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WEIGHTED PROPORTIONAL LOSSES SOLUTION

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ABSTRACT. We propose and characterize a new solution for problems with asymmetric bargaining power among the agents that we named *weighted proportional losses solution*. It is specially interesting when agents are bargaining under restricted probabilistic uncertainty. The weighted proportional losses assigns to each agent losses proportional to her ideal utility and also proportional to her bargaining power. This solution is always individually rational, even for 3 or more agents and it can be seen as the normalized *weighted equal losses* solution. When bargaining power among the agents is equal, the weighted proportional losses solution becomes the Kalai-Smorodinsky solution. We characterize our solution in the basis of *restricted monotonicity* and *restricted concavity*. A consequence of this result is an alternative characterization of Kalai-Smorodinsky solution which includes contexts with some kind of uncertainty. Finally we show that weighted proportional losses solution satisfies desirable properties as are *strong Pareto optimality* for 2 agents and *continuity* also fulfilled by Kalai-Smorodinsky solution, that are not satisfied either by weighted or asymmetric Kalai-Smorodinsky solutions.

INTRODUCTION

In his seminal paper, Nash (1950)[13] defined a *bargaining solution* as a function that produces, for each problem in the class, an alternative in that problem. Nash's objective was to develop a theory that would help to predict the compromise the agents would reach. He characterized a solution, now called the Nash solution, and showed that this solution is the only that satisfies a certain list of axioms. Nash limited his attention to the two-person case but his solution, and his characterization,can easily be extended to the n-person case.

A central axiom in Nash's characterization is *Independence of Irrelevant Alternatives* axiom, which says that whether a point is the solution, all contractions of the set will not affect the solution provided this point remains feasible. This axiom has been criticized in the subsequent years.

Kalai and Smorodinsky 1975, [10] proposed a new axiom, the *Individual Monotonicity*, which says that whether the feasible set enlarges, the utilities of all agents should improve or at least should not be smaller. These authors felt that *Independence of Irrelevant Alternatives* was not fully justified and, in their opinion, if the alternatives to the solution for one agent are worse than the alternatives of the other agent, this should profit of this situation. This means that the alternatives that the agents have if the set enlarges matters to reach an agreement.

Other solutions proposed after Nash are: the *Egalitarian* solution (Kalai 1977b[9]), that is the maximal point in the set of equal coordinates; *Dictatorial* solutions, that is the maximal point of the set with the maximal "dictator agent" coordinate;

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the *Perles-Maschler* solution (Perles and Maschler in 1981[14]) by means of *super-additivity* axiom, this axiom means that the solution of the sum of the sets is equal or greater than the sum of the solutions of each set. This property is very close to *set concavity* axiom, which says that if the feasible set is uncertain, the agents would prefer to reach an agreement before uncertainty is resolved; *Utilitarian* solution, that is the maximization of the sum of all utilities and *Equal losses* solution, that means the point on the Pareto frontier where all agents make the same concessions.

In the literature are reflected some approaches that analyze asymmetric bargaining problems, that is, bargaining situations where the bargaining power is not the same among the agents; the first contribution is Kalai (1977a)[8] with the *Nonsymmetric Nash Solution*, that arise from symmetric Nash solution through replications. In this work Kalai shows that this replications do not work for the case of monotonic solutions because after replications the replicated monotonic solutions keep the symmetric property.

Kalai (1977b) [9] proposes the *Proportional* solution as an asymmetric *Egalitarian* solution (E^{α}), we will illustrate the solution in the figure 1.

There exist two versions of the Kalai-Smorodinsky solution (KS) that can reflect asymmetries in the bargaining power that agents have (or other difference among agents expressed in relative terms). The first, *Weighted KS* solution (KS^{α}), was introduced by Thomson (1994)[21] and consists in an asymmetric one parameter function that generalizes the KS solution. The second, *Asymmetric KS* solution was introduced by Dubra (2001)[4] and consists in a *lexicographic* extension of the weighted KS solution (lKS^{α}).

There have been several attempts in the literature to characterize asymmetric solutions by means of a *reference point*, which could in some cases be *endogenous* and in other cases *exogenous*. There exist two approaches in this way, solutions defined by the maximal point in the set on a straight line passing though the *disagreement point* and the reference point; and solutions defined by the maximal point in the set on the straight line that connect the reference point with the *utopia point*. Salonen[18] shows a solution on the straight line that connect the disagreement point with a endogenous reference point and Salonen[19] shows a new solution on the straight line that connect the same endogenous reference point with the utopia point. Gupta and Livne[6] show a solution on the straight line that connect a exogenous reference point with the utopia point and Gupta[5] shows a solution for multiple issue bargaining using an endogenous reference point. Anbarci[1] shows the concept of *gravity center* as a endogenous reference point and later define two concepts of solutions, one on the straight line that connect the disagreement point with the gravity center and another on the straight line that connect the gravity center with the utopia point. Anbarci and Bigelow (1994)[2] characterize a new solution is the point in the Pareto frontier when the line through the disagreement point, d, divides S_+ into two subsets of equal area. All these solutions depends critically of the shape's set, and this is the reason why they do not satisfy any property that deal with uncertainty, except for the solution of Gupta and Livne [6], but this has a exogenous reference point and furthermore assume by means of *Limited Sensitivity to Changes in the Conflict Point* axiom that if the conflict situation changes, the sole change being in the conflict point, and if the change induces no change in the ideal point, then the solution does not change.

