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Monotone methods for equilibrium selection under perfect foresight dynamics

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This paper studies a dynamic adjustment process in a large society of forward-looking agents where payoffs are given by a supermodular normal form game. The stationary states of the dynamics correspond to the Nash equilibria of the stage game. It is shown that if the stage game has a monotone potential maximizer, then the corresponding stationary state is uniquely linearly absorbing and globally accessible for any small degree of friction. A simple example of a unanimity game with three players is provided where there are multiple globally accessible states for a small friction.

Keywords. Equilibrium selection, perfect foresight dynamics, supermodular game, strategic complementarity, stochastic dominance, potential, monotone potential.

JEL classification. C72, C73.

1. Introduction

Supermodular games capture the key concept of strategic complementarity in various economic phenomena. Examples include oligopolistic competition, the adoption of new technologies, bank runs, currency crises, and economic development. Strategic complementarity plays an important role in particular in Keynesian macroeconomics

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(Cooper 1999). From a theoretical viewpoint, supermodular games have appealing properties due to their monotone structure (Topkis 1979, Milgrom and Roberts 1990, Vives 1990, and Athey 2001).

A salient feature of supermodular games is that they typically admit multiple strict Nash equilibria due to the presence of strategic complementarities, which raises the question as to which equilibrium is likely to be played. To address the problem of equilibrium selection, game theory has so far proposed two major lines of approach besides the classic one of Harsanyi and Selten (1988). One is to consider the stability of Nash equilibria in transition dynamics (Kandori et al. 1993, Young 1993; KMRY for short). The other is to embed the original game in a static incomplete information game and examine the robustness of equilibrium outcomes to a small amount of uncertainty (Carlsson and van Damme 1993, Kajii and Morris 1997). Early papers using these two approaches studied 2×2 coordination games and established a connection between the approaches through risk-dominance due to Harsanyi and Selten (1988): the risk-dominant equilibrium is played most of the time in the long run in stochastic evolutionary dynamics as shown by KMRY, and it is the unique rationalizable outcome in slightly perturbed incomplete information games, called global games, as shown by Carlsson and van Damme (1993). Beyond 2×2 games, however, the connection fails. The incomplete information approach, on the one hand, has provided general results for larger classes of games, by using the concepts of p-dominance and the (generalized) potential function (Frankel et al. 2003, Kajii and Morris 1997, and Morris and Ui 2005[1]). For the stochastic evolutionary dynamic approach à la KMRY, on the other hand, the results obtained so far apply only to restricted classes of games.[2] Kim (1996) shows that these approaches predict different outcomes in some binary action games with more than two players.

In the present paper, we study an alternative to KMRY, the perfect foresight dynamics first introduced by Matsui and Matsuyama (1995) for 2×2 games.[3] Matsui and Matsuyama (1995) formalize a dynamic adjustment process in a large society where agents make irreversible decisions (e.g., career or sector choices as considered in Matsuyama 1991) and examine the possibility that forward-looking expectations destabilize strict Nash equilibria. They demonstrate that in 2×2 coordination games, if the degree of friction in action revisions is sufficiently small, then the belief that all agents will switch from the risk-dominated action to the risk-dominant one can become self-fulfilling, whereas the belief in the reverse switch cannot. Our first purpose in this paper is to develop a general theory of stability under perfect foresight dynamics for supermodular games. The second is to derive sufficient conditions for the stability of Nash equilibria for broader classes of supermodular games than 2×2 games, thereby extending the connection between the dynamic stability approach and the incomplete information approach. Specifically, we show that for games with monotone potentials,

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[1]See also Morris and Shin (2003) for an extensive survey.

[2]See, among others, Kandori and Rob (1995), Young (1998), and Durieu et al. (2006).

[3]This class of dynamics is studied also by Matsuyama (1991) but with nonlinear payoff functions in the context of development economics. See also Matsuyama (1992) and Kaneda (1995).

our condition coincides with the condition for robustness to incomplete information (Morris and Ui 2005).

We employ the following framework. The society consists of N large populations of infinitesimal agents, who are repeatedly and randomly matched to play an N-player normal form game. There are frictions: each agent must make a commitment to a particular action for a random time interval. Opportunities to revise actions follow Poisson processes that are independent across agents. The dynamic process thus exhibits inertia in that the action distribution in the society changes continuously. Unlike in standard evolutionary games, each agent forms his belief about the future path of the action distribution and, when given a revision opportunity, takes an action to maximize his expected discounted payoff. A *perfect foresight path* is defined to be a feasible path of action distributions along which every revising agent takes a best response to the future course of play. While the stationary states of these dynamics correspond to the Nash equilibria of the stage game, there may also exist a perfect foresight path that escapes from a strict Nash equilibrium when the degree of friction, defined as the discounted average duration of the commitment, is sufficiently small. We say that a Nash equilibrium a^* is *linearly absorbing* if the feasible path converging linearly to a^* is the unique perfect foresight path whenever the initial state is close enough to a^*; a^* is *globally accessible* if for any initial state, there exists a perfect foresight path converging to a^*.[4] If a Nash equilibrium is both linearly absorbing and globally accessible, then self-fulfilling expectations cannot destabilize this equilibrium, whereas from any other equilibrium, expectations may lead the society to this equilibrium; that is to say, it is the unique equilibrium that is robust to the possibility of self-fulfilling prophecies.

In this paper, we consider supermodular games and games that have a monotonic relationship with supermodular games, by employing methods of analysis based on monotonicity and comparison. An underlying observation is that a perfect foresight path is characterized as a fixed point of the best response correspondence defined on the set of feasible paths. We observe that if the stage game is supermodular, this correspondence is monotone with respect to the partial order over feasible paths induced by the stochastic dominance order. We then compare the perfect foresight paths of two different stage games that are comparable in terms of best responses and show the following analogue of the comparison theorem from the theory of monotone dynamical systems (Smith 1995):[5] if at least one of the two games is supermodular, then the order

[4]Since there may exist multiple perfect foresight paths from a given initial state, it is possible that a state is globally accessible but not linearly absorbing. Indeed, we provide an example where there exist multiple globally accessible states when the friction is small; by definition, none of them is linearly absorbing.

[5]In the case of Euclidean space, the comparison theorem says that if two dynamical systems are ordered with respect to the partial order of the Euclidean space and at least one of them is a cooperative monotone system, then, when their initial conditions are ordered, any two solutions that these systems generate are ordered as well.

Hofbauer and Sandholm (2002, 2007) show that when the underlying game is supermodular, the perturbed best response dynamics form a monotone dynamical system. The perfect foresight dynamics, on the other hand, cannot be considered as a dynamical system due to the multiplicity of perfect foresight paths.

of best responses between the games is preserved in the perfect foresight dynamics. This fact allows us to transfer stability properties from one game to the other.

We apply our monotone methods to the class of games with monotone potentials introduced by Morris and Ui (2005), who show that a monotone potential maximizer (MP-maximizer) is robust to incomplete information (Kajii and Morris 1997).[6] A normal form game is said to have a *monotone potential* if it is comparable (in terms of best responses) to a potential game, and the action profile that maximizes the potential is said to be an *MP-maximizer*. Monotone potential games include both potential games and, interestingly, games with a $\mathbf{p}$-dominant equilibrium with $\sum_i p_i < 1$. By invoking the potential game results due to Hofbauer and Sorger (2002), our main result shows that if the stage game or the monotone potential is supermodular, then an MP-maximizer is globally accessible for any small degree of friction and (generically) linearly absorbing for any degree of friction. Our result unifies and extends previous results using potential maximization (Hofbauer and Sorger 1999, 2002) and p-dominance (Oyama 2002), as done by Morris and Ui (2005) for the robustness of equilibria to incomplete information.

The concept of a perfect foresight path requires that agents optimize against their beliefs about the future path of the action distribution and that these beliefs coincide with the actual path. Relaxing the latter requirement, Matsui and Oyama (2006) introduce a model of rationalizable foresight dynamics, where while the rationality of the agents as well as the structure of the society is common knowledge, beliefs about the future path are not necessarily coordinated among the agents. It is instead assumed that the agents form their beliefs in a rationalizable manner: in particular, they may misforecast the future. A *rationalizable foresight path* is a feasible path along which every revising agent optimizes against another rationalizable foresight path. We show that in supermodular games, a linearly absorbing and globally accessible state is the unique state from which no rationalizable foresight path escapes. That is, our stability results for supermodular games hold also under the less demanding assumption of rationalizable foresight.

We briefly review existing results in the literature on perfect foresight dynamics. Oyama (2002) appeals to the notion of p-dominance to identify (in a single population setting) a class of games where one can explicitly characterize the set of perfect foresight paths relevant for stability considerations, showing that a p-dominant equilibrium with $p < 1/2$ is selected.[7] Hofbauer and Sorger (2002), Kojima (2006), and Kojima and Takahashi (forthcoming) obtain related results based on other generalizations of the risk-dominance concept in a multiple population setting.[8] Hofbauer and Sorger (1999, 2002) establish the selection of the unique potential maximizer in potential games, both in a single population setting and in a multi-population setting.[9] Their results rely on the

[6]Morris and Ui (2005) show more generally that a generalized potential maximizer is robust to incomplete information. A monotone potential induces a generalized potential in the case considered here. Frankel et al. (2003) show that under certain conditions, a local potential maximizer (LP-maximizer) is selected in global games with strategic complementarities. In games with marginal diminishing returns, an LP-maximizer is an MP-maximizer.

[7]Tercieux (2006) considers set-valued stability concepts and obtains a similar result.

[8]Kim (1996) establishes a similar result for binary games with many identical players.

[9]To be precise, they show that a unique potential maximizer a^* is *absorbing* (and globally accessible for

relationship between the perfect foresight paths and the solutions to, roughly, an associated "dynamic potential maximization" problem as well as the Hamiltonian structure of the dynamics when the stage game is a potential game.

The paper is organized as follows. Section 2 introduces perfect foresight dynamics for general finite N-player games and provides a characterization of perfect foresight paths as the fixed points of the best response correspondence defined on the set of feasible paths. Section 3 studies monotone properties of perfect foresight dynamics and proves our comparison theorem. It also examines the relationship between the stability concepts under perfect foresight and those under rationalizable foresight. Section 4 considers games with monotone potentials and establishes the selection of an MP-maximizer. Section 5 concludes.

2. Perfect foresight dynamics

2.1 *Stage game*

Let $G = (I,(A_i)_{i\in I},(u_i)_{i\in I})$ be a normal form game with $N \geq 2$ players, where $I = \{1,2,\dots,N\}$ is the set of players, $A_i = \{0,1,\dots,n_i\}$ is the finite set of actions of player $i \in I$, and $u_i\colon \prod_{i\in I} A_i \to \mathbb{R}$ is the payoff function of player i. We denote $\prod_{i\in I} A_i$ by A and $\prod_{j\neq i} A_j$ by A_{-i}.

Denote by $\mathbb{R}_+$ the set of all nonnegative real numbers and by $\mathbb{R}_{++}$ the set of all positive real numbers. The set of mixed strategies for player i is denoted by

$$\Delta(A_i) = \Big\{x_i = (x_{i0}, x_{i1}, \dots, x_{in_i}) \in \mathbb{R}_+^{n_i+1} \;\Big|\; \sum_{h\in A_i} x_{ih} = 1\Big\},$$

which is identified with the n_i-dimensional simplex. We sometimes identify each action in A_i with the element of $\Delta(A_i)$ that assigns one to the corresponding coordinate. The polyhedron $\prod_{i\in I}\Delta(A_i)$ is a subset of the n-dimensional real space endowed with the sup norm $|\cdot|$, where $n = \sum_{i\in I}(n_i+1)$. For $x \in \prod_i \Delta(A_i)$ and $\varepsilon > 0$, $B_\varepsilon(x)$ denotes the ε-neighborhood of x relative to $\prod_i \Delta(A_i)$, i.e., $B_\varepsilon(x) = \{y \in \prod_i \Delta(A_i) \mid |y-x| < \varepsilon\}$.

Payoff functions $u_i(h,\cdot)$ are extended to $\prod_{j\neq i}\Delta(A_j)$, and $u_i(\cdot)$ to $\prod_{j\in I}\Delta(A_j)$, i.e.,

$$u_i(h, x_{-i}) = \sum_{a_{-i}\in A_{-i}} \Big(\prod\nolimits_{j\neq i} x_{ja_j}\Big)\, u_i(h, a_{-i})$$

for $x_{-i} \in \prod_{j\neq i}\Delta(A_j)$, and

$$u_i(x) = \sum_{h\in A_i} x_{ih}\, u_i(h, x_{-i})$$

for $x \in \prod_{j\in I}\Delta(A_j)$. Let $br^i(x_{-i})$ be the set of best responses to $x_{-i} \in \prod_{j\neq i}\Delta(A_j)$ in pure

small friction): i.e., any perfect foresight path, which may or may not be unique, from a neighborhood of a^* must converge to a^*. It is not known whether a potential maximizer is linearly absorbing. In supermodular games, as we show, absorption and linear absorption are equivalent.

strategies, i.e.,

$$br^i(x_{-i}) = \operatorname*{argmax}_{h\in A_i} u_i(h, x_{-i})$$
$$= \{h \in A_i \mid u_i(h, x_{-i}) \geq u_i(k, x_{-i}) \text{ for all } k \in A_i\}.$$

An element $x^* \in \prod_i \Delta(A_i)$ is a Nash equilibrium if for all $i \in I$ and all $h \in A_i$,

$$x^*_{ih} > 0 \Rightarrow h \in br^i(x^*_{-i}),$$

and x^* is a strict Nash equilibrium if for all $i \in I$ and all $h \in A_i$,

$$x^*_{ih} > 0 \Rightarrow \{h\} = br^i(x^*_{-i}).$$

Let $\Delta(A_{-i})$ be the set of probability distributions on A_{-i}. We sometimes extend $u_i(h, \cdot)$ to $\Delta(A_{-i})$. For $\pi_i \in \Delta(A_{-i})$, we write $u_i(h, \pi_i) = \sum_{a_{-i}\in A_{-i}} \pi_i(a_{-i}) u_i(h, a_{-i})$ and $br^i(\pi_i) = \operatorname{argmax}_{h\in A_i} u_i(h, \pi_i)$.

