{F}^N, \mathbb{P}^W \otimes \mathbb{P}^N)$. That is, $(\Omega^W, \mathcal{F}^W, \mathbb{P}^W)$ is the canonical Wiener space and $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$ is the canonical Lévy space of the compound Poisson process with Lévy measure ν. Also, in the remaining we assume that W and Z are the canonical processes. For details we recommend Solé, Utzet and Vives (2007).

The elements of this space will be written as $\omega = (\omega^W, \omega^N)$. In particular ω^W will be a continuous trajectory null a the origin and ω^N is a sequence of pairs of jump instants and jump amplitudes

$$\omega^N := ((t_1, x_1), (t_2, x_2), (t_3, x_3), \dots).$$

From Solé, Utzet and Vives (2007) and Petrou (2006) we have

$$D^M_{t,0} = D^{\bar{M}}_{t,0} = \frac{1}{\sigma} D^W_t \mathbb{1}_{\{\sigma>0\}} \tag{8}$$

where D^W_t denotes the classical Malliavin derivative with respect to the Brownian motion W (see for example Nualart (1995)).

In order to obtain the probabilistic interpretations of operators $D^M_{t,x}$ and $D^{\bar{M}}_{t,x}$ for $x \neq 0$ we consider the following transformation.

Given $(t, x) \in [0, T] \times \mathbb{R}_0$, we can add to any ω^N a jump of size x at instant t, call the new element

$$\omega^N_{t,x} := ((t_1, x_1), (t_2, x_2), (t_3, x_3), (t, x) \dots)$$

and write $\omega_{t,x} := (\omega^W, \omega^N_{t,x})$. So $\forall\,(t, x) \in [0, T] \times \mathbb{R}_0$, we can define the operator $T_{t,x}F := F(\omega_{t,x})$. As it is shown in Solé, Utzet, Vives (2007) (Proposition 4.8.) this is a well defined operator.

In the same reference is defined the operator $\Psi_{t,x}F := \frac{T_{t,x}F-F}{x}, \quad x \neq 0$. Combining results from Solé, Utzet and Vives (2007) and Alòs, León and Vives (2007a) it is easy to show that for $F \in L^2(\Omega)$,

$$\Psi F \in L^2([0, T] \times \mathbb{R}_0, \mu) \text{ and } F \in Dom\ D^W \Leftrightarrow F \in \mathbb{D}^{1,2},$$

and in this case $D^M_{t,x}F = \Psi_{t,x}F, \quad x \neq 0$.

Moreover, for all $F \in \mathbb{D}^{1,2}$,

$$D^{\bar{M}}_{t,x}F = T_{t,x}F - F, \quad x \neq 0. \tag{9}$$

In the remaining of this paper, we will denote $D^N_{t,x} = T_{t,x} - Id$, of course only defined on $[0, T] \times \mathbb{R}_0$. Observe that we have proved

$$D^{\bar{M}}_{t,x} = \mathbb{1}_{\{\sigma>0\}} \mathbb{1}_{\{0\}}(x) \frac{1}{\sigma} D^W_t + \mathbb{1}_{\mathbb{R}_0}(x) D^N_{t,x}, \tag{10}$$

which follows from (7), (8) and (9).

Observe also that it is immediate from (9), to see that

$$D^N_{t,x}F = FD^N_{t,x}G + GD^N_{t,x}F + D^N_{t,x}FD^N_{t,x}G. \tag{11}$$

Finally remember from Section 2 in Solé, Utzet and Vives (2007), that if $\mathbb{D}^W$ and $\mathbb{D}^N$ are the domains of D^W and D^N respectively, we have $\mathbb{D}^{1,2} = \mathbb{D}^W \cap \mathbb{D}^N$.

2.4 Skorohod-type integrals

Let δ^M and $\delta^{\bar{M}}$ be the duals of the operators D^M and $D^{\bar{M}}$, respectively. It means,

$$E(F\delta^M(u)) = E \int_0^T \int_{\mathbb{R}} u(t, x) D^M_{t,x} F \mu(dt, dx)$$

and

$$E(F\delta^{\bar{M}}(v)) = E\int_0^T\int_{\mathbb{R}} v(t,x)D^{\bar{M}}_{t,x}F\bar{\mu}(dt,dx), \tag{12}$$

for $F\in\mathbb{D}^{1,2}$, $u\in Dom\ \delta^M$ and $v\in\delta^{\bar{M}}$. Sometimes we will write $\delta_t(u)$ instead of $\delta(u1\!\!1_{[0,t]})$.