We propose a new solution for asymmetric bargaining problems by means of mutual concession among the agents concept using an endogenous reference point.

We also introduce two new axioms, one that we use for characterizing our solution in 2-person bargaining situations and other that we introduce in the context of n-person bargaining situations.

The first, which is call *restricted concavity*, is a straightforward transformation of *concavity* axiom proposed by Myerson (1981). Restricted concavity is a weaker condition than the usual version and also is related with the bargaining under one way of *probabilistic uncertainty*. This axiom says that if bargaining take place now but the feasible set will be known only later, but the utopia point of all possible sets are the same, it would be preferable for the agents to reach a compromise now instead of waiting until the uncertainty is resolved. The second axiom is called *utopia continuity*, it says that if a sequence of sets tend to the utopia point (free conflict point), the solutions of each set it should also tend to the utopia point.

Section I introduces notation and the main definitions, section II shows our solution concept and its characterization for 2 agents, section III extends the result for $n \geq 3$ agents providing also its characterization and, finally, section IV summarizes the main highlights and conclusions.

1. Definitions and Notations

The following notation is used. Let $x, y \in \mathbb{R}^2$, $y \geq x$; $y_1 \geq x_1$ and $y_2 \geq x_2$; $y \geqq x$ if $y_1 > x_1$ and $y_2 > x_2$ or $y_1 = x_1$ and $y_2 = x_2$; $y > x$ if $y \geq x$ and $y \neq x$; $y \gg x$ if $y_1 > x_1$ and $y_2 > x_2$.

We consider the classical bargaining problem introduced by Nash. (For a survey see Thomson (1994) [21]). A subset $\mathbf{S} \subset \mathbb{R}^\mathbf{2}$ called **feasible set** if it is convex, compact and comprehensive (for each $x \in S$ and each $y \in \mathbb{R}^2$, such that if $y \leq x$, then $y \in S$). The **convex comprehensive hull** of a set $S \subseteq \mathbb{R}^2$, $\mathbf{cch(S)}$, is the smallest convex and comprehensive set containing S. The **disagreement point**, $\boldsymbol{d}$, is the outcome that the agents get if they do not reach an agreement. Furthermore $d \in S$ and a set S is *non-degenerate* if $x \in S$ such that $x \gg d$.

A **bargaining problem** is a pair (S, d), when S is a feasible set and d is the disagreement point.

Let $\mathbf{\Sigma}$ the **set of bargaining problems**. A **solution** is a function $f\colon \Sigma \to \mathbb{R}^2$ such that, for each $(S, d) \in \Sigma$, $f(S, d) \in S$. The point $f(S, d)$ is the **solution point of** $(\mathbf{S}, \mathbf{d})$.

Without loss of generality we consider the domain $\mathbf{\Sigma_0}$ (problems whose disagreement point is zero). For problems with disagreement different of zero, $\mathbf{\Sigma_d}$, we can make this transformation. By **translation invariance** (which says that the addition of constants to utility functions should be accompanied by the corresponding translation of the solution outcome), if $S \in \Sigma_d$, then $S' = S - \{d\} = \{x \in \mathbb{R}^n | x = s - d; s \in S\} \in \Sigma_0$. For simplicity, we write S instead of (S, d).

The set of **individually rational points of S** is defined as $\mathbf{S}_+ = \mathbf{S} \cap \mathbb{R}^\mathbf{2}_+$. Given S and $d = 0$, we restrict our attention to sets contained in $\mathbb{R}^\mathbf{2}_+$, we omit for simplicity the subscript $+$. For a given S, the point $\mathbf{m(S)}$ is the **utopia point**, where $m_i(S)$ defines the maximum outcome for the player i on the set S.

The **strong Pareto** frontier of S is the set $SPO(S) \equiv \{x \in S | \text{if } y > x \text{ then } y \notin S\}$ and the **weak Pareto** frontier is the set $WPO(S) \equiv \{x \in S | \text{if } y \gg x \text{ then } y \notin S\}$.

It is widely accepted in the literature that every solution should satisfy at least the following axioms that give to the solution desirable properties: *Pareto optimality, translation invariance* and *individual rationality.* We already introduced Pareto optimality and translation invariance. Individual rationality says that at the solution outcome, all agents' payoffs should be at least as large as at d. The classic bargaining solutions consider that all agents have the same bargaining power, under this assumption, this solutions often satisfy *symmetry*, which says that for each $\{i, j\}, f_i(S) = f_j(S)$.

Nash (1950) required that solution satisfied **independence of irrelevant alternatives**: if $S \subset T$ and $f(T) \in S$, then $f(S) = f(T)$. The Nash solution is the only one that satisfies Pareto optimality, translation invariance , symmetry, individual rationality and independence of irrelevant alternatives axioms (Nash 1950 [13]).

To reflect relative differences among each agent i, we define a parameter α; $\alpha \in \Delta^{n-1} = \left\{\beta \in \mathbb{R}^n_+ : \sum_{i=1}^n \beta_i = 1\right\}$. The Nash solution expression that consider asymmetry in the bargaining problems in the class Σ_0 is defined by:
The **weighted Nash solution with weights** $\alpha \in \mathbf{\Delta^{n-1}}$ is defined by setting, for each $S \in \Sigma_0$:

$$N^\alpha(S) = argmax \left\{(u_1)^{\alpha_1}(u_2)^{\alpha_2} | u \in S\right\}. \tag{1.1}$$

When $\alpha_i = \frac{1}{n}$, N^α is the standard Nash solution.