2.2 *Perfect foresight paths*

Given an N-player normal form game, which we call the stage game, we consider the following dynamic societal game. Society consists of N large populations of infinitesimal agents, one for each role in the stage game. In each population, agents are identical and anonymous. At each point in time, one agent is selected randomly from each population and matched to form an N-tuple and play the stage game. Agents cannot switch actions at every point in time. Instead, every agent must make a commitment to a particular action for a random time interval. Time instants at which each agent can switch actions follow a Poisson process with the arrival rate $\lambda > 0$. The processes are independent across agents. We choose without loss of generality the unit of time in such a way that $\lambda = 1$.[10]

The action distribution in population $i \in I$ at time $t \in \mathbb{R}_+$ is denoted by

$$\phi_i(t) = (\phi_{i0}(t), \phi_{i1}(t), \ldots, \phi_{in_i}(t)) \in \Delta(A_i),$$

where $\phi_{ih}(t)$ is the fraction of agents who are committed to action $h \in A_i$ at time t. Let $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_N(t)) \in \prod_i \Delta(A_i)$. Due to the assumption that the switching times follow independent Poisson processes with arrival rate $\lambda = 1$, $\phi_{ih}(\cdot)$ is Lipschitz continuous with Lipschitz constant 1, which implies in particular that it is differentiable at almost all $t \geq 0$. Moreover, its speed of adjustment is bounded: $-\phi_{ih}(t) \leq \dot{\phi}_{ih}(t) \leq 1 - \phi_{ih}(t)$, where $\sum_{h\in A_i} \dot{\phi}_{ih}(t) = 0$. We call such a path $\phi(\cdot)$ a feasible path.

Definition 2.1. A path $\phi : \mathbb{R}_+ \to \prod_i \Delta(A_i)$ is *feasible* if it is Lipschitz continuous and for all $i \in I$ and almost all $t \geq 0$ there exists $\alpha_i(t) \in \Delta(A_i)$ such that

$$\dot{\phi}_i(t) = \alpha_i(t) - \phi_i(t). \tag{2.1}$$

[10]An alternative interpretation can be given as follows. Each agent exits from his population according to the Poisson process with parameter λ and is replaced by his successor. Agents make once-and-for-all decisions upon entry, i.e., an agent cannot change his action once it is chosen.

In (2.1), $\alpha_i(t) \in \Delta(A_i)$ denotes the action distribution of the agents in population i who have a revision opportunity during the short time interval $[t, t+dt)$. In particular, if for some action profile $a = (a_i)_{i\in I} \in A$, $\alpha_i(t) = a_i$ for all $i \in I$ and all $t \geq 0$, then the resulting feasible path, which converges linearly to a, is called a *linear path* to a.

Denote by Φ^i the set of feasible paths for population i, and let $\Phi = \prod_i \Phi^i$ and $\Phi^{-i} = \prod_{j\neq i} \Phi^j$. For $x \in \prod_i \Delta(A_i)$, the set of feasible paths starting from x is denoted by $\Phi_x = \prod_i \Phi^i_x$. For each $x \in \prod_i \Delta(A_i)$, Φ_x is convex and compact in the topology of uniform convergence on compact intervals.[11]

An agent in population i anticipates the future evolution of the action distribution, and, if given the opportunity to switch actions, commits to an action that maximizes his expected discounted payoff. Since the duration of the commitment has an exponential distribution with mean 1, the expected discounted payoff of committing to action $h \in A_i$ at time t with a given anticipated path $\phi \in \Phi$ is represented by

$$\begin{aligned} V_{ih}(\phi)(t) &= (1+\theta)\int_0^\infty \int_t^{t+s} e^{-\theta(z-t)} u_i(h, \phi_{-i}(z))\, dz\, e^{-s}\, ds \\ &= (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} u_i(h, \phi_{-i}(s))\, ds, \end{aligned}$$

where $\theta > 0$ is a common discount rate (relative to $\lambda = 1$). We view the discounted average duration of a commitment, $\theta/\lambda = \theta$, as the *degree of friction*. Note that V is well-defined whenever $\theta > -1$.

Given a feasible path $\phi \in \Phi$, let $BR^i(\phi)(t)$ be the set of best responses in pure strategies to $\phi_{-i} = (\phi_j)_{j\neq i}$ at time t, i.e.,

$$BR^i(\phi)(t) = \arg\max_{h\in A_i} V_{ih}(\phi)(t).$$

Note that for each $i \in I$, the correspondence $BR^i : \Phi \times \mathbb{R}_+ \to A_i$ is upper semi-continuous since V_i is continuous.

A perfect foresight path is a feasible path along which each agent optimizes against the correctly anticipated future path.

Definition 2.2. A feasible path ϕ is a *perfect foresight path* if for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$,

$$\dot{\phi}_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \in BR^i(\phi)(t).$$

Note that $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$ (i.e., $\alpha_{ih}(t) > 0$ in (2.1)) implies that action h is taken by some positive fraction of the agents in population i having a revision opportunity during the short time interval $[t, t+dt)$. The definition says that such an action must be a best response to the path ϕ itself.

As we observe in Remark 2.2, a perfect foresight path from $x \in \Delta$ is equivalent to a Nash equilibrium of an N-player differential game in which each population $i \in I$ acts as a single player, who chooses a feasible path from the set Φ^i_x and whose payoff is given by the sum of discounted values of u_i.

[11] One can instead use the topology induced by the discounted sup norm.

2.3 *Best response correspondence*

For a given initial state $x \in \prod_i \Delta(A_i)$, a best response path for population i to a feasible path $\phi \in \Phi_x$ is a feasible path $\psi_i \in \Phi_x^i$ along which every agent takes an optimal action against ϕ. This defines the *best response correspondence* $\beta_x^i \colon \Phi_x \to \Phi_x^i$, which maps each feasible path $\phi \in \Phi_x$ to the set of best response paths for population i:

$$\beta_x^i(\phi) = \{\psi_i \in \Phi_x^i \mid \dot{\psi}_{ih}(t) > -\psi_{ih}(t) \Rightarrow h \in BR^i(\phi)(t) \quad \text{a.e.}\}.$$

Let $\beta_x \colon \Phi_x \to \Phi_x$ be defined by $\beta_x(\phi) = \prod_i \beta_x^i(\phi)$. We denote by $\beta \colon \Phi \to \Phi$ the extension of β_x to Φ, i.e., $\beta(\phi) = \beta_{\phi(0)}(\phi)$ for $\phi \in \Phi$.

A perfect foresight path ϕ with $\phi(0) = x$ is a fixed point of $\beta_x \colon \Phi_x \to \Phi_x$, i.e., $\phi \in \beta_x(\phi)$. The existence of perfect foresight paths follows, due to Kakutani's fixed point theorem, from the fact that β_x is a nonempty-, convex-, and compact-valued upper semi-continuous correspondence. This fact can be shown by either of the two characterizations given below.

Remark 2.1. For a given feasible path $\phi \in \Phi_x$, a best response path $\psi \in \beta_x(\phi)$ is a Lipschitz solution to the differential inclusion

$$\dot{\psi}(t) \in F(\phi)(t) - \psi(t) \quad \text{a.e.}, \qquad \psi(0) = x,$$

where $F \colon \Phi \times \mathbb{R}_+ \to \prod_i \Delta(A_i)$ is defined by

$$F_i(\phi)(t) = \{\alpha_i \in \Delta(A_i) \mid \alpha_{ih} > 0 \Rightarrow h \in BR^i(\phi)(t)\}, \tag{2.2}$$

which is the convex hull of $BR^i(\phi)(t)$. Since $F(\phi)(\cdot)$ is convex- and compact-valued, and upper semi-continuous, the existence theorem for differential inclusions (see, e.g., Aubin and Cellina 1984, Theorem 2.1.4) implies the nonemptiness of the set of solutions, $\beta_x(\phi)$. The convexity of $\beta_x(\phi)$ is obvious. Furthermore, we can show that $\beta_x(\phi)$ is compact and depends upper semi-continuously on ϕ. For these properties of β_x, we need only the upper semi-continuity of BR^i, which is in turn implied by the continuity of V_i.

Lemma 2.1. *β_x is compact-valued and upper semi-continuous.*

Proof. Since the values are contained in the compact set Φ_x, it is sufficient to show that β_x has a closed graph. Let $\{\phi^k\}_{k=1}^\infty$ and $\{\psi^k\}_{k=1}^\infty$ be such that $\psi^k \in \beta_x(\phi^k)$, and assume that $\phi^k \to \phi$ and $\psi^k \to \psi$ as $k \to \infty$. Take any $i \in I$, $h \in A_i$, and $t \geq 0$ such that $\dot{\psi}_{ih}(t) > -\psi_{ih}(t)$. We want to show that $h \in BR^i(\phi)(t)$.

Observe that for any $\varepsilon > 0$, there exists $\bar{k}$ such that for all $k \geq \bar{k}$,

$$\dot{\psi}_{ih}^k(t_k) > -\psi_{ih}^k(t_k)$$

for some $t_k \in (t - \varepsilon, t + \varepsilon)$. Take a sequence $\{\varepsilon_\ell\}_{\ell=1}^\infty$ such that $\varepsilon_\ell > 0$ and $\varepsilon_\ell \to 0$ as $\ell \to \infty$. Then, we can take a subsequence $\{\psi^{k_\ell}\}_{\ell=1}^\infty$ of $\{\psi^k\}_{k=1}^\infty$ such that $\dot{\psi}_{ih}^{k_\ell}(t_\ell) > -\psi_{ih}^{k_\ell}(t_\ell)$ holds for some $t_\ell \in (t - \varepsilon_\ell, t + \varepsilon_\ell)$. By assumption, $h \in BR^i(\phi^{k_\ell})(t_\ell)$ for all ℓ. Now let $\ell \to \infty$. Since $BR^i(\cdot)(\cdot)$ is upper semi-continuous, we have $h \in BR^i(\phi)(t)$. □

REMARK 2.2. The correspondence β_x^i is actually the best response correspondence for an associated differential game, as constructed in Hofbauer and Sorger (2002). Given the stage game G, the discount rate $\theta > 0$, and an initial state $x \in \prod_i \Delta(A_i)$, the associated differential game is an N-player normal form game in which the set of actions for player $i \in I$ is Φ_x^i and the payoff function for player i is given by

$$J_i(\phi) = \int_0^\infty e^{-\theta t} u_i(\phi(t))\, dt.$$

As shown by Hofbauer and Sorger (2002), the perfect foresight paths are precisely the Nash equilibria of this game, due to the following fact.

LEMMA 2.2. *For a feasible path $\phi \in \Phi_x$,*

$$\beta_x^i(\phi) = \underset{\psi_i \in \Phi_x^i}{\arg\max}\, J_i(\psi_i, \phi_{-i}).$$

PROOF. Follows from Lemma 3.1 in Hofbauer and Sorger (2002). □

The continuity of J_i, the quasi-concavity of $J_i(\cdot, \phi_{-i})$, and the compactness of Φ_x^i therefore imply the desired properties of β_x^i.

2.4 *Stability concepts*

The constant path $\bar{\phi}$ given by $\bar{\phi}(t) = x^* \in \prod_i \Delta(A_i)$ for all $t \geq 0$ is a perfect foresight path if and only if x^* is a Nash equilibrium of the stage game. Nevertheless, there may exist another perfect foresight path starting at x^* that converges to a different Nash equilibrium; that is to say, self-fulfilling beliefs may enable the society to escape from a Nash equilibrium. When the degree of friction $\theta > 0$ is sufficiently small, this may happen even from a strict Nash equilibrium. In fact, in 2×2 coordination games, there exists a perfect foresight path from the risk-dominated equilibrium to the risk-dominant equilibrium for small $\theta > 0$, but not vice versa. This motivates the following stability concepts.

DEFINITION 2.3. (i) $x^* \in \prod_i \Delta(A_i)$ is *absorbing* if there exists $\varepsilon > 0$ such that any perfect foresight path from any $x \in B_\varepsilon(x^*)$ converges to x^*.

(ii) $a^* \in A$ is *linearly absorbing* if there exists $\varepsilon > 0$ such that for any $x \in B_\varepsilon(a^*)$, the linear path to a^* is a unique perfect foresight path from x.

(iii) $x^* \in \prod_i \Delta(A_i)$ is *accessible* from $x \in \prod_i \Delta(A_i)$ if there exists a perfect foresight path from x that converges to x^*. x^* is *globally accessible* if it is accessible from any x.

If x^* is absorbing and the current state is close enough to x^*, then along any (not necessarily unique) perfect foresight path, the behavior pattern of the society converges to x^*. Linear absorption is a stronger concept than absorption:[12] if a^* is linearly absorbing and the current state is close enough to a^*, then the perfect foresight path is unique,

[12]No example is known of a state that is absorbing but not linearly absorbing. We show that in supermodular games, any absorbing state is a strict Nash equilibrium and is also linearly absorbing.

	0	1
0	a,a	c,b
1	b,c	d,d

FIGURE 2.1. A 2×2 coordination game.

along which every agent at every revision opportunity takes the action prescribed in a^*. If a (linearly) absorbing state is also globally accessible, then it is the unique (linearly) absorbing state; if a globally accessible state is also absorbing, then it is the unique globally accessible state.

A globally accessible state is not necessarily absorbing, as there are generally multiple perfect foresight paths from a given initial state. We present a (nondegenerate) example in Section 4.3.4 that has two globally accessible states for small θ; by definition, neither of them is absorbing.

Any absorbing or globally accessible state is a Nash equilibrium of the stage game, which follows from the following proposition.

PROPOSITION 2.1. *If $x^* \in \prod_i \Delta(A_i)$ is the limit of a perfect foresight path, then x^* is a Nash equilibrium.*

PROOF. Suppose that x^* is the limit of a perfect foresight path ϕ^*. Let $\bar{\phi}$ be the constant path at x^*, i.e., $\bar{\phi}(t) = x^*$ for all $t \geq 0$. Let ϕ^t be the feasible path defined by $\phi^t(s) = \phi^*(s+t)$ for all $s \geq 0$. Then $\{\phi^t\}_{t\geq 0}$ converges to $\bar{\phi}$ as $t \to \infty$.