Due to equality (7) we have that this two Skorohod type integrals satisfy

$$\delta^M(u) = \delta^{\bar{M}}(hu), \tag{13}$$

and this property, together with (7), allows us to translate the properties of δ^M to $\delta^{\bar{M}}$. See Section 6 of [16] for a presentation of the main properties of δ^M.

The following lemma will be useful for our purposes. A version of this lemma in the pure jump case is given in Di Nunno et al (2004), Theorem 3.13.

Lemma 1 *Let $F\in\mathbb{D}^{1,2}$ and $u\in Dom\ \delta^{\bar{M}}$ such that $u\cdot(F+D^{\bar{M}}F\cdot 1\!\!1_{\mathbb{R}_0})\in L^2(\Omega\times[0,T]\times\mathbb{R},\mathbb{P}\otimes\bar{\mu})$. Then*

$$u\cdot(F+D^{\bar{M}}F\cdot 1\!\!1_{\mathbb{R}_0})\in Dom\delta^{\bar{M}} \Leftrightarrow F\delta^{\bar{M}}(u)-\int_{[0,T]\times\mathbb{R}} u(t,x)D^{\bar{M}}_{t,x}F\bar{\mu}(dt,dx)\in L^2(\Omega)$$

and in this case

$$\delta^{\bar{M}}(u\cdot F) = F\delta^{\bar{M}}(u)-\delta^{\bar{M}}(u\cdot D^{\bar{M}}F\cdot 1\!\!1_{\mathbb{R}_0})-\int_{[0,T]\times\mathbb{R}} u(t,x)D^{\bar{M}}_{t,x}F\bar{\mu}(dt,dx).$$

Proof: This result follows using relations (7), (10) and (13) and applying Lemma 2.4. and Proposition 2.5. in Alòs, León and Vives (2007a) to the random field $\frac{u}{h}$. Remark that, in our case, F doesn't need to be bounded. This is a consequence of the fact that if G is a bounded random variable of $L^2(\Omega^N)$ and ν is finite, we have that $G\in\mathbb{D}^N$ and D^NG is also bounded. ■

In order to give the relation between $\delta^{\bar{M}}$ and the pathwise integral with respect to N, we consider the following two sets

Definition 2 *We define $\mathbb{L}^{1,2} := L^2([0,T]\times\mathbb{R};\mathbb{D}^{1,2})$.*

Remark that if $u=\{u(s,y):(s,y)\in[0,T]\times\mathbb{R}\}$ is a random field of $\mathbb{L}^{1,2}$ we have, in particular, that u and $D^{\bar{M}}u$ are in $L^2(\mathbb{P}\otimes\bar{\mu})$ and $L^2(\mathbb{P}\otimes\bar{\mu}\otimes\bar{\mu})$ respectively.

Definition 3 *We define $\mathbb{L}^{1,2}_{-}$ as the subset of $\mathbb{L}^{1,2}$ of random fields u such that the following $\mathbb{P}\otimes\bar{\mu}$-a.s. left-limits exists and belong to $L^2(\mathbb{P}\otimes\bar{\mu})$:*

$$u^-(s,y) = \lim_{r\uparrow s, x\uparrow y} u(r,x),$$

$$D^-u(s,y) = \lim_{r\uparrow s, x\uparrow y} D^{\bar{M}}_{s,y}u(r,x).$$

Proposition 4 *Assume that u is a random field belonging to $\mathbb{L}^{1,2}_{-}$. Let be $T^-u := u^- + D^-u$. Assume*

$$\int_0^T\int_{\mathbb{R}_0} |u^-(s,x)|N(ds,dx)\in L^2(\Omega), \tag{14}$$

where $\int_0^T \int_{\mathbb{R}_0} u(s,x)N(ds,dx)$ *is the classical path-by-path integral defined by* $\sum_{\Delta Z_t \neq 0} u(t, \Delta Z_t)$.

Then, $T^- u = u^- + D^- u \in Dom\ \delta^{\bar{M}}$, *and in this case,*

$$\delta^{\bar{M}}((u^- + D^- u) \cdot \mathbb{1}_{\mathbb{R}_0}) = \int_0^T \int_{\mathbb{R}_0} u^-(s,x)\tilde{N}(ds,dx) - \int_0^T \int_{\mathbb{R}_0} D^- u(s,x)\nu(dx)ds,$$

or equivalently,

$$\delta^{\bar{M}}(T^- u \cdot \mathbb{1}_{\mathbb{R}_0}) = \int_0^T \int_{\mathbb{R}_0} u^-(s,x)N(ds,dx) - \int_0^T \int_{\mathbb{R}_0} T^- u(s,x)\nu(dx)ds.$$

Proof:

Assume as a first step that $u \in \mathbb{L}_-^{1,2}$ is bounded. Then $D^{\bar{M}} u$, u^- and $D^- u$ are also bounded. In particular (14) is true.