The Kalai-Smorodinsky solution assigns to each $S \in \Sigma$ the unique maximal element in $L(0, m(S)) \cap S$, where Σ is the set of all convex and comprehensive sets; $m_i(S) = \max\{x_i | x \in WPO\}$, for each $i \in \{1, 2\}$; and $L(x, y)$ is the line passing though $x, y \in \mathbb{R}^2$. **Individual monotonicity** axiom, which says that if $S \subset T$ and $m_j(S) = m_j(T)$ for each $j \neq i$; then $f_i(S) \leqq f_i(T)$. The Kalai-Smorodinsky solution is the only solution that satisfies *weak Pareto optimality, translation invariance, symmetry,individual rationality* and *individual monotonicity* (Kalai-Smorodinsky 1975[10])

Strong Pareto optimality and **restricted monotonicity** characterize a large family of solutions that define all monotonic solutions that connect the disagreement point with utopia point (Peters and Tijs 1985b[15]). If two bargaining pairs have the same disagreement point and utopia point, and if the set of feasible utility pairs in the first problem contains that of the second problem, then in the first problem an individually monotonic solution assigns larger utilities to the players than in the second one. Restricted monotonicity was used to emphasize the importance of comprehensiveness, since if this assumption is not imposed, *weak Pareto optimality, symmetry and restricted monotonicity* are incompatible (Roth 1979 [17]).

To accommodate asymmetries in the bargaining power of the agents, Thomson (1994)[21] generalizes the KS solution as follows:
The **weighted Kalai-Smorodinky solution with weights** $\alpha \in$ **int** $\{\mathbf{\Delta^{n-1}}\}$ is defined by setting, for each $S \in \Sigma_0$:

$$KS^\alpha(S) = \left\{x \in \mathbb{R}^2_+ | x_2 = \frac{(1-\alpha)m_2(S)}{\alpha m_1(S)} x_1\right\} \cap WPO(S). \tag{1.2}$$

This weighted Kalai-Smorodinky solution can be interpreted as a weighted proportional gains solution, in the sense that the solution is proportional to the utopia point of each agent considering a weight or asymmetry α among the agents.

Dubra (2001)[4] points out that Thomson's solution is not strong Pareto optimal for all possible sets in the bargaining domain. Hence, using a lexicographic extension of KS^α, he defines a new one parameter asymmetric solution that satisfies SPO and that he call asymmetric KS solution.
The **asymmetric Kalai-Smorodinky solution with weights** $\alpha \in$ **int** $\{\mathbf{\Delta^{n-1}}\}$ is defined by setting, for each $S \in \Sigma_0$:

$$lKS^\alpha(S) = \left\{x \in \mathbb{R}^2_+ | x \geq KS^\alpha(S)\right\} \cap SPO(S). \tag{1.3}$$

A limitation of this solution is that it does not satisfy **continuity**, and this feature is not in the line of Kalai-Smorodinsky solution. This solution proposed by Dubra is strong Pareto optimal at a cost of losing continuity. In the next section we propose a new solution that considers asymmetries in the bargaining power of the agents and yield an strong Pareto optimal solution maintaining continuity, being KS solution a special symmetric case.

The **equal losses solution** equalize across the agents the losses from the ideal point (Chun 1988 [3]). This equal losses solution does not satisfy **individual rationality** for $n \geq 3$ players, since the line that intersect the set starting from the utopia point could not find the set depending of the shape of this set. We adapt the equal losses in a weighted version as follow:

The **weighted losses solution with weights** $\alpha \in \mathbf{\Delta^{n-1}}$ is defined by setting, for each $S \in \Sigma_0$:

$$EL^{\alpha}(S) = argmax\left\{x \in S | m_1(S) - \alpha x_1 = m_2 - (1-\alpha)x_2\right\}. \quad (1.4)$$

This weighted Losses solution does not satisfy individual rationality when $n \geq 3$ agents, but for 2-person bargaining problems it does.
There is a **rational equal-losses** that ensures individual rationality (Herrero and Marco 1993[7]). As we can see later, lexicographical solutions can not deal with any kind of **probabilistic uncertainty**.