Take any $i \in I$ and any $h \in A_i$ with $x^*_{ih} > 0$. Then, there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to \infty$ as $k \to \infty$ and $h \in BR^i(\phi^*)(t_k) = BR^i(\phi^{t_k})(0)$ for any k since ϕ^* is a perfect foresight path that converges to x^*. Let $k \to \infty$. By the upper semi-continuity of $BR^i(\cdot)(0)$, we have $h \in BR^i(\bar{\phi})(0) = br^i(x^*_{-i})$. □

2.5 2×2 *case*

Before starting our analysis of general supermodular games, we illustrate in a simple 2×2 game example how our stability concepts allow us to discriminate among strict Nash equilibrium outcomes. Consider the symmetric 2×2 coordination game given in Figure 2.1, where $a > b$ and $d > c$ so that $\mathbf{0} = (0,0)$ and $\mathbf{1} = (1,1)$ are strict Nash equilibria. Note that this game is supermodular.

Assume that $a - b < d - c$ so that $\mathbf{1}$ risk-dominates $\mathbf{0}$. In the following, we review the result by Matsui and Matsuyama (1995) that $\mathbf{1}$ is the unique state that is linearly absorbing and globally accessible for any sufficiently small $\theta > 0$.

Define the function $\Delta u : [0,1] \to \mathbb{R}$ by

$$\Delta u_i(x_{j1}) = u_i(1,(1-x_{j1},x_{j1})) - u_i(0,(1-x_{j1},x_{j1})),$$

where $\Delta u_i(x_{j1})$ denotes the payoff difference between the actions 1 and 0 for agents in population i when the fraction x_{j1} of agents in $j \neq i$ play action 1. Note that Δu_i is (strictly) increasing by the supermodularity of the underlying game and satisfies

$$\int_0^1 \Delta u_i(\xi)\,d\xi > 0$$

by the assumption that **1** is a risk-dominant equilibrium.

To verify the global accessibility of **1** for small θ, let the initial condition be $x^0 = \mathbf{0}$ and consider the linear path ϕ from **0** to **1**, given by $\phi_{j1}(t) = 1 - e^{-t}$, along which every agent switches to action 1 at his first revision opportunity. Consider an agent in i who is given a revision opportunity at time 0, and call him agent 0. Against the path ϕ, the difference in the expected discounted payoffs that agent 0 obtains during a commitment is computed as

$$\begin{aligned}\Delta V_i(\phi)(0) &= V_{i1}(\phi)(0) - V_{i0}(\phi)(0)\\ &= (1+\theta)\int_0^\infty e^{-(1+\theta)s}\Delta u_i\left(1-e^{-s}\right)ds\\ &= (1+\theta)\int_0^1 \Delta u_i(\xi)(1-\xi)^\theta\,d\xi\\ &\to \int_0^1 \Delta u_i(\xi)\,d\xi > 0 \quad \text{as } \theta \to 0,\end{aligned} \tag{2.3}$$

which implies that this agent has an incentive to switch to action 1 provided that θ is sufficiently small. Due to the supermodularity, this in turn implies that **1** is globally accessible for small θ. To see this, observe that for any linear path ϕ' given by $\phi'_{j1}(t) = 1 - (1 - \phi'_{j1}(0))e^{-t}$ where $\phi'_{j1}(0) \in [0,1]$, we have $\phi_{j1}(t) \leq \phi'_{j1}(t)$ for all $t \geq 0$ and hence $\Delta V_i(\phi)(0) \leq \Delta V_i(\phi')(0)$, since Δu_i is increasing. (This observation is extended to general supermodular games in Section 3.2.)

Along the path ϕ, all the agents are playing action 0 at time 0, but will eventually switch to action 1 in the future. For agent 0, the decision is whether to already commit to action 1, which is suboptimal in the present but will become optimal in the future, or to stay with the currently optimal action 0, postponing the switch to action 1 to a next or later revision opportunity. In the presence of positive time discounting $\theta > 0$, the agent assigns larger weights to lower values of $\xi \in [0,1]$ in computing $\Delta V_i(\phi)$. In the limit as $\theta \to 0$, however, all the ξ's receive an equal weight, as seen in (2.3). Since **1** is a risk-dominant equilibrium and hence we have $\int_0^1 \Delta u_i(\xi)\,d\xi > 0$, this implies that, when θ is small, the relative return from the future coordination on **1** is large enough, so that the agent chooses to commit to action 1.

To verify the linear absorption of **1**, let the initial condition be $x^0 = \mathbf{1}$ and consider the linear path ψ from **1** to **0**, given by $\psi_{j1}(t) = e^{-t}$, along which every agent switches to action 0 at his first revision opportunity. Consider again an agent who is given a revision opportunity at time 0. Against ψ,

$$\begin{aligned}\Delta V_i(\psi)(0) &= (1+\theta)\int_0^\infty e^{-(1+\theta)s}\Delta u_i\left(e^{-s}\right)ds\\ &= (1+\theta)\int_0^1 \Delta u_i(\xi)\xi^\theta\,d\xi\\ &> \int_0^1 \Delta u_i(\xi)\,d\xi > 0,\end{aligned}$$

which implies that the agent has an incentive to stick to action 1 for any $\theta > 0$ (observe that $\Delta V_i(\psi)(0)$ is increasing in θ, as Δu_i is increasing). That is, since the risk-dominated action 0 does not perform well enough to compensate the loss from not playing the currently optimal action 1, the agent chooses action 1 at this revision opportunity, intending to switch to action 0 at a future opportunity. However, this is the case for all the agents revising at time 0, so that the escaping path ψ cannot be a perfect foresight path. Moreover, since, due to the supermodularity, the path ψ is the best scenario for action 0 to be played, action 1 is the best response to any feasible path from **1**. It thus follows that the only perfect foresight path from **1** is the constant path at **1**. Since the argument above remains valid when the initial condition lies in a small neighborhood of **1**, we conclude that **1** is linearly absorbing.

3. Supermodularity and monotonicity

In a supermodular game, the actions are ordered so that each player's marginal payoff to any increase in his action is nondecreasing in other players' actions. In this section, we first identify monotone properties of the perfect foresight dynamics for supermodular stage games. In particular, we observe the monotonicity of the best response correspondence β with respect to a partial order on Φ induced by the stochastic dominance relation over mixed strategies. We then prove a comparison theorem for the perfect foresight paths associated with two different stage games that are comparable in terms of best responses. This theorem implies that if at least one of the two games is supermodular, then one game inherits stability properties from the other. Finally, we show that for supermodular games, stability under perfect foresight is equivalent to that under rationalizable foresight (Matsui and Oyama 2006).

3.1 *Supermodular games*

For $x_i, y_i \in \Delta(A_i)$, we write $x_i \precsim y_i$ if y_i stochastically dominates x_i, i.e.,

$$\sum_{k=h}^{n_i} x_{ik} \le \sum_{k=h}^{n_i} y_{ik}$$

for all $h \in A_i$. For $x, y \in \prod_i \Delta(A_i)$, we write $x \precsim y$ if $x_i \precsim y_i$ for all $i \in I$ and $x_{-i} \precsim y_{-i}$ if $x_j \precsim y_j$ for all $j \neq i$. Moreover, we define $\phi_i \precsim \psi_i$ for $\phi_i, \psi_i \in \Phi^i$ by $\phi_i(t) \precsim \psi_i(t)$ for all $t \ge 0$; $\phi \precsim \psi$ for $\phi, \psi \in \Phi$ by $\phi_i \precsim \psi_i$ for all $i \in I$; and $\phi_{-i} \precsim \psi_{-i}$ for $\phi_{-i}, \psi_{-i} \in \Phi^{-i}$ by

$\phi_j \precsim \psi_j$ for all $j \neq i$. Note that if $\phi(0) \precsim \psi(0)$ and $\dot{\phi}(t)+\phi(t) \precsim \dot{\psi}(t)+\psi(t)$ for almost all $t \geq 0$, then $\phi \precsim \psi$. This can be verified by observing that

$$\phi(t)=e^{-t}\phi(0)+\int_0^t e^{s-t}\left(\dot{\phi}(s)+\phi(s)\right)ds \tag{3.1}$$

for all $t \geq 0$.

The game G is *supermodular* if whenever $h<k$, the difference $u_i(k,a_{-i})-u_i(h,a_{-i})$ is nondecreasing in $a_{-i} \in A_{-i}$, i.e., if $a_{-i} \leq b_{-i}$, then

$$u_i(k,a_{-i})-u_i(h,a_{-i}) \leq u_i(k,b_{-i})-u_i(h,b_{-i}).$$

It is well known that this property extends to mixed strategies: if $h<k$ and $x_{-i} \precsim y_{-i}$, then

$$u_i(k,x_{-i})-u_i(h,x_{-i}) \leq u_i(k,y_{-i})-u_i(h,y_{-i}).$$

The expected discounted payoff function V_i preserves this property, implying that BR^i is monotone with respect to the partial order on Φ.

Lemma 3.1. *Suppose that the stage game is supermodular. For $\phi,\psi \in \Phi$, if $\phi_{-i} \precsim \psi_{-i}$, then for all $t \geq 0$,*

$$V_{ik}(\phi)(t)-V_{ih}(\phi)(t) \leq V_{ik}(\psi)(t)-V_{ih}(\psi)(t)$$

for $h<k$, and

$$\begin{aligned}\min BR^i(\phi)(t) &\leq \min BR^i(\psi)(t)\\ \max BR^i(\phi)(t) &\leq \max BR^i(\psi)(t).\end{aligned}$$

Proof. Suppose $\phi_{-i} \precsim \psi_{-i}$ and fix any t. If $k>h$, then

$$\begin{aligned}V_{ik}(\phi)(t)-V_{ih}(\phi)(t) &= (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)}\{u_i(k,\phi_{-i}(s))-u_i(h,\phi_{-i}(s))\}\,ds\\ &\leq (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)}\{u_i(k,\psi_{-i}(s))-u_i(h,\psi_{-i}(s))\}\,ds\\ &= V_{ik}(\psi)(t)-V_{ih}(\psi)(t).\end{aligned}$$

Next, let $k=\min BR^i(\phi)(t)$. For any $h<k$,

$$V_{ik}(\psi)(t)-V_{ih}(\psi)(t) \geq V_{ik}(\phi)(t)-V_{ih}(\phi)(t)>0$$

since $h \notin BR^i(\phi)(t)$. Hence, if $\ell \in BR^i(\psi)(t)$, then $\ell \geq k=\min BR^i(\phi)(t)$. We thus have $\min BR^i(\psi)(t) \geq \min BR^i(\phi)(t)$.

The other claim that $\max BR^i(\phi)(t) \leq \max BR^i(\psi)(t)$ can be proved similarly. □

The next proposition establishes the monotonicity of the best response correspondence β^i over Φ. For $\phi \in \Phi$, a feasible path $\phi_i^- \in \beta^i(\phi)$ is the smallest element of $\beta^i(\phi)$ if $\phi_i^- \precsim \phi_i'$ for all $\phi_i' \in \beta^i(\phi)$, and $\phi_i^+ \in \beta^i(\phi)$ is the largest element of $\beta^i(\phi)$ if $\phi_i' \precsim \phi_i^+$ for all $\phi_i' \in \beta^i(\phi)$.

Proposition 3.1. *Suppose that the stage game is supermodular. For* $\phi \in \Phi$, $\beta^i(\phi)$ *has smallest element* $\min \beta^i(\phi)$ *and largest element* $\max \beta^i(\phi)$. *If* $\phi_i(0) \precsim \psi_i(0)$ *and* $\phi_{-i} \precsim \psi_{-i}$, *then*

$$\min \beta^i(\phi) \precsim \min \beta^i(\psi)$$
$$\max \beta^i(\phi) \precsim \max \beta^i(\psi).$$

Proof. Take ϕ and ψ such that $\phi_i(0) = x_i$, $\psi_i(0) = y_i$, $x_i \precsim y_i$, and $\phi_{-i} \precsim \psi_{-i}$. First, we construct $\phi_i^- = \min \beta^i(\phi)$; the construction of $\max \beta^i(\phi)$ is similar. Define

$$\alpha_i(t) = \min BR^i(\phi)(t),$$

where the right hand side is considered to be a mixed strategy. Note that α_i is lower semi-continuous, and hence measurable, since $BR^i(\phi)(\cdot)$ is an upper semi-continuous correspondence. Then, the unique solution ϕ_i^- to

$$\dot{\phi}_i^-(t) = \alpha_i(t) - \phi_i^-(t) \quad \text{a.e.}, \qquad \phi_i^-(0) = x_i$$

is given by

$$\phi_i^-(t) = e^{-t} x_i + \int_0^t e^{s-t} \alpha_i(s)\, ds.$$

By construction, $\phi_i^- \in \beta^i(\phi)$, and $\phi_i^- \precsim \phi_i'$ for all $\phi_i' \in \beta^i(\phi)$, i.e., ϕ_i^- is the smallest element of $\beta^i(\phi)$.

On the other hand, any path $\psi_i' \in \beta^i(\psi)$ is given by

$$\psi_i'(t) = e^{-t} y_i + \int_0^t e^{s-t} \alpha_i'(s)\, ds$$

for some $\alpha_i' : \mathbb{R}_+ \to \Delta(A_i)$ such that $\alpha_i'(t) \in F_i(\psi)(t)$ for almost all $t \geq 0$, where $F_i(\psi)$ is defined by (2.2). Since $\phi_{-i} \precsim \psi_{-i}$, it follows from Lemma 3.1 that

$$\min BR^i(\phi)(t) \leq \min BR^i(\psi)(t),$$

and hence $\alpha_i(t) \precsim \alpha_i'(t)$ for almost all t. Together with the assumption that $x_i \precsim y_i$, this implies that $\phi_i^- \precsim \psi_i'$ (recall (3.1)), thereby completing the proof that $\min \beta^i(\phi) \precsim \min \beta^i(\psi)$. □

3.2 *Comparison theorem*

Fix the set of players, I, and the set of action profiles, A. Consider two games $G = (I, (A_i)_{i\in I}, (u_i)_{i\in I})$ and $G' = (I, (A_i)_{i\in I}, (v_i)_{i\in I})$ such that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\min br^i_{v_i}(\pi_i) \leq \min br^i_{u_i}(\pi_i), \tag{3.2}$$

or that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\max br^i_{v_i}(\pi_i) \leq \max br^i_{u_i}(\pi_i), \tag{3.3}$$

where $br^i_{u_i}(\pi_i)$ and $br^i_{v_i}(\pi_i)$ are the sets of best responses to π_i in the games G and G', respectively. In this subsection, we study the relationship between the perfect foresight paths for the stage game G and those for G'. Note that the state space $\prod_i \Delta(A_i)$ is common to both cases. We show that if G or G' is supermodular, then the perfect foresight dynamics preserve the order of best responses between G and G', and therefore G inherits stability properties from G'.