We begin considering the following partition of $[0,\infty) \times \mathbb{R}$:

$$0 = s_0 < s_1 < \cdots < s_n < \infty = s_{n+1}$$

$$-\infty = x_0 < x_1 < \cdots < x_m < \infty = x_{m+1}.$$

Then we can define

$$u^{n,m}(s,x) = \sum_{i=0}^{n} \sum_{j=0}^{m} u(s_i, x_j) \mathbb{1}_{(s_i, s_{i+1}]}(s) \mathbb{1}_{(x_j, x_{j+1}]}(x).$$

Using Lemma 1, we have that for all n and m,

$$\begin{aligned} &\delta^{\bar{M}}(u^{n,m} \cdot \mathbb{1}_{\mathbb{R}_0}) + \delta^{\bar{M}}(Du^{n,m} \cdot \mathbb{1}_{\mathbb{R}_0}) \qquad (15) \\ = &\sum_{i=0}^{n} \sum_{j=0}^{m} u(s_i, x_j) \delta^{\bar{M}}(\mathbb{1}_{(s_i, s_{i+1}]} \mathbb{1}_{(x_j, x_{j+1}]} \mathbb{1}_{\mathbb{R}_0}) - \sum_{i=0}^{n} \sum_{j=0}^{m} \int_0^T \int_{\mathbb{R}_0} \mathbb{1}_{(s_i, s_{i+1}]}(s) \mathbb{1}_{(x_j, x_{j+1}]}(x) D_{s,x}^{\bar{M}} u(s_i, x_j) \nu(dx)ds. \end{aligned}$$

First of all observe that if $r, s \in]s_i, s_{i+1}]$ and $x, y \in]x_j, x_{j+1}]$, then $(D_{s,y} u^{n,m})(r,x) = u(s_i, x_j, \omega_{s,y}) - u^{n,m}(r,x)$ and $(D_{s,y} u)(r,x) = u(r,x,\omega_{s,y}) - u(r,x)$ almost surely go to the same limit whatever n and m goes to infinity or $r \uparrow s$ and $x \uparrow y$. By the theorem hypothesis this limit is $D^- u$.

Observe now that being u bounded, and having $u^{n,m}$ the same bound, $D^- u$ and u^- are also L^2-limits. So, using that $\delta^{\bar{M}}$ is a closed operator, the left hand side in (15) goes to $\delta^{\bar{M}}((u^- + D^- u) \cdot \mathbb{1}_{\mathbb{R}_0})$ in L^2 if we prove that the terms on the right hand side converge in L^2 to the limits defined by the proposition.

For the first term in the right hand side, observe that $\delta^{\bar{M}}$ coincides with the path by path integral because the integrand is deterministic. Then, using u is bounded and the dominated convergence theorem we obtain the expected L^2-limit. For the second term we have also a direct application of dominated convergence theorem.

In order to prove the non-bounded case observe that we can assume that u is positive, because the formula that we want to prove is linear. Then, for the general case, we simply have to apply the result separately to the positive and negative part.

So, let $u \geq 0$ and $u_K = u \wedge K$. Of course, $u_K \leq u$ and u_K converges increasingly to u. We have, as a consequence of the first step, that

$$\delta^{\tilde{M}}((T^- u_K) \cdot 1\!\!1_{\mathbb{R}_0}) = \int_0^T \int_{\mathbb{R}_0} u_K^-(s,x) N(ds,dx) - \int_0^T \int_{\mathbb{R}_0} T^- u_K(s,x) \nu(dx) ds.$$

Being u^- and $T^- u$ in L^2, we have that u_K^- and $T^- u_K$ go up to u^- and $T^- u$ in L^2, respectively. So, hypothesis (14), the monotone convergence theorem and the closeness of the operator $\delta^{\tilde{M}}$ yields the result. ∎

Remark 5 *Observe that $T^- u = u^-$ when u is adapted to the filtration generated by N. Therefore in such a case*

$$\int_0^t \int_{\mathbb{R}_0} u^-(s,y) \tilde{N}(ds,dy) = \delta^{\tilde{M}}(u^-(\cdot,\cdot) 1\!\!1_{[0,t]\times\mathbb{R}_0}(\cdot,\cdot)).$$

That is, in this case, the pathwise and Skorohod integrals with respect to $\tilde{N}$ are the same.