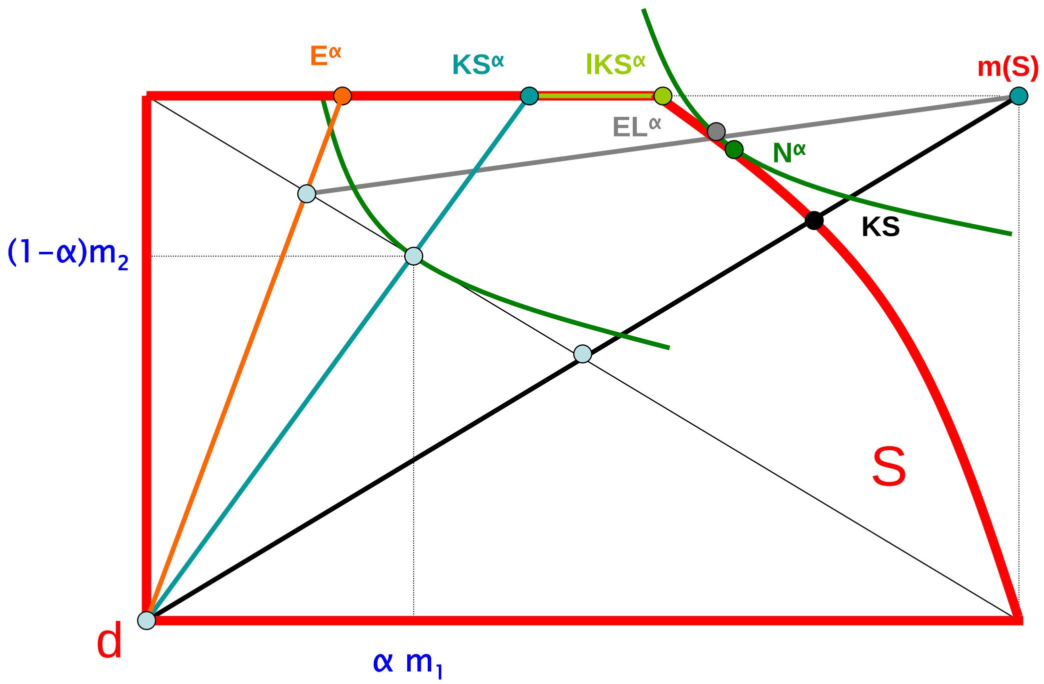

FIGURE 1. Solutions for 2 agents bargaining problem

2. New Solution and Characterization

In the line of mutual concessions philosophy, we propose a new solution for asymmetric bargaining problems that we name **weighted proportional losses** solution. In figure 2 is this weighted proportional losses solution graphically represented. Let $m(S)$ the utopia point and $edr(S)$ the endogenous reference point given by $(\alpha m_1(S), (1-\alpha)m_2(S))$.

Definition 1. The **weighted proportional losses solution with weights** $\alpha \in \mathbf{\Delta^{n-1}}$ is defined by setting, for each $S \in \Sigma_0$:

$$PL^{\alpha}(S) = argmax\left\{x \in S | x = (1-\lambda)edr(S) + \lambda m(S)\right\}. \tag{2.1}$$

This point is yielded by the distribution of bargaining power on the smallest convex and comprehensive set given an utopia point. In terms of losses, we can define the weighted proportional losses as follow:

$$PL^{\alpha}(S) = argmax\left\{x \in S | \frac{\alpha(m_1(S) - x_1)}{m_1(S)} = \frac{(1-\alpha)(m_2(S) - x_2)}{m_2(S)}\right\} \tag{2.2}$$

It is a normalized version of weighted losses solution, in the sense that the solution outcome is proportional to the utopia point of the bargaining set and for this reason the solution also satisfies **scale invariance** axiom. We provide a characterization for $n \geq 3$ that does not involve this axiom, but note that the property is still satisfied by PL^{α}. This happen because for $n = 2$, this generalization of the equal losses solution (weighted losses solution) and also the symmetric case (equal losses solution), are individually rational and therefore we need to include scale invariance in the characterization to achieve uniqueness in the proof of the theorem 1. In contrast, with $n \geq 3$, those solutions do not satisfy individual rationality, then we do not need add scale invariance in the characterization to get uniqueness.

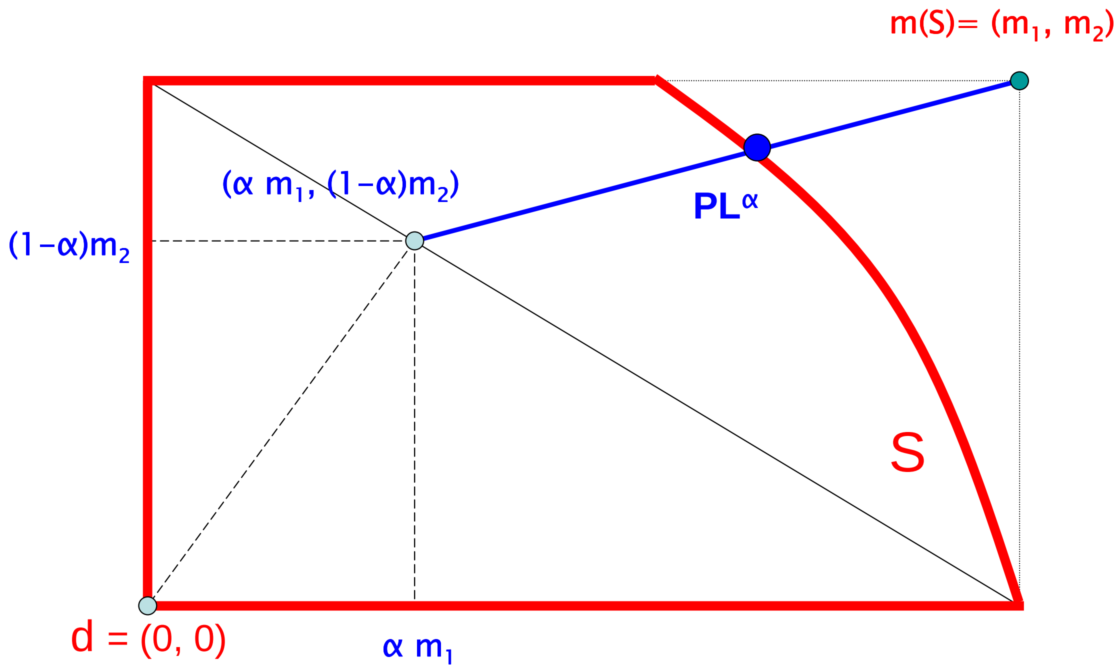

FIGURE 2. The weighted proportional losses solution for 2 agents bargaining problem

• We will see later that PL^{α} solution outcome satisfies SPO for $n = 2$ agents and not for $n \geq 3$ as is the case for the Kalai-Smorodinsky solution. We state formally the axiom of SPO:

Strong Pareto Optimality: For every $S \in \Sigma_0$, then $f(S) \in SPO(S)$.