To specify the payoff functions, we denote by $BR^i_{u_i}(\phi)(t)$ (respectively $BR^i_{v_i}(\phi)(t)$) the set of best responses for population i to a feasible path ϕ at time t when the stage game is G (respectively G'). Note that this set can be written as

$$BR^i_{u_i}(\phi)(t) = br^i_{u_i}(\pi^t_i(\phi))$$

with a probability distribution $\pi^t_i(\phi) \in \Delta(A_{-i})$ that is given by

$$\pi^t_i(\phi)(a_{-i}) = (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} \left(\prod\nolimits_{j\neq i} \phi_{ja_j}(s)\right) ds.$$

Thus, if (3.2) is satisfied, then for any $\phi \in \Phi$ and any $t \geq 0$,

$$\min BR^i_{v_i}(\phi)(t) \leq \min BR^i_{u_i}(\phi)(t),$$

while if (3.3) is satisfied, then for any $\phi \in \Phi$ and any $t \geq 0$,

$$\max BR^i_{v_i}(\phi)(t) \leq \max BR^i_{u_i}(\phi)(t).$$

The following lemma is a key to our comparison theorem. The proof relies on a fixed point argument together with the monotonicity of BR^i.

Lemma 3.2. *Let $x, y \in \prod_i \Delta(A_i)$ be such that $y \precsim x$.*

(i) Suppose that G and G' satisfy (3.2) *and that G or G' is supermodular. If a feasible path $\phi \in \Phi_x$ is such that for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$,*

$$\dot\phi_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \geq \min BR^i_{u_i}(\phi)(t), \tag{3.4}$$

then there exists a perfect foresight path $\psi^* \in \Phi_y$ for G' such that $\psi^* \precsim \phi$.

(ii) Suppose that G and G' satisfy (3.3) *and that G or G' is supermodular. If a feasible path $\psi \in \Phi_y$ is such that for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$,*

$$\dot\psi_{ih}(t) > -\psi_{ih}(t) \Rightarrow h \leq \max BR^i_{v_i}(\psi)(t),$$

then there exists a perfect foresight path $\phi^* \in \Phi_x$ for G such that $\psi \precsim \phi^*$.

Proof. We show only (i). Given $x, y \in \prod_i \Delta(A_i)$ with $y \precsim x$ and $\phi \in \Phi_x$ satisfying (3.4), define the convex and compact subset $\tilde\Phi_y \subset \Phi_y$ by

$$\tilde\Phi_y = \{\psi \in \Phi_y \mid \psi \precsim \phi\}.$$

Let $\beta_{G'}$ be the best response correspondence for the stage game G'. We define a convex- and compact-valued and upper semi-continuous correspondence $\tilde{\beta}_{G'}: \tilde{\Phi}_y \to \tilde{\Phi}_y$ by

$$\tilde{\beta}_{G'}(\psi) = \beta_{G'}(\psi) \cap \tilde{\Phi}_y \qquad (\psi \in \tilde{\Phi}_y).$$

We want to show that $\tilde{\beta}_{G'}(\psi)$ is nonempty for any $\psi \in \tilde{\Phi}_y$. Then, it follows from Kakutani's fixed point theorem that $\tilde{\beta}_{G'}$ has a fixed point $\psi^* \in \tilde{\beta}_{G'}(\psi^*) \subset \tilde{\Phi}_y$, which is a perfect foresight path for G' and satisfies $\psi^* \precsim \phi$.

For $\psi \in \tilde{\Phi}_y$, take any $i \in I$, $h \in A_i$, and $t \geq 0$ such that $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$. If G is supermodular, then

$$h \geq \min BR^i_{u_i}(\phi)(t) \geq \min BR^i_{u_i}(\psi)(t) \geq \min BR^i_{v_i}(\psi)(t),$$

where the second inequality follows from the supermodularity of G and Lemma 3.1, and the third inequality follows from the assumption of (3.2). If G' is supermodular, then

$$h \geq \min BR^i_{u_i}(\phi)(t) \geq \min BR^i_{v_i}(\phi)(t) \geq \min BR^i_{v_i}(\psi)(t),$$

where the second inequality follows from the assumption of (3.2), and the third inequality follows from the supermodularity of G' and Lemma 3.1. Therefore, in each case, we have

$$h \geq \min BR^i_{v_i}(\psi)(t) \tag{3.5}$$

for all h such that $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$.

Now let $\psi' \in \Phi_y$ be given by

$$\dot{\psi}'_i(t) = \min BR^i_{v_i}(\psi)(t) - \psi'_i(t) \quad \text{a.e.}, \qquad \psi'_i(0) = y_i$$

for all $i \in I$. By construction, we have $\psi' \in \beta_{G'}(\psi)$. We also have

$$\dot{\psi}'_i(t) + \psi'_i(t) = \min BR^i_{v_i}(\psi)(t) \precsim \dot{\phi}_i(t) + \phi_i(t) \tag{3.6}$$

for all i and almost all t, since (3.5) holds for all h such that $\dot{\phi}_{ih}(t) + \phi_{ih}(t) > 0$. From (3.6) along with $\psi'(0) = y \precsim x = \phi(0)$, it follows that $\psi' \precsim \phi$ (recall (3.1)), i.e., $\psi' \in \tilde{\Phi}_y$. Therefore, we have $\psi' \in \tilde{\beta}_{G'}(\psi) = \beta_{G'}(\psi) \cap \tilde{\Phi}_y$, which implies the nonemptiness of $\tilde{\beta}_{G'}(\psi)$. □

As a corollary, we have the following result, which is an analogue of the comparison theorem from the theory of differential equations (Walter 1970) or monotone (cooperative) dynamical systems (Smith 1995) and of the comparative statics theorem (Milgrom and Roberts 1990).

THEOREM 3.1. *Let $x, y \in \prod_i \Delta(A_i)$ be such that $y \precsim x$.*

(i) Suppose that G and G' satisfy (3.2) and that G or G' is supermodular. For any perfect foresight path ϕ^* for G with $\phi^*(0) = x$, there exists a perfect foresight path ψ^* for G' with $\psi^*(0) = y$ such that $\psi^* \precsim \phi^*$.

(ii) *Suppose that G and G' satisfy (3.3) and that G or G' is supermodular. For any perfect foresight path ψ^* for G' with $\psi^*(0)=y$, there exists a perfect foresight path ϕ^* for G with $\phi^*(0)=x$ such that $\psi^* \precsim \phi^*$.*

Suppose that G or G' is supermodular. This theorem implies that if G is comparable (in terms of best responses) to G', then G inherits stability properties from G'. First, assume that G and G' satisfy (3.2) and that the action profile $\max A=(n_i)_{i\in I}$ is (linearly) absorbing in G'. Take any state $x \in B_\varepsilon(\max A)$ for a sufficiently small $\varepsilon > 0$ and any perfect foresight path ϕ^* for G with $\phi^*(0)=x$. By Theorem 3.1(i), there exists a perfect foresight path ψ^* for G' with $\psi^*(0)=x$ such that $\psi^* \precsim \phi^*$. By the assumption that $\max A$ is (linearly) absorbing in G', ψ^* converges (linearly) to $\max A$, so that ϕ^* also converges (linearly) to $\max A$. This implies that $\max A$ is (linearly) absorbing in G as well.

Second, assume that G and G' satisfy (3.3) and that $\max A$ is globally accessible in G'. Take any state $x \in \prod_i \Delta(A_i)$. By the assumption that $\max A$ is globally accessible in G', there exists a perfect foresight path ψ^* for G' with $\psi^*(0)=x$ that converges to $\max A$. By Theorem 3.1(ii), there exists a perfect foresight path ϕ^* for G with $\phi^*(0)=x$ such that $\psi^* \precsim \phi^*$. Since ψ^* converges to $\max A$, ϕ^* also converges to $\max A$. This implies that $\max A$ is globally accessible in G as well.

Note that by reversing the orders of the actions, the above arguments can be applied to $\min A$.

A candidate for the game G' is a potential game. Such a case is considered, with some refinement, in Section 4.

Lemma 3.2 with $G'=G$ (i.e., $v_i = u_i$ for all $i \in I$) yields the following corollary. We say that a feasible path ϕ is a *superpath* if

$$\dot{\phi}_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \geq \min BR^i(\phi)(t)$$

for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$; a feasible path ψ is a *subpath* if

$$\dot{\psi}_{ih}(t) > -\psi_{ih}(t) \Rightarrow h \leq \max BR^i(\psi)(t)$$

for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$.

Lemma 3.3. *Suppose that the stage game is supermodular. Let $x, y \in \prod_i \Delta(A_i)$ be such that $y \precsim x$.*

(i) *If there exists a superpath ϕ with $\phi(0)=x$, then there exists a perfect foresight path ψ^* with $\psi^*(0)=y$ such that $\psi^* \precsim \phi$.*

(ii) *If there exists a subpath ψ with $\psi(0)=y$, then there exists a perfect foresight path ϕ^* with $\phi^*(0)=x$ such that $\psi \precsim \phi^*$.*

This lemma is used to prove the following propositions.

Proposition 3.2. *Suppose that the stage game is supermodular. If $x^* \in \prod_i \Delta(A_i)$ is absorbing, then it is a strict Nash equilibrium.*

	0	1
0	1, 1	1, 1
1	0, 0	1, 1

FIGURE 3.1. A globally accessible, non-strict Nash equilibrium.

PROOF. In light of Proposition 2.1, it is sufficient to show that any Nash equilibrium that is not a strict Nash equilibrium is not absorbing. Suppose that x^* is a non-strict Nash equilibrium. We show the existence of an escaping path from x^*.

Let a'_i (respectively a''_i) be the smallest (respectively the largest) in $br^i(x^*_{-i})$ for each player i, and let $a' = (a'_i)_{i\in I}$ and $a'' = (a''_i)_{i\in I}$, which are considered to be mixed strategy profiles. Note that $a' \precsim x^* \precsim a''$ and, by the definition of a non-strict Nash equilibrium, $a' \neq a''$, so that a' or a'' is different from x^*. Let us assume that $a' \neq x^*$.

Now denote by $\bar{\phi}$ the constant path such that $\bar{\phi}(t) = x^*$ for all t. Note that $BR^i(\bar{\phi})(t) = br^i(x^*_{-i})$, so that $\min BR^i(\bar{\phi})(t) = a'_i$ for all t. Let ϕ be the feasible path starting from x^* and converging linearly to a', i.e.,

$$\phi(t) = e^{-t}x^* + (1-e^{-t})a'.$$

This path satisfies $\phi \precsim \bar{\phi}$, $\phi \neq \bar{\phi}$, and $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$ only for $h = a'_i$. We also have

$$a'_i = \min BR^i(\bar{\phi})(t) \geq \min BR^i(\phi)(t),$$

where the inequality follows from Lemma 3.1. This means that ϕ is a superpath. Therefore, it follows from Lemma 3.3 that there exists a perfect foresight path ψ^* from x^* such that $\psi^* \precsim \phi$, which does not converge to x^*. □

The next proposition shows the equivalence of absorption and linear absorption for supermodular games. A proof is given in Section A.1, in the Appendix.

PROPOSITION 3.3. *Suppose that the stage game is supermodular. If $a^* \in A$ is absorbing, then it is linearly absorbing.*

A globally accessible state need not in general be a strict Nash equilibrium. Even in the class of strict supermodular games, there are degenerate games for which a non-strict, pure-strategy Nash equilibrium is globally accessible. In the game given by Figure 3.1, the non-strict Nash equilibrium $(0, 1)$ is globally accessible for any degree of friction. It is an open problem whether every globally accessible state must be a pure Nash equilibrium in generic supermodular games.

3.3 *Stability under rationalizable foresight*

The concept of a perfect foresight path requires that agents maximize their future discounted payoffs against their beliefs about the future path of the action distribution and that these beliefs coincide with the actual path. Relaxing the latter requirement, Matsui and Oyama (2006) introduce the model of rationalizable foresight dynamics. In this model, while the rationality of agents as well as the structure of the society is common knowledge, beliefs about the future path are not necessarily coordinated among agents. It is assumed instead that agents form their beliefs in a rationalizable manner: in particular, they may misforecast the future. In this subsection, we consider stability under rationalizable foresight dynamics and show that in supermodular games, an absorbing and globally accessible state under perfect foresight dynamics is uniquely absorbing under rationalizable foresight dynamics as well.

Following Matsui and Oyama (2006), we define rationalizable foresight paths as follows. First let Ψ^0 be the set of all feasible paths, Φ. Then for each positive integer k, define Ψ^k to be

$$\Psi^k = \{\psi \in \Psi^{k-1} \,\big|\, \forall\, i \in I, \forall\, h \in A_i, \text{a.a. } t \geq 0 : [\dot{\psi}_{ih}(t) > -\psi_{ih}(t) \\ \Rightarrow \exists\, \psi' \in \Psi^{k-1} : \psi'(s) = \psi(s)\, \forall\, s \in [0,t] \text{ and } h \in BR^i(\psi')(t)]\}.$$

Along a path in Ψ^k, an agent with a revision opportunity at time t takes a best response to some path in Ψ^{k-1} while knowing the past history up to time t.[13] Let $\Psi^* = \bigcap_{k=0}^{\infty} \Psi^k$.

Definition 3.1. A path in Ψ^* is a *rationalizable foresight path*.

Our concept of rationalizable foresight path differs from rationalizability in the associated differential game defined in Remark 2.2. The former incorporates the feature of societal games that different agents in a population can have different beliefs and a single agent can have different beliefs at different revision opportunities, while for the latter, each population acts as a single player, who makes his decision only at time zero.

Along every rationalizable foresight path, each agent optimizes against some, possibly different, rationalizable foresight path. We state this without a proof, as it is essentially the same as Proposition 3.3 in Matsui and Oyama (2006).

