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Transitive regret

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Preferences may arise from regret, i.e., from comparisons with alternatives forgone by the decision maker. We ask whether regret-based behavior is consistent with nonexpected utility theories of transitive choice and show that the answer is no. If choices are governed by ex ante regret and rejoicing, then nonexpected utility preferences must be intransitive.

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1. Introduction

Standard models of choice assume that decision makers act as if they maximize a preference relation over sets of options and these preferences are assumed to be independent of the environment. There are, however, good reasons to challenge this assumption. Preferences may depend on the decision maker's holding (reference point), on other people's holdings (envy), or on the choice set itself.

One such model is regret theory (Bell 1982 and Loomes and Sugden 1982). According to this theory the decision maker anticipates his future feelings about the choice he is about to make and acts according to these feelings. This approach is natural when the decision maker has to choose between two (or more) random variables. Once the uncertainty is resolved, he will know what outcome he received, but also what outcome he could have received had he chosen an alternative option. This comparison may lead to rejoicing—if his actual outcome is better than the alternative—or regret.

Formally, let X and Y be two random variables with money outcomes. Let $\psi(x, y)$ measure the regret or rejoicing a person feels when observing that he won x while the alternative choice would have landed him y. Choosing X over Y thus leads, ex ante, to a lottery $\Psi(X, Y)$ where the outcomes are $\psi(x, y)$. Choice is based on regret and rejoicing if there is a functional V over regret/rejoice lotteries such that X is chosen over Y if and only if $V(\Psi(X, Y)) > 0$.

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The question we ask is simple: What functionals V and regret/rejoice functions ψ are consistent with transitive choice? That is, when is it true that if $V(\Psi(X, Y)) > 0$ and $V(\Psi(Y, Z)) > 0$, then $V(\Psi(X, Z)) > 0$ as well? If regret is separable across events, that is, if $V(\Psi(X, Y)) = \sum_i V_i(\psi(x_i, y_i), s_i)$, then the possibility of having a violation of transitivity is well known (see Bell 1982, Loomes and Sugden 1982, and Fishburn 1989). In fact, in that case, violations of transitivity *must* be observed unless $\psi(x, y) = u(x) - u(y)$, which means that the original preferences are expected utility.[1] The main result of our paper is that regret-based transitive choice implies expected utility and this conclusion does not depend on V being linear in probabilities or even separable across states.

To see why this result is not obvious, consider the following intuition. For equiprobable partition $S_1, \ldots, S_n$, transitivity implies that for any vector of outcomes $(x_1, \ldots, x_n)$ and any permutation π of $\{1, \ldots, n\}$,

$$(x_1, S_1; \ldots; x_n, S_n) \sim (x_{\pi(1)}, S_1; \ldots; x_{\pi(n)}, S_n)$$

(see Proposition 1 below). Separability of V implies that the regret evaluation of $(x_i, x_{\pi(i)})$ in event S_i does not depend on what happens in event S_j, $j \neq i$. Therefore, any regret pair (x, y) can be evaluated through a lottery and its permutation as above. Without separability this cannot be done, as the evaluation of the regret pair (x, y) depends on the rest of the lottery.

One can read the result of the paper in two different ways. It offers a necessary and sufficient condition for a functional to be expected utility without making any references to mixture spaces (see Kreps 1988 for summary of terms and basic results). But the real contribution is the impossibility result that shows that regret is inherently intransitive. If so, then one must either conclude that (i) regret, despite its clear psychological appeal, cannot be used in standard economic models; (ii) models of regret that are richer than in Bell (1982) and Loomes and Sugden (1982) are necessary—for example, as is done in Sarver (2008) or by defining regret with respect to foregone distributions rather than foregone outcomes (see Machina 1987 and Starmer 2000 for some steps in this direction); (iii) models of intransitive preferences *must* be incorporated into economics as in Fishburn and LaValle (1988), Loomes and Sugden (1987), or Hayashi (2008).[2]

The paper is organized as follows. The model and the main result are presented in the next section. Section 3 offers an outline of the proof, while the details of the proof appear in the Appendix.

2. The model and main result

Let $\mathcal{L}$ be the set of real finite-valued random variables over (S, Σ, P) with $S = [0, 1]$, Σ being the standard Borel σ algebra on S, $\mathrm{P} = \mu$ being the Lebesgue measure, and the set of outcomes being the bounded interval $[\underline{x}, \bar{x}]$. The decision maker has a preference relation $\succeq$ over $\mathcal{L}$. In the sequel, we denote events by S_i and T_i.

[1] For this observation, see Sugden (2004, p. 739). We offer a formal proof of this claim in Lemma 7 below as we are not aware of one in the literature.

[2] See also Starmer (2000) for further references.

Definition 1. The continuous function $\psi:[\underline{x},\bar{x}]\times[\underline{x},\bar{x}]\to\Re$ is a *regret function* if for all x, $\psi(x,x)=0$, $\psi(x,y)$ is strictly increasing in x and strictly decreasing in y.

If in some event X yields x and Y yields y, then $\psi(x,y)$ is a measure of the decision maker's *ex post* feelings (of regret if $x<y$ or rejoicing if $x>y$) about the choice of X over Y. This leads to the next definition.

Definition 2. Let $X,Y\in\mathcal{L}$, where $X=(x_1,S_1;\dots;x_n,S_n)$ and $Y=(y_1,S_1;\dots;y_n,S_n)$. The *regret lottery* evaluating the choice of X over Y is

$$\Psi(X,Y)=(\psi(x_1,y_1),p_1;\dots;\psi(x_n,y_n),p_n),$$

where $p_i=\mathrm{P}(S_i)$, $i=1,\dots,n$. Denote the set of regret lotteries by $\mathcal{R}=\{\Psi(X,Y):X,Y\in\mathcal{L}\}$.