Strong Pareto optimality axiom implies efficiency, it means that a solution will be in the frontier of the set that is a attainable point where none of the agents can improve their utility without a loss of utility for the other agent.

• Other well known axiom deal with the possibility that the bargaining set could be normalized by means of a positive affine transformation. This axiom is *scale invariance*; formally:

Scale Invariance: For each $S \in \Sigma_0$, $\lambda f(S) = f(\lambda S)$.

Scale invariance was introduced by Nash (1950)[13] and is one of the four axioms that characterize Nash solution. When one solution satisfies scale invariance, the bargaining set can be normalized in another with disagreement $\{0\}$ and utopia point $\{1\}$, and the solution on this set is in relative terms equivalent to the solution outcome in the original set. In other words, it does no matter if agents change the scale of units in which they are bargaining. Hence scale invariance implies interpersonal comparison of utility, thing that usually can be observed when agents try to reach compromises.

• The concept of monotonicity is very clear in bargaining situations, since is very convincing to argue that if the feasible set increases, the payoff of any agent should not decrease. In this kind of axioms we have:

Restricted Monotonicity: For each pair $S, T \in \Sigma_0$, if $S \subseteq T$ and $m(S) = m(T)$; then $f(S) \leq f(T)$.

Restricted monotonicity is weaker than *individual monotonicity* used by Kalai and Smorodinsky that was introduced by Rosenthal (1976)[16]. This axiom means that an expansion of the feasible set leaving unaffected the utopia point benefits all agents.

Peters and Tijs (1985) [15] characterize the big family of all individually monotonic bargaining solution by means of scale invariance, strong Pareto optimality and restricted monotonicity.

• We are also interested in contexts with uncertainty, there exist in the literature several kinds of uncertainty, for example probabilistic and non probabilistic. We focus in probabilistic uncertainty, that is, when we can relate each event with a probability that this event happen. Myerson(1981)[12] introduced **set concavity** axiom, which meaning is closely related with the fact to be bargaining under probabilistic uncertainty; when this axiom is satisfied, the agents are willing to reach an agreement before to know the set because all them benefit from this early agreement (see Thomson's survey[21]).

In this context, as well as KS solution, we are interested in properties that remain unaffected for a given utopia point and, as in the case of monotonicity, we introduce a **straightforward transformation of *concavity* axiom** that is a weaker condition than usual version and it is related with the concept of probabilistic uncertainty for sets with the same utopia point, therefore, this axioms is desirable when the agents are bargaining under **restricted probabilistic uncertainty**. The PL^{α} it also satisfies *proportional concavity*, that is, if there exist changes in all utopia points but they are in the same proportion, PL^{α} still maintain the concavity property.

Restricted Concavity: For each $\lambda \in [0, 1]$ and for each pair $S, T \in \Sigma_0$, such that $m(S) = m(T)$, $f(\lambda S + (1 - \lambda)T) \geqq \lambda f(S) + (1 - \lambda)f(T)$.

Restricted concavity axiom means that if bargaining take place now but the feasible set will be known only later, and we know the utopia point of all possible sets are the same, it would be preferable for the agents to reach a compromise now instead of speculate waiting the resulting set. A necessary condition for this is that both benefit from early agreement.

Now we are in position to state our main result in the next theorem:

Theorem 1. *For $n = 2$, a solution satisfies strong Pareto optimality, scale invariance, restricted monotonicity, and restricted concavity, if and only if is a weighted proportional losses solution.*

Proof. **Step 1**. Let $\alpha \in \Delta^n$. We show that PL^{α} satisfies the axioms of the theorem.

•We show that PL^{α} is an individually monotonic bargaining solution as defined Peters and Tijs (1985)[15]: Let $\nabla = conv\{(1, 0), (0, 1), (1, 1)\}$. Let $\gamma : [1, 2] \to \nabla$ and $\tau \in [1, 2]$, then $\gamma(\tau) = (2 - \tau)(\alpha, 1 - \alpha) + (\tau - 1)(1, 1)$ is a map satisfying the property that Peters and Tijs (1985)[15] called (C). Let Λ be the family of maps satisfying (C), then $\gamma_{PL^{\alpha}} \in \Lambda$. By Peters and Tijs (1985)[15] proportional losses solution is an individual monotonic solution and then satisfies strong Pareto optimality, scale invariance and restricted monotonicity.