Proposition 3.4. *A feasible path $\psi \in \Phi$ is contained in Ψ^* if and only if for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$ such that $\dot{\psi}_{ih}(t) > -\psi_{ih}(t)$, there exists $\psi' \in \Psi^*$ such that $\psi'(s) = \psi(s)$ for all $s \in [0,t]$ and $h \in BR^i(\psi')(t)$.*

As in one-shot games, we have the following relationship between perfect and rationalizable foresight paths. This is verified by observing that every perfect foresight path is contained in each Ψ^k.

Lemma 3.4. *A perfect foresight path is a rationalizable foresight path.*

We define absorption under rationalizable foresight analogously to that under perfect foresight.[14]

Definition 3.2. $x^* \in \prod_i \Delta(A_i)$ is *absorbing* under *rationalizable foresight* if there exists $\varepsilon > 0$ such that any rationalizable foresight path from any $x \in B_\varepsilon(x^*)$ converges to x^*.

[13] Since the environment is stationary and $BR^i(\phi)(t)$ depends only on the behavior of ϕ after time t, in the definition of Ψ^k one can equivalently take ψ' as a path in Ψ^{k-1} that satisfies only $\psi'(t) = \psi(t)$.

[14] We can define global accessibility under rationalizable foresight in a similar manner. Due to Lemma 3.4, it is weaker than that under perfect foresight.

An absorbing state under rationalizable foresight is also absorbing under perfect foresight due to Lemma 3.4, but not vice versa in general (see Examples 3.1 and 3.2 in Matsui and Oyama 2006). As noted, in rationalizable foresight dynamics, agents are allowed to have different beliefs at various revision opportunities, and these beliefs need only be rationalizable and thus may be misforecasts about the actual course of future play. Hence, it is generally easier for the action distribution to escape from an equilibrium state under rationalizable foresight than under perfect foresight. Nevertheless, for supermodular games we can show that absorption under perfect foresight is in fact equivalent to that under rationalizable foresight.

THEOREM 3.2. *Suppose that the stage game is supermodular. Then $x^* \in \prod_i \Delta(A_i)$ is absorbing under rationalizable foresight if and only if it is absorbing under perfect foresight.*

Therefore, in supermodular games, an absorbing and globally accessible state under perfect foresight is the unique state that is absorbing under rationalizable foresight.

The "if" part of this theorem follows from the lemma below, an analogue of the fact in general supermodular games that the set of rationalizable strategy profiles has smallest and largest elements, and these elements are pure-strategy Nash equilibria (Milgrom and Roberts 1990). For $x \in \prod_i \Delta(A_i)$, let $\Psi^k_x = \Psi^k \cap \Phi_x$ and $\Psi^*_x = \bigcap_{k=0}^{\infty} \Psi^k_x$. Note that $\Psi^*_x = \Psi^* \cap \Phi_x$, i.e., Ψ^*_x is the set of rationalizable foresight paths from x.

LEMMA 3.5. *Suppose that the stage game is supermodular. Then Ψ^*_x has smallest and largest elements, and these elements are perfect foresight paths.*

PROOF. We show that Ψ^*_x has a smallest element and that it is a perfect foresight path. Let ϕ^0 be the smallest feasible path from x (i.e., the linear path from x to $\min A$) and ϕ^k the smallest best response path to ϕ^{k-1}, which is given by

$$\dot\phi^k_i(t) = \min BR^i(\phi^{k-1})(t) - \phi^k_i(t) \quad \text{a.e.}, \qquad \phi^k_i(0) = x_i.$$

Then $\{\phi^k\}_{k=0}^{\infty}$ is an increasing sequence in the compact set Φ_x, so that $\{\phi^k\}_{k=0}^{\infty}$ converges to some $\phi^* \in \Phi_x$. By the upper semi-continuity of β_x, ϕ^* is a perfect foresight path, and hence, an element of Ψ^*_x by Lemma 3.4.

It suffices to show that ϕ^* is a lower bound of Ψ^*_x. We show that ϕ^k is a lower bound of Ψ^k_x ($\supset \Psi^*_x$) for all k. It follows that the limit ϕ^* is also a lower bound of Ψ^*_x.

First, ϕ^0 is a lower bound of Ψ^0_x. Then, suppose that ϕ^{k-1} is a lower bound of Ψ^{k-1}_x. Fix any $\psi \in \Psi^k_x$, and take any i and any t such that ϕ^k_i and ψ_i are differentiable at t. For any h such that $\dot\psi_{ih}(t) > -\psi_{ih}(t)$, we have $h \in BR^i(\psi')(t)$ for some $\psi' \in \Psi^{k-1}_x$. Since $\phi^{k-1} \precsim \psi'$ by assumption, it follows from the supermodularity and Lemma 3.1 that $\min BR^i(\phi^{k-1})(t) \le \min BR^i(\psi')(t) \le h$. Therefore, we have $\dot\phi^k_i(t) + \phi^k_i(t) \precsim \dot\psi_i(t) + \psi_i(t)$ for almost all t, which implies that $\phi^k \precsim \psi$. Hence, ϕ^k is a lower bound of Ψ^k_x. □

PROOF OF THEOREM 3.2. "If" part: Take any rationalizable foresight path ψ from x sufficiently close to x^*. By Lemma 3.5, there exist perfect foresight paths ϕ and ϕ' from x such that $\phi \precsim \psi \precsim \phi'$. If x^* is absorbing under perfect foresight, then both ϕ and ϕ' converge to x^*, and therefore, ψ also converges to x^*.

"Only if" part: Follows from Lemma 3.4. □

REMARK 3.1. All the results in this section, as well as Lemma 2.1, hold in more general settings (after appropriate modifications, replacing "$\phi_{-i} \precsim \psi_{-i}$" with "$\phi \precsim \psi$", and br^i with BR^i) where $V_i(\cdot)(\cdot)\colon \Phi \times \mathbb{R}_+ \to \mathbb{R}^{n_i+1}$ is continuous and $V_i(\cdot)(t)\colon \Phi \to \mathbb{R}^{n_i+1}$ is supermodular, i.e., if $\phi \precsim \psi$, then

$$V_{ik}(\phi)(t) - V_{ih}(\phi)(t) \le V_{ik}(\psi)(t) - V_{ih}(\psi)(t)$$

for $k > h$. Examples of such functions include the expected discounted payoffs induced by the stage game where the payoff to an agent in population i taking action $h \in A_i$ is given by a continuous function $g_{ih}\colon \prod_i \Delta(A_i) \to \mathbb{R}$. Note here that the payoff function for an agent in population i may depend on the action distribution within population i itself and may not be N-linear in $\prod_i \Delta(A_i)$. Such payoff functions can describe random matching models within a single population, considered in Matsui and Matsuyama (1995), Hofbauer and Sorger (1999), and Oyama (2002), as well as models with nonlinear payoffs, considered in Matsuyama (1991, 1992) and Kaneda (1995). In alternative settings, V_i may depend on the past behavior of ϕ.

4. Games with monotone potentials

This section applies the monotonicity argument developed in the previous section to games with monotone potentials introduced by Morris and Ui (2005). Suppose that the games G and G' satisfy (3.2) or (3.3). Roughly speaking, G has a monotone potential if G' can be chosen to be a potential game, and the action profile $\max A$ is a monotone potential maximizer of G if it is the unique potential maximizer of G'. For potential games, Hofbauer and Sorger (2002) show that the unique potential maximizer is absorbing and globally accessible for any small degree of friction. Therefore we can conclude from Theorem 3.1 and the subsequent discussion that if G or G' is supermodular, then $\max A$ is absorbing (if (3.2) is satisfied) and globally accessible (if (3.3) is satisfied) for any small degree of friction in the stage game G.

For the precise definition, which is given in the following subsection, two remarks are in order. First, when G' is a potential game, a condition weaker than both (3.2) and (3.3) is sufficient for the global accessibility result. Morris and Ui's (2005) definition of monotone potential employs this weaker version (Definition 4.1), while (3.2) corresponds to what we call strict monotone potential (Definition 4.2). Second, in order to define the concept for action profiles a^* other than $\max A$ or $\min A$, we need to divide the set of actions for each player i into two parts: the actions below a_i^* and those above a_i^*.

4.1 *Monotone potential maximizer*

Fix an action profile $a^* \in A$. Let $A_i^- = \{h \in A_i \mid h \le a_i^*\}$ and $A_i^+ = \{h \in A_i \mid h \ge a_i^*\}$. For a function $f\colon A \to \mathbb{R}$, a probability distribution $\pi_i \in \Delta(A_{-i})$, and a nonempty set of actions $A_i' \subset A_i$, let

$$br_f^i(\pi_i|A_i') = \arg\max_{h \in A_i'} f(h, \pi_i),$$

where $f(h,\pi_i)=\sum_{a_{-i}\in A_{-i}}\pi_i(a_{-i})f(h,a_{-i})$. We employ the following simplified version of monotone potential.[15]

Definition 4.1. The action profile $a^*\in A$ is a *monotone potential maximizer*, or *MP-maximizer*, of G if there exists a function $v\colon A\to\mathbb{R}$ with $v(a^*)>v(a)$ for all $a\neq a^*$ such that for all $i\in I$ and all $\pi_i\in\Delta(A_{-i})$,

$$\min br^i_v(\pi_i|A_i^-)\le\max br^i_{u_i}(\pi_i|A_i^-)\tag{4.1}$$

and

$$\max br^i_v(\pi_i|A_i^+)\ge\min br^i_{u_i}(\pi_i|A_i^+).\tag{4.2}$$

Such a function v is called a *monotone potential function* for a^*.

In addition, we introduce a slight refinement of the notion of an MP-maximizer.[16]

Definition 4.2. The action profile $a^*\in A$ is a *strict monotone potential maximizer*, or *strict MP-maximizer*, of G if there exists a function $v\colon A\to\mathbb{R}$ with $v(a^*)>v(a)$ for all $a\neq a^*$ such that for all $i\in I$ and all $\pi_i\in\Delta(A_{-i})$,

$$\min br^i_v(\pi_i|A_i^-)\le\min br^i_{u_i}(\pi_i|A_i^-)\tag{4.3}$$

and

$$\max br^i_v(\pi_i|A_i^+)\ge\max br^i_{u_i}(\pi_i|A_i^+).\tag{4.4}$$

Such a function v is called a *strict monotone potential function* for a^*.

Recall that in a potential game, the best response correspondences are exactly equal to those in the corresponding common interest game whose payoffs are given by the potential function. In defining a monotone potential function v for a game G, equalities are replaced with inequalities, so that the best responses in G are bounded by those (restricted to A^- and A^+) in the potential game G_v. With supermodularity in G or G_v, this suffices to allow G to inherit properties of potential games.

A (strict) MP-maximizer is a (strict) Nash equilibrium. A strict MP-maximizer is always an MP-maximizer, but the converse is not true. In a degenerate game (with at least two action profiles) where payoffs are constant for each player, all the action profiles become MP-maximizers, while none of them is a strict MP-maximizer. For a generic choice of payoffs, an MP-maximizer is a strict MP-maximizer. For supermodular games, a strict MP-maximizer is unique if it exists, due to Theorems 4.1 and 4.2, given subsequently.

The notion of an MP-maximizer unifies several existing concepts. A unique weighted potential maximizer is a strict MP-maximizer. A (strict) **p**-dominant equilibrium with $\sum_{i\in I}p_i<1$ is a (strict) MP-maximizer. For games with diminishing marginal returns, the notion of an MP-maximizer reduces to that of a local potential maximizer (Morris 1999 and Morris and Ui 2005). See Section 4.3 for details.

[15] In Morris and Ui (2005) a monotone potential function is defined on a given partition of A.

[16] Morris (1999) introduces a version of MP-maximizer that is stronger than our concept of a strict MP-maximizer: if a^* is an MP-maximizer in the sense of Morris (1999), then it is a strict MP-maximizer, but not vice versa in general.

4.2 *Results*

For a function $f: A \to \mathbb{R}$, a feasible path ϕ, and a nonempty set of actions $A'_i \subset A_i$, let

$$BR^i_f(\phi|A'_i)(t) = \underset{h\in A'_i}{\arg\max}(1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} f(h,\phi_{-i}(s))\,ds,$$

where $f(h,x_{-i}) = \sum_{a_{-i}\in A_{-i}} \left(\prod_{j\neq i} x_{j a_j}\right) f(h,a_{-i})$ for $x_{-i} \in \prod_{j\neq i}\Delta(A_j)$. Let $G_v = (I,(A_i)_{i\in I},(v)_{i\in I})$ be the potential game in which all players have the common payoff function v. We have the following two theorems. Their proofs are given in Sections A.2 and A.3, in the Appendix.

Theorem 4.1. *Suppose that the stage game G has an MP-maximizer a^* with a monotone potential function v. If G or G_v is supermodular, then there exists $\bar{\theta} > 0$ such that a^* is globally accessible for all $\theta \in (0,\bar{\theta})$.*

Theorem 4.2. *Suppose that the stage game G has a strict MP-maximizer a^* with a strict monotone potential function v. If G or G_v is supermodular, then a^* is linearly absorbing for all $\theta > 0$.*

In particular, a strict MP-maximizer is the unique linearly absorbing (and globally accessible) state for any small degree of friction, if G or G_v is supermodular.

Note that, in order for Theorems 4.1 and 4.2 to apply, it is sufficient that at least one of the games G and G_v be supermodular. Indeed, in the special case considered in Section 4.3.1, the original game G need not be a supermodular game, and Section 4.3.3 presents a numerical example of a non-supermodular game that has a supermodular monotone potential function.

Given an MP-maximizer a^* and a monotone potential v, observe that the restricted games $G^-_v = (I,(A^-_i)_{i\in I},(v)_{i\in I})$ and $G^+_v = (I,(A^+_i)_{i\in I},(v)_{i\in I})$ are potential games with the unique potential maximizer a^*. The proofs of Theorems 4.1 and 4.2 utilize this observation to apply results on potential games by Hofbauer and Sorger (2002).