For brevity we refer to ψ and Ψ as regret function and regret lottery, respectively, even though they encompass both regret and rejoicing.

Definition 3. The preference relation $\succeq$ is *regret based* if there is a regret function ψ and a continuous functional V that is defined over regret lotteries such that for any $X,Y\in\mathcal{L}$,

$$X\succeq Y \quad \text{if and only if} \quad V(\Psi(X,Y))\geq 0.$$

The main result of this paper is the following.

Theorem 1. *Let $\succeq$ be a complete, transitive, continuous, and monotonic preference relation over the set $\mathcal{L}$ of random variables. The relation $\succeq$ is regret based if and only if it is expected utility.*

This theorem implies, in particular, the known result that the regret models of Bell (1982), Loomes and Sugden (1982), and Sugden (1993) are intransitive.[3] We take this result a step further and show that this intransitivity is not caused by separability across events, but is the result of regret itself.[4]

Recently, Sarver (2008) presented a nonexpected utility model of regret that is transitive, but it departs from the standard regret model of Bell (1982) and Loomes and Sugden (1982). In Sarver's model, the decision maker chooses between menus of lotteries and a lottery from the selected menu. At the time these two choices are made, the decision maker is uncertain about the utility of different outcomes. Later, after uncertainty is

[3]An important exception is the case where the choice set consists of statistically independent random variables, and for the two lotteries $(x_1,p_1;\dots;x_n,p_n)$ and $(y_1,q_1;\dots;y_m,q_m)$, the probability of the regret $\psi(x_i,y_j)$ is p_iq_j (see Machina 1987, pp. 138–140 and Starmer 2000, pp. 355–356). For example, Hong (1983) weighted utility theory is consistent with this form of regret.

[4]Gul's (1991) model of disappointment is transitive and nonexpected utility. The comparison in this model is between the outcome of a lottery and the lottery itself, rather than between possible outcomes of a pair of lotteries.

resolved, the decision maker may experience ex post regret if the selected lottery turns out to be inferior to another lottery that is also in the menu he selected. This induces a transitive, nonexpected utility preference relation over menus of lotteries in the initial period. However, this is not inconsistent with Theorem 1. First, if menus are singletons, then Sarver's model reduces to expected utility. Second, the source of uncertainty is different. In our model, the decision maker does not know which state of nature will hold and, therefore, he does not know what outcome he will receive. In Sarver's model, the decision maker does not know his future preferences and regret may emerge from realizing that given his (now) known preferences, he chose the wrong option.

Theorem 1 is proved as follows. It is well known that expected utility is regret based (with $\psi(x, y) = u(x) - u(y)$ and $V(\Psi(X, Y)) = \sum_i p_i \psi(x_i, y_i)$). That any transitive regret-based preferences must be expected utility is proved in a sequence of steps summarized below.

Step 1. Preferences satisfy the equivalence condition (Loomes and Sugden 1982, p. 818). That is, if X and Y have the same distribution, then $X \sim Y$ (Section 3.1, Proposition 1).

Step 2. The indifference curve of V through zero, $\{R : V(R) = 0\}$, is linear in probabilities (Section 3.3, Lemmas 3–5).

Step 3. There exists V as in Definition 3 that is linear in probabilities for all regret lotteries R (Section 3.4, Lemma 6).

Step 4. The preference relation $\succeq$ is expected utility (Section 3.4, Lemma 7).

3. Proof of the theorem

3.1 *Probabilistic equivalence*

When preferences are regret based, the decision maker cares about what events will happen as this will tell him what are the alternative outcomes he could have received had he chosen differently. When the decision maker learns that the number 4 on a die yields \$100 under X and \$150 under Y, the fact that these two outcomes are linked to the same state of the world is important, but the state itself is not. Consequently, only the probabilities of the underlying states are relevant for regret between X and Y. As long as the probability of the number 1 is the same as that of 4, it makes no difference whether the regret $\psi(100, 150)$ is obtained when the number is 1 or 4. This is why regret lotteries are evaluated with respect to their probabilities and not with respect to the generating events.

Proposition 1 shows that this observation, together with transitivity, has a significant implication to the evaluation of random variables. To see this, consider a box with n balls, numbered $1, \ldots, n$. Draw one ball at random, and let $X = (x_1, S_1; \ldots; x_n, S_n)$, where S_i is the event "ball i is drawn." Let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation of the n numbers and let $\pi(X) \equiv (x_{\pi(1)}, S_1; \ldots; x_{\pi(n)}, S_n)$. If $X \succ \pi(X)$, then according to the discussion in the last paragraph, it should also be the case that $\pi(X) \succ \pi^2(X)$, $\pi^2(X) \succ \pi^3(X), \ldots, \pi^{n!-1}(X) \succ \pi^{n!}(X)$. By transitivity, we obtain that $X \succ \pi^{n!}(X) = X$, a contradiction.

For $X \in \mathcal{L}$, let F_X be the distribution of X, that is, $F_X(x) = \mathrm{P}(X \leq x)$.