• PL^α also satisfies restricted concavity: Let $S, T \in \Sigma_0$ and $C(S,T,\lambda) \equiv (\lambda S + (1-\lambda)T)$. We know that $\lambda PL^\alpha(S) + (1-\lambda)PL^\alpha(T) \in C(S,T,\lambda)$ and $PL^\alpha(\lambda S + (1-\lambda)T) \in C(S,T,\lambda)$. By *restricted concavity*, $PL^\alpha(\lambda S + (1-\lambda)T) \geqq \lambda PL^\alpha(S) + (1-\lambda)PL^\alpha(T)$. Suppose by contradiction that $PL^\alpha(\lambda S + (1-\lambda)T) < \lambda PL^\alpha(S) + (1-\lambda)PL^\alpha(T)$. By definition of the weighted proportional losses, if $x \in S$, there is no $y \in S$ such that $y > x$. Thus, if $\lambda PL^\alpha(S) + (1-\lambda)PL^\alpha(T) \in C(S,T,\lambda)$, then $PL^\alpha(\lambda S + (1-\lambda)T, \alpha) \not< \lambda PL^\alpha(S) + (1-\lambda)PL^\alpha(T)$.

Step 2. We prove uniqueness.

• Let a solution f satisfying strong Pareto optimality, scale invariance, restricted monotonicity and restricted concavity. By Peters and Tijs (1985)[15], f is an individually monotonic solution, then there is a monotonic curve $\gamma : [1,2] \to \nabla$ that connect the disagreement point with the utopia point. We prove that γ is a line.
Let $S \in \Sigma_0$ with $m(S) = (1,1)$, $f(S)$ is the intersection among the strong Pareto boundary of S and γ curve, then there is $t \in [1,2]$ such that $f(S) = \gamma(t)$.
Let $V_t = conv\{(0,0),(1,0),(1,t-1),(t-1,1),(0,1)\}$, then $f(V_t) = f(S) = \gamma(t)$. Let $\lambda \in [0,1]$, since f is restricted concavity, $f(\lambda cch(V_t) + (1-\lambda)V_t) \geqq \lambda f(cch(V_t)) + (1-\lambda)f(V_t)$. By the definition of V_t, $f(\lambda cch(V_t) + (1-\lambda)V_t) = \lambda f(cch(V_t)) + (1-\lambda)f(V_t)$. This implies that γ is a line that connect $f(cch(V_t))$ and $f(V_t)$. Let $2 \geq s \geq t \geq 1$, let $V_s = conv\{(0,0),(1,0),(1,s-1),(s-1,1),(0,1)\}$, analogously V_t we prove that for $s < 1$, the curve between $f(V_s)$ and $f(cch(V_s))$ is a line. □

A special case occurs when $\alpha_i = \frac{1}{n}$ that yield a particular outcome of weighted proportional losses solution, if the agents have the same bargaining power, the solution becomes symmetric and then weighted proportional losses solution becomes Kalai-Smoradinsky solution.

• There exist a weaker axiom than *symmetry* to reflect this fact, it is **midpoint domination**. (Moulin (1983) [11]; also see Thomson (1994)[21], Sect 4.1). It says that for each problem, each agent's payoff should be at least as large as the average of its dictatorial outcomes. This average can be interpreted as an equal-probability lottery over these outcomes. *Midpoint domination* is formally defined in the $\mathbb{R}^2$ domain as follow.

Midpoint Domination: For each $S \in \Sigma_0$, $f(S) \geqq [\sum_{i=1}^{2} f^{D_i}(S)]/2$ where $f^{D_i}(S)$ is the dictatorial not lexicographic solution for player i

The next corollary shows an alternative characterization of Kalai-Smorodinsky solution in terms of restricted probabilistic uncertainty, since the KS solution satisfies RCAV axiom (but not *concavity* itself as pointed by Thomson (1994) [21]).

Corollary 1. *The Kalai-Smorodinsky solution is the only solution that satisfies strong Pareto optimality, midpoint domination, restricted monotonicity and restricted concavity.*

This weighted proportional losses solution belongs to a family of solutions that are all straight lines on the feasible set starting in the utopia point and finishing on the *cch* boundary of the set. The solution outcome is the intersection of this straight lines with the pareto frontier. We named this family **dual reference functions solutions**. Conceptually our dual reference functions solutions are related with Thomson's[20] reference functions solutions, that are all straight lines on the feasible set that start in the disagreement point. Kalai-Smorodinky soltion , is a **itself dual** function, since it could be considered as **proportional gains** from the disagreement point d as **proportional losses** from utopia point $m(S)$. The weighted Kalai-Smorodinky solution introduced by Thomson (1994) [21] is focused in the sense of proportional gains from disagreement point, while PL^α solution is focused in the sense of proportional concessions from the utopia point, yielding another properties as strong pareto optimality in $\mathbb{R}^2$ and restricted concavity.

All axioms of theorem 1 are tight because none is implied by the others together and if we do not use one of them, we obtain at least other solution. We have that, given an $\alpha \in \Delta^n$, the only solution that satisfies strong Pareto optimality, scale invariance, restricted monotonicity and restricted concavity in $\mathbb{R}^2$ is the weighted Proportional Losses solution.