The proofs proceed as follows. Suppose that a^* is an MP-maximizer with a monotone potential function v. Observe (for the case where $a^* = \max A$) that (4.1) is weaker than (3.3). We thus need feasible paths ϕ^- and ϕ^+ such that

$$\dot{\phi}^-_i(t) = \min BR^i_v(\phi^-|A^-_i)(t) - \phi^-_i(t) \quad \text{a.e.}, \qquad \phi^-_i(0) = \min A_i$$
$$\dot{\phi}^+_i(t) = \max BR^i_v(\phi^+|A^+_i)(t) - \phi^+_i(t) \quad \text{a.e.}, \qquad \phi^+_i(0) = \max A_i$$

for all $i \in I$, and $\lim_{t\to\infty}\phi^-(t) = \lim_{t\to\infty}\phi^+(t) = a^*$. Notice that ϕ^- (respectively ϕ^+) is a feasible path on $\prod_i \Delta(A^-_i)$ (respectively $\prod_i \Delta(A^+_i)$), and actually a perfect foresight path for the stage game G^-_v (respectively G^+_v).

To obtain these paths, we use the fact that if the stage game is a potential game, then any solution to a certain optimal control problem is a perfect foresight path, and when the friction $\theta > 0$ is sufficiently small, it converges to the potential maximizer a^*. Fix such a small θ. We show that a minimal (respectively maximal) solution to the optimal control problem associated with G^-_v (respectively G^+_v) satisfies the above conditions.

Then, an argument similar to that in the proof of Lemma 3.2 allows us to show that if G or G_v is supermodular, then for any $x \in \prod_i \Delta(A_i)$ there exists a perfect foresight path ϕ^* with $\phi^*(0) = x$ such that $\phi^- \precsim \phi^* \precsim \phi^+$. Since ϕ^- and ϕ^+ converge to a^*, ϕ^* also converges to a^*. This implies that a^* is globally accessible for a small friction.

Next, suppose that a^* is a strict MP-maximizer with a strict monotone potential function v. Take any perfect foresight path ϕ^* starting from a state sufficiently close to a^*. As in the proof of Lemma 3.2, we can show that if G or G_v is supermodular, then there exist feasible paths ϕ^- and ϕ^+ starting from states sufficiently close to a^* such that $\phi^- \precsim \phi^* \precsim \phi^+$ and that ϕ^- and ϕ^+ are perfect foresight paths for the restricted games G_v^- and G_v^+, respectively. Since a^*, the potential maximizer of G_v^- and G_v^+, is absorbing in G_v^- and G_v^+, ϕ^- and ϕ^+ converge to a^*, and therefore, ϕ^* also converges to a^*. In the case where G is supermodular, this implies that a^* is linearly absorbing in G, due to Proposition 3.3. In the case where G_v is supermodular, a^* is linearly absorbing in G_v^- and G_v^+, so that ϕ^- and ϕ^+ converge linearly to a^*. Therefore, ϕ^* also converges linearly to a^*, implying the linear absorption of a^* in G.

4.3 *Examples*

This subsection provides special cases of games with monotone potentials. An example is also presented in Section 4.3.4 in which there are multiple globally accessible states for small frictions, so that the game admits no monotone potential.[17]

4.3.1 **p**-*dominance* Let $\mathbf{p} = (p_1, \ldots, p_N) \in [0,1)^N$. The notion of **p**-dominance (Kajii and Morris 1997) is a many-player, many-action generalization of risk-dominance.

Definition 4.3. (i) An action profile $a^* \in A$ is a **p**-*dominant equilibrium* of G if for all $i \in I$, $a_i^* \in br^i(\pi_i)$ holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a_{-i}^*) \geq p_i$.

(ii) An action profile a^* is a *strict* **p**-*dominant equilibrium* of G if for all $i \in I$, $\{a_i^*\} = br^i(\pi_i)$ holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a_{-i}^*) > p_i$.

A **p**-dominant equilibrium with low enough **p** is an MP-maximizer with a monotone potential function that is supermodular (with an appropriate re-ordering of actions).

Lemma 4.1. *If a^* is a (strict)* **p**-*dominant equilibrium with $\sum_{i\in I} p_i < 1$, then a^* is a (strict) MP-maximizer with the (strict) monotone potential v given by*

$$v(a) = \begin{cases} 1 - \sum_{i\in I} p_i & \text{if } a = a^* \\ -\sum_{i\in C(a)} p_i & \text{otherwise,} \end{cases}$$

where $C(a) = \{i \in I \mid a_i = a_i^*\}$.

A proof is given in Section A.4, in the Appendix.

[17] Morris (1999) presents a symmetric 4×4 game that has no robust equilibrium, and hence no monotone potential, while by the result of Takahashi (2008), this game has an absorbing and globally accessible state for small frictions.

By relabeling actions so that $a_i^* = \max A_i$ for all $i \in I$, we can make v supermodular. Therefore, we have the following result as a corollary to Theorems 4.1 and 4.2, which generalizes a result for symmetric two-player games by Oyama (2002, Theorem 3).

Corollary 4.1. *(i) A* $\mathbf{p}$*-dominant equilibrium with* $\sum_{i\in I} p_i < 1$ *is globally accessible for any small degree of friction.*

(ii) A strict $\mathbf{p}$*-dominant equilibrium with* $\sum_{i\in I} p_i < 1$ *is linearly absorbing for any degree of friction.*

In particular, a strict $\mathbf{p}$-dominant equilibrium with $\sum_{i\in I} p_i < 1$ is the unique linearly absorbing (and globally accessible) state for any small degree of friction.[18]

Remark 4.1. Hofbauer and Sorger (2002) consider the following concept of $\frac{1}{2}$-dominance and show that for games with linear incentives, it implies linear absorption and global accessibility for small frictions. An action profile $a^* \in A$ is said to be $\frac{1}{2}$-dominant if for all $i \in I$, $\{a_i^*\} = br^i(x_{-i})$ holds for all $x_{-i} \in \prod_{j\neq i} \Delta(A_j)$ such that $x_{j a_j^*} \geq \frac{1}{2}$ for all $j \neq i$. For two-player games, $\frac{1}{2}$-dominance is equivalent to strict $\mathbf{p}$-dominance with $p_i < \frac{1}{2}$ for any $i \in I$, so that Corollary 4.1 covers their result. For games with more than two players, there is no obvious relation. Note the difference between π_i and x_{-i} in the definitions, where π_i is a correlated probability distribution over the opponents' action profiles, while x_{-i} is a profile of probability distributions over each opponent's actions.

4.3.2 *Local potential maximizer* We consider a simplified version of the notion of local potential maximizer introduced by Morris and Ui (2005) as well as its refinement.

Definition 4.4. (i) An action profile $a^* \in A$ is a *local potential maximizer*, or *LP-maximizer*, of G if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$, there exists a function $\mu_i\colon A_i \to \mathbb{R}_+$ such that if $h < a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(h)\big(v(h+1,a_{-i}) - v(h,a_{-i})\big) \leq u_i(h+1,a_{-i}) - u_i(h,a_{-i}),$$

and if $h > a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(h)\big(v(h-1,a_{-i}) - v(h,a_{-i})\big) \leq u_i(h-1,a_{-i}) - u_i(h,a_{-i}).$$

Such a function v is called a *local potential function* for a^*.

(ii) An action profile a^* is a *strict local potential maximizer*, or *strict LP-maximizer*, of G if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$, there exists a function $\mu_i\colon A_i \to \mathbb{R}_{++}$ such that if $h < a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(h)\big(v(h+1,a_{-i}) - v(h,a_{-i})\big) \leq u_i(h+1,a_{-i}) - u_i(h,a_{-i}),$$

[18] Kojima and Takahashi (forthcoming) give an alternative proof for this result, which does not rely on a potential function.

and if $h > a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(h)\big(v(h-1, a_{-i}) - v(h, a_{-i})\big) \leq u_i(h-1, a_{-i}) - u_i(h, a_{-i}).$$

Such a function v is called a *strict local potential function* for a^*.

An LP-maximizer is a strict LP-maximizer if one can take strictly positive numbers for the weights μ_i.[19]

The game G is said to have *diminishing marginal returns* if for all $i \in I$, all $h \neq 0, n_i$, and all $a_{-i} \in A_{-i}$,

$$u_i(h, a_{-i}) - u_i(h-1, a_{-i}) \geq u_i(h+1, a_{-i}) - u_i(h, a_{-i}).$$

In games with diminishing marginal returns, the MP-maximizer conditions reduce to the LP-maximizer conditions.

Lemma 4.2. *If the game G has a (strict) LP-maximizer a^* with a (strict) local potential function v and if G or G_v has diminishing marginal returns, then a^* is a (strict) MP-maximizer with the same function v.*

A proof is given in Section A.4, in the Appendix. We have the following result as a corollary of Theorems 4.1 and 4.2.

Corollary 4.2. *(i) Suppose that the stage game G has an LP-maximizer a^* with a local potential function v. If G or G_v has diminishing marginal returns and if G or G_v is supermodular, then a^* is globally accessible for any small degree of friction.*

(ii) Suppose that the stage game G has a strict LP-maximizer a^* with a strict local potential function v. If G or G_v has diminishing marginal returns and if G or G_v is supermodular, then a^* is linearly absorbing for any degree of friction.

In particular, a strict LP-maximizer is the unique linearly absorbing (and globally accessible) state for any small degree of friction if G or G_v has diminishing marginal returns and G or G_v is supermodular.

4.3.3 *Young's example* Consider the 3×3 game given in Figure 4.1(a), taken from Young (1993). Oyama (2002) shows by direct computation that $(2,2)$ is linearly absorbing and globally accessible for a small degree of friction. In fact, $(2,2)$ is a strict MP-maximizer with a strict monotone potential function that is supermodular (Figure 4.1(b)), while the original game is not supermodular (for any ordering of actions). Therefore, our results, Theorems 4.1 and 4.2, apply also to this game.

Note that $(1,1)$ is stochastically stable (Young 1993), while it is neither absorbing nor globally accessible when the friction is small.

[19] Morris (1999) and Frankel et al. (2003) give a slightly different definition of LP-maximizer, which is weaker than strict LP-maximizer.

	0	1	2
0	6, 6	0, 5	0, 0
1	5, 0	7, 7	5, 5
2	0, 0	5, 5	8, 8

(a) Original game

	0	1	2
0	6	5	0
1	5	7	5
2	0	5	8

(b) Monotone potential function

FIGURE 4.1. Young's example.

4.3.4 *Unanimity game with multiple globally accessible states* Consider the three-player unanimity game: for each player $i \in \{1,2,3\}$, $A_i = \{0,1\}$ and

$$u_i(a) = \begin{cases} y_i & \text{if } a = \mathbf{0} \\ z_i & \text{if } a = \mathbf{1} \\ 0 & \text{otherwise,} \end{cases}$$

where $y_i, z_i > 0$ and $\mathbf{0} = (0,0,0), \mathbf{1} = (1,1,1) \in A$. Now let $y_1 = 6 + c > 0$, $y_2 = y_3 = 1$, and $z_1 = z_2 = z_3 = 2$ (see Figure 4.2). This game is a modified version of an example in Morris and Ui (2005, Subsection 7.2).[20] If $c > 0$, then $\mathbf{0}$ is globally accessible for a small friction, while if $c < 2\sqrt{6}$, then $\mathbf{1}$ is globally accessible for a small friction (see an earlier version of this paper, Oyama et al. 2006, Section 5). Therefore, if $0 < c < 2\sqrt{6}$, the game has two globally accessible states simultaneously when the friction is small. Note that $\mathbf{0}$ (respectively $\mathbf{1}$) has the higher Nash product if $c > 2$ (respectively $c < 2$).

On the other hand, one can show that if $c \leq 0$, then $\mathbf{1}$ is absorbing for any degree of friction, while if $c \geq 2\sqrt{6}$, then $\mathbf{0}$ is absorbing for any degree of friction.

5. CONCLUSION

In this paper, we study perfect foresight dynamics à la Matsui and Matsuyama (1995) for supermodular games and generalizations thereof (games that have a monotone relation, in terms of best responses, with supermodular games), and elucidate the induced monotone structure of the dynamics. We prove, in particular, the stability of a monotone potential maximizer, which is known to be robust to incomplete information (Morris and Ui 2005), thus demonstrating that the prediction obtained by the dynamic stability approach based on perfect foresight dynamics agrees with that obtained by the incomplete information approach in monotone potential games. We also show that for supermodular games, stability in perfect foresight dynamics coincides with stability under the less demanding assumption of rationalizable foresight.

We conclude by noting that, beyond the agreement in the formal results, there is a parallelism at a conceptual level between the two approaches. Perfect foresight dynamics, as well as incomplete information games, fall into the class of *interaction games*

[20] One can verify that $\mathbf{0}$ is not an MP-maximizer for any c, while $\mathbf{1}$ is an MP-maximizer (and hence, robust to incomplete information) if and only if $c < -2$. In the case where $c \geq -2$, nothing seems to be known about the robustness of equilibria.

	0	1
0	$6+c,1,1$	$0,0,0$
1	$0,0,0$	$0,0,0$

0

	0	1
0	$0,0,0$	$0,0,0$
1	$0,0,0$	$2,2,2$

1

FIGURE 4.2. Multiple globally accessible states.

(Morris 1999, Morris and Shin 2006), in which a type or player interacts with various subsets of the set of all types/players, and total payoffs are given by the payoffs from different interactions with different weights. Under perfect foresight dynamics, forward-looking agents make irreversible decisions, so that each agent interacts with agents who will make decisions in the future as well as those who have made decisions in the past. A recent paper by Takahashi (2008) proves that perfect foresight dynamics can formally be understood as a static incomplete information game by identifying the time axis with the type space. With this interpretation, global accessibility and linear absorption in perfect foresight dynamics, roughly, correspond respectively to robustness (Kajii and Morris 1997) and contagion/infection (Morris et al. 1995) in the incomplete information game literature. We leave for future research an investigation of the relationship among these concepts beyond the class of games we study in this paper.

APPENDIX

A.1 *Proof of Proposition 3.3*

Suppose that the stage game is supermodular and that $a^* \in A$ is absorbing (recall from Proposition 3.2 that in supermodular games, any absorbing state is a pure-strategy state). We first show that a perfect foresight path from a^* is unique. Denote by $\bar{\phi}$ the constant path at a^*.

LEMMA A.1. *Suppose that the stage game is supermodular. If a^* is absorbing, then $\bar{\phi}$ is the unique perfect foresight path from a^*.*

PROOF. Suppose that a^* is absorbing. Let ϕ^- and ϕ^+ be the smallest and the largest perfect foresight paths from a^*, respectively (these exist, as demonstrated in Lemma 3.5, due to the supermodularity of the stage game). We show that ϕ^- is nonincreasing in the sense that $\phi^-(s) \precsim \phi^-(t)$ if $t \leq s$; a dual argument shows that ϕ^+ is nondecreasing. Then, ϕ^- and ϕ^+ must be constant at a^*; otherwise, ϕ^- or ϕ^+ would not converge to a^*, contradicting the absorption of a^*.