Proposition 1 (Probabilistic equivalence). *Let $\succeq$ be a continuous and transitive regret-based preference relation over $\mathcal{L}$. For any two random variables $X, Y \in \mathcal{L}$, if $F_X = F_Y$, then $X \sim Y$.*

Loomes and Sugden (1987) and Fishburn and LaValle (1988) use cycles as above to justify violations of transitivity. In Fishburn and LaValle (1988), a fair die is rolled and payments are made according to the number shown. Consider the random variables X_1 and $X_2 = \pi(X_1)$ given by

	S_1	S_2	S_3	S_4	S_5	S_6
X_1	\$1,000	\$500	\$600	\$700	\$800	\$900
X_2	\$900	\$1,000	\$500	\$600	\$700	\$800

As in five of six cases X_1 yields \$100 more than X_2, Fishburn and LaValle suggest that preferring X_1 to X_2 is natural. But of course, using such a permutation five more times leads to a nontransitive cycle.

The converse of Proposition 1 is not true. As is demonstrated by the following example, there are nontransitive regret-based preferences that satisfy probabilistic equivalence.

Example 1. For two random variables X and Y, find comonotonic X' and Y' with the same distributions as X and Y. Formally, for $X = (x_1, S_1; \ldots; x_n, S_n)$ and $Y = (y_1, T_1; \ldots; Y_m, T_m)$, find $X' = (x'_1, E_1; \ldots; x'_\ell, E_\ell)$ and $Y' = (y'_1, E_1; \ldots; y'_\ell, E_\ell)$ such that $x'_1 \leq \cdots \leq x'_\ell$, $y'_1 \leq \cdots \leq y'_\ell$, $F_X = F_{X'}$, and $F_Y = F_{Y'}$. Observe that X' and Y' depend on both X and Y. Define now $X \succeq Y$ if and only if $V(X', Y') \geq 0$, where $V(X', Y') = \sum \mathrm{P}(E_i)(x'_i - y'_i)^3$. In other words, $\succeq$ is regret based with respect to the probability distribution functions. As such, it satisfies probabilistic equivalence.

Let $\mathrm{P}(E_1) = \mathrm{P}(E_2) = \mathrm{P}(E_3) = \frac{1}{3}$. The random variables X, Y, Z are given by

	E_1	E_2	E_3
X	8	19	30
Y	9	20	28
Z	10	18	29

Clearly $V(X, Y) = V(Y, Z) = V(Z, X) = 6$, hence $X \succ Y$, $Y \succ Z$, but $Z \succ X$. ◊

3.2 *Preliminary results*

We assume that outcomes are in a finite interval $[\underline{x}, \bar{x}]$. Let $\underline{r} = \psi(\underline{x}, \bar{x})$ and $\bar{r} = \psi(\bar{x}, \underline{x})$. By the continuity of the regret functional, $-\infty < \underline{r} < 0 < \bar{r} < \infty$. As $\psi(x, y)$ is continuous, increasing in x, and decreasing in y, it follows that the set of regret lotteries $\mathcal{R}$ defined in Definition 2 is the set of finite-valued lotteries with outcomes in the interval $[\underline{r}, \bar{r}]$. The following monotonicity properties of V are inherited from the monotonicity of $\succeq$.

LEMMA 1. *Let R and R' be two distinct regret lotteries such that R dominates R' by first-order stochastic dominance (FOSD).*

(i) If $V(R) = 0$, then $V(R') < 0$.

(ii) If $V(R') = 0$, then $V(R) > 0$.

The next lemma permits a selection of regret lotteries that are skew symmetric in regret and rejoicing.

LEMMA 2. *(i) If $\psi(x, y) = \psi(x', y')$, then $\psi(y, x) = \psi(y', x')$.*

(ii) The equality $\psi(x, y) = -\psi(y, x)$ is without loss of generality.

We will assume throughout that $\psi(x, y) = -\psi(y, x)$ and that $\Psi(X, Y) = -\Psi(Y, X) \equiv (-\psi(y_1, x_1), p_1; \dots; -\psi(y_n, x_n), p_n)$. Moreover, $\underline{r} = -\bar{r}$.

3.3 *The indifference curve through zero is linear*

A regret lottery R is generated by a permutation if there exists a random variable $X = (x_1, S_1; \dots; x_n, S_n)$, $\mathrm{P}(S_i) = 1/n$, and a permutation π of X such that $\Psi(X, \pi(X)) = R$. By Proposition 1, if R is generated by a permutation, then $V(R) = 0$. The next lemma shows that the subset of $\{R : V(R) = 0\}$ that is generated by permutations is convex.

LEMMA 3. *If R and R' are generated by permutations, then so is $\frac{1}{2}R + \frac{1}{2}R'$.*

As R and R' are generated by permutations, we have $V(R) = V(R') = 0$ and, by Lemma 3, $V(\frac{1}{2}R + \frac{1}{2}R') = 0$. As is shown by the next example, one cannot guarantee that every regret lottery $R = (r_1, 1/n; \dots; r_n, 1/n)$ such that $V(R) = 0$ is generated by a permutation.

EXAMPLE 2. Consider an expected value maximizer whose choice set consists of random variables with prizes in the interval $[-3, 3]$. This individual's regret function is $\psi(x, y) = x - y$ and he is indifferent between X and Y defined below, where $\mathrm{P}(S_i) = 0.2$:

$$X = (3, S_1; 3, S_2; -1, S_3; -1, S_4; -1, S_5)$$
$$Y = (-3, S_1; -3, S_2; 3, S_3; 3, S_4; 3, S_5).$$

As $X \sim Y$, $V(\Psi(X, Y)) = V(6, 0.2; 6, 0.2; -4, 0.2; -4, 0.2; -4, 0.2) = 0$. But there does not exist a random variable $\hat{Z}$ with outcomes in the interval $[-3, 3]$ and a permutation π such that $\Psi(X, Y) = \Psi(\hat{Z}, \pi(\hat{Z}))$. To see why, observe that the rejoicing 6 must be generated by the outcomes -3 and 3. From outcome 3, only regret is possible, and as the only regret level is -4, the outcome 3 must be paired with -1. From outcome -1, one cannot generate rejoicing 6 or have regret -4.[5] $\diamondsuit$

[5]If, instead, we had assumed that the set of outcomes was $(-\infty, \infty)$, then any $R = (r_1, 1/n; \dots; r_n, 1/n)$ such that $V(R) = 0$ would be generated by a permutation, leading to a simpler proof of Theorem 1.

