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Equilibrium concepts in the large-household model

Tao Zhu
Department of Economics, Cornell University, and Department of Economics, University of Glasgow

This paper formulates equilibrium concepts in the large (nonatomic) household model under the team notion, characterizes a class of equilibrium allocations, explores whether an equilibrium allocation in the large-household model is a limit of equilibrium allocations in the finite-household model, and establishes existence of equilibrium allocations generated by generalized Nash bargaining.

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1. Introduction

Search models play a dominant role in labor economics and a prominent role in monetary economics. In such models, meeting-specific shocks are obvious sources of heterogeneity in wealth. Because such heterogeneity precludes closed-form solutions, efforts have been made to create models in which equilibria have degenerate distributions of wealth. One such model is the so-called *large-household* model initiated by Merz (1995) in labor economics and by Shi (1997) in monetary economics. In this model, each household consists of a nonatomic measure of agents and each agent from a household meets someone from outside the household—a firm in Merz (1995) and an agent from another household in Shi (1997). If all households start with the same wealth, then it is feasible for all households to experience the same distribution of meeting outcomes, and, by a law of large numbers argument, to end up with the same wealth. But *feasibility* should not be confused with *equilibrium*.

There are two conceivable interpretations of the multi-member household in this literature. A household can be viewed as a team in the sense of Marschak and Radner (1972): each member is a decision maker, but all share the same objective function. Alternatively, a household can be viewed as a programmer and the members as automata: the programmer, the unique decision maker, chooses what the members do in pairwise meetings. With the team notion, we can maintain, as in the standard single-agent household, subgame perfection of the pairwise split-the-surplus game. Indeed, we can adopt an off-the-shelf equilibrium concept to provide a unified treatment—one that

Tao Zhu: tz34@cornell.edu
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applies independent of the size of the household and one that makes the single-agent household a special case. In contrast, with the automata notion, Nash equilibrium is the only solution concept for the split-the-surplus game.[1] The result is a huge multiplicity of equilibria. (Think about two players in an ultimatum game played through automata.)[2]

Without doubt, the team notion is superior. Although not made explicit, it seems to be used in the initial papers (Merz 1995, Shi 1997). Those papers, however, do not correctly define an equilibrium. In particular, they do not correctly describe meeting outcomes obtained by agents of an off-equilibrium household, one whose wealth is not average wealth. In Merz (1995) and Shi (1997), each agent of such a household is assumed to obtain the same outcome as an agent of an in-equilibrium household. But in a money model this is obviously problematic because an off-equilibrium buyer may be too poor to afford the assumed outcome. Rauch (2000), in a lengthy comment on Shi (1997), pointed out the relevance of the household's wealth, but his suggested correction is incomplete. He misses the following loop. The end-of-meeting wealth of the household depends on the distribution of meeting outcomes obtained by its agents. For each agent of the household, the meeting outcome depends on the agent's evaluation of each feasible outcome in the meeting. That evaluation, in turn, depends on the household's end-of-meeting wealth, which the agent takes as given (because the household is large). This loop constitutes a mapping: the household's pre-meeting wealth is a parameter of the mapping and the end-of-meeting wealth is a fixed point (if one exists).

In this paper, I focus on the team notion in the context of a money model. In Section 2, I set up the model so that feasibility of degeneracy holds for any size household with the same number (or measure) of buyers and sellers. In particular, search is directed—a buyer always meets a seller. This setup permits me to treat *large* as a limit of *finite*. Then, in Section 3, I define equilibrium and give a refinement: no defection by the household as a whole, which I call team optimality. In Section 4, I characterize a class of equilibrium allocations. In Section 5, I use an ϵ-equilibrium approach to explore the sense in which an equilibrium allocation in the large-household setting is a limit of equilibrium allocations in the finite-household setting. In Section 6, I prove existence of equilibrium allocations generated by generalized Nash bargaining. A more detailed discussion of the problems in the literature is given in Section 7.

The most important existence result is for equilibrium in the large-household model under the team optimality refinement (Proposition 4). I do not develop a general fixed-point proof, which, because of the loop described above, would involve two layers of fixed points. My proof is constructive. The basic idea is to let any off-equilibrium buyer/seller spend/acquire as much as possible, so any off-equilibrium household after search has average wealth. If the derivative of the household's value function at the average wealth is defined, then all agents in present meetings, in and off equilibrium, have

[1]See Abreu and Rubinstein (1988, p. 1265) for a discussion of the solution concept in the context of repeated games.

[2]The origin of the household construct can be traced back to Lucas (1980, 1990). The setup in Lucas (1990) is exactly designed to obtain the representative household. Lucas does not give an explicit interpretation of the household. In the Walrasian models he studies, the above two notions turn out to give the same results.

the same linear payoff function of money with coefficient equal to the discounted value of that derivative. As a result, I can solve for that derivative from an equation that does not depend on the value function. Then I can use the derivative to construct the value function. (In this construction, the linear function of money implied by *large* plays the exact role of the linear function of money implied by quasi-linearity in the single-agent household model of Lagos and Wright 2005.) The in-equilibrium trade of this equilibrium is the one that appears in some of the literature. In that sense, my formulation justifies that part of the literature.

More recent papers (e.g., Shi 1999, Head and Shi 2003, and Wang and Shi 2006) seem to use the automata notion. With this notion, there are many equilibria (see Appendix B), a problem those papers fail to recognize. A basic message to users of the large-household model is that a choice has to be made. With the automata notion, existence is easy but multiplicity is a problem. With the team notion, existence may be challenging. My constructive proof does not seem general enough to apply to all existing versions of the large-household model.

2. The physical environment

Time is discrete. There is a nonatomic measure of infinitely lived households. All households are ex ante identical. Each household is identified with a probability space $(I, \mathscr{I}, \mu)$: there are a set of buyers and a set of sellers in the household, both indexed by I. The set I is either finite with n elements or uncountably infinite. If the former, then μ is uniform over I and the household is referred to as a *finite household*; if the latter, then μ is nonatomic and the household is referred to as a *large household*.[3]

There is one produced and perishable good per date. At each date, agents from different households are matched in pairs; matching is random, but a buyer always meets a seller.[4] In each meeting, the buyer can consume but not produce the good; the seller can produce but not consume the good; and the good produced must be consumed by the end of the meeting.[5] Agents from the same household share the same objective. The period return is

$$\int_{i\in I} u(q_b^i)\mu(di) - \int_{i\in I} c(q_s^i)\mu(di),$$

where q_b^i is the consumption of buyer i from the household, and q_s^i is the production of seller i from the household. Each agent as an independent decision-maker (or player) maximizes expected discounted utility with discount factor $\beta \in (0,1)$. As is standard, u

[3] A household with n buyers and n sellers can be viewed as the n-fold replica of a household with 1 buyer and 1 seller; a large household can be viewed as the limit of a finite household as $n \to \infty$. This is analogous to treating a large economy as the limit of a sequence of replica economies (see Hildenbrand 1974, Ch. 2).

[4] Search is not directed in Rauch (2000) and Shi (1997) so the household must be large to make degeneracy feasible.

[5] Inspired by Kiyotaki and Wright (1989), Rauch (2000) and Shi (1997) consider multiple types of goods and households. In this setting, one can further assume that buyers pool goods together after search. It is easy to adapt the formulations and results below to these variants.

is bounded, $u' > 0$, $u'' < 0$, $u(0) = 0$, and $u'(0) = \infty$; it is without loss of generality to set $c(q) = q$.

There is another durable, divisible, and intrinsically useless object called money. The stock of money is constant. Each household starts at date 0 with one unit of money. There is an upper bound $M > 1$ on the household's money holdings. Some results (Propositions 3 and 4) assume a finite M. When that is the case, M is nonbinding in the sense given below.

In each pairwise meeting, agents may exchanges the good for money. The trading outcome is described below. Agents from a household are anonymous to agents from other households, so each agent's trading history is unknown to agents of other households (but known to agents of the same household). Agents from the same household cannot communicate in pairwise meetings.

Two more assumptions about the physical environment are made. First, within each household, any money held is evenly redistributed among its buyers at the start of each date. (Note that sellers hold zero.) This assumption is explicitly made in Rauch (2000) and Shi (1997). Here it permits me to simplify the individual agent's state space (see footnote 6). Second, in each meeting, one agent's money holding and his household's pre-meeting money holding are common knowledge to the relevant pair. This assumption is consistent with the treatment in Rauch (2000) and Shi (1997). Here it permits me to avoid dealing with asymmetric information.

3. Allocation, trading mechanism, and equilibrium

My first goal is to examine the conditions under which an allocation can be an equilibrium allocation under some trading mechanism. An allocation prescribes a trade for each pairwise meeting conditional on certain factors. A *trading mechanism* specifies for each pairwise meeting sets of actions for agents and a mapping from actions to trading outcomes. The matching process and the trading mechanism imply a game so one can define equilibrium. An allocation is an *equilibrium allocation* if its prescribed trades coincide with the trades implied by some equilibrium strategy profile.

I focus on *stationary allocations*, which prescribe the trade of a meeting conditional only on the states of the pair in the meeting. An agent is said to be in state m if his household's pre-meeting money holding is m. A generic pairwise trade is denoted by (q, l); q is the transfer of the good (from a seller to a buyer), $l n^{-1}$ is the transfer of money (from a buyer to a seller) when the household is finite, and l is the transfer of money when the household is large. Therefore, an allocation A is a pair of real-valued functions $(q(.), l(.))$ on $[0, M]^2$, where $(q(m_b, m_s), l(m_b, m_s))$ is the prescribed trade between a buyer in state m_b and a seller in state m_s. (It is important to remember the following convention: q and l are scalars; $q(.)$ and $l(.)$ are functions over $[0, M]^2$; and $q(m_b, m_s)$ and $l(m_b, m_s)$ are evaluations of $q(.)$ and $l(.)$ at (m_b, m_s).)

While allocations need not be those generated by surplus-splitting rules standard in the literature (Nash bargaining, the ultimatum game, price taking, etc.), they satisfy two properties that allocations generated by those rules satisfy. Specifically, if an allocation A is an equilibrium allocation under some trading mechanism, then its prescribed pairwise trade, in equilibrium or off equilibrium, satisfies *sequential individual rationality* (SIR)—the trade weakly dominates autarky for each agent, and *pairwise efficiency* (PE)—the trade is on the pairwise Pareto frontier.

Given $A = (q(.), l(.))$, I study a two-stage trading mechanism, denoted T^A. When a buyer in state $m_b$meets a seller in state m_s, T^A describes the following two stages of actions.

Stage 1 First the buyer announces a number in $\{0,1\}$ and then the seller announces a number in $\{0,1\}$. If both announce 1, then they move to stage 2; otherwise, the trade is $(0,0)$ and the meeting is over. (This stage is for SIR.)

Stage 2 First the buyer proposes a trade (q,l) and then the seller announces a number in $\{0,1\}$. If 1 is announced, then (q,l) is carried out and the meeting is over; otherwise $(q(m_b,m_s), l(m_b,m_s))$, the trade prescribed by A, is carried out and the meeting is over. (This stage is for PE.)

Throughout, I assume that each agent in each stage, when indifferent between saying 1 and 0, says 1.