For each $i \in I$, denote by $\underline{a}_i$ the smallest action among h's such that $\min BR^i(\psi)(t) = h$ for some $t \geq 0$. Note that $\underline{a}_i \leq a_i^*$, since $\phi^- \precsim \bar{\phi}$, and hence $\min BR^i(\phi^-)(t) \leq \min BR^i(\bar{\phi})(t) \leq a_i^*$ for all $t \geq 0$ by Lemma 3.1. Then, define for each $i \in I$ a sequence $T_i^{\underline{a}_i}, \dots, T_i^{a_i^*}$ by

$$T_i^h = \inf\{t \geq 0 \mid \min BR^i(\phi^-)(t) \leq h\}$$

for $h = \underline{a}_i, \dots, a_i^*$. Note that $0 = T_i^{a_i^*} \le T_i^{a_i^*-1} \le \cdots \le T_i^{\underline{a}_i+1} \le T_i^{\underline{a}_i} < \infty$. Now define $\alpha\colon \mathbb{R}_+ \to \prod_i \Delta(A_i)$ by

$$\alpha_{ih}(t) = 1 \quad \text{if } t \in [T_i^h, T_i^{h-1}),$$

where $T_i^{\underline{a}_i-1} = \infty$, and let ϕ be the feasible path given by

$$\dot\phi_i(t) = \alpha_i(t) - \phi_i(t) \quad \text{a.e.,} \qquad \phi_i(0) = a_i^*$$

for all $i \in I$. Observe that ϕ is nonincreasing and that $\phi \precsim \phi^-$.

We show that ϕ is a superpath. Take any $i \in I$, $h \in A_i$, and $t \ge 0$ such that $\dot\phi_{ih}(t) > -\phi_{ih}(t)$. By the definition of ϕ, $t \in [T_i^h, T_i^{h-1})$. Then,

$$h \ge \min BR^i(\phi^-)(T_i^h) \ge \min BR^i(\phi)(T_i^h) \ge \min BR^i(\phi)(t),$$

where the first inequality follows from the definition of T_i^h, the second from the fact that $\phi \precsim \phi^-$, and the third from the fact that ϕ is nonincreasing. This means that ϕ is a superpath. It therefore follows from Lemma 3.3 that there exists a perfect foresight path ψ^* from a^* such that $\psi^* \precsim \phi$.

On the other hand, ϕ^- is the smallest perfect foresight path from a^*. Therefore, we must have $\phi^- \precsim \psi^*$, so that $\psi^* = \phi = \phi^-$. We conclude that ϕ^- is nonincreasing. □

We now show that a^* is linearly absorbing. Note that

$$BR^i(\bar\phi)(t) = \{a_i^*\} \text{for all } i \in I \text{ and all } t \ge 0, \tag{A.1}$$

since a^* is a strict Nash equilibrium by Proposition 3.2.

Proof of Proposition 3.3. Suppose that a^* is absorbing. For $\varepsilon \in [0,1]$, let $x_\varepsilon^- = \varepsilon \min A + (1-\varepsilon)a^*$ and $x_\varepsilon^+ = \varepsilon \max A + (1-\varepsilon)a^*$. In order to show the linear absorption of a^*, it is sufficient to prove that there exists $\bar\varepsilon > 0$ such that the smallest perfect foresight path from $x_{\bar\varepsilon}^-$, ϕ^-, and the largest perfect foresight path from $x_{\bar\varepsilon}^+$, ϕ^+, satisfy $BR^i(\phi^-)(t) = BR^i(\phi^+)(t) = \{a_i^*\}$ for all $i \in I$ and $t \ge 0$. Then, for any perfect foresight path ϕ from $B_\varepsilon(a^*)$, which satisfies $\phi^- \precsim \phi \precsim \phi^+$ by Lemma 3.3, we have $BR^i(\phi)(t) = \{a_i^*\}$ for all $i \in I$ and $t \ge 0$, so that ϕ converges linearly to a^*.

Take any sequence $\{\varepsilon^k\}_{k=0}^\infty$ such that $\varepsilon^0 > \varepsilon^1 > \cdots > 0$ and $\lim_{k\to\infty} \varepsilon^k = 0$, and let $\phi^{k,-}$ and $\phi^{k,+}$ be the smallest perfect foresight path from $x_{\varepsilon^k}^-$ and the largest perfect foresight path from $x_{\varepsilon^k}^+$, respectively. Here, we assume that ε^0 is small enough so that both $\phi^{0,-}$ and $\phi^{0,+}$ converge to a^*. We show only that for some k, $\min BR^i(\phi^{k,-})(t) \ge a_i^*$ for all $i \in I$ and all $t \ge 0$; a dual argument shows that for some k', $\max BR^i(\phi^{k',+})(t) \le a_i^*$ for all $i \in I$ and all $t \ge 0$. Then, setting $\bar\varepsilon = \min\{\varepsilon^k, \varepsilon^{k'}\}$ completes the proof. Note that $\phi^{0,-} \precsim \phi^{1,-} \precsim \cdots \precsim \bar\phi$ and that $\{\phi^{k,-}\}_{k=0}^\infty$ converges, as $k \to \infty$, to some perfect foresight path from a^*, which must be $\bar\phi$ by Lemma A.1.

Seeking a contradiction, suppose that for each k, there exists T^k such that $\min BR^{\tilde\imath}(\phi^{k,-})(T^k) < a_i^*$ for some $\tilde\imath \in I$, where $\tilde\imath$ can be taken independently of k due to the finiteness of I. Since a^* is absorbing (and a strict Nash equilibrium), there exists $\bar T$ such that $\min BR^{\tilde\imath}(\phi^{0,-})(t) = a_{\tilde\imath}^*$ for all $t \ge \bar T$. Since $\phi^{0,-} \precsim \phi^{k,-}$ ($\precsim \bar\phi$), it follows that for all k, $\min BR^{\tilde{\iota}}(\phi^{k,-})(t)=a^*_{\tilde{\iota}}$ for all $t\geq\bar{T}$. Therefore, it must be true that $T^k<\bar{T}$ for all k, so that there exists a convergent subsequence of $\{T^k\}_{k=0}^{\infty}$ with some limit T^*. By the lower semi-continuity of $\min BR^{\tilde{\iota}}$, we have $\min BR^{\tilde{\iota}}(\bar{\phi})(T^*)<a^*_{\tilde{\iota}}$, which contradicts (A.1). □

A.2 *Proof of Theorem 4.1*

Suppose that a^* is an MP-maximizer with a monotone potential function v. Let $A'_i\subset A_i$ denote a set of actions for player i that contains a^*_i. This set is taken to be $A^-_i=\{h\in A_i\mid h\leq a^*_i\}$ or $A^+_i=\{h\in A_i\mid h\geq a^*_i\}$. For the potential game $G'_v=(I,(A'_i)_{i\in I},(v)_{i\in I})$ with the unique potential maximizer $a^*\in A'$, consider the following optimal control problem with a given initial state $z\in\prod_i\Delta(A'_i)$:

$$\text{maximize } J(\phi)=\int_0^{\infty}e^{-\theta t}v(\phi(t))\,dt \quad \text{subject to } \phi\in\Phi'_z, \tag{A.2}$$

where Φ'_z is the set of feasible paths defined on $\prod_i\Delta(A'_i)$ with the initial state z. The state z is taken as $\min A=(0,\dots,0)$ or $\max A=(n_1,\dots,n_N)$.

Lemma A.2. *There exists $\bar{\theta}>0$ such that for any $\theta\in(0,\bar{\theta})$ and any $z\in\prod_i\Delta(A'_i)$, any optimal solution to the optimal control problem (A.2) converges to a^*.*

Proof. Apply Lemma 1 in Hofbauer and Sorger (1999) and Lemmas 4.2 and 4.3 in Hofbauer and Sorger (2002) to the restricted potential game G'_v. □

Lemma A.3. *Let X be a nonempty compact set endowed with a preorder $\precsim$. Suppose that for all $x\in X$, the set $L_x=\{y\in X\mid y\precsim x\}$ is closed. Then X has a minimal element.*

Proof. Take any totally ordered subset of X, and denote it by X'. Since $\{L_x\}_{x\in X'}$ consists of nonempty closed subsets of a compact set and has the finite intersection property, $L^*=\bigcap_{x\in X'}L_x$ is nonempty. Any element $x^*\in L^*$ is a lower bound of X' in X. Therefore, it follows from Zorn's lemma that X has a minimal element. □

Lemma A.4. *For any $z\in\prod_i\Delta(A'_i)$ there exist optimal solutions ϕ^- and ϕ^+ to the optimal control problem (A.2) such that*

$$\dot{\phi}^-_i(t)=\min BR^i_v(\phi^-|A'_i)(t)-\phi^-_i(t),$$
$$\dot{\phi}^+_i(t)=\max BR^i_v(\phi^+|A'_i)(t)-\phi^+_i(t)$$

for all $i\in I$ and almost all $t\geq 0$.

Proof. Fix $z\in\prod_i\Delta(A'_i)$. We show only the existence of ϕ^-; the existence of ϕ^+ is shown similarly. Since the functional J is continuous on Φ'_z, the set of optimizers is a nonempty, closed, and hence compact subset of Φ'_z. Hence a minimal optimal solution (with respect to the order $\phi\precsim\psi$ defined by $\phi(t)\precsim\psi(t)$ for all $t\geq 0$) exists by Lemma A.3. Let ϕ^- be such a minimal solution.

Take any $i \in I$ and consider the feasible path ϕ_i given by $\phi_i(0) = z_i$ and

$$\dot{\phi}_i(t) = \min BR^i_v(\phi^-|A'_i)(t) - \phi_i(t)$$

for almost all $t \geq 0$. Since, by Lemma 2.2, for almost all $t \geq 0$ there exists $\alpha_i(t)$ in the convex hull of $BR^i_v(\phi^-|A'_i)(t)$ such that

$$\dot{\phi}^-_i(t) = \alpha_i(t) - \phi^-_i(t),$$

we have $\phi_i \precsim \phi^-_i$. On the other hand, since ϕ_i is a best response to ϕ^-_{-i} for the game G'_v by construction, we have

$$J(\phi_i, \phi^-_{-i}) \geq J(\phi^-) = \max_{\psi \in \Phi'_z} J(\psi)$$

by Lemma 2.2, meaning that the path (ϕ_i, ϕ^-_{-i}) is also optimal. Hence, the minimality of ϕ^- implies $\phi^-_i(t) = \phi_i(t)$ for all $t \geq 0$. Therefore, we have

$$\dot{\phi}^-_i(t) = \min BR^i_v(\phi^-|A'_i)(t) - \phi^-_i(t)$$

for almost all $t \geq 0$, as claimed. □

Lemma A.5. *There exists $\bar{\theta} > 0$ such that the following holds for all $\theta \in (0, \bar{\theta})$: there exists a feasible path ϕ^- such that*

$$\dot{\phi}^-_i(t) = \min BR^i_v(\phi^-|A^-_i)(t) - \phi^-_i(t) \quad a.e., \qquad \phi^-_i(0) = \min A_i$$

for all $i \in I$ and $\lim_{t\to\infty} \phi^-(t) = a^*$; there exists a feasible path ϕ^+ such that

$$\dot{\phi}^+_i(t) = \max BR^i_v(\phi^+|A^+_i)(t) - \phi^+_i(t) \quad a.e., \qquad \phi^+_i(0) = \max A_i$$

for all $i \in I$ and $\lim_{t\to\infty} \phi^+(t) = a^*$.

Proof. Follows from Lemmas A.2 and A.4. □

Proof of Theorem 4.1. Suppose that v is a monotone potential function for a^*. Take ϕ^- and ϕ^+ as in Lemma A.5. In what follows, we fix a sufficiently small $\theta > 0$ so that both ϕ^- and ϕ^+ converge to a^*.

Now fix any $x \in \prod_i \Delta(A_i)$. Note that $\phi^- \precsim \phi^+$ and $\phi^-(0) \precsim x \precsim \phi^+(0)$. Consider the best response correspondence β_G for the stage game G. Let $\tilde{\Phi}_x = \{\phi \in \Phi_x \mid \phi^- \precsim \phi \precsim \phi^+\}$. We show, as in the proof of Lemma 3.2, that $\tilde{\beta}_G(\phi) = \beta_G(\phi) \cap \tilde{\Phi}_x$ is nonempty for any $\phi \in \tilde{\Phi}_x$. Then, since $\tilde{\Phi}_x$ is convex and compact, it follows from Kakutani's fixed point theorem that there exists a fixed point $\phi^* \in \tilde{\beta}_G(\phi^*) \subset \tilde{\Phi}_x$, which is a perfect foresight path in G and satisfies $\phi^- \precsim \phi^* \precsim \phi^+$. Since both ϕ^- and ϕ^+ converge to a^*, ϕ^* also converges to a^*.