In the lasts two results does not appear the contribution of W in the integrals. This is because on $\mathbb{R}_0$ the operator $\delta^{\tilde{M}}$ agrees with the Skorohod-type integral with respect to $\tilde{N}$, as the following result explains

Lemma 6 *Let δ^W and δ^N the adjoint operators of D^W and D^N, respectively, and $u \in Dom\ \delta^{\tilde{M}}$. Then u also belongs to $Dom\ \delta^W \cap Dom\ \delta^N$ and*

$$\delta^{\tilde{M}}(u) = \sigma \delta^W(u_{\cdot,0}) + \delta^N(u 1\!\!1_{\mathbb{R}_0}).$$

Proof: This result is implied by (12) and (10). ∎

2.5 The anticipating Itô's formula

The basic tool for our results is the following anticipative Itô formula. Remember that the process X is introduced in (1) and Y is the future average volatility, which is an anticipative process, even σ is adapted.

Theorem 7 *Let $\sigma^2 \in \mathbb{L}^{1,2}$ and $F : [0,T] \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$, a function in $C^{1,2,2}([0,T] \times \mathbb{R} \times [0,\infty))$ such that there exists a positive constant C satisfying that for all $t \in [0,T]$, F and its partial derivatives evaluated in (t, X_t, Y_t) are bounded by C. Then,*

$$\begin{aligned} F(t,X_t,Y_t) \quad - \quad & F(0,X_0,Y_0) \\ = \quad & \int_0^t \partial_s F(s,X_s,Y_s) ds + \int_0^t \partial_x F(s,X_s,Y_s)(r - \frac{\sigma_s^2}{2} - \lambda k) ds \\ & + \delta_t^{W,B}(\partial_x F(.,X_.,Y_.)\sigma_.) - \int_0^t \partial_y F(s,X_s,Y_s)\sigma_s^2 ds \\ & + \rho \int_0^t \partial_{xy}^2 F(s,X_s,Y_s)\Lambda_s ds + \frac{1}{2}\int_0^t \partial_{xx}^2 F(s,X_s,Y_s)\sigma_s^2 ds \\ & + \delta_t^N(T^- u \cdot 1\!\!1_{\mathbb{R}_0}) + \int_0^t \int_{\mathbb{R}_0} T^- u(s,x)\nu(dx) ds \end{aligned}$$

where $\delta^{W,B}$ is the Skorohod integral with respect the Brownian motion $\rho W_s + \sqrt{1-\rho^2} B_s$, $\Lambda_s := (\int_s^T D_s^W \sigma_r^2 dr)\sigma_s$ and $u(s,x) := F(s, X_{s-} + x, Y_s) - F(s, X_{s-}, Y_s)$.

Proof:

The proof is as in Alòs, León and Vives (2007b) combined by Proposition 4 to treat the sum of jump terms.

We apply it to the random field $u(s,x) = F(s, X_{s-}+x, Y_s) - F(s, X_{s-}, Y_s)$. Here, the independence between Z, B and W, the fact that Y is a continuous process and the fact that Z is a compound Poisson process with a finite number of jumps on every compact time interval play a key role.

Indeed, let T_i these jump instants. Then

$$F(T_{i+1}, X_{T_{i+1}}, Y_{T_{i+1}}) - F(T_i, X_{T_i}, Y_{T_i}) = \int_{T_i}^{T_{i+1}^-} dF(s, X_s, Y_s) + F(T_{i+1}, X_{T_{i+1}}, Y_{T_{i+1}}) - F(T_{i+1}, X_{T_{i+1}^-}, Y_{T_{i+1}}).$$

The first term yields a standard Itô formula concerning continuous process, so Alòs, León and Vives (2007b) results apply and we get the six first terms in the right hand side of the Theorem 7 formula. On other hand, the sum of second terms is the path by path integral

$$\int_0^t \int_{\mathbb{R}_0} u(s,x) N(ds, dx).$$

Remark here that $F,\ X_-, Y$ are left continuous so $u = u^-$. Then using Proposition 4 we get the last sum is equal to:

$$\delta_t^N(T^- u) + \int_0^t \int_{\mathbb{R}_0} T^- u(s,x)\nu(dx)ds.$$

■

3 The Hull and White formula

Now we have the following extension of the Hull and White formula

Theorem 8 *Let σ and X be as in Theorem 7. Then*

$$\begin{aligned} V_t &= E(BS(t, X_t, v_t)|\mathcal{F}_t) + \frac{\rho}{2} E\left(\int_t^T e^{-r(s-t)} \partial_x G(s, X_s, v_s)\Lambda_s ds|\mathcal{F}_t\right) \\ &+ E\left(\int_t^T \int_{\mathbb{R}_0} e^{-r(s-t)}(T^- BS(s, X_{s-}+y, v_s) - T^- BS(s, X_{s-}, v_s) ds\nu(dy)|\mathcal{F}_t\right) \\ &- \lambda k E\left(\int_t^T e^{-r(s-t)} \partial_x BS(s, X_s, v_s) ds|\mathcal{F}_t\right). \end{aligned}$$

where $G = (\partial_{xx}^2 - \partial_x)BS$.

Remark 9 *Remember that in the case that σ only depends on the filtration generated by W, we have $T^- = Id$. Consequently, in this case, we obtain the Hull and White formula given in Alòs, León and Vives (2007b).*

Proof: This proof is similar to the one of the Theorem 4.2. in Alòs, León and Vives (2007b). Notice that $BS(T, X_T, v_T) = V_T$. Then, from (2) we have

$$e^{-rt}V_t = E(e^{-rT}BS(T, X_T, v_T)|\mathcal{F}_t).$$

Now, our idea is to apply the Itô formula (Theorem 7) to the process $e^{-rt}BS(t, X_t, v_t)$. As the derivatives of $BS(t, x, \sigma)$ are not bounded we will make use of an approximating argument, changing v_t by

$$v_t^\delta := \sqrt{\frac{1}{T-t}(Y_t + \delta)},$$

and $BS(t, x, \sigma)$ by $BS_n(t, x, \sigma) := BS(t, x, \sigma)\psi_n(x)$, where $\psi_n(X_t) := \phi\left(\frac{1}{n}x\right)$, for some $\phi \in \mathcal{C}_b^2$ such that $\phi(x) = 1$ for all $x < 1$ and $\phi(x) = 0$ for all $x > 2$. Now, applying Theorem 7 between t and T to function

$$F : (t, x, y) \mapsto e^{-rt}BS_n\left(t, x, \sqrt{\frac{y+\delta}{T-t}}\right)$$

and grouping terms according with the type of derivative we obtain:

$$\begin{aligned}
& e^{-rT}BS_n(T, X_T, v_T^\delta) \\
= \; & e^{-rt}BS_n(t, X_t, v_t^\delta) + \int_t^T e^{-rs}\mathcal{L}_{BS}(\sigma_s)BS_n(s, X_s, v_s^\delta)ds \\
& -\frac{1}{2}\int_t^T e^{-rs}\partial_\sigma BS_n(s, X_s, v_s^\delta)\frac{(\sigma_s^2 - (v_s^\delta)^2)}{v_s^\delta(T-s)}ds \\
& -\lambda k\int_t^T e^{-rs}\partial_x BS_n(s, X_s, v_s^\delta)ds \\
& +\int_t^T e^{-rs}\partial_x BS_n(s, X_s, v_s^\delta)\sigma_s(\rho dW_s + \sqrt{1-\rho^2}dB_s) \\
& +\frac{\rho}{2}\int_t^T e^{-rs}\partial_{\sigma x}^2 BS_n(s, X_s, v_s^\delta)\frac{1}{v_s^\delta(T-s)}\Lambda_s ds \\
& +\delta_t^N(e^{-rs}(T^-BS_n(s, X_{s-}+y, v_s^\delta) - T^-BS_n(s, X_{s-}, v_s^\delta))) \\
& +\int_t^T\int_{\mathbb{R}_0} e^{-rs}(T^-BS_n(s, X_{s-}+y, v_s^\delta) - T^-BS_n(s, X_{s-}, v_s^\delta))ds\nu(dy).
\end{aligned}$$

Notice that $\mathcal{L}_{BS}(\sigma_s)BS_n(s, X_s, v_s^\delta) = \left(\mathcal{L}_{BS}(\sigma_s)BS(s, X_s, v_s^\delta)\right)\psi_n(X_s) + A_n(s)$, where

$$\begin{aligned}
A_n(s) \;=\; & \frac{1}{2}\sigma_s^2\left[2\partial_x BS(s, X_s, v_s^\delta)\psi_n'(X_s)\right. \\
& \left.+BS(s, X_s, v_s^\delta)\left(\psi_n''(X_s) - \psi_n'(X_s)\right)\right] + rBS(s, X_s, v_s^\delta)\psi_n'(X_s).
\end{aligned}$$