The problem is that the outcomes in X and Y are far apart. However, as is shown by the next example, one can find in Example 2 a random variable Z whose outcomes are sufficiently close to both X and Y such that $X \sim Z \sim Y$, and the regret lotteries $\Psi(X, Z)$ and $\Psi(Z, Y)$ are generated by permutations.

EXAMPLE 3. Using the notation of Example 2, let $Z = (0, S_1; 0, S_2; 1, S_3; 1, S_4; 1, S_5)$. Thus

$$\Psi(X, Z) = \Psi(Z, Y) = (3, 0.2; 3, 0.2; -2, 0.2; -2, 0.2; -2, 0.2).$$

Define

$$\hat{Z} = (3, S_1; 0, S_2; -3, S_3; -1, S_4; 1, S_5)$$
$$\pi(\hat{Z}) = (0, S_1; -3, S_2; -1, S_3; 1, S_4; 3, S_5).$$

Then $\Psi(\hat{Z}, \pi(\hat{Z})) = \Psi(X, Z) = \Psi(Z, Y)$. ◊

This idea is formalized below.

LEMMA 4. *Let $X \sim Y$, where $X = (x_1, S_1; \dots; x_n, S_n)$, $Y = (y_1, S_1; \dots; y_n, S_n)$, and $\mathrm{P}(S_i) = 1/n$. Then there is a sequence $X = Z_1 \sim Z_2 \sim \cdots \sim Z_k = Y$ such that for every $\ell = 1, \dots, k-1$, there is a regret lottery $\hat{Z}_\ell$ and a permutation π_ℓ so that $\Psi(Z_\ell, Z_{\ell+1}) = \Psi(\hat{Z}_\ell, \pi_\ell(\hat{Z}_\ell))$.*

Thus, even if a regret lottery $R = (r_1, 1/n; \dots; r_n, 1/n)$ with $V(R) = 0$ is not generated by a permutation, one can find a sequence of random variables $Z_1 \sim \cdots \sim Z_k$ such that each $\Psi(Z_\ell, Z_{\ell+1})$ is generated by a permutation and $R = \Psi(Z_1, Z_k)$. This is used to prove that the set $\{R : V(R) = 0\}$ is convex.

LEMMA 5. *If $V(R) = V(R') = 0$, then $V(\frac{1}{2}R + \frac{1}{2}R') = 0$.*

3.4 *V is linear in probabilities and $\succeq$ is expected utility*

The following lemma establishes that all indifference curves of V are linear.

LEMMA 6. *(i) There is a function $v : [-\bar{r}, \bar{r}] \to \Re$ such that $V(R) \gtreqless 0$ if and only if $\mathrm{E}[v(R)] \gtreqless 0$.*

(ii) Moreover, v is strictly increasing with $v(0) = 0$ and $v(\psi(x, y)) = -v(\psi(y, x))$ for all x, y.

We now use the function v to create a function u on outcomes that will turn out to be the von Neuman–Morgenstern utility claimed by Theorem 1.

LEMMA 7. *There exists an increasing function $u : [\underline{x}, \bar{x}] \to \Re$ such that*

$$v(\psi(x, y)) = u(x) - u(y).$$

From the last two lemmas, we have for $X = (x_1, S_1; \dots; x_n, S_n)$ and $Y = (y_1, S_1; \dots; y_n, S_n)$, where $\mathrm{P}(S_i) = p_i$,

$$\begin{aligned} X \succeq Y &\iff V(\Psi(X, Y)) \ge 0 \\ &\iff \sum_i p_i v(\psi(x_i, y_i)) \ge 0 \\ &\iff \sum_i p_i [u(x_i) - u(y_i)] \ge 0 \\ &\iff \mathrm{E}[u(X)] \ge \mathrm{E}[u(Y)], \end{aligned}$$

which is the claim of the theorem.

Appendix

Proof of Proposition 1. Let $X = (x_1, S_1; \dots; x_n, S_n)$ and $Y = (y_1, S_1'; \dots; y_n, S_n')$ be such that $F_X = F_Y$.

Case 1. $S_i = S_i'$ and $\mathrm{P}(S_i) = 1/n$, $i = 1, \dots, n$. Then there is a permutation $\hat{\pi}$ such that $Y = \hat{\pi}(X)$. Obviously, $\Psi(X, \hat{\pi}(X)) = \Psi(\hat{\pi}^i(X), \hat{\pi}^{i+1}(X))$. Hence, as there exists $m \le n!$ such that $\hat{\pi}^m(X) = X$, it follows by transitivity that for all i, $X \sim \hat{\pi}^i(X)$. In particular, $X \sim Y$.

Case 2. For all i, j, $\mathrm{P}(S_i \cap S_j')$ is a rational number. Let N be a common denominator of all these fractions. Random variables X and Y can now be written as in Case 1 with equiprobable events $T_1, \dots, T_N$.