Take any $\phi \in \tilde{\Phi}_x$. Suppose first that the original game G is supermodular. Then, we have

$$\min BR^i_v(\phi^-|A^-_i)(t) \leq \max BR^i_{u_i}(\phi^-|A^-_i)(t) \leq \max BR^i_{u_i}(\phi|A^-_i)(t),$$

where the first inequality follows from the assumption that v is a monotone potential, and the second inequality follows from the supermodularity of u_i and Lemma 3.1. Similarly, we have

$$\max BR^i_v(\phi^+|A_i^+)(t) \geq \min BR^i_{u_i}(\phi^+|A_i^+)(t) \geq \min BR^i_{u_i}(\phi|A_i^+)(t).$$

Suppose next that the potential game G_v is supermodular. Then, we have

$$\min BR^i_v(\phi^-|A_i^-)(t) \leq \min BR^i_v(\phi|A_i^-)(t) \leq \max BR^i_{u_i}(\phi|A_i^-)(t),$$

where the first inequality follows from the supermodularity of v and Lemma 3.1, and the second inequality follows from the assumption that v is a monotone potential. Similarly, we have

$$\max BR^i_v(\phi^+|A_i^+) \geq \max BR^i_v(\phi|A_i^+)(t) \geq \min BR^i_{u_i}(\phi|A_i^+)(t).$$

Therefore, in each case, we have

$$\max BR^i_{u_i}(\phi|A_i^-)(t) \geq \min BR^i_v(\phi^-|A_i^-)(t)$$
$$\min BR^i_{u_i}(\phi|A_i^+)(t) \leq \max BR^i_v(\phi^+|A_i^+)(t)$$

for all $i \in I$ and all $t \geq 0$, so that there exists $h \in BR^i_{u_i}(\phi)(t)$ such that

$$\min BR^i_v(\phi^-|A_i^-)(t) \leq h \leq \max BR^i_v(\phi^+|A_i^+)(t).$$

Define

$$\tilde{F}_i(\phi)(t) = F_i(\phi)(t) \cap \left[\min BR^i_v(\phi^-|A_i^-)(t), \max BR^i_v(\phi^+|A_i^+)(t)\right],$$

where

$$F_i(\phi)(t) = \{\alpha_i \in \Delta(A_i) \mid \alpha_{ih} > 0 \Rightarrow h \in BR^i_{u_i}(\phi)(t)\},$$

and $[\alpha_i, \alpha_i'] = \{\alpha_i'' \in \Delta(A_i) \mid \alpha_i \precsim \alpha_i'' \precsim \alpha_i'\}$ denotes the order interval. Then the differential inclusion

$$\dot{\psi}(t) \in \tilde{F}(\phi)(t) - \psi(t), \qquad \psi(0) = x$$

has a solution ψ as in Remark 2.1. Since $\tilde{F}_i(\phi)(t) \subset F_i(\phi)(t)$, we have $\psi \in \beta_G(\phi)$. By the construction of ϕ^-, ϕ^+, and ψ, we have $\phi^- \precsim \psi \precsim \phi^+$. Thus, we have $\psi \in \tilde{\beta}_G(\phi) = \beta_G(\phi) \cap \tilde{\Phi}_x$, implying the nonemptiness of $\tilde{\beta}_G(\phi)$. □

A.3 *Proof of Theorem 4.2*

Suppose that a^* is a strict MP-maximizer with a strict monotone potential function v. For a nonempty set of actions $A_i' \subset A_i$ that contains a_i^*, consider the potential game $G_v' = (I, (A_i')_{i\in I}, (v)_{i\in I})$.

LEMMA A.6 (Hofbauer and Sorger 2002). *Suppose that G'_v is a potential game with a unique potential maximizer $a^* \in A'$. Then, a^* is absorbing for all $\theta > 0$.*

PROOF OF THEOREM 4.2. Suppose that v is a strict monotone potential function with the strict MP-maximizer a^*, and let $A_i^- = \{h \in A_i \mid h \leq a_i^*\}$ and $A_i^+ = \{h \in A_i \mid h \geq a_i^*\}$. By Lemma A.6, a^* is absorbing in each of the restricted potential games $G_v^- = (I, (A_i^-)_{i\in I}, (v)_{i\in I})$ and $G_v^+ = (I, (A_i^+)_{i\in I}, (v)_{i\in I})$. Let

$$x_\varepsilon^- = \varepsilon \min A + (1-\varepsilon)a^*$$
$$x_\varepsilon^+ = \varepsilon \max A + (1-\varepsilon)a^*$$

for $\varepsilon \in [0,1]$.

Choose a small $\varepsilon > 0$ so that any perfect foresight path for G_v^- from x_ε^- and for G_v^+ from x_ε^+ converges to a^*. Fix any state $x \in \prod_i \Delta(A_i)$ close to a^* satisfying

$$x_\varepsilon^- \precsim x \precsim x_\varepsilon^+,$$

and let ϕ^* be any perfect foresight path from x in the original game G.

In the following, we find perfect foresight paths ϕ^- and ϕ^+ for G_v^- and G_v^+, respectively, such that $\phi^-(0) = x_\varepsilon^-$, $\phi^+(0) = x_\varepsilon^+$, and $\phi^- \precsim \phi^* \precsim \phi^+$. Then, since a^* is absorbing both in G_v^- and in G_v^+, ϕ^- and ϕ^+ converge to a^*, and thus ϕ^* also converges to a^*. In the case where G is supermodular, this implies that a^* is linearly absorbing in G by Proposition 3.3. In the case where G_v is supermodular, a^* is linearly absorbing in G_v^- and in G_v^+, by Proposition 3.3, so that ϕ^- and ϕ^+ linearly converge to a^*, and therefore, ϕ^* also converges linearly to a^*, implying the linear absorption of a^* in G. We show only the existence of ϕ^-; the existence of ϕ^+ is proved similarly.

Let $\tilde{\Phi}_{x_\varepsilon^-} = \{\phi \in \Phi_{x_\varepsilon^-} \mid \phi \precsim \phi^* \text{ and } \phi(t) \in \prod_i \Delta(A_i^-) \text{ for all } t \geq 0\}$. Consider the best response correspondence $\beta_{G_v^-}$ for the stage game G_v^-. We show that $\tilde{\beta}_{G_v^-}(\phi) = \beta_{G_v^-}(\phi) \cap \tilde{\Phi}_{x_\varepsilon^-}$ is nonempty for any $\phi \in \tilde{\Phi}_{x_\varepsilon^-}$. Then, since $\tilde{\Phi}_{x_\varepsilon^-}$ is convex and compact, it follows from Kakutani's fixed point theorem that there exists a fixed point $\phi^- \in \tilde{\beta}_{G_v^-}(\phi^-) \subset \tilde{\Phi}_{x_\varepsilon^-}$, as desired.

Take any $\phi \in \tilde{\Phi}_{x_\varepsilon^-}$. If G is supermodular, then

$$\min BR_v^i(\phi|A_i^-)(t) \leq \min BR_{u_i}^i(\phi|A_i^-)(t) \leq \min BR_{u_i}^i(\phi^*|A_i^-)(t),$$

where the first inequality follows from the assumption that v is a strict monotone potential and the second inequality follows from the supermodularity of u_i and Lemma 3.1.

If G_v is supermodular, then

$$\min BR_v^i(\phi|A_i^-)(t) \leq \min BR_v^i(\phi^*|A_i^-)(t) \leq \min BR_{u_i}^i(\phi^*|A_i^-)(t),$$

where the first inequality follows from the supermodularity of v and Lemma 3.1, and the second inequality follows from the assumption that v is a strict monotone potential.

Therefore, in each case, we have

$$\min BR_v^i(\phi|A_i^-)(t) \leq \min BR_{u_i}^i(\phi^*|A_i^-)(t),$$

so that there exists $h \in BR^i_v(\phi|A^-_i)(t)$ such that

$$h \leq \min BR^i_{u_i}(\phi^*|A^-_i)(t).$$

Then, there exists a best response ψ to ϕ in the game G^-_v such that $\psi(0) = x^-_\varepsilon$ and $\psi \precsim \phi^*$, which can be constructed as in the proof of Proposition 3.1. □

A.4 *Proofs for Section 4.3*

Proof of Lemma 4.1. Let v be given as in the lemma. We show only the conditions (4.1) and (4.3) for A^-_i; (4.2) and (4.4) are proved similarly. Fix any $i \in I$ and $\pi_i \in \Delta(A_{-i})$. If $a^*_i = \min A_i$, then (4.1) and (4.3) are satisfied. Then consider the case of $a^*_i > \min A_i$. Observe that $v(h,\pi_i) = \sum_{a_{-i}\in A_{-i}} \pi_i(a_{-i})v(h,a_{-i})$ is constant for all $h < a^*_i$, so that $\min br^i_v(\pi_i|A^-_i)$ is either $\min A_i$ or a^*_i. It is sufficient to consider the case where $a^*_i = \min br^i_v(\pi_i|A^-_i)$.

Since

$$\begin{aligned} v(a^*_i,\pi_i) - v(\min A_i,\pi_i) &= \pi_i(a^*_{-i})\cdot(1-p_i) + \sum_{a_{-i}\neq a^*_{-i}} \pi_i(a_{-i})\cdot(-p_i) \\ &= \pi_i(a^*_{-i}) - p_i, \end{aligned}$$

it follows from $a^*_i = \min br^i_v(\pi_i|A^-_i)$ that $\pi_i(a^*_{-i}) > p_i$.

Therefore, if a^* is a **p**-dominant equilibrium, then $a^*_i \in br^i_{u_i}(\pi_i|A^-_i)$, i.e., $a^*_i = \max br^i_{u_i}(\pi_i|A^-_i)$; if a^* is a strict **p**-dominant equilibrium, then $\{a^*_i\} = br^i_{u_i}(\pi_i|A^-_i)$, i.e., $a^*_i = \min br^i_{u_i}(\pi_i|A^-_i)$. □

Proof of Lemma 4.2. (i) Suppose that a^* is an LP-maximizer with a local potential function v. We show that if G or G_v has diminishing marginal returns, then a^* is an MP-maximizer with this function v. Fix any $i \in I$ and $\pi_i \in \Delta(A_{-i})$. We show that $\max br^i_v(\pi_i|A^-_i) \leq \max br^i_{u_i}(\pi_i|A^-_i)$. Let $\overline{a}_i = \max br^i_v(\pi_i|A^-_i)$. It is sufficient to consider the case where $\overline{a}_i > \min A_i$.

Since a^* is an LP-maximizer, for all $h < \overline{a}_i$ there exists $\mu_i(h) \geq 0$ such that

$$\mu_i(h)\bigl(v(h+1,a_{-i}) - v(h,a_{-i})\bigr) \leq u_i(h+1,a_{-i}) - u_i(h,a_{-i})$$

for all $a_{-i} \in A_{-i}$, so that we have

$$\mu_i(h)\bigl(v(h+1,\pi_i) - v(h,\pi_i)\bigr) \leq u_i(h+1,\pi_i) - u_i(h,\pi_i)$$

for all $h < \overline{a}_i$. On the other hand, we have

$$v(\overline{a}_i,\pi_i) - v(\overline{a}_i - 1,\pi_i) \geq 0$$

by the definition of $\overline{a}_i$.

Suppose first that G has diminishing marginal returns. Then, we have

$$\begin{aligned} u_i(h+1,\pi_i) - u_i(h,\pi_i) &\geq u_i(\overline{a}_i,\pi_i) - u_i(\overline{a}_i - 1,\pi_i) \\ &\geq \mu_i(\overline{a}_i - 1)\bigl(v(\overline{a}_i,\pi_i) - v(\overline{a}_i - 1,\pi_i)\bigr) \\ &\geq 0 \end{aligned}$$

for any $h < \overline{a}_i$. Hence, we have

$$u_i(\overline{a}_i, \pi_i) - u_i(h, \pi_i) \geq 0$$

for all $h < \overline{a}_i$, which implies that $\overline{a}_i \leq \max br^i_{u_i}(\pi_i | A_i^-)$.

Suppose next that G_v has diminishing marginal returns. Then, we have

$$\begin{aligned} u_i(h+1, \pi_i) - u_i(h, \pi_i) &\geq \mu_i(h)\big(v(h+1, \pi_i) - v(h, \pi_i)\big) \\ &\geq \mu_i(h)\big(v(\overline{a}_i, \pi_i) - v(\overline{a}_i - 1, \pi_i)\big) \\ &\geq 0 \end{aligned}$$

for any $h < \overline{a}_i$. Hence, we have

$$u_i(\overline{a}_i, \pi_i) - u_i(h, \pi_i) \geq 0$$

for all $h < \overline{a}_i$, which implies that $\overline{a}_i \leq \max br^i_{u_i}(\pi_i | A_i^-)$.

(ii) Suppose that a^* is a strict LP-maximizer with a strict local potential function v. We show that if G or G_v has diminishing marginal returns, then a^* is a strict MP-maximizer with the same function v. Fix any $i \in I$ and $\pi_i \in \Delta(A_{-i})$. We show that $\min br^i_v(\pi_i | A_i^-) \leq \min br_{u_i}(\pi_i | A_i^-)$. Let $\underline{a}_i = \min br^i_v(\pi_i | A_i^-)$. It is sufficient to consider the case where $\underline{a}_i > \min A_i$.

Since a^* is a strict LP-maximizer, for all $h < \underline{a}_i$ there exists $\mu_i(h) > 0$ such that

$$\mu_i(h)\big(v(h+1, a_{-i}) - v(h, a_{-i})\big) \leq u_i(h+1, a_{-i}) - u_i(h, a_{-i})$$

for all $a_{-i} \in A_{-i}$, so that we have

$$\mu_i(h)\big(v(h+1, \pi_i) - v(h, \pi_i)\big) \leq u_i(h+1, \pi_i) - u_i(h, \pi_i)$$

for all $h < \underline{a}_i$. On the other hand, we have

$$v(\underline{a}_i, \pi_i) - v(\underline{a}_i - 1, \pi_i) > 0$$

by the definition of $\underline{a}_i$.

Suppose first that G has diminishing marginal returns. Then we have

$$\begin{aligned} u_i(h+1, \pi_i) - u_i(h, \pi_i) &\geq u_i(\underline{a}_i, \pi_i) - u_i(\underline{a}_i - 1, \pi_i) \\ &\geq \mu_i(\underline{a}_i - 1)\big(v(\underline{a}_i, \pi_i) - v(\underline{a}_i - 1, \pi_i)\big) \\ &> 0 \end{aligned}$$

for any $h < \underline{a}_i$. Hence we have

$$u_i(\underline{a}_i, \pi_i) - u_i(h, \pi_i) > 0$$

for all $h < \underline{a}_i$, which implies that $\underline{a}_i \leq \min br^i_{u_i}(\pi_i | A_i^-)$.

Suppose next that G_v has diminishing marginal returns. Then we have

$$\begin{aligned} u_i(h+1,\pi_i) - u_i(h,\pi_i) &\geq \mu_i(h)\big(v(h+1,\pi_i) - v(h,\pi_i)\big) \\ &\geq \mu_i(h)\big(v(\underline{a}_i,\pi_i) - v(\underline{a}_i - 1,\pi_i)\big) \\ &> 0 \end{aligned}$$

for any $h < \underline{a}_i$. Hence we have

$$u_i(\underline{a}_i,\pi_i) - u_i(h,\pi_i) > 0$$

for all $h < \underline{a}_i$, which implies that $\underline{a}_i \leq \min br^i_{u_i}(\pi_i|A_i^-)$. □

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