In the implied game, I restrict attention to equilibria such that each agent does not condition his actions on calendar time or his private information about past meetings that he and the members of the same household experience. Denote by $f_b = (f_{b1}, f_{b2})$ a buyer's strategy and by $f_s = (f_{s1}, f_{s2})$ a seller's strategy in such an equilibrium, where the subscript 1 denotes a stage-1 action and the subscript 2 a stage-2 action. Then effectively f_b and f_s can be expressed as the following mappings:

$$f_{b1} : [0,M]^2 \to \{0,1\},\ f_{b2} : [0,M]^2 \to \mathbb{R}^2_+ \tag{1}$$

$$f_{s1} : [0,M]^2 \times \{0,1\} \to \{0,1\},\ f_{s2} : [0,M]^2 \times \mathbb{R}^2_+ \to \{0,1\}. \tag{2}$$

(When the buyer in state m_b meets the seller in state m_s, $f_{b1}(m_b,m_s)$ is the buyer's stage-1 announcement; $f_{s1}(m_b,m_s,n)$ is the seller's stage-1 announcement following the buyer's announcement n; $f_{b2}(m_b,m_s)$ is the buyer's stage-2 proposal; and $f_{s2}(m_b,m_s,q,l)$ is the seller's stage-2 announcement following the proposal (q,l).[6]) Moreover, I restrict attention to symmetric equilibria (i.e., ones in which all buyers/sellers from all households choose the same f_b/f_s).

Definition 1. Given the trading mechanism T^A, an *equilibrium* is a strategy profile represented by $f = (f_b, f_s)$ (see (1) and (2)) such that it is optimal for one agent to follow the actions indicated by f currently and in the future, provided that all other agents, including those from the same household, follow the actions indicated by f.

Remark 1. By design, if A is an equilibrium allocation under T^A then it satisfies SIR and PE. There is a class of mechanisms with such a property. An arbitrary mechanism in

[6]In the absence of the special money-redistribution assumption, the individual agent's state is a probability distribution describing how money is distributed among the household members, and so the domains of A and f pertain to a set of such distributions, instead of the interval $[0,M]$.

the class, denoted T, consists of three stages in each meeting. Stage 0 of T may consist of many substages, and each sequence of actions in stage 0 leads to a proposed trade (q_0, l_0). Stage 1 of T is exactly stage 1 of T^A. In stage 2 of T, with pre-determined probability $p \in \{0,1\}$ the buyer proposes a new trade and the seller announces a number in $\{0,1\}$; with probability $1-p$ the seller proposes a trade and the buyer makes an announcement. If the newly-proposed trade is followed by 1, then it is carried out; otherwise (q_0, l_0), the trade proposed in stage 0, is carried out. (An ultimatum game is equivalent to such a T in which one agent's action dictates the stage-0 proposed trade. Also, T^A is equivalent to such a T—by simply adding stage 0 in the obvious way.) We can make the same tie-breaking assumption, and define equilibrium as above. Then by design, if A is an equilibrium allocation under T, it satisfies SIR and PE. Also, it is not hard to verify that if A is an equilibrium allocation under T then it is an equilibrium allocation under T^A. This justifies the focus on T^A.[7]

Given the initial distribution of money, symmetry implies that any Definition-1 equilibrium is *degenerate* in that all in-equilibrium households hold one unit of money at the start of each date. From now on, I refer to a household with one unit of money as a *regular* household, an agent from a regular household as a regular agent, and a meeting between two regular agents as a regular meeting.

As is standard, the notion of equilibrium ensures individual optimality (i.e., no beneficial unilateral deviation), but need not ensure team optimality (i.e., no beneficial joint deviation by the household as a whole). When members of the same household can agree on a beneficial joint deviation, there is no problem of implementation. While those members do not communicate in search, if they communicate before search, they can make an agreement in advance. So I introduce a refinement to rule out such deviations.[8] Because equilibrium is degenerate, I restrict joint deviations to those that occur when meeting regular agents.

Definition 2. A Definition-1 equilibrium f is *strong* if agents from the same household cannot improve by any joint deviation when meeting regular agents.

Remark 2. As is shown below (Propositions 5 and 6 (i)), in some equilibria there exist such beneficial joint deviations. If one has in mind that agents from the same household communicate before search, then strong equilibrium is the suitable equilibrium notion. But if one has the opposite in mind, then non-strong equilibrium should not be thrown out, and one may even view existence of non-strong equilibrium as an interesting implication of the team model.

[7]This result resembles a result in Kocherlakota (1998), who studies a class of trading mechanisms that ensure SIR. He shows that an equilibrium allocation under such a trading mechanism is an equilibrium allocation under the direct trading mechanism associated with the allocation.

[8]The relationship between equilibrium and this refinement is analogous to the relationship between a person-by-person satisfactory team decision function and the best team decision function in Marschak and Radner (1972).

Given an allocation $A=(q(.),l(.))$, by a trivial application of Blackwell's sufficient conditions, there is a unique bounded function on $[0,M]$ satisfying

$$v(m)=u(q(m,1))-q(1,m)+\beta v(g(m)) \quad \text{with } g(m)=m-l(m,1)+l(1,m). \tag{3}$$

This function is called *the value function* associated with the allocation A because when A is an equilibrium allocation, v is the value function on the household's money holdings. (In that equilibrium, if the household starts with m, then with probability one, each of its buyers consumes $q(m,1)$, each of its sellers produces $q(1,m)$, and the household ends up with $g(m)$.)

In what follows, I restrict attention to $A=(q(.),l(.))$ satisfying (C1) $q(1,1)>0$ and (C2) the associated value function v is *nondecreasing*, *continuous*, and *concave*. If A is an equilibrium allocation, then by (C1) the equilibrium is *monetary*. Monotonicity of v is equivalent to free disposal of money. In the large-household setting, I identify the payoff to an individual agent from his own action in any pairwise meeting as the marginal contribution to the agent's objective function. (Because the measure of the agent is infinitesimal, so is the payoff. This treatment resembles the way that Aumann and Shapley (1974) define the Shapley value of a nonatomic agent.) Given the concavity and continuity of v, I can express the marginal contribution in terms of the left and right derivatives of v (cf. Rockafellar 1970, p. 213, Theorem 23.1). As is usual in dynamic models, these properties of A cannot be ensured in equilibrium by assumptions on the primitives. In Section 4, I study conditions for an A with such properties to be an equilibrium allocation; in Section 6, I show existence of equilibrium allocations with such properties.

4. Characterization of equilibrium allocations

Fix an allocation $A=(q(.),l(.))$ satisfying (C1)–(C2) and suppose that it is an equilibrium allocation under the trading mechanism T^A. In order to describe the conditions that ensure no deviation from trades prescribed by A, I first describe how an agent in arbitrary state m evaluates a trade in a meeting, taking as given that all other agents from the same household obtain the trades prescribed by A currently and that v in (3) is the value function defined on the household's money holdings.

When the household is finite and the agent in state m is a buyer, with probability one each of the n sellers from his household trades $(q(1,m),l(1,m))$, and each of the other $n-1$ buyers from his household trades $(q(m,1),l(m,1))$. Therefore, if the agent trades $(0,0)$, then the value of his objective function is $u(q(m,1))(1-n^{-1})-q(1,m)+\beta v(g(m)+l(m,1)n^{-1})$ (the first term is the contribution to the objective function from the other buyers' consumption, the second term is the contribution from the sellers' production, and the last term is the contribution from the household's end-of-meeting money holding). It follows that the additional contribution to his objective function from trading (q,l) is

$$B_0(q,l,m)=u(q)n^{-1}-\beta[v(g(m)+l(m,1)n^{-1})-v(g(m)+l(m,1)n^{-1}-l\,n^{-1})], \tag{4}$$

and this contribution per unit of the buyer's measure in the set I (which is n^{-1}) is

$$B(q,l,m)=u(q)-n\beta[v(g(m)+l(m,1)n^{-1})-v(g(m)+l(m,1)n^{-1}-l\,n^{-1})]. \tag{5}$$

The analogous expression applies when the agent in state m is a seller. The household's end-of-meeting money holding is $g(m)-l(1,m)n^{-1}$ if the seller trades $(0,0)$, so the additional contribution to his objective function from (q,l) is

$$S_0(q,l,m)=-qn^{-1}+\beta[v(g(m)-l(1,m)n^{-1}+l\,n^{-1})-v(g(m)-l(1,m)n^{-1})], \tag{6}$$

and this contribution per unit of his measure in I is

$$S(q,l,m)=-q+n\beta[v(g(m)-l(1,m)n^{-1}+l\,n^{-1})-v(g(m)-l(1,m)n^{-1})]. \tag{7}$$

When the household is large, as indicated above I define the payoff from trading (q,l) as the marginal contribution to the agent's objective function. As $n\to\infty$, the limit of the additional contribution $B(q,l,m)$ in (5) (respectively $S(q,l,m)$ in (7)) defines the marginal contribution when the agent in state m is a buyer (respectively seller). The limit of $B(q,l,m)$ in (5), still denoted $B(q,l,m)$, is

$$B(q,l,m)=u(q)-\beta w_b(l,m) \tag{8}$$

with

$$w_b(l,m)=\min\{l,l(m,1)\}v'_+(g(m))+\max\{0,l-l(m,1)\}v'_-(g(m)),$$

and the limit of $S(q,l,m)$ in (7), still denoted $S(q,l,m)$, is

$$S(q,l,m)=-q+\beta w_s(l,m) \tag{9}$$

with

$$w_s(l,m)=\min\{l,l(1,m)\}v'_-(g(m))+\max\{l-l(1,m),0\}v'_+(g(m)),$$

where $v'_-(z)$ is the left derivative and $v'_+(z)$ is the right derivative of v at z.[9]

I have assumed that when M is finite it is *nonbinding*. Precisely, this means that for any feasible transfer of money from a buyer in state $m_b\in[0,M]$ to a seller in state $m_s\in[0,M]$, given all other agents follow the trades prescribed by A, the buyer's and seller's households end the date with holdings less than M. That is, when the household is finite, $g(m_b)+l(m_b,1)n^{-1}<M$ and $g(m_s)-l(1,m_s)n^{-1}+m_bn^{-1}<M$;[10] when the household is large, $g(m_b)<M$ and $g(m_s)<M$. With this assumption, I can apply the payoffs in (4)–(9) with no restriction.

[9]Note that if $v'_-(g(m))=v'_+(g(m))$, then we have $w_b(l,m)=w_s(l,m)=l\,v'(g(m))$. To see (8), write $v(g(m)+l(m,1)n^{-1})-v(g(m)+l(m,1)n^{-1}-l\,n^{-1})$ as $v(g(m)+l(m,1)n^{-1})-v(g(m))+v(g(m))-v(g(m)+l(m,1)n^{-1}-l\,n^{-1})$. A similar treatment leads to (9).

[10]If the buyer in state m_b transfers all his holding m_bn^{-1} to the seller in state m_s, then the seller's household ends up with $g(m_s)-l(1,m_s)n^{-1}+m_bn^{-1}$. If the buyer transfers zero, then the buyer's household ends up with $g(m_b)+l(m_b,1)n^{-1}$.