Case 3. There exist i and j, such that $\mathrm{P}(S_i \cap S_j')$ is irrational. Any random variable $Z = (z_1, T_1; \dots; z_n, T_n)$ is the limit of $Z^k = (z_1^k, T_1^k; \dots; z_{2^k}^k, T_{2^k}^k)$, where for all k and ℓ, $\mathrm{P}(z_\ell^k) = 2^{-k}$. This case follows by continuity from Case 2. □

Proof of Lemma 1. Let R and R' be two regret lotteries. As usual, R dominates R' by FOSD if and only if there is a list of probabilities $p_1, \dots, p_n$ adding up to 1 such that $R = (r_1, p_1; \dots; r_n, p_n)$ and $R' = (r_1', p_1; \dots; r_n', p_n)$, and for all i, $r_i \ge r_i'$.

From the continuity of ψ, we know that for every $r \in [\underline{r}, \bar{r}]$ there exist $x, y \in [\underline{x}, \bar{x}]$ such that $r = \psi(x, y)$. Hence there are $X, Y \in \mathcal{L}$ such that $\Psi(X, Y) = R$. By the continuity and monotonicity of ψ, we can find X' and Y' such that $x_i' \le x_i$, $y_i' \ge y_i$, $\psi(x_i', y_i') = r_i'$ for each i, and $\Psi(X', Y') = R'$. Either X strictly dominates X' by FOSD or Y' strictly dominates Y by FOSD (or both). Monotonicity of $\succeq$ implies that $X \succeq X'$ and $Y' \succeq Y$ with at least one of these preferences being strict.

(i) If $V(R) = 0$, then $X \sim Y$. By transitivity, $X' \prec Y'$ and hence $V(R') = V(\Psi(X', Y')) < 0$.

(ii) If $V(R') = 0$, then $X' \sim Y'$. By transitivity, $X \succ Y$ and, therefore, $V(R) = V(\Psi(X, Y)) > 0$. □

Proof of Lemma 2. (i) Let S_1 and S_2 be two disjoint events, where $P(S_1) = P(S_2) = 0.5$. Define the lotteries $X = (x, S_1; y, S_2)$, $Y = (y, S_1; x, S_2)$, $X' = (x', S_1; y', S_2)$, and $Y' = (y', S_1; x', S_2)$. Let $r = \psi(x, y) = \psi(x', y')$. Then

$$\Psi(X, Y) = (r, 0.5; \psi(y, x), 0.5)$$
$$\Psi(X', Y') = (r, 0.5; \psi(y', x'), 0.5).$$

By Proposition 1, $X \sim Y$ and $X' \sim Y'$; thus, we have $V(\Psi(X, Y)) = V(\Psi(X', Y')) = 0$. But if $\psi(y, x) \neq \psi(y', x')$, then $\Psi(X, Y)$ either dominates or is dominated by $\Psi(X', Y')$, contradicting Lemma 1.

(ii) Recall that $\psi(x, x) = 0$. Let $f : [\underline{r}, \bar{r}] \to [-\bar{r}, \bar{r}]$ be defined as

$$f(r) = \begin{cases} -\psi(y, x) & \text{if } r < 0 \text{ and } x < y \text{ is such that } \psi(x, y) = r \\ r & \text{if } r \geq 0. \end{cases}$$

By the first part of this lemma, the value of $f(r)$ for $r < 0$ does not depend on the choice of x, y in the above definition; hence f is well defined. Monotonicity of ψ implies that f is strictly increasing. We can, therefore, define

$$V^*(r_1, p_1; \ldots; r_n, p_n) = V(f^{-1}(r_1), p_1; \ldots; f^{-1}(r_n), p_n).$$

Let

$$\psi^*(x, y) = \begin{cases} \psi(x, y) & \text{if } x \geq y \\ f(\psi(x, y)) & \text{if } x < y. \end{cases}$$

Now

$$\begin{aligned} X \succeq Y \quad &\iff \quad V(\Psi(X, Y)) \geq V(\Psi(Y, X)) \\ &\iff \quad V^*(\Psi^*(X, Y)) \geq V^*(\Psi^*(Y, X)), \end{aligned}$$

where $\Psi^*(X, Y)$ is obtained from $\Psi(X, Y)$ by replacing $\psi(x, y)$ with $\psi^*(x, y)$. □

Proof of Lemma 3. In the sequel, random variables Q with m (not necessarily distinct) outcomes are of the form $(q_1, S_1^m; \ldots; q_m, S_m^m)$ for some canonical partition where $P(S_i^m) = 1/m$, $i = 1, \ldots, m$. For Q and Q' with m outcomes each, let

$$\langle Q, Q' \rangle = (q_1, S_1^{2m}; \ldots; q_m, S_m^{2m}; q'_1, S_{m+1}^{2m}; \ldots; q'_m, S_{2m}^{2m}),$$

where $P(S_i^{2m}) = 1/(2m)$.

Let R and R' be generated by permutations π of $X = (x_1, S_1; \ldots; x_n, S_n)$ and π' of $Y = (y_1, S'_1; \ldots; y_n, S'_n)$, respectively, where $P(S_i) = P(S'_i) = 1/n$, $i = 1, \ldots, n$. That is, $R = \Psi(X, \pi(X))$ and $R' = \Psi(Y, \pi'(Y))$. (The assumption that X and Y are of the same length is without loss of generality.) Define $Z = \langle X, Y\rangle$ and $\pi^* : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ by

$$\pi^*(i) = \begin{cases} \pi(i) & \text{if } i \le n \\ \pi'(i-n)+n & \text{if } i > n \end{cases}$$

to obtain $\Psi(Z, \pi^*(Z)) = \Psi(\langle X, Y\rangle, \pi^*\langle X, Y\rangle) = \frac{1}{2}R + \frac{1}{2}R'$. □

Proof of Lemma 4. All random variables in this proof have n outcomes on the equiprobable events $S_1, \ldots, S_n$. For $Z = (z_1, S_1; \ldots; z_n, S_n)$ and $Z' = (z_1', S_1; \ldots; z_n', S_n)$, define $\|Z - Z'\| = \max_i |z_i - z_i'|$.