Also note that the classical relation between the *Gamma*, the *Vega* and the *Delta* gives us that

$$\partial_\sigma BS(s, x, \sigma)\frac{1}{\sigma(T-s)} = (\partial_{xx}^2 - \partial_x)BS(s, x, \sigma).$$

Then we can write

$$
\begin{aligned}
& e^{-rT} BS_n(T, X_T, v_T^\delta) \\
= \; & e^{-rt} BS_n(t, X_t, v_t^\delta) + \int_t^T e^{-rs} \left[(\mathcal{L}_{BS}(\sigma_s) BS)(s, X_s, v_s^\delta) \psi_n(X_s) + A_n(s) \right] ds \\
& - \frac{1}{2} \int_t^T e^{-rs} (\partial_{xx}^2 - \partial_x) BS(s, X_s, v_s^\delta) \psi_n(X_s) (\sigma_s^2 - (v_s^\delta)^2) ds \\
& - \lambda k \int_t^T e^{-rs} \partial_x BS_n(s, X_s, v_s^\delta) ds \\
& + \int_t^T e^{-rs} \partial_x BS_n(s, X_s, v_s^\delta) \sigma_s (\rho dW_s + \sqrt{1-\rho^2} dB_s) \\
& + \frac{\rho}{2} \int_t^T e^{-rs} \left[\left(\partial_x (\partial_{xx}^2 - \partial_x) BS \right) (s, X_s, v_s^\delta) \psi_n(X_s) + (\partial_{xx}^2 - \partial_x) BS(s, X_s, v_s^\delta) \psi_n'(X_s) \right] \Lambda_s ds \\
& + \delta_t^N (e^{-rs} (T^- BS_n(s, X_{s-} + y, v_s^\delta) - T^- BS_n(s, X_{s-}, v_s^\delta))) \\
& + \int_t^T \int_{\mathbb{R}_0} e^{-rs} (T^- BS_n(s, X_{s-} + y, v_s^\delta) - T^- BS_n(s, X_{s-}, v_s^\delta)) ds \nu(dy)
\end{aligned}
$$

Hence, taking into account that $\mathcal{L}_{BS}(\sigma_s) = \mathcal{L}_{BS}(v_s^\delta) + \frac{1}{2}(\sigma_s^2 - (v_s^\delta)^2) \left(\partial_{xx}^2 - \partial_x \right)$ it follows that (using the fact that $\mathcal{L}_{BS}(v_s^\delta) BS(s, X_s, v_s^\delta) = 0$)

$$
\begin{aligned}
& e^{-rT} BS_n(T, X_T, v_T^\delta) = e^{-rt} BS_n(t, X_t, v_t^\delta) + \int_t^T e^{-rs} A_n(s) ds \\
& - \lambda k \int_t^T e^{-rs} \partial_x BS_n(s, X_s, v_s^\delta) ds \\
& + \int_t^T e^{-rs} \partial_x BS_n(s, X_s, v_s^\delta) \sigma_s (\rho dW_s + \sqrt{1-\rho^2} dB_s) \\
& + \frac{\rho}{2} \int_t^T e^{-rs} \left[\left(\partial_x (\partial_{xx}^2 - \partial_x) BS \right) (s, X_s, v_s^\delta) \psi_n(X_s) + (\partial_{xx}^2 - \partial_x) BS(s, X_s, v_s^\delta) \psi_n'(X_s) \right] \Lambda_s ds \\
& + \delta_t^N (e^{-rs} (T^- BS_n(s, X_{s-} + y, v_s^\delta) - T^- BS_n(s, X_{s-}, v_s^\delta))) \\
& + \int_t^T \int_{\mathbb{R}_0} e^{-rs} (T^- BS_n(s, X_{s-} + y, v_s^\delta) - T^- BS_n(s, X_{s-}, v_s^\delta)) ds \nu(dy)
\end{aligned}
$$