Now consider the meeting between the buyer in state m_b and the seller in state m_s. For A to be an equilibrium allocation under T^A, SIR must hold; that is,

$$B(q(m_b,m_s),l(m_b,m_s),m_b)\geq 0 \tag{10}$$

$$S(q(m_b,m_s),l(m_b,m_s),m_s)\geq 0 \tag{11}$$

(otherwise at least one agent says 0 in stage 1). Also, PE must hold; that is,

$$(q(m_b,m_s),l(m_b,m_s))\in \arg\max_{(q,l)} B(q,l,m_b) \tag{12}$$
$$\text{subject to } 0\leq l\leq m_b \text{ and } S(q,l,m_s)\geq S(q(m_b,m_s),l(m_b,m_s),m_s)$$

(otherwise the buyer can offer a trade in stage 2 leading to pairwise improvement). Let the strategy profile $f^A=(f_b^A,f_s^A)$ be given by

$$f_{b1}^A(m_b,m_s)=1,\ f_{b2}^A(m_b,m_s)=(q(m_b,m_s),l(m_b,m_s)), \tag{13}$$

$$f_{s1}^A(m_b,m_s,n)=1\ \forall n\in\{0,1\}, \tag{14}$$

$$f_{s2}^A(m_b,m_s,q,l)=1\Leftrightarrow[0\leq l\leq m_b \text{ and } S(q,l,m_s)\geq S(q(m_b,m_s),l(m_b,m_s),m_s)]. \tag{15}$$

If A satisfies (10)–(12) (i.e., (10)–(12) hold for all (m_b,m_s)), then by the one-stage-deviation principle, f^A is a Definition-1 equilibrium.

In the equilibrium f^A, for a buyer in state m the set of admissible trades when meeting a regular seller is

$$\bar{\Gamma}_b(m)=\{(0,0)\}\cup\{(q,l):0\leq l\leq m, S(q,l,1)\geq S(q(m,1),l(m,1),1)\}. \tag{16}$$

(This buyer obtains either no trade (by saying 0 at stage 1), or any trade giving the seller a payoff no less than the trade prescribed by A.) Also, in the equilibrium f^A, for a seller in state m the set of admissible trades when meeting a regular buyer is

$$\underline{\Gamma}_s(m)=\{(0,0)\}\cup\{(q(1,m),l(1,m))\}. \tag{17}$$

(This seller obtains either no trade or the trade prescribed by A.) Letting $[K]^I$ denote the product set $\prod_{i\in I}K_i$ with $K_i=K$ for all i, it follows that $[\bar{\Gamma}_b(m)\times\underline{\Gamma}_s(m)]^I$ is the set of admissible joint trades for agents from the household with m. For each μ-measurable $\gamma(m)=\{(q_b^i,l_b^i),(q_s^i,l_s^i)\}_{i\in I}$ with $q_b^i,l_b^i,q_s^i,l_s^i\geq 0$, $l_b^i\leq m$, and $l_s^i\leq 1$, define

$$W(\gamma(m))=\int_{i\in I}u(q_b^i)\mu(di)-\int_{i\in I}q_s^i\mu(di)+\beta v\left(m-\int_{i\in I}l_b^i\mu(di)+\int_{i\in I}l_s^i\mu(di)\right). \tag{18}$$

It is clear that f^A is strong if and only if for all m,

$$v(m)=\max W(\gamma(m)) \text{ subject to } \gamma(m)\in[\bar{\Gamma}_b(m)\times\underline{\Gamma}_s(m)]^I. \tag{19}$$

To summarize, we have the following result.

Proposition 1. *Let A be an allocation satisfying (C1) and (C2). When the household is finite, let the additional payoff functions B and S be as given in (5) and (7); when the household is large, let the marginal payoff functions B and S be as given in (8) and (9).*

(i) f^A (see (13)–(15)) is a Definition-1 equilibrium if and only if A satisfies (10)–(12).

(ii) f^A is a Definition-2 strong equilibrium if and only if A satisfies (10)–(12) and (19).

Now consider T, an arbitrary mechanism as described in Remark 1. Suppose that A is an equilibrium allocation under T. Then it satisfies (10)–(12). We can define strong equilibrium as in Definition 2, so that the equilibrium is strong if and only if for all m, $v(m)$ is the maximum of $W(.)$ (see (18)) over $\Gamma(m)$, the set of admissible joint trades for agents from the household with m when meeting regular agents. Letting

$$\underline{\Gamma}_b(m) = \{(0,0)\} \cup \{(q(m,1), l(m,1)\} \tag{20}$$

$$\bar{\Gamma}_s(m) = \{(0,0)\} \cup \{(q,l) : 0 \le l \le 1, B(q,l,1) \ge B(q(1,m), l(1,m), 1)\}, \tag{21}$$

we can set $\Gamma(m) = [\bar{\Gamma}_b(m) \times \underline{\Gamma}_s(m)]^I$ if $p = 1$ in stage 2 and $\Gamma(m) = [\underline{\Gamma}_b(m) \times \bar{\Gamma}_s(m)]^I$ if $p = 0$. (Here we may throw out some obviously dominated trades. In (20), a buyer in state m obtains either no trade or the trade prescribed by A. In (21), a seller in state m obtains either no trade or any trade giving a regular buyer a payoff no less than the trade prescribed by A.) Although in general p matters, the next proposition gives conditions independent of p to assure the equilibrium is strong.

Proposition 2. *Let A be an allocation satisfying (C1) and (C2), and let the associated value function be strictly increasing and differentiable. Let the household be large. If A satisfies (10)–(12), then for all m, $v(m) = \max W(\gamma(m))$ subject to $\gamma(m) \in [\bar{\Gamma}_b(m) \times \bar{\Gamma}_s(m)]^I$.*

By this result, if the value function has the asserted properties then agents from the large household with arbitrary m do not jointly deviate (in the equilibrium supporting A under T) even if they can choose joint trades from $[\bar{\Gamma}_b(m) \times \bar{\Gamma}_s(m)]^I$, so the equilibrium is strong.

All proofs are in Appendix A. I sketch the proof of Proposition 2 to illustrate the roles of the properties of the value function and the largeness of the household. I first establish (i) $B(q,l,m) < B(q(m,1), l(m,1), m)$ for $(q,l) \in \bar{\Gamma}_b(m) \setminus \underline{\Gamma}_b(m)$, and (ii) $S(q,l,m) < S(q(1,m), l(1,m), m)$ for $(q,l) \in \bar{\Gamma}_s(m) \setminus \underline{\Gamma}_s(m)$. Here I use the strict monotonicity of v. Next, I show that $g(m)$ is the household's end-of-meeting holding implied by some optimal $\gamma(m)$. With this result, I rule out all possible compositions of $\gamma(m)$ that may lead to $W(\gamma(m)) \neq v(m)$. A trick here is to combine the inequalities in (10), (11), (i), and (ii) and cancel out the v terms. This trick uses the differentiability of v and the largeness of the household, which imply $v'(g(m))$ is the only v term in those inequalities.

5. An approximation result

In this section, I take an ϵ-equilibrium approach to explore whether an equilibrium allocation A in the large-household setting is a limit of equilibrium allocations in the finite-household setting.[11] Given the trading mechanism T^A, I consider two notions of ϵ-equilibrium when the household is finite.

Definition 3. An *ϵ-equilibrium* is a strategy profile represented by f (see (1) and (2)) such that for one agent, the (expected lifetime) payoff from any sequence of his own actions when meeting regular agents does not exceed by ϵ the payoff from the sequence of actions indicated by f, provided that all other agents, including those from the same household, follow the actions indicated by f.[12]

Definition 4. A *strengthened ϵ-equilibrium* is a strategy profile represented by f such that for agents from the same household, the (expected lifetime) payoff from any sequence of their joint actions when meeting regular agents does not exceed by ϵ the payoff from the sequence of joint actions indicated by f, provided that agents outside the household follow the actions indicated by f.

In the rest of this section, I adopt the following notation. (a) The individual agent's payoff functions B in (5) and S in (7) are denoted B_n and S_n, respectively. (b) The strategy profile f^A in (13)–(15) with $S = S_n$ is denoted f_n^A. (c) The sets of admissible trades $\bar{\Gamma}_b$ in (16), $\underline{\Gamma}_s$ in (17), $\underline{\Gamma}_b$ (20), and $\bar{\Gamma}_s$ in (21) with $B = B_n$ and $S = S_n$ are denoted $\bar{\Gamma}_b^n$, $\underline{\Gamma}_s^n$, $\bar{\Gamma}_s^n$, and $\underline{\Gamma}_b^n$, respectively. (d) The finite household's payoff function W in (18) is denoted W_n. (e) The large-household counterparts of the objects in (a)–(d) are denoted as before.

The next proposition gives the main result of this section.

Proposition 3. *Let A be an allocation satisfying (C1) and (C2). Let M (the upper bound on the household's money holdings) be finite. Let $\epsilon > 0$.*

(i) If f^A is a Definition-1 equilibrium when the household is large, then there exists N such that f_n^A is a Definition-3 ϵn^{-1}-equilibrium when the household is finite and $n > N$.

(ii) If f^A is a Definition-2 strong equilibrium when the household is large, then there exists N such that f_n^A is a Definition-4 strengthened ϵ-equilibrium when the household is finite and $n > N$.

In Proposition 3 (i), I am interested in ϵn^{-1}-equilibrium instead of ϵ-equilibrium because when n increases, the measure of an agent in the household decreases, so that in any pairwise meeting the payoff from any action of the agent decreases (see (4) and (6)). The proof of Proposition 3 is built on the following lemma.

[11] As an alternative, one may address this issue by studying whether there exists a sequence of equilibrium allocations in the finite-household setting that converges (in some sense) to A as $n \to \infty$. I do not have any general result of this sort.

[12] Alternatively, in the definition one can let each sequence of the agent's actions start in an arbitrary meeting; that is, the meeting partner need not be a regular agent. Defining an ϵ-equilibrium in this way does not affect anything substantial.

LEMMA 1. *Let A be an allocation satisfying (C1) and (C2). Let the household be finite. Let $\epsilon > 0$.*

(i) If for all m, all $(q_b, l_b) \in \bar{\Gamma}_b^n(m)$, and all $(q_s, l_s) \in \underline{\Gamma}_s^n(m)$,

$$B_n(q_b, l_b, m) - B_n(q(m,1), l(m,1), m) < \epsilon(1-\beta) \quad (22)$$
$$S_n(q_s, l_s, m) - S_n(q(1,m), l(1,m), m) < \epsilon(1-\beta), \quad (23)$$

then f_n^A is a Definition-3 ϵn^{-1}-equilibrium.

(ii) If for all m and all $\gamma(m) \in [\bar{\Gamma}_b^n(m) \times \underline{\Gamma}_s^n(m)]^I$,

$$W_n(\gamma(m)) - v(m) < \epsilon(1-\beta), \quad (24)$$

then f_n^A is a Definition-4 strengthened ϵ-equilibrium.

With the lemma in hand, the key to Proposition 3 (i) is the upper bound in (22). The basic idea is as follows. For any fixed m, when n is sufficiently large, any trade in $\bar{\Gamma}_b^n(m)$ is not far away from some trades in $\bar{\Gamma}_b(m)$, and, therefore, if this upper bound is violated by (q_b, l_b), then from this (q_b, l_b) some value of (q, l) in $\bar{\Gamma}_b(m)$ with $B(q, l, m) > B(q(m,1), l(m,1), m)$ can be constructed. A similar idea works for the upper bound in (24), the key to Proposition 3 (ii). The finiteness of M in the proposition ensures uniformity.