The proof follows from Claims 1 and 2.

Claim 1. *Let $X \sim Y$. For any $\delta > 0$, there exist $Z_1, \ldots, Z_k$ such that $X = Z_1 \sim \cdots \sim Z_k = Y$ and $\|Z_{\ell-1} - Z_\ell\| \le \delta$, $\ell = 2, \ldots, k$.*

Proof. We construct the sequence $Z_1, \ldots$ inductively. Suppose that $X \neq Y$ and that we have already defined $X = Z_1 \sim \cdots \sim Z_\ell$ such that $\|Z_{i-1} - Z_i\| \le \delta$, $i = 2, \ldots, \ell$. If $Z_\ell = Y$, we are through. Otherwise, define $L_+^\ell = \{i : z_i^\ell > y_i\}$ and $L_-^\ell = \{i : z_i^\ell < y_i\}$. As $Z_\ell \sim Y$ and $Z_\ell \neq Y$, both L_+^ℓ and L_-^ℓ are nonempty. Let

$$\delta_+^\ell = \min_{i \in L_+^\ell} \{z_i^\ell - y_i\}$$

$$\delta_-^\ell = \min_{i \in L_-^\ell} \{y_i - z_i^\ell\}.$$

Define $f_\ell(\theta)$ such that $Z_\ell \sim Z_{\ell+1}(\theta) \equiv (z_1^{\ell+1}(\theta), S_1; \ldots; z_n^{\ell+1}(\theta), S_n)$, where

$$z_i^{\ell+1}(\theta) = \begin{cases} z_i^\ell - \theta & \text{if } i \in L_+^\ell \\ z_i^\ell + f_\ell(\theta) & \text{if } i \in L_-^1 \\ z_i^\ell & \text{otherwise.} \end{cases}$$

By continuity and monotonicity of $\succeq$, $f_\ell(\theta)$ is well defined (for small θ), continuous, and increasing. Its inverse exists and is continuous. Define $\theta^\ell = \min\{\delta, \delta_+^\ell, f_\ell^{-1}(\delta_-^\ell)\}$ and let $Z_{\ell+1} = Z_{\ell+1}(\theta^\ell)$. Note that $Z_1, \ldots, Z_{\ell+1}$ satisfy the hypothesis of the claim.

If $\theta^\ell = \delta$, then $\|Z_{\ell+1} - Y\| \le \|Z_\ell - Y\| - \delta$. If $\theta^\ell = \delta_+^\ell$, then $|L_+^{\ell+1}| \le |L_+^\ell| - 1$. If $\theta^\ell = f_\ell^{-1}(\delta_-^\ell)$, then $|L_-^{\ell+1}| \le |L_-^\ell| - 1$. Thus, this process terminates in a finite number of steps with $Z_k = Y$. ◁

Claim 2. *There exists $\varepsilon_n > 0$ such that if for all i, $|r_i| < \varepsilon_n$, then there exist a random variable $\hat{Z}$ and a permutation π such that $R = (r_1, 1/n; \ldots; r_n, 1/n)$ satisfies $R = \Psi(\hat{Z}, \pi(\hat{Z}))$.*

Proof. The domain of outcomes is $[\underline{x}, \bar{x}]$. Let

$$\hat{z}_1 = \frac{\bar{x} + \underline{x}}{2}$$

$$\delta_n = \frac{\bar{x} - \underline{x}}{2n} = \frac{\hat{z}_1 - \underline{x}}{n} = \frac{\bar{x} - \hat{z}_1}{n} > 0.$$

Thus, $\hat{z}_1 + n\delta_n = \bar{x}$ and $\hat{z}_1 - n\delta_n = \underline{x}$.

The function ψ is continuous on the compact segment $[\underline{x}, \bar{x}]$; therefore, for any $\delta_n > 0$, there exists $\varepsilon_n > 0$ such that $|\psi(x,y)| < \varepsilon_n$ implies $|x-y| < \delta_n$. Thus, with $|r_i| < \varepsilon_n$ we can construct $\hat{Z}$ such that

	S_1	S_2	S_3	S_4	$\cdots$	S_{n-1}	S_n
$\hat{Z}$	$\hat{z}_1$	$\hat{z}_2$	$\hat{z}_3$	$\hat{z}_4$	$\cdots$	$\hat{z}_{n-1}$	$\hat{z}_n$
$\pi(\hat{Z})$	$\hat{z}_2$	$\hat{z}_3$	$\hat{z}_4$	$\hat{z}_5$	$\cdots$	$\hat{z}_n$	$\hat{z}_1$
$\Psi(\hat{Z}, \pi(\hat{Z}))$	r_1	r_2	r_3	r_4	$\cdots$	r_{n-1}	$\psi(\hat{z}_n, \hat{z}_1)$