(1) If we drop *strong Pareto optimality*, there exist the weighted Kalai-Smorodinsky solution (KS^α), introduced by Thomson (1994) [21] , since by the definition of KS^α is easy to see that this solution satisfies the rest of axioms of Theorem 1, since it is a monotonic straight line that connect disagreement point with a point on the frontier of the set depending of utopia and bargaining power of each agent, this is the reason for which satisfies *scale invariance, restricted monotonicity and restricted concavity.*
(2) If we drop *scale invariance*, there exist the equal losses solution, characterized by Chun (1988) [3], and weighted losses solution that also satisfy *strong Pareto optimality, restricted monotonicity and restricted concavity* for 2 agents problems.
(3) If we drop *restricted concavity*, we have the asymmetric Kalai-Smorodinsky solution (lKS^α) that was introduced by Dubra (2001) [4] , that is a lexicographic extension of KS^α solution in order to be strong Pareto optimality, but renouncing to satisfy continuity axiom that in this context yield a renounce of restricted concavity.
(4) If we drop *restricted monotonicity*, an utilitarian solution appears defined as follow: Let $S, \in \Sigma_0$, a utilitarian solution $UT(S) = argmax\left\{\frac{x_1}{m_1} + \frac{x_2}{m_2} \mid x \in S\right\}$. Note that this version of utilitarian solution satisfy *scale invariance* because the outcome is the tangent point with a straight whose slope is the same that the efficient frontier of the minimum convex comprehensive hull of the set and also satisfies *restricted concavity.* Myerson (1981) [12] proved that all utilitarian solution satisfies concavity, this means that for all $S, T \in \Sigma_0$, $f(\lambda S + (1-\lambda)T) \geqq \lambda f(S) + (1-\lambda) f(T)$. In words, the solution of a convex combination of two sets is greater or equal of the convex combination among the solutions of these sets. In some cases the utilitarian solution yield a set, in this cases we choose an specific element of it. We take a selection of this utilitarian solution, $lUT(S) = argmin\left\{dist(x, PL^\alpha(S) \mid x \in UT(S)\right\}$, if $UT(S)$ is a single point, then $lUT(S) = UT(S)$, it is easy to see that this solution satisfies all axioms of theorem 1 except restricted monotonicity.

3. Weighted Proportional Losses for $n \geq 3$ agents

We also provide a characterization of PL^α solution for $n \geq 3$ agents given an $\alpha \in \Delta^{n-1} = \left\{\beta \in \mathbb{R}^n_+ : \sum_{i=1}^n \beta_i = 1\right\}$. We drop scale invariance, replace strong Pareto optimality by the weak version, weak Pareto optimality, we also add another axiom that we call **utopia continuity** with the formal definition:

Utopia Continuity: Given $\left\{S^k\right\}_{k\geq 1} \subset \Sigma_0$, such that $dist(S^k, m(S)) \to 0$ in the Hausdorff topology, then $dist(f(S^k), m(S)) \to 0$.

Utopia continuity is an axiom which meaning is that if a sequence of sets tend to the *utopia point* (free conflict point), the solutions of each set it should also tend to the utopia point.

With these modifications we have that:

Theorem 2. *For $n \geq 3$, a solution satisfies weak Pareto optimality, restricted monotonicity, utopia continuity, and restricted concavity, axioms if and only if is a weighted proportional loses solution.*

Proof. Let $\alpha \in \Delta^{n-1}$, the proof proceeds in several steps.

Step 1. **Weak Pareto optimality, restricted monotonicity, and utopia continuity**: By the definition of PL^α are satisfied.

• **Restricted concavity**: Suppose by contradiction that $\lambda PL^\alpha(S)+(1-\lambda)PL^\alpha(T) \in (\lambda S + (1-\lambda)T)$ and $PL^\alpha(\lambda S + (1-\lambda)T) \in (\lambda S + (1-\lambda)T,)$. By *restricted concavity*, $PL^\alpha(\lambda S + (1-\lambda)T) \geqq \lambda PL^\alpha(S) + (1-\lambda)PL^\alpha(T)$. Suppose $PL^\alpha(\lambda S + (1-\lambda)T) < \lambda PL^\alpha(S) + (1-\lambda)PL^\alpha(T)$. By definition of PL^α, there is no $y \in S$ such that $y > x$. Then, if $\lambda PL^\alpha(S) + (1-\lambda)PL^\alpha(T) \in (\lambda S + (1-\lambda)T)$, no have $PL^\alpha(\lambda S + (1-\lambda)T) \not< \lambda PL^\alpha(S) + (1-\lambda)PL^\alpha(T)$.

Step 2. To prove uniqueness, let $S \in \Sigma_0$, let $V_t = \left\{x \in \mathbb{R}^n_+ | \sum_{i=1}^n \frac{x_i}{m_i} \leq t\right\}$. Let $s,t \in [1,n]$ and $cch(V_t) = cch(V_s)$. By restricted concavity $f(\lambda cch(V_t)+(1-\lambda)V_t) \geqq \lambda f(cch(V_t)+(1-\lambda)f(V_t)$. By definition of $(\lambda cch(V_t)+(1-\lambda)V_t)$, $f(\lambda) = \lambda f$. Suppose this line do not cross utopia point, then there is a point $\bar{x} \leqq m(S)$ in this line such that for some j, $x_j = m_j$. Also there is a (V_k) such that $f(V_k) = \bar{x}$. Let $p > k$, by restricted monotonicity, $f(V_p \geqq f(V_k)$. If $f(V_p = f(V_k)$, this is a contradiction with utopia continuity. Then if for $i \neq j$, $f_i(V_p > f(V_k)$ this is also a contradiction with restricted concavity.