Lemma 1 still holds if $\bar{\Gamma}_b^n(m)$ is replaced by $\underline{\Gamma}_b^n(m)$ and $\underline{\Gamma}_s^n(m)$ is replaced by $\bar{\Gamma}_s^n(m)$. By this result, we can adapt Proposition 3 to an arbitrary mechanism as described in Remark 1 (refer to the discussion following Proposition 1).

6. Generalized Nash bargaining

We say that an allocation $A = (q(.), l(.))$ satisfying (C1) and (C2) is *generated by generalized Nash bargaining* if there exists some $\lambda \in (0,1]$ such that for all (m_b, m_s),

$$(q(m_b, m_s), l(m_b, m_s)) \in \arg\max_{(q,l)} [B(q, l, m_b)]^{\lambda} [S(q, l, m_s)]^{1-\lambda} \text{ subject to } 0 \le l \le m_b. \quad (25)$$

That is, if the buyer in state m_b and the seller in state m_s take the meeting-specific Pareto frontier to be the one implied by the individual payoff functions $B(., m_b)$ and $S(., m_s)$, then the trade $(q(m_b, m_s), l(m_b, m_s))$ is the Nash bargaining solution with λ as the buyer's bargaining weight. If A satisfies (C1), (C2), and (25), then by Proposition 1, f^A is a Definition-1 equilibrium.[13] In the rest of this section, I establish the existence of such allocations.

[13] If $\lambda = 1$, then A is also an equilibrium allocation when agents play the ultimatum game in pairwise meetings (buyers make offers). If $\lambda < 1$ and if one applies a suitable version of the Rubinstein–Ståhl alternating-offer game in the meeting where the Pareto frontier is determined by $B(., m_b)$ and $S(., m_s)$, then after taking the limit one obtains $(q(m_b, m_s), l(m_b, m_s))$ as the trading outcome.

First, I consider the large-household setting. If the transfer of money is l, $w_b(l, m_b) = l\omega_b$, and $w_s(l, m_s) = l\omega_s$ (see (8) and (9)), then the Nash solution implies that

$$y(l, \omega_b, \omega_s) = \underset{q \geq 0}{\arg\max}[u(q) - \omega_b l]^{\lambda}[-q + \omega_s l]^{1-\lambda}$$

is the transfer of the good. This satisfies

$$\lambda u'(y(l, \omega_b, \omega_s))[-y(l, \omega_b, \omega_s) + \omega_s l] = (1-\lambda)[u(y(l, \omega_b, \omega_s)) - \omega_b l]. \tag{26}$$

I use this function $y(.)$ to construct two equilibrium allocations satisfying (C1), (C2), and (25). For the construction to go through, I need the following assumption:

(U) Either $\lambda = 1$, or $\lambda < 1$ and $u''u'' \geq u'u'''$.

The first equilibrium in the large-household setting is strong. Let y_λ satisfy $\beta u'(y_\lambda) \geq 1$ and

$$1 = \frac{\beta[\lambda u'(y_\lambda) + (1-\lambda)]}{1 - (1-\lambda)u''(y_\lambda)[u(y_\lambda) - \omega]/[u'(y_\lambda)u'(y_\lambda)]}, \tag{27}$$

where

$$\omega = \frac{\lambda u'(y_\lambda)y_\lambda + (1-\lambda)u(y_\lambda)}{\lambda u'(y_\lambda) + (1-\lambda)}. \tag{28}$$

The existence and uniqueness of y_λ follow from (U).[14] (Letting $(\omega_b, \omega_s, l) = (\omega, \omega, 1)$ in (26) and comparing to (28), we see $y(1, \omega, \omega) = y_\lambda$, which turns out to be the equilibrium regular-meeting output.[15]) Let A be defined by

$$q(m_b, m_s) = y(m_b, \omega, \omega) \text{ and } l(m_b, m_s) = m_b. \tag{29}$$

The value function associated with A in (29) is

$$v(m) = u(y(m, \omega, \omega)) - y_\lambda + \beta(1-\beta)^{-1}[u(y_\lambda) - y_\lambda]. \tag{30}$$

Here I also need the bound on the household's money holding to satisfy $u'(y(M, \omega, \omega)) \geq 1$. The existence of such an M follows from $u'(y_\lambda) \geq \beta^{-1}$ (implied by (27)) and the continuity of the mapping $m \to y(m, \omega, \omega)$.

Proposition 4. *Suppose that (U) holds. Let $M > 1$ satisfy $u'(y(M, \omega, \omega)) \geq 1$. Let the household be large. Then the allocation A given by (29) satisfies (C1), (C2), and (25), and f^A is a Definition-2 strong equilibrium.*

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[14]Substituting ω in (28) into (27), we obtain $[\lambda u' + (1-\lambda)][\beta\lambda u' - \beta\lambda + \beta - 1] = -\lambda(1-\lambda)(u''/u')(u - y)$. Now notice that $\lambda - \beta\lambda + \beta - 1 \leq 0$, that $-(1-\lambda)u''/u'$ is nondecreasing (implied by (U)), and that u is strictly concave.

[15]In dealing with symmetric Nash bargaining ($\lambda = 0.5$), Rauch (2000) provides a regular-meeting output comparable to $y_{0.5}$; see Section 6 for how Rauch obtains his result. Also, in a seemingly programmer-automata setup, Shi (1999) provides a regular-meeting output comparable to y_1; see Appendix B for how Shi obtains his result.

The remarkable feature of the Proposition-4 allocation is $l(m,1)=m$ and $l(1,m)=1$ so that $g(m)=1$ all m; that is, the household's end-of-meeting money holding does not depend on its pre-meeting holding. Consequently, the associated value function is completely determined by its derivative at 1, $v'(1)$. As shown in the proof, $v'(1)=\omega/\beta$, but ω is completely determined by (27) and (28) (which obviously do not depend on $v(.)$), so my construction can go through. The restriction on M ensures that $l(m,1)=m$ holds for all m.

The Proposition-4 equilibrium resembles the equilibrium in the Lagos–Wright model. In Lagos and Wright (2005), agents trade in a centralized market after random matching, and preferences over centralized-trade goods are quasi-linear. For an internal solution in the centralized market, the agent must enter the centralized market with money holdings that are not too large. In that case, the assumed quasi-linear preferences imply that the value function for the agent's end-of-meeting money holdings is affine, and that, in turn, implies that in a pairwise meeting, the buyer's and seller's payoff functions are quasi-linear, linear in end-of-meeting holdings. Moreover, these functions have the same linear coefficient, provided that the sum of the buyer's and seller's holdings is consistent with an internal solution in the centralized market. In the Proposition-4 equilibrium, the buyer's and seller's payoffs are linear in each agent's end-of-meeting holdings (so the individual agent's payoff functions are quasi-linear). The upper bound on the household's money holding ensures that all households have the same end-of-meeting money holding, which in turn ensures that the linear coefficients regarding money for the buyer and seller in a meeting are identical. Finally, (U) has the same uses in Lagos and Wright (2005) as it does here.

The second equilibrium in the large-household setting is not strong. Let $\tilde{y}_\lambda$ satisfy $\beta u'(\tilde{y}_\lambda)\geq 1$ and

$$1=\frac{\beta\lambda u'(\tilde{y}_\lambda)}{1-(1-\lambda)u''(\tilde{y}_\lambda)u(\tilde{y}_\lambda)/[u'(\tilde{y}_\lambda)u'(\tilde{y}_\lambda)]},\tag{31}$$

where

$$\tilde{\omega}=\frac{\lambda u'(\tilde{y}_\lambda)\tilde{y}_\lambda+(1-\lambda)u(\tilde{y}_\lambda)}{\lambda u'(\tilde{y}_\lambda)}.\tag{32}$$

The existence and uniqueness of $\tilde{y}_\lambda$ again follow from (U). (Letting $(\omega_b,\omega_s,l)=(0,\tilde{\omega},1)$ in (26) and comparing to (32), we see $y(1,0,\tilde{\omega})=\tilde{y}_\lambda$, which turns out to be the equilibrium regular-meeting output.) Let A be defined by

$$q(m_b,m_s)=y(\min\{m_b,1\},0,\tilde{\omega})\text{ and }l(m_b,m_s)=m_b.\tag{33}$$

The value function associated with A in (33) is

$$v(m)=u(y(\min\{m,1\},0,\tilde{\omega}))-\tilde{y}_\lambda+\beta(1-\beta)^{-1}[u(\tilde{y}_\lambda)-\tilde{y}_\lambda].\tag{34}$$

Proposition 5. *Suppose that (U) holds. Let the household be large. Then the allocation A given by (33) satisfies (C1), (C2), and (25) (but f^A is not Definition-2 strong).*

The value function associated with the Proposition-4 allocation is constant over $[1,M]$ (so $v'_+(1)=0$). Here again $g(m)=1$, so the value function is completely determined by $v'_-(1)$. As shown in the proof, $v'_-(1)=\tilde{\omega}/\beta$, but $\tilde{\omega}$ is completely determined by (31) and (32), so my construction can go through. The reason that the function v is flat is simple. Let $m \geq 1$. Given that all buyers from the same household spend m, the suitable value of $v'_{-}(1)$ induces a seller in state m to acquire 1. Given that all sellers from the same household acquire 1, $v'_{+}(1) = 0$ induces a buyer in state m to spend m, even though spending $m > 1$ gets the same amount of good as spending 1. In the equilibrium f^A, if all buyers from the household with $m > 1$ offer $(\tilde{y}_\lambda, 1)$ to regular sellers and all sellers from the household accept $(\tilde{y}_\lambda, 1)$ from regular buyers, then the household obtains a higher payoff than it does by following f^A. So this equilibrium is not strong.

Next I turn to two results for the finite-household setting. Both results can be extended to general bargaining powers; I restrict to $\lambda = 1$ (the buyer has all bargaining power) for the sake of simplicity.

Proposition 6. *Let $\lambda = 1$. Let the household be finite.*

(i) If $n > 1$ and $M > n/(n-1)$, then there exists an allocation A with $l(1,1) = 1$ satisfying (C1), (C2), and (25) (but f^A is not Definition-2 strong).

(ii) If A with $l(1,1) = 1$ satisfies (C1), (C2), and (25), then $n > 1$.

The allocation in Proposition 6 (i) resembles the one in Proposition 5. In particular, it has $l(m_b, m_s) = m_b$ and its associated value function is constant over $[1, M]$. (The condition $M > n/(n-1)$ ensures nonbindingness when M is finite.) Proposition 6 (ii) says that $n > 1$ is necessary for the existence of an equilibrium with the regular-meeting transfer of money equal to one, the type of equilibrium in Propositions 4, 5, and 6 (i). The message is that the feasibility of degeneracy does not assure the existence of a certain type of degenerate equilibrium.[16]

The proof of Proposition 6 (i) is not completely constructive, because now $v'_{-}(1)$ is not sufficient to determine $v(.)$. I use a fixed-point argument to determine $v(.)$ over $[0,1]$. I have attempted to adapt this argument to establish a strong equilibrium allocation for large n, but I have not been able to obtain a positive increment of v from 1 to some $z > 1$. The key behind Proposition 6 (ii) is as follows. When $n = 1$, $l(1,1) = 1$ gives rise to the dependence of the current payoff to agents in state m on the value function in the form of $v(m) = u(\beta v(m))$ for m in a neighborhood of 0, so v is a strictly concave transformation of itself in the neighborhood. But this is impossible.