Outcome $\hat{z}_1$ is chosen to be the midpoint between $\underline{x}$ and $\bar{x}$, and each $\hat{z}_{\ell+1}$ is chosen so that $\psi(\hat{z}_\ell, \hat{z}_{\ell+1}) = r_\ell$, $\ell = 1, 2, \ldots, n-1$. As $|r_\ell| < \varepsilon_n$, we have $|\hat{z}_\ell - \hat{z}_{\ell+1}| < \delta_n$ and each $\hat{z}_\ell \in [\underline{x}, \bar{x}]$. As $V(R) = V(\Psi(\hat{Z}, \pi(\hat{Z}))) = 0$, it must be that $\psi(\hat{z}_n, \hat{z}_1) = r_n$. Otherwise, R either dominates or is dominated by $\Psi(\hat{Z}, \pi(\hat{Z}))$, contradicting Lemma 1. Thus, $R = \Psi(\hat{Z}, \pi(\hat{Z}))$. ◁

This completes the proof of Lemma 4. □

Proof of Lemma 5. For $R = (r_1, 1/n; \ldots; r_n, 1/n)$ and $R' = (r'_1, 1/n; \ldots; r'_n, 1/n)$ such that $\psi(R) = \psi(R') = 0$, let X, Y, X', Y' be such that $\Psi(X,Y) = R$ and $\Psi(X', Y') = R'$. By Lemma 4, there exist sequences $X = Z_1 \sim \cdots \sim Z_k = Y$ and $X' = Z'_1 \sim \cdots \sim Z'_k = Y'$ such that for all $\ell = 1, \ldots, k-1$ there exist $\hat{Z}_\ell, \pi_\ell, \hat{Z}'_\ell, \pi'_\ell$ satisfying $\Psi(\hat{Z}_\ell, \pi_\ell(\hat{Z}_\ell)) = \Psi(Z_\ell, Z_{\ell+1})$ and $\Psi(\hat{Z}'_\ell, \pi'_\ell(\hat{Z}'_\ell)) = \Psi(Z'_\ell, Z'_{\ell+1})$.[6] Thus, for each $\ell = 1, 2, \ldots, k-1$, the pair of regret lotteries $\Psi(Z_\ell, Z_{\ell+1})$ and $\Psi(Z'_\ell, Z'_{\ell+1})$ satisfies the hypothesis of Lemma 3. Therefore,

$$V\Big(\tfrac{1}{2}\Psi(Z_\ell, Z_{\ell+1}) + \tfrac{1}{2}\Psi(Z'_\ell, Z'_{\ell+1})\Big) = 0.$$

Note that $\frac{1}{2}\Psi(Z_\ell, Z_{\ell+1}) + \frac{1}{2}\Psi(Z'_\ell, Z'_{\ell+1}) = \Psi(\langle Z_\ell, Z'_\ell\rangle, \langle Z_{\ell+1}, Z'_{\ell+1}\rangle)$, where $\langle\cdot,\cdot\rangle$ is defined in the proof of Lemma 3. Consequently,

$$\langle X, X'\rangle = \langle Z_1, Z'_1\rangle \sim \cdots \sim \langle Z_k, Z'_k\rangle = \langle Y, Y'\rangle.$$

Hence

$$V\big(\Psi(\langle X, X'\rangle, \langle Y, Y'\rangle)\big) = 0,$$

but

$$\Psi(\langle X, X'\rangle, \langle Y, Y'\rangle) = \tfrac{1}{2}R + \tfrac{1}{2}R'$$

and we obtain $V(\frac{1}{2}R + \frac{1}{2}R') = 0$.

As each $X \in \mathcal{L}$ is the limit of a sequence $\{X^k\}$, where for each k, $X_k = (x^k_1, 1/n_k; \ldots; x^k_{n_k}, 1/n_k)$, the lemma now follows by continuity for all R and R' such that $V(R) = V(R') = 0$. □

[6] We use the same k in both sequences without loss of generality, as the sequences may become stationary from a certain point on.

Proof of Lemma 6. Recall that $V(\delta_{\bar{r}}) > 0 > V(\delta_{-\bar{r}})$, where δ_t is the constant lottery yielding t.

(i) For a regret lottery R such that $V(R) > 0$, let $\alpha(R)$ be defined by $V(\alpha(R)R + (1-\alpha(R))\delta_{-\bar{r}}) = 0$, and for R such that $V(R) < 0$, let $\alpha(R)$ be defined by $V(\alpha(R)R + (1-\alpha(R))\delta_{\bar{r}}) = 0$. By Lemma 1 and the continuity of V, $\alpha(R)$ is well defined and $\alpha(R) < 1$. Let α^* satisfy $V(\alpha^*\delta_{\bar{r}} + (1-\alpha^*)\delta_{-\bar{r}}) = 0$.

We show first that α is a continuous function. Let $R_k \to R_0$ and suppose that $\alpha(R_k) \to \alpha'$.[7] Suppose without loss of generality that for all k, $V(R_k) \geq 0$. By the continuity of V,

$$V(\alpha' R_0 + (1-\alpha')\delta_{-\bar{r}}) = \lim_k V\big(\alpha(R_k)R_k + (1-\alpha(R_k))\delta_{-\bar{r}}\big) = 0,$$

hence $\alpha' = \alpha(R_0)$.

Define now

$$U(R) = \begin{cases} \dfrac{\alpha^*}{\alpha(R)} - \alpha^* & \text{if } V(R) > 0 \\ 0 & \text{if } V(R) = 0 \\ 1 - \alpha^* - \dfrac{1-\alpha^*}{\alpha(R)} & \text{if } V(R) < 0. \end{cases}$$

For R such that $V(R) \neq 0$, $\alpha(R) < 1$; hence $U(R) \gtreqless 0$ if and only if $V(R) \gtreqless 0$. The continuity of $\alpha(\cdot)$ implies that $U(R)$ is continuous. We show next that U is linear. That is, for all R and R', $U(\frac{1}{2}R + \frac{1}{2}R') = \frac{1}{2}U(R) + \frac{1}{2}U(R')$.