Now, taking conditional expectations we obtain that

$$
\begin{aligned}
& E \left(e^{-rT} BS_n(T, X_T, v_T^\delta) \middle| \mathcal{F}_t \right) \\
= \; & E \left\{ e^{-rt} BS_n(t, X_t, v_t^\delta) + \int_t^T e^{-rs} A_n(s) ds \right. \\
& - \lambda k \int_t^T e^{-rs} \partial_x BS_n(s, X_s, v_s^\delta) ds \\
& + \frac{\rho}{2} \int_t^T e^{-rs} \left[\left(\partial_x (\partial_{xx}^2 - \partial_x) BS \right) (s, X_s, v_s^\delta) \psi_n(X_s) + (\partial_{xx}^2 - \partial_x) BS(s, X_s, v_s^\delta) \psi_n'(X_s) \right] \Lambda_s ds \\
& \left. + \int_t^T \int_{\mathbb{R}_0} e^{-rs} (T^- BS_n(s, X_{s-} + y, v_s^\delta) - T^- BS_n(s, X_{s-}, v_s^\delta)) ds \nu(dy) \middle| \mathcal{F}_t \right\}.
\end{aligned}
$$

Let us remark that continuity of BS_n, v^δ and left continuity of X_- imply that $(T^- BS_n(s, X_{s-} + y, v_s^\delta) = BS_n(s, X_{s-} + y, T^- v^\delta(s, y))$.

Finally we obtain the result proceeding as in the proof of Theorem 3 in Alòs, León and Vives (2007b). That is, letting first $n \uparrow \infty$, then $\delta \downarrow 0$ and using the dominated convergence theorem. ■

Remark 10 *The additional term given by $T^{-}BS$ can be detailed as following. Suppose that $\sigma_r^2 = f(W_u, Z_u, u \leq r)$. Then we can define*

$$\tilde{v}_s^2 = \lim_{t \uparrow s, y \uparrow x} \frac{1}{T-t} \int_t^T T_{s,x}(\sigma_r^2) dr.$$

But for $r > s$, $T_{s,x}(\sigma_r^2) = f(W_u, Z_u + x\mathbb{1}_{\{s \leq u\}}, u \leq r)$ and

$$\tilde{v}_s^2 = \lim_{t \uparrow s} \frac{1}{T-t} \int_t^T f(W_u, Z_u + x\mathbb{1}_{\{s \leq u\}}, u \leq r) dr = \frac{1}{T-s} \int_s^T \tilde{\sigma}_r^2 dr$$

where $\tilde{\sigma}_r^2 = f(W_u, Z_u + x\mathbb{1}_{\{s \leq u\}}, u \leq r)$.

For example, consider the following pure volatility jump case described in Álvarez (2007). See also Espinosa and Vives (2006). Let $T_1, \ldots, T_n, \ldots$ the jump instants and $\Delta_{T_i} Z$ the jump size of process Z, with $T_0 = 0$. Assume that the dynamic of σ is given by

$$\sigma_t^2 = \sum_{i=0}^{N_t} \sigma_i^2 \mathbb{1}_{[T_i, T_{i+1}[}(t),$$

with $\sigma_i^2 = \sigma_{i-1}^2 + f(\Delta_{T_i} Z)$, for a certain function f. In this case, we have $\tilde{\sigma}_r^2 = \sigma_r^2 + f(x)\mathbb{1}_{\{r \geq s\}}(r)$ and so, the explicit computation of $\tilde{v}_s$ gives $\tilde{v}_s^2 = v_s^2 + f(x)$.

4 Short time behaviour of the implied volatility

In this section we will show that the short time behaviour of the at-the-money implied volatility is the same as in the case where the volatility σ is independent of the filtration of Z, even the Hull and White formula is different in the last case (see Remark 9). This is a fact that must be taken in account for pricing and hedging.

Let $I_t(X_t)$ denote the implied volatility process. By definition it satisfies $V_t = BS(t, X_t, I_t(X_t))$. Assume that $\sigma \in \mathbb{L}^{1,2}$ is as in model (1). Proceeding as in Alòs, León and Vives (2007b), the derivative of the implied volatility with respect the log-strike $Z = \log K$ is

$$\frac{\partial I_t}{\partial Z}(x_t^*) = - \left. \frac{E(\int_t^T (\partial_x F(s, X_s, v_s) - \frac{1}{2} F(s, X_s, v_s)) ds | \mathcal{F}_t)}{\partial_\sigma BS(t, x_t^*, I_t(x_t^*))} \right|_{X_t = x_t^*}, \text{ a.s.}$$

where

$$\begin{aligned} F(s, X_s, v_s) := \quad & \frac{\rho}{2} e^{-r(s-t)} \partial_x G(s, X_s, v_s) \Lambda_s - \lambda k e^{-r(s-t)} \partial_x BS(s, X_s, v_s) \\ + \quad & \int_{\mathbb{R}} e^{-r(s-t)} \left(BS(s, X_{s-} + y, \tilde{v}_s) - BS(s, X_{s-} + y, \tilde{v}_s) \right) \nu(dy). \end{aligned}$$

Now, in order to study the limit of $\frac{\partial I_t}{\partial Z}(x_t^*)$ as $T \downarrow t$ we need to introduce the following hypotheses:

(H1) $\sigma \in \mathbb{L}_W^{2,4} = L^4([0,T] \times \mathbb{R}; \mathbb{D}_W^{2,4})$.