7. Comparison to the literature

As indicated above, aside from details, Shi (1997) and Rauch (2000) make all the important assumptions I make about the physical environment.[17] They both consider symmetric Nash bargaining ($\lambda = 0.5$). Shi (1997), who initiated the use of the large-household model for money applications, describes the household's problem in terms

[16] Wallace and Zhu (2004, Section 2) deliver the same message in a standard single-agent household model.

[17] Rauch (2000) and Shi (1997) also study an endogenous buyer–seller ratio and lump-sum money transfers. I can extend the above formulations to these variants, but I can extend the above existence proof to deal only with the variant with money transfers.

of sequences of the household's choices. In his formulation, each household takes as given that the regular-meeting trade is the trade that its buyers and sellers will make—independent of the household's start-of-date money holding. However, such trade is not feasible for a household with $m<1$, which leaves $v(m)$ for $m<1$ undefined. It also implies that $v(m)=v(1)$ for $m\geq 1$. In a comment on Shi (1997), Rauch (2000) points out the relevance of the household's start-of-date holding. He proposes an alternative formulation.

Following Shi (1997), Rauch describes the household problem in sequence form. He proposes a special Lagrangian

$$\mathscr{L}(\{m_t\},\{\omega_t\})=\sum_t \beta^t\{F(m_t,\omega_t)+\omega_t[m_{t+1}-m_t-\Delta(m_t,\omega_t)]\},$$

where m_t is the household's money holding at the start of t, $F(m_t,\omega_t)$ is the return to the household from consumption and production at t, $\Delta(m_t,\omega_t)$ is the net money inflow to the household at t, and ω_t is the Lagrangian multiplier associated with the constraint $m_{t+1}=m_t+\Delta(m_t,\omega_t)$.[18] Notably, Rauch treats ω_t as a function of m_t (see Rauch 2000, (22) and (23)).[19] This function cannot be arbitrary. It ought to be determined by equilibrium conditions. But Rauch provides no such conditions, so his formulation is incomplete.[20]

We can complete Rauch's formulation by distinguishing the Lagrangian multipliers in $\mathscr{L}$ from a function describing the individual agent's marginal value of money. For instance, let $\hat{\omega}(.)$ be such a function so that if an agent's household starts with m at $t+1$, then $\hat{\omega}(m)$ is the agent's marginal value of money at a date-t meeting. Then for each m_0, let

$$\hat{\mathscr{L}}(\{m_t\},\{\omega_t\})=\sum_t \beta^t\{F(m_t,\hat{\omega}(m_{t+1}))+\omega_t[m_{t+1}-m_t-\Delta(m_t,\hat{\omega}(m_{t+1}))]\}. \tag{35}$$

The maximum of the expected discounted utility for a household with m_0 is $v(m_0)=\max\hat{\mathscr{L}}(\{m_t\},\{\omega_t\})$ subject to (L) $m_{t+1}\in\{x: x=m_t+\Delta(m_t,\hat{\omega}(x))\}$, all t. The constraint (L) captures the loop described in the introduction. This constraint is nontrivial—given $\hat{\omega}(.)$ and m_t, the mapping $x\mapsto m_t+\Delta(m_t,\hat{\omega}(x))$ need not have a fixed point.

Now the equilibrium condition is $\hat{\omega}(.)=\beta v'(.)$. The existence of an equilibrium means the existence of such a $\hat{\omega}(.)$, from which we can construct an equilibrium allocation as follows. Fix m and set $m_0=m$ in (35) and then set an optimal m_1 to be $g(m)$.

[18] This Lagrangian is recovered from Rauch (2000, (10)–(15) and (17)) in the case that the buyer's ratio is exogenous. Although $\{m_t\}$ is not included as a choice in Rauch (2000, (10)), it should be; otherwise, when the buyer's ratio is exogenous, the problem is absent choices, which is not the case according to the context. By Rauch (2000, (1)), ω_t is taken by the individual agent as his marginal value of money in date-t Nash bargaining; it affects F and Δ in the way given in Rauch (2000, (6) and (7)).

[19] Rauch (2000, (22) and (23)) ignores the effect of the change in m_t on ω_t. As it turns out, this inconsistency, among others, leads him to obtain a regular-meeting output comparable to $y_{0.5}$ given by (27).

[20] Completeness cannot be obtained by assuming that ω_t is a free variable as in the usual Lagrangian formulation, because here the period return F (also the money flow Δ) depends on ω_t. For comparison, think about the Lagrangian in the planner's version of the growth model where the period return does not depend on the multipliers.

Then let $(q(m_b,m_s),l(m_b,m_s))\in\arg\max[u(q)-\hat{\omega}(g(m_b))l][-q+\hat{\omega}(g(m_s))l]$ subject to $0\le l\le m_s$ (recall that Rauch deals with symmetric Nash bargaining).

8. Concluding remarks

The equilibrium concepts developed in the money model in this paper can be applied to the labor search model of Merz (1995) and its variant in Den Haan et al. (2000). They can also be adapted to deal with the large firm's decision problem in the labor search literature. The large firm has many job positions, and the wage in each position is determined by bargaining with a worker. But in the literature, the firm takes the prevailing wage as given (see Pissarides 2000, Ch. 3.1). This treatment seems problematic, and it actually influences the initial formulation of the large-household model in monetary economics (see the discussion in Shi 1998, p. 327). Following the approach used above, it could be assumed that the wage in each position is determined by bargaining between a firm's agent and a worker, while taking as given the bargaining outcome between other agents of the firm and workers.

Appendix

A. Proofs

Proof of Proposition 2. Let Π be the set of $\pi=(\pi_b,\pi_b^*,\pi_b^0,\pi_s,\pi_s^*,\pi_s^0)\in\mathbb{R}_+^6$ with $\pi_b+\pi_b^*+\pi_b^0=1$ and $\pi_s+\pi_s^*+\pi_s^0=1$. Fix m. By the concavity of u and v and Jensen's inequality, for any $\gamma(m)$ there exist $\pi\in\Pi$ and $(q_b,l_b,q_s,l_s)\in[\bar{\Gamma}_b(m)\setminus\underline{\Gamma}_b(m)]\times[\bar{\Gamma}_s(m)\setminus\underline{\Gamma}_s(m)]\equiv K$ such that

$$\pi_b u(q_b)+\pi_b^* u(q_b^*)-\pi_s q_s-\pi_s^* q_s^*+\beta v(m-\pi_b l_b-\pi_b^* l_b^*+\pi_s l_s+\pi_s^* l_s^*)$$
$$\equiv w(\pi,q_b,l_b,q_s,l_s)\ge W(\gamma(m)),$$

where $(q_b^*,l_b^*)=(q(m,1),l(m,1))$ and $(q_s^*,l_s^*)=(q(1,m),l(1,m))$. So if $v(m)$ is the maximum of $w(.)$ over $\Pi\times K$, then $v(m)$ is the maximum of $W(.)$ over $[\bar{\Gamma}_b(m)\times\bar{\Gamma}_s(m)]^I$. Denote by $(\bar{\pi},q_b,l_b,q_s,l_s)$ a maximizer of $w(.)$ with $\bar{\pi}=(\bar{\pi}_b,\bar{\pi}_b^*,\bar{\pi}_b^0,\bar{\pi}_s,\bar{\pi}_s^*,\bar{\pi}_s^0)$. Now it suffices to show $\bar{\pi}_b^*=\bar{\pi}_s^*=1$, which clearly implies $w(\bar{\pi},q_b,l_b,q_s,l_s)=v(m)$. I first establish some intermediate results, mainly properties of $\bar{\pi}$. In what follows, let $\hat{w}(\pi)\equiv w(\pi,q_b,l_b,q_s,l_s)$ and $h(\pi)\equiv m-\pi_b l_b-\pi_b^* l_b^*+\pi_s l_s+\pi_s^* l_s^*$.

Claim 1. *(i) $B(q_b,l_b,m)<B(q_b^*,l_b^*,m)$; (ii) if $\bar{\pi}_b>0$, then $l_b\ne l_b^*$; (iii) $S(q_s,l_s,m)<S(q_s^*,l_s^*,m)$; and (iv) if $\bar{\pi}_s>0$ then $l_s\ne l_s^*$.*

Proof. For part (i), first note that by (12), $B(q_b,l_b,m)\le B(q_b^*,l_b^*,m)$. Suppose that $B(q_b,l_b,m)=B(q_b^*,l_b^*,m)$. By the strict monotonicity of v and $g(m)<M$, $v'(g(m))>0$. So it must be that $q_b\ne q_b^*$and $l_b\ne l_b^*$. But then any interior linear combination of (q_b,l_b) and (q_b^*,l_b^*), denoted (q,l), satisfies $B(q,l,m)>B(q_b^*,l_b^*,m)$ and $S(q,l,1)\ge S(q_b^*,l_b^*,1)$ (recall that u is strictly concave), contradicting (12). For part (ii), suppose $l_b=l_b^*$. Then part (i) implies $u(q_b)<u(q_b^*)$, so that $(\bar{\pi},q_b,l_b,q_s,l_s)$ is not a maximizer of $w(.)$. Analogously, we can establish parts (iii) and (iv). ◁

Next, without loss of generality, we can assume (A1) if $l_b^* = 0$ then $\bar{\pi}_b^0 = 0$, (A2) if $l_s^* = 0$ then $\bar{\pi}_s^0 = 0$, (A3) if $\bar{\pi}_b^0 > 0$ then $h(\bar{\pi}) \leq g(m)$, and (A4) if $\bar{\pi}_s^0 > 0$ then $h(\bar{\pi}) \geq g(m)$.

For (A1), suppose $\bar{\pi}_b^0 > 0$. Then $\hat{w}(\bar{\pi}) = \hat{w}(\pi)$, where $\pi = (\bar{\pi}_b, \bar{\pi}_b^* + \bar{\pi}_b^0, 0, \bar{\pi}_s, \bar{\pi}_s^*, \bar{\pi}_s^0)$, so we can replace $\bar{\pi}$ by this π. Analogously, we can justify (A2); now use $\pi = (\bar{\pi}_b, \bar{\pi}_b^*, \bar{\pi}_b^0, \bar{\pi}_s, \bar{\pi}_s^* + \bar{\pi}_s^0, 0)$. For (A3), suppose $h(\bar{\pi}) > g(m)$. Then by (10) and the concavity of v, $d\hat{w}(\pi(x))/dx \geq 0$ at $h(\pi(x)) \geq g(m)$, where $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^* + x, \bar{\pi}_b^0 - x, \bar{\pi}_s, \bar{\pi}_s^*, \bar{\pi}_s^0)$, so we can replace $\bar{\pi}$ by the $\pi(x)$ for which $h(\pi(x)) = g(m)$. Analogously, we can justify (A4); now use (11) and $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^*, \bar{\pi}_b^0, \bar{\pi}_s, \bar{\pi}_s^* + x, \bar{\pi}_s^0 - x)$.