By Lemma 5 and the continuity of V we have the following conclusion.

Conclusion 1. If $V(R) = V(R') = 0$, then for all $\alpha \in [0,1]$, $V(\alpha R + (1-\alpha)R') = 0$.

For arbitrary regret lotteries R and R', consider the three dimensional simplex Δ of lotteries over R, R', $\delta_{\bar{r}}$, $\delta_{-\bar{r}}$. Take a linear transformation K of Δ such that $K(\delta_{-\bar{r}}) = (0,0,-1)$, $K(\delta_{\bar{r}}) = (0,0,(1-\alpha^*/\alpha^*))$, $K(R) = (x^*, y^*, z^*)$, $K(R') = (x', y', z')$, and, by Conclusion 1, $V(x,y,z) = 0$ if and only if $z = 0$. It follows that for $z > 0$, $\alpha(x,y,z)$ solves

$$\alpha z - (1-\alpha) = 0 \quad \Longrightarrow \quad \alpha(x,y,z) = \frac{1}{z+1}$$

and for $z < 0$, $\alpha(x,y,z)$ solves

$$\alpha z + (1-\alpha)\frac{1-\alpha^*}{\alpha^*} = 0 \quad \Longrightarrow \quad \alpha(x,y,z) = \frac{1-\alpha^*}{1-\alpha^*-\alpha^* z}.$$

In both cases, $U(x,y,z) = \alpha^* z$.

Define now a preference relation $\succeq^*$ on regret lotteries by $R \succeq^* R'$ if and only if $U(R) \geq U(R')$. Since U is continuous, so is $\succeq^*$, and since U is linear, $\succeq^*$ satisfies the independence axiom. Therefore, there is a function v such that $U(R) \gtreqless 0$ if and only if $\mathrm{E}[v(R)] \gtreqless 0$. The lemma follows since $U(R) \gtreqless 0$ if and only if $V(R) \gtreqless 0$.

[7] If $\alpha(R_k)$ does not have a limit, then we take a subsequence that has a limit.

(ii) Suppose that $v(\cdot)$ is not strictly increasing. Then there exists $r_1 < r_2$ such that $v(r_1) \geq v(r_2)$. Take $R = (r_1, p_1; r_2, p_2; \dots; r_n, p_n)$ such that $V(R) = 0$. The continuity of V implies that such an R exists. Construct $R' = (r_1, p_1 - \varepsilon; r_2, p_2 + \varepsilon; \dots; r_n, p_n)$. Clearly R' dominates R by FOSD, but $0 = V(R) \geq V(R')$, contradicting Lemma 1. The fact that $v(0) = 0$ follows from $V(0, 1) = 0$.

Finally, let S_1 and S_2 be two disjoint events where $\mathrm{P}(S_1) = \mathrm{P}(S_2) = 0.5$. Define $X = (x, S_1; y, S_2)$ and $Y \equiv (y, S_1; x, S_2)$. By Proposition 1, $X \sim Y$. Thus $v(\psi(x, y)) = -v(\psi(y, x))$. □

Proof of Lemma 7. The following claim follows from a theorem in Aczèl (1966) and is mentioned, without an explicit proof, in Sugden (2004, p. 739).

Claim 3. *If $G(x, y) + G(y, z) = G(x, z)$ for all $x < y < z$, then there exists a function $g : \Re \to \Re$ such that $G(x, y) = g(x) - g(y)$.*

Proof. Define

$$H(x,y) = \begin{cases} G(x,y) & \text{if } x < y \\ 0 & \text{if } x = y \\ -G(y,x) & \text{if } x > y. \end{cases}$$

It may be verified that for all x, y, z,

$$H(x, y) + H(y, z) = H(x, z).$$

Therefore, Aczèl (1966, Theorem 1, p. 223) implies that there exists $g : \Re \to \Re$ such that $H(x, y) = g(x) - g(y)$. ◁

Select $x_1 < x_2 < x_3$ and $p, q > 0$, $p \neq q$, $p + q < \frac{1}{3}$. Define lotteries X and Y as follows:

	S_1	S_4	S_7	S_2	S_5	S_8	S_3	S_6	S_9
$\mathrm{P}(S_i)$	p	p	p	q	q	q	$\frac{1}{3} - p - q$	$\frac{1}{3} - p - q$	$\frac{1}{3} - p - q$
X	x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
Y	x_3	x_1	x_2	x_2	x_3	x_1	x_1	x_2	x_3

Proposition 1 implies $X \sim Y$, as each of these lotteries gives x_1, x_2, and x_3 with probability $\frac{1}{3}$ each. Thus, $V(\Psi(X, Y)) = 0$ and, by Lemma 6, $\mathrm{E}[v(\Psi(X, Y))] = 0$. As $v(\psi(x, y)) = -v(\psi(y, x))$ and $v(\psi(x, x)) = 0$ (see Lemma 6), it follows that

$$[q - p]v(\psi(x_1, x_2)) + [q - p]v(\psi(x_2, x_3)) + [p - q]v(\psi(x_1, x_3)) = 0.$$

Since $p \neq q$, we obtain for all $x_1 < x_2 < x_3$, $v(\psi(x_1, x_2)) + v(\psi(x_2, x_3)) = v(\psi(x_1, x_3))$. By Claim 3, there exists a function $u : \Re \to \Re$ such that $v(\psi(x_1, x_2)) = u(x_1) - u(x_2)$. Monotonicity of u follows from the monotonicity of $\succeq$. □

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