(H2) There exists a constant $\delta > -\frac{1}{2}$ such that, for all $0 < t < s < r < T$,

$$E\left(\left(D_s^W \sigma_r\right)^2 \middle| \mathcal{F}_t\right) \leq C\,(r-s)^{2\delta}\,, \tag{16}$$

$$E\left(\left(D_\theta^W D_s^W \sigma_r\right)^2 \middle| \mathcal{F}_t\right) \leq C\,(r-s)^{2\delta}\,(r-\theta)^{-2\delta}\,. \tag{17}$$

(H3) For every fixed $t > 0$, $\sup_{s,r,\theta \in [t,T]} E\left(\left(\sigma_s \sigma_r - \sigma_\theta^2\right)^2 \middle| \mathcal{F}_t\right) \to 0$ as $T \to t$.

Theorem 11 *Under the Hypotheses (H1)-(H3) we have:*

1. *Assume that δ in (H2) is nonnegative and that there exists a $\mathcal{F}_t$-measurable random variable $D_t^{W,+}\sigma_t$ such that, for every $t > 0$,*

$$sup_{s,r \in [t,T]} \left| E\left(\left(D_s^W \sigma_r - D_t^{W,+}\sigma_t\right) \middle| \mathcal{F}_t\right)\right| \to 0, \tag{18}$$

a.s. as $T \to t$. Then

$$\lim_{T \to t} \frac{\partial I_t}{\partial Z}(x_t^*) = \frac{1}{\sigma_t}\left(\lambda k + \rho \frac{D_t^{W,+}\sigma_t}{2}\right). \tag{19}$$

2. *Assume that δ in (H2) is negative and that there exists a $\mathcal{F}_t$-measurable random variable $L_t^{\delta,+}\sigma_t$ such that, for every $t > 0$,*

$$\frac{1}{(T-t)^{2+\delta}} \int_t^T \int_s^T E\left(D_s^W \sigma_r \middle| \mathcal{F}_t\right) drds - L_t^{\delta,+}\sigma_t \to 0, \tag{20}$$

a.s. as $T \to t$. Then

$$\lim_{T \to t} (T-t)^{-\delta} \frac{\partial I_t}{\partial Z}(x_t^*) = \frac{\rho}{\sigma_t} L_t^{\delta,+}\sigma_t. \tag{21}$$

Proof: We can write

$$\begin{aligned}
& -\partial_\sigma BS(t, x_t^*, I_t(x_t^*)) \frac{\partial I_t}{\partial Z}(x_t^*) \\
= \; & \frac{\rho}{2} E\Big(\int_t^T e^{-r(s-t)}(\partial_x - \frac{1}{2})\partial_x(\partial_{xx}^2 - \partial_x)BS(s, X_s, v_s)\Lambda_s ds|\mathcal{F}_t)|_{X_t = x_t^*} \\
+ \; & E\Big(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)}(\partial_x - \frac{1}{2})[BS(s, X_s + y, \tilde{v}_s) - BS(s, X_s, \tilde{v}_s)]\nu(dy)ds|\mathcal{F}_t)|_{X_t = x_t^*} \\
& -\lambda k E\Big(\int_t^T e^{-r(s-t)}(\partial_x - \frac{1}{2})\partial_x BS(s, X_s, v_s)ds|\mathcal{F}_t)|_{X_t = x_t^*} = T_1 + T_2 + T_3.
\end{aligned}$$

The term T_2 is $O(T-t)$ due to the fact that the following majoration is uniform on σ :

$$|BS(t,x,\sigma)| + |\partial_x BS(t,x,\sigma)| \leq 2e^x + K$$

Now the result follows as in Alòs, León and Vives (2007b) (Proposition 6 and Theorem 7). ∎

5 Conclusion

As a conclusion, let us stress that the presence of jumps in a stochastic volatility model has a relevance. An additional term appears in the Hull and White formula. Nevertheless the correlation between price jumps and the stochastic volatility has no influence on the short time behaviour of the implied volatility.

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