Claim 2. *(i) If $l_b < l_b^*$ and $\bar{\pi}_b > 0$ then $h(\bar{\pi}) \leq g(m)$; (ii) if $l_b > l_b^*$ and $\bar{\pi}_b > 0$ then $h(\bar{\pi}) \geq g(m)$; (iii) if $l_s > l_s^*$ and $\bar{\pi}_s > 0$ then $h(\bar{\pi}) \leq g(m)$; and (iv) if $l_s < l_s^*$ and $\bar{\pi}_s > 0$ then $h(\bar{\pi}) \geq g(m)$.*

Proof. For part (i), suppose $h(\bar{\pi}) > g(m)$. But then by Claim 1 (i) and the concavity of v, $d\hat{w}(\pi(x))/dx > 0$ at $h(\pi(x)) > g(m)$, where $\pi(x) = (\bar{\pi}_b - x, \bar{\pi}_b^* + x, \bar{\pi}_b^0, \bar{\pi}_s, \bar{\pi}_s^*, \bar{\pi}_s^0)$. The same argument rules out $h(\pi) < g(m)$ in part (ii). Analogously, we can establish parts (iii) and (iv); now use Claim 1 (iii) and $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^*, \bar{\pi}_b^0, \bar{\pi}_s - x, \bar{\pi}_s^* + x, \bar{\pi}_s^0)$. $\lhd$

Claim 3. $h(\bar{\pi}) = g(m)$.

Proof. First suppose $h(\bar{\pi}) < g(m)$. Then by (A4), $\bar{\pi}_s^0 = 0$; by Claim 1 (ii) and Claim 2 (ii), if $\bar{\pi}_b > 0$ then $l_b < l_b^*$; by Claim 1 (iv) and Claim 2 (iv), if $\bar{\pi}_s > 0$ then $l_s > l_s^*$. But then it must be $h(\bar{\pi}) \geq g(m)$, a contradiction. Analogously, we can rule out $h(\bar{\pi}) > g(m)$; now use (A3), Claim 1 (ii), Claim 2 (i), Claim 1 (iv), and Claim 2 (iii). $\lhd$

Next, without loss of generality, we can assume (A5) $\bar{\pi}_b^0 \bar{\pi}_s^0 = 0$.

For (A5), suppose $\bar{\pi}_b^0 \bar{\pi}_s^0 > 0$. Then first by (A1) and (A2), $l_b^* l_s^* > 0$. Then by Claim 3, (10), and (11), $d\hat{w}(\pi(x))/dx \geq 0$, where $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^* + x l_s^*, \bar{\pi}_b^0 - x l_s^*, \bar{\pi}_s, \bar{\pi}_s^* + x l_b^*, \bar{\pi}_s^0 - x l_b^*)$ (note that $h(\pi(x)) = g(m)$), so we can replace $\bar{\pi}$ by the $\pi(x)$ for which $x = \min\{\bar{\pi}_b^0 / l_s^*, \bar{\pi}_s^0 / l_b^*\}$.

Claim 4. *(i) If $\bar{\pi}_b \bar{\pi}_s > 0$ then $(l_b - l_b^*)(l_s - l_s^*) < 0$; (ii) if $\bar{\pi}_b^0 \bar{\pi}_b > 0$ then $l_b < l_b^*$; and (iii) if $\bar{\pi}_s^0 \bar{\pi}_s > 0$ then $l_s < l_s^*$.*

Proof. For part (i), suppose $(l_b - l_b^*)(l_s - l_s^*) > 0$. But then by Claim 3 and Claim 1 (i) and (iii), $d\hat{w}(\pi(x))/dx > 0$, where $\pi(x) = (\bar{\pi}_b - x, \bar{\pi}_b^* + x, \bar{\pi}_b^0, \bar{\pi}_s - x\delta, \bar{\pi}_s^* + x\delta, \bar{\pi}_s^0)$ with $\delta = (l_b - l_b^*)/(l_s - l_s^*)$ (note $h(\pi(x)) = g(m)$). For part (ii), first by Claim 1 (ii), $l_b \neq l_b^*$. Suppose $l_b > l_b^*$, but then by Claim 3, (10), and Claim 1 (i), $d\hat{w}(\pi(x))/dx > 0$, where $\pi(x) = (\bar{\pi}_b - x l_b^*/l_b, \bar{\pi}_b^* + x, \bar{\pi}_b^0 + x l_b^*/l_b - x, \bar{\pi}_s, \bar{\pi}_s^*, \bar{\pi}_s^0)$ (note $h(\pi(x)) = g(m)$). Analogously, we can establish part (iii); now use Claim 1 (iv), Claim 3, (11), Claim 1 (iii), and $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^*, \bar{\pi}_b^0, \bar{\pi}_s - x l_s^*/l_s, \bar{\pi}_s^* + x, \bar{\pi}_s^0 + x l_s^*/l_s - x)$. $\lhd$

Finally, I show $\bar{\pi}_b^* = \bar{\pi}_s^* = 1$ by ruling out the following three cases.

(a) $\bar{\pi}_b^0 = \bar{\pi}_s^0 = 0$ and $\bar{\pi}_b + \bar{\pi}_s > 0$. If $\bar{\pi}_b \bar{\pi}_s = 0$ then $h(\bar{\pi}) = g(m)$ cannot hold. If $\bar{\pi}_b \bar{\pi}_s > 0$, then by Claim 4 (i), $h(\bar{\pi}) = g(m)$ cannot hold either.

(b) $\bar{\pi}_b^0 > 0$. By (A5), $\bar{\pi}_s^0 = 0$. If $\bar{\pi}_b + \bar{\pi}_s = 0$, then $h(\bar{\pi}) = g(m)$ cannot hold. If $\bar{\pi}_b > 0$, then by Claim 4 (ii), $l_b < l_b^*$ so $h(\bar{\pi}) = g(m)$ can hold only if $\bar{\pi}_s > 0$ and $l_s < l_s^*$, which contradicts Claim 4 (i). So the remaining possibility is $\bar{\pi}_b = 0$ and $\bar{\pi}_s > 0$. Now $h(\bar{\pi}) = g(m)$ only if $l_s < l_s^*$. But then by (10), Claim 1 (iii), and (A1), $d\hat{w}(\pi(x))/dx > 0$, where $\pi(x) = (\bar{\pi}_b, \bar{\pi}_b^* + x(l_s^* - l_s), \bar{\pi}_b^0 - x(l_s^* - l_s), \bar{\pi}_s - x l_b^*, \bar{\pi}_s^* + x l_b^*, \bar{\pi}_s^0)$ (note $h(\pi(x)) = g(m)$).

(c) $\bar{\pi}_s^0 > 0$. We can rule out this case analogously to case (b); here use (A5), Claim 4 (iii) and (i), (11), Claim 1 (i), (A2), and $\pi(x) = (\bar{\pi}_b - x l_s^*, \bar{\pi}_b^* + x l_s^*, \bar{\pi}_b^0, \bar{\pi}_s, \bar{\pi}_s^* + x(l_b^* - l_b), \bar{\pi}_s^0 - x(l_b^* - l_b))$. □

Proof of Lemma 1. For part (i), let $v_n^b(m)$ be the supremum of the (expected lifetime) payoffs to the buyer from sequences of his actions starting from a date when his household holds m, provided that all other agents follow f^A. Note that the function $v_n^b : [0, M] \to \mathbb{R}$ is bounded above (recall that u and v are bounded). Without loss of generality we can assume that $v_n^b(m)$ is attained, so that

$$\begin{aligned} v_n^b(m) - v(m) &= [B_n(q_b, l_b, m)n^{-1} + \beta v_n^b(m_+) - \beta v(m_+)] - B_n(q(m,1), l(m,1), m)n^{-1} \\ &< \epsilon n^{-1}(1-\beta) + \beta[v_n^b(m_+) - v(m_+)], \end{aligned}$$

where $(q_b, l_b) \in \bar{\Gamma}_b^n(m)$ is the buyer's present-date trade in meeting a regular seller, m_+ is the implied household's end-of-meeting holding (i.e., $m_+ = g(m) + l(m,1)n^{-1} - l_b n^{-1}$), and the inequality follows from the hypothesis. Iterating this and using the boundedness of v_n^b, we obtain $v_n^b(m) - v(m) < \epsilon n^{-1}$, as desired. If the agent concerned is a seller, we define v_n^s analogously and by a similar argument obtain $v_n^s(m) - v(m) < \epsilon n^{-1}$.

For part (ii), let $v_n^h(m)$ be the (expected lifetime) payoffs to the household from sequences of joint actions of the household members starting from a date when the household holds m, provided that all agents outside the household follow f^A. Note that the function $v_n^h : [0, M] \to \mathbb{R}$ is bounded above. Without loss of generality, we can assume that $v_n^h(m)$ is attained, so that

$$\begin{aligned} v_n^h(m) - v(m) &= [W_n(\gamma(m)) + \beta v_n^h(m_+) - \beta v(m_+)] - v(m) \\ &< \epsilon(1-\beta) + \beta[v_n^h(m_+) - v(m_+)], \end{aligned}$$

where $\gamma(m) \in [\bar{\Gamma}_b^n(m) \times \underline{\Gamma}_s^n(m)]^I$ is the household's present-date trades in meeting regular agents, m_+ is the implied household's end-of-meeting holding, and the inequality follows from the hypothesis. Iterating this and using the boundedness of v_n^h, we obtain $v_n^h(m) - v(m) < \epsilon$, as desired. □

Proof of Proposition 3. First let A satisfy (C1) and (C2) and let M be finite.

Claim 1. *For all m, $B_n(q, l, m) - B_n(q_b^*, l_b^*, m) \le B(q, l, m) - B(q_b^*, l_b^*, m)$, where $(q_b^*, l_b^*) \equiv (q(m,1), l(m,1))$;*

Claim 2. *For all m, $S_n(q, l, m) - S_n(q_s^*, l_s^*, m) \le S(q, l, m) - S(q_s^*, l_s^*, m)$, where $(q_s^*, l_s^*) \equiv (q(1,m), l(1,m))$;*

Claim 3. *For all $\sigma > 0$, there exists N such that for all $n > N$, $m \in [0, M]$, $q \geq 0$, and $l \in [0, m]$, $S_n(q, l, 1) - S_n(q_b^*, l_b^*, 1) < S(q, l, 1) - S(q_b^*, l_b^*, 1) + \sigma$, where $(q_b^*, l_b^*) \equiv (q(m, 1), l(m, 1))$.*

To see these claims, by the concavity of v and the definitions of v'_+ and v'_-, we have (i) $\forall z \geq 0$, if $\delta > 0$ then $v'_+(z) \geq [v(z+\delta) - v(z)]\delta^{-1}$; (ii) $\forall z > 0$, if $\delta \in (0, z]$ then $[v(z) - v(z-\delta)]\delta^{-1} \geq v'_-(z)$; and (iii) $\forall \sigma > 0$, $\exists \delta_\sigma < 1$ such that $\forall \delta \in (0, \delta_\sigma)$, $v'_+(1) < [v(1+\delta) - v(1)]\delta^{-1} + \sigma$ and $v'_-(1) > [v(1) - v(1-\delta)]\delta^{-1} - \sigma$. Applying (i) and (ii) to (5), (7), (8), and (9), we obtain Claims 1 and 2; applying (i)–(iii) to (7) and (9), we obtain Claim 3. Here note that $n[v(z + \iota n^{-1}) - v(z)] = [v(z+\delta) - v(z)]\delta^{-1}\iota$ for $\delta = \iota n^{-1}$. For Claim 3, we go through all six ordering relationships among l, $l(1,1)$, and l_b^*, and use $v(z_2) - v(z_1) = [v(z_2) - v(1)] + [v(1) - v(z_1)]$.

Next let (10)–(12) hold for the large household. By (11) and (C1), $v'_-(1) > 0$. Fix $\epsilon > 0$. Let $\bar{\epsilon} \equiv \epsilon(1-\beta)$. Let $y_\epsilon, \iota_\epsilon > 0$ satisfy (a1) $y_\epsilon = \beta v'_+(1)\iota_\epsilon$ if $v'_+(1) > 0$ and $y_\epsilon = \beta v'_-(1)\iota_\epsilon$ if $v'_+(1) = 0$, (a2) $2u(y_\epsilon) < \bar{\epsilon}$, and (a3) $2\beta[v(\iota_\epsilon) - v(0)] < \bar{\epsilon}$. Fix N such that $\forall n > N$ Claim 3 holds with $\sigma = y_\epsilon$. Fix $n > N$ and m, and let (q_b^*, l_b^*) and (q_s^*, l_s^*) be as given above. For $q \geq 0$, $l \in [0, m]$, and $\pi = (\pi_b, \pi_b^*, \pi_s^*) \in \mathbb{R}_+^3$ with $\pi_b + \pi_b^* \leq 1$ and $\pi_s^* \leq 1$, define $w(\pi, q, l) \equiv \pi_b u(q) + \pi_b^* u(q_b^*) - \pi_s^* q_s^* + \beta v(m - \pi_b l - \pi_b^* l_b^* + \pi_s^* l_s^*)$.

For part (i) of the proposition, fix $(q_b, l_b) \in \bar{\Gamma}_b^n(m)$ and $(q_s, l_s) \in \underline{\Gamma}_s^n(m)$. It suffices to show that (22) and (23) hold. For (23), suppose to the contrary that $S_n(q_s, l_s, m) - S_n(q_s^*, l_s^*, m) \geq \bar{\epsilon}$, so (q_s, l_s) cannot be (q_s^*, l_s^*) and must be $(0,0)$. But then setting $(q, l) = (0, 0)$ in Claim 2 gives $S(0, 0, m) - S(q_s^*, l_s^*, m) \geq \bar{\epsilon}$, which contradicts (11) (for the large household). For (22), suppose to the contrary that $\Delta \equiv B_n(q_b, l_b, m) - B_n(q_b^*, l_b^*, m) \geq \bar{\epsilon}$. Again (q_b, l_b) can neither be (q_b^*, l_b^*) nor $(0,0)$ (here use Claim 1 and (10)). So $(q_b, l_b) \in \bar{\Gamma}_b^n(m) \setminus \underline{\Gamma}_b^n(m)$, which implies $S_n(q_b, l_b, 1) \geq S_n(q_b^*, l_b^*, 1)$; then by Claim 3, $S(q_b, l_b, 1) + y_\epsilon > S(q_b^*, l_b^*, 1) \equiv S^*$. By $\Delta \geq \bar{\epsilon}$, setting $(q, l) = (q_b, l_b)$ in Claim 1 gives $B(q_b, l_b, m) - B(q_b^*, l_b^*, m) \geq \bar{\epsilon}$, so by (10), $u(q_b) \geq \bar{\epsilon}$. By (a2) (and concavity of u and $u(0) = 0$), $B(q_b, l_b, m) - B(q_b - y_\epsilon, l_b, m) < \bar{\epsilon}/2$, so $B(q_b - y_\epsilon, l_b, m) > B(q_b^*, l_b^*, m)$. But this contradicts (12) because $S(q_b - y_\epsilon, l_b, 1) - S(q_b, l_b, 1) = y_\epsilon$ so that $S(q_b - y_\epsilon, l_b, 1) > S^*$.

Next let (19) hold for the large household. For part (ii) of the proposition, fix $\gamma(m) \in [\bar{\Gamma}_b^n(m) \times \underline{\Gamma}_s^n(m)]^I$. It suffices to show (24) holds. Suppose to the contrary that $W_n(\gamma(m)) - v(m) \geq \bar{\epsilon}$. Note that there exist π and $(q_b, l_b) \in \bar{\Gamma}_b^n(m) \setminus \underline{\Gamma}_b^n(m)$ such that $w(\pi, q_b, l_b) \geq W_n(\gamma(m))$. So $w(\pi, q, l) \geq v(m) + \bar{\epsilon}$, and note as above that $S(q_b, l_b, 1) + y_\epsilon > S^*$. First suppose $q_b - q_b^* > y_\epsilon$. By (a2), $w(\pi, q_b - y_\epsilon, l_b) > v(m)$. But this contradicts (19) because $S(q_b - y_\epsilon, l_b, 1) > S^*$. So it must be that $q_b - q_b^* \leq y_\epsilon$. Next suppose $\delta \equiv \beta[w_s(l_b^*, 1) - w_s(l_b, 1)] > y_\epsilon$ (see (9)). Let $\beta[w_s(l, 1) - w_s(l_b, 1)] = y_\epsilon$. By (a1), $l - l_b \leq \iota_\epsilon$; then by (a3), $w(\pi, q_b, l) > v(m)$. But this contradicts (19) because $S(q_b, l, 1) > S^*$. So it must be that $\delta \leq y_\epsilon$. Next suppose $l_b^* - l_b > \iota_\epsilon$. Then $q_b^* \geq q_b$. Given $\delta \leq y_\epsilon$, $l_b^* - l_b > \iota_\epsilon$ and (a1) imply $v'_+(1) = 0$ and $l_b^* > l(1,1)$, so that $(q_b^*, l(1,1)) \in \Gamma_b(m)$. It follows that $g(m) \geq 1$ and $q_s^* = 0$; otherwise, the large household with m has obvious beneficial joint deviations (recall that $v'_-(1) > 0$). But then $w(\pi, q_b, l_b)$ cannot exceed $v(m)$, a contradiction. So it must be that $l_b^* - l_b \leq \iota_\epsilon$. But this, $q_b - q_b^* \leq y_\epsilon$, (a2), and (a3) imply $w(\pi, q_b, l_b) < v(m) + \bar{\epsilon}$, a contradiction. □

Proof of Proposition 4. First I claim that $u(y(m,\omega,\omega))$ is strictly increasing and concave and differentiable in m, and $\beta v'(1)=\omega$. To see this, set $\omega_b=\omega_s=\omega$ in (26). It is clear that $l\mapsto y(l,\omega,\omega)$ is strictly increasing and so is $l\mapsto u(y(l,\omega,\omega))$. By the implicit function theorem, $l\mapsto y(l,\omega,\omega)$ is continuously differentiable and so is $l\mapsto u(y(l,\omega,\omega))$. Differentiating (26) with respect to l at $(\omega_b,\omega_s)=(\omega,\omega)$ and substituting $-y(l,\omega,\omega)+\omega l=(1-\lambda)[u(y(l,\omega,\omega))-\omega l]/[\lambda u'(y(l,\omega,\omega))]$ (which is obtained by rewriting (26) with $(\omega_b,\omega_s)=(\omega,\omega)$), we have

$$y'(l,\omega,\omega)=\frac{\lambda u'(y)\omega+(1-\lambda)\omega}{u'(y)-(1-\lambda)u''(y)[u(y)-\omega l]/u'(y)},\tag{36}$$

where $y'(l,\omega,\omega)$ denotes the derivative of $y(.)$ with respect to its first argument and $y=y(l,\omega,\omega)$. By (36) and (U), some algebra confirms $u''y'+u'y''\leq 0$, so $l\mapsto u(y(l,\omega,\omega))$ is concave. By (30), $v'(1)=u'(y(1,\omega,\omega))y'(1,\omega,\omega)$. Using this and (36) with $l=1$ and (27), we have $\beta v'(1)=\omega$ (recall $y_\lambda=y(1,\omega,\omega)$). Next, by $y_\lambda>0$ and the above claim, A in (29) satisfies (C1) and (C2). Note that $g(m_b)=g(m_s)=1$. Substituting this and $\beta v'(1)=\omega$ into (25) and referring to (8) and (9), we see that A satisfies (25) if and only if (i) $\lambda u'(q(m_b,m_s))[-q(m_b,m_s)+\omega m_b]=(1-\lambda)[u(q(m_b,m_s))-\omega m_b]$ and (ii) $u'(q(m_b,m_s))\geq 1$. But (i) follows from (26) and (ii) from $u'(y(M,\omega,\omega))\geq 1$. Finally, by Propositions 1 and 2 and the claim, f^A is a strong equilibrium. □

Proof of Proposition 5. First I claim that $u(y(m,0,\tilde{\omega}))$ is continuous, concave, and strictly increasing in m, $\beta v'_-(1)=\tilde{\omega}$, and $v'_+(1)=0$. The continuity, monotonicity, and fact that $v'_+(1)=0$ are obvious. Differentiating (26) with respect to l at $(\omega_b,\omega_s)=(0,\tilde{\omega})$ and substituting $-y(l,0,\tilde{\omega})+\tilde{\omega}l=(1-\lambda)u(y(l,0,\tilde{\omega}))/[\lambda u'(y(l,0,\tilde{\omega}))]$ (which is obtained by rewriting (26) with $(\omega_b,\omega_s)=(0,\tilde{\omega})$), we have

$$y'(l,0,\tilde{\omega})=\frac{\lambda u'(y)\tilde{\omega}}{u'(y)-(1-\lambda)u''(y)u(y)/u'(y)},\tag{37}$$

where $y'(l,0,\tilde{\omega})$ denotes the derivative of $y(.)$ with respect to its first argument and $y=y(l,0,\tilde{\omega})$. By (37) and (U), some algebra confirms $u''y'+u'y''\leq 0$ so $l\mapsto u(y(l,0,\tilde{\omega}))$ is concave. By (34), $v'_-(1)=u'(y(1,0,\tilde{\omega}))y'(1,0,\tilde{\omega})$. Using this, (37) with $l=1$, and (31), we have $\beta v'_-(1)=\tilde{\omega}$ (recall $\tilde{y}_\lambda=y(1,0,\tilde{\omega})$). Next, by $\tilde{y}_\lambda>0$ and the above claim, A in (33) satisfies (C1) and (C2). Note that $g(m_b)=g(m_s)=1$. Substituting this, $\beta v'_-(1)=\tilde{\omega}$, and $\beta v'_+(1)=0$ into (25) and referring to (8) and (9), we see that A satisfies (25) if and only if $\lambda u'(q(m_b,m_s))[-q(m_b,m_s)+\tilde{\omega}m_b]=(1-\lambda)u(q(m_b,m_s))$. But the equality follows from (26). □

Proof of Proposition 6. When the household is finite, A satisfies (25) with $\lambda=1$ (refer to (5) and (7)) if and only if

$$q(m_b,m_s)=n\beta[v(g(m_s)-l(1,m_s)n^{-1}+l(m_b,m_s)n^{-1})-v(g(m_s)-l(1,m_s)n^{-1})]\tag{38}$$

$$\begin{aligned}[l(m_b,m_s)=m_b]\Rightarrow[u'(q(m_b,m_s))v'_-(g(m_s)-l(1,m_s)n^{-1}+l(m_b,m_s)n^{-1})\\ \geq v'_+(g(m_b)+l(m_b,1)n^{-1}-l(m_b,m_s)n^{-1})]\end{aligned}\tag{39}$$

$$\begin{aligned}[l(m_b,m_s)<m_b]\Rightarrow[u'(q(m_b,m_s))v'_+(g(m_s)-l(1,m_s)n^{-1}+l(m_b,m_s)n^{-1})\\ \leq v'_-(g(m_b)+l(m_b,1)n^{-1}-l(m_b,m_s)n^{-1})].\end{aligned}\tag{40}$$

Also notice that $g(1)=1$, which is used frequently below.

Now I prove part (i) of the proposition. Let Δ satisfy $u(\beta\Delta)<\Delta$, and let $\mathbf{V}$ be the set of functions $v:[0,M]\to\mathbb{R}$ that are continuous, nondecreasing, concave, and satisfy $v(1)-v(0)\le\Delta$. Let $K=[1-n^{-1},1]$ and let $\mathbf{W}$ be the set of functions $w:K\to\mathbb{R}$ with $w=v$ on K for some $v\in\mathbf{V}$. For $w\in\mathbf{W}$ and $m\in[0,1]$, let

$$y(m,w)=n\beta[w(1-n^{-1}+mn^{-1})-w(1-n^{-1})]. \tag{41}$$

Then let $Gw:K\to\mathbb{R}$ be defined by

$$Gw(m)=u(y(m,w))-y(1,w)+\beta w(1). \tag{42}$$

Suppose that w is a fixed point of G with $y(1,w)>0$. Let the allocation A be defined by

$$q(m_b,m_s)=y(\min\{m_b,1\},w),\ l(m_b,m_s)=m_b.$$

The value function associated with this A is

$$v(m)=u(y(\min\{m,1\},w))-y(1,w)+\beta w(1). \tag{43}$$

Note that by $w=Gw$, $v=w$ on K. Clearly A satisfies (C1) and (C2). Substituting $g(m_b)=g(m_s)=1$ and $v'_+(1)=0$ (all implied by this A) into (38)–(40), we immediately see that A satisfies (38)–(40). It is also easy to verify that f^A is not a Definition-2 strong equilibrium.

It remains to find a fixed point of G with $y(1,w)>0$. To this end, let $\mathbf{W}$ be equipped with the sup norm topology. I claim (i) $\mathbf{W}$ is compact and convex and (ii) $G\mathbf{W}\subset\mathbf{W}$. For (i), it is obvious that $\mathbf{W}$ is convex and closed. Fix w and let $v\in\mathbf{V}$ satisfy $v=w$ on K. By $v(1)-v(0)\le\Delta$, $w'_+(1-n^{-1})$ must be bounded above by Δ. This and concavity of w imply that $\mathbf{W}$ is equicontinuous and hence compact (by the Arzelà–Ascoli theorem). For (ii), fix w. It is obvious that Gw is nondecreasing, continuous and concave. For this w, let v be defined by (43). It is obvious that $v=Gw$ on K and that $v\in\mathbf{V}$ if $v(1)-v(0)\le\Delta$. But by $y(1,w)\le n\beta[w(1)-w(1-n^{-1})]\le\beta\Delta$ and $y(0,w)\ge 0$, we have $v(1)-v(0)\le u(\beta\Delta)<\Delta$.

Because $\mathbf{W}$ is compact, $y(.,.)$ is uniformly continuous on $[0,1]\times\mathbf{W}$ and so $w_j\to w$ implies $y(.,w_j)\to y(.,w)$ uniformly, that is, $G:\mathbf{W}\to\mathbf{W}$ is continuous. To rule out the trivial fixed point of G (the zero function), I introduce a sequence of auxiliary mappings. Specifically, define $w^i:K\to\mathbb{R}$ by $w^i(m)=w(m)+m/i$. Then let $G^iw:K\to\mathbb{R}$ be defined by $G^iw(m)=Gw^i(m)$. Choose a sufficiently large integer i so that $\forall w\in\mathbf{W}$, $G^iw\in\mathbf{W}$. Because G^i is continuous, by Sauder's fixed point theorem, it has a fixed point w_i; that is, $w_i=G^iw_i=Gw_i^i$. Because $\mathbf{W}$ is compact, the sequence $\{w_i\}$ has a convergent subsequence. To simplify the notation, denote this subsequence by $\{w_i\}$ and let w be its limit point. Because w is also the limit point of $\{w_i^i\}$ and G is continuous and $w_i=Gw_i^i$, it follows that w is a fixed point of G. Because w_i^i is concave and strictly increasing, it follows that $w_i=Gw_i^i$ is concave and nondecreasing, and hence that $w'_{i-}(1)$ is defined and nonnegative. It follows from $Gw_i^i=w_i$ and (41) and (42) that $w'_{i-}(1)=\beta u'(y(1,w_i^i))[w'_{i-}(1)+1/i]$, which implies $y(1,w_i^i)\ge y_1$ (y_1 is defined by (27) and note that $\beta u'(y_1)=1$). By the continuity of $y(1,.)$, $y(1,w)\geq y_1$. (It can actually be shown that $y(1,w)=y_1$.)

Next I prove part (ii) of the proposition. Suppose that A satisfies the hypotheses but $n=1$. Setting $(m_b,m_s)=(1,1)$ in (38) and using $l(1,1)=1$ and $q(1,1)>0$, we have $v(1)>v(0)$. So by the continuity and concavity of v, v is strictly increasing in a neighborhood of 0. By continuity, we can find $\bar{m}\in(0,1)$ close to 0 so that $v(\bar{m})$ is close to $v(0)$ and $\forall m\in[0,\bar{m}]$, $q(m,1)$ is close to 0 (again use (38) and note that $l(m,1)\leq m$) and $u'(q(m,1))>1$.

Let $z\equiv\min\{m: v(m)=v(1)\}$. First consider $z\geq 1$. I claim that $\forall m\in[0,\bar{m}]$, $m=l(m,1)$ and $l(1,m)=1$ so that $g(m)=1$. By this claim and (38) and $l(1,1)=1$, $q(m,1)=\beta v(m)-\beta v(0)$ and $q(1,m)=\beta v(1)-\beta v(0)$, so that $v(m)=u(q(m,1))+\beta v(0)$. Then by $q(0,1)=0$ and $u(0)=0$, we have $v(0)=0$, so that $v(m)=u(q(m,1))$ or $v(m)=u(\beta v(m))$. But this cannot hold for all $m\in[0,\bar{m}]$, because u is strictly concave and v is concave.

To see the claim, fix $m\in[0,\bar{m}]$. First suppose $l(1,m)<1$. Setting $(m_b,m_s)=(1,m)$ in (40) and using $l(1,1)=1$, we have

$$u'(q(1,m))v'_+(g(m))\leq v'_-(2-l(1,m)). \tag{44}$$

Setting $(m_b,m_s)=(1,1)$ in (39) gives $u'(q(1,1))v'_-(1)\geq v'_+(1)$. Comparing this with (44) and using $l(1,m)<1$ and $q(1,m)<q(1,1)$ (implied by (38) and $l(1,m)<l(1,1)$), we have $g(m)\geq 1$ so $l(m,1)<m$. Then setting $(m_b,m_s)=(m,1)$ in (40) gives $u'(q(m,1))v'_+(l(m,1))\leq v'_-(g(m))$, which by $u'(q(m,1))>1$ implies $l(m,1)\geq g(m)$, contradicting $g(m)\geq 1$ because $l(m,1)<m\leq\bar{m}<1$. So $l(1,m)=1$. Now suppose $l(m,1)<m$. Then again $u'(q(m,1))v'_+(l(m,1))\leq v'_-(g(m))$ so $l(m,1)\geq g(m)$. Given $l(1,m)=1$, $l(m,1)<m$ implies $g(m)>1$, contradicting $l(m,1)\geq g(m)$ because $m<1$.

In case $z<1$, $v'_+(z)=0$. Using this and by an argument similar to the one establishing the above claim, we can verify that $\forall m\in[0,\bar{m}]$, $l(m,1)=m$ and $l(1,m)\in[z,1]$, which again imply $v(m)=u(\beta v(m))$. □

B. The programmer-automata interpretation

Here I study the physical environment in Section 2. But now I consider the automata notion of the household (see the introduction). As indicated above, this seems to be the notion used in Shi (1999) and most of the recent literature. This literature specifically assumes that the household gives an order to each agent regarding what to do in a meeting. Following this literature, I restrict attention to the ultimatum game (buyers make offers) in pairwise meetings.

First I describe the automata game. In a generic household, the programmer at the start of a date chooses for each buyer an offering program and for each seller a responding program. A generic offering program is a pair of real-valued functions $(q(.),l(.))$ on $[0,M]^2$. A generic responding program is a function $\sigma(.)$ from $[0,M]^2\times\mathbb{R}^2_+$ to $\{0,1\}$. To see how the programs work, consider a meeting between a buyer from a household whose start-of-date money holding is m_b and a seller from a household whose start-of-date money holding is m_s. Let $(q(.),l(.))$ be the offering program carried by the buyer and let $\sigma(.)$ be the responding program carried by the seller. Then the buyer's offer is

$(q(m_b, m_s), l(m_b, m_s))$. If $0 \le l(m_b, m_s) \le m_b$ and $\sigma(m_b, m_s, (q(m_b, m_s), l(m_b, m_s)) = 1$, then this offer is carried out; otherwise, there is no trade.

I restrict attention to equilibria such that all buyers/sellers (from all households) are given the same offering/responding program forever. So an equilibrium is an offering program and a responding program such that it is optimal for a programmer to choose these programs currently and in future, provided that all other programmers choose these programs. Any equilibrium is degenerate in the same sense as in the main text.

Now let the household be large. I show that there exist a continuum of equilibria. Let y_1 be the one given by (27) and let $M > 1$ satisfy $u'(My_1) \ge 1$. Fix $y \in (0, y_1]$. Let $(q(m_b, m_s), l(m_b, m_s)) = (m_b y, m_b)$, for all (m_b, m_s). Let $\sigma(q, l, m_b, m_s) = 1$ if and only if $q \le ly$ and $0 \le l \le m_b$, for all (m_b, m_s). If all programmers choose $(q(.), l(.))$ and $\sigma(.)$, then the value function defined on the household's money holdings is the unique continuous, strictly increasing, and strictly concave function satisfying $v(m) = u(my) - y + \beta v(1)$. By $u'(My) \ge 1$, this v also satisfies

$$v(m) = \max_{0 \le l \le m, 0 \le \rho \le 1} u(ly) - \rho y + \beta v(m + \rho - l). \tag{45}$$

By (45), one programmer cannot gain from any deviation if the other programmers choose $(q(.), l(.))$ and $\sigma(.)$. So $(q(.), l(.))$ and $\sigma(.)$ constitute an equilibrium. In this equilibrium, $(y, 1)$ is the regular-meeting trade (in the same sense as in the main text), and the value function for the household's money holdings is concave.

In contrast, with the team notion, if A with $l(1,1) = 1$ is an equilibrium allocation generated by the ultimatum game and its associated value function is concave, then it can be shown that the regular-meeting output must be y_1. Shi (1999) formulates a value function v comparable to the one in (45). He argues that the buyer is able to extract all trade surplus, so the regular-meeting output y must be $\beta v'(1)$, which, in turn, implies $y = y_1$. With the automata notion, this argument is not valid. Now the split-the-surplus game is a simultaneous-move game between programmers, who choose programs for the buyer and seller, so the buyer, or more accurately, his programmer, may not be able to extract all the surplus even though the buyer makes a take-it-or-leave-it offer in the meeting.

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