□

All axioms of theorem 2 are not implied by the others together and if we do not use one of them, we obtain at least other solution. We have that, given an $\alpha \in \Delta^{n-1}$, the only solution that satisfies weak Pareto optimality, restricted monotonicity, restricted concavity and utopia concavity in $\mathbb{R}^n$ is the weighted proportional losses solution.

(1) Given an $\alpha \in \Delta^{n-1}$, if we drop weak Pareto optimality, there exist at least another solution that satisfies restricted monotonicity, restricted concavity and utopia continuity, this solution is the non-efficient PL^α and this solutions yield the same outcome that proportional losses solution less an $\epsilon \in \mathbb{R}^n$ such that $NePL^\alpha(S) \ll PL^\alpha(S)$.
(2) If we drop restricted concavity, we obtain the asymmetric Kalai-Smorodinsky solution (lKS^α) that is a lexicographic extension of KS^α introduced by Dubra (2001) [4].
(3) If we drop restricted monotonicity, we have the same utilitarian solution that we use in the independence of axioms for $\mathbb{R}^2$.
(4) If we drop utopia continuity, we have the weighted Kalai-Smorodinsky solution (KS^α) introduced by Thomson (1994) [21].

In the figure 3 are represented several solutions for a n-person bargaining problem and we can see several interesting things, the first one is that PL^α is a solution based in losses that has not rationality problems for n-person bargaining problems as happen with EL^α solution that could not intersect the bargaining set, instead PL^α solution is the convex combination between a endogenous reference point on the smallest convex comprehensive hull, that in the figure 3 is the triangular surface, with the utopia point. This is the reason why PL^{α} will always intersect the bargaining set for n-person bargaining problem.

We can also see that if we have and expansion of the feasible set maintaining the same utopia point, and the new frontier of the set it is very close to the utopia point, KS^{α} and other solutions as **proportional solution** (asymmetric egalitarian, Kalai 19877[**?**]) E^{α} can not satisfies utopia continuity axiom, is for this reason that if the frontier of the set tend to the utopia point their solutions for these sets do not tend to the utopia point.

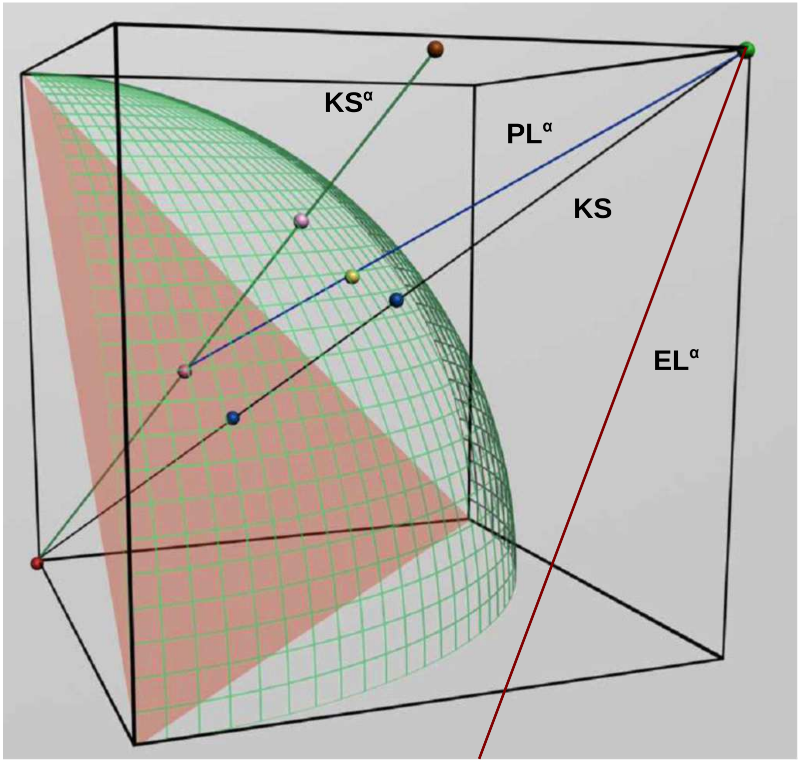

FIGURE 3. Solutions for 3 agents bargaining problem

4. CONCLUSIONS

It is interesting to observe that PL^{α} solution depends both on *disagreement point* and on the *balance of the power among the agents* (α), and in addition to this features, it also depends on the *utopia point*, since this solution is a mutual concession from the free conflict point in relative terms. In our opinion both *disagreement point dependence* and *bargaining power dependence* are features that should be take in account in a bargaining context, because is straightforward to think that if one agent has a worse disagreement point, the rest of the agents could push it to take advantage from this situation; on the other hand, if one agent has more bargaining power, he could obtain a better outcome.

Although geometrically the Gupta and Livne [6] solution looks very similar to PL^{α}, conceptually they are very different, not only because Gupta and Livne use an exogenous reference point but also because Gupta and Livne solution does not take in account differences in the disagreement point, however, PL^{α} solution takes in account the disagreement point to fix the endogenous reference point.

In our opinion, individual monotonicity is an essential axiom that should satisfy a solution in a context of bargaining (when the agents should reach an agreement). We have showed that PL^{α} satisfies desirable properties as *strong Pareto optimality* in $\mathbb{R}^2$ and *continuity* that are also satisfied by Kalai-Smorodinsky solution, but are not satisfied either by the others versions of weighted or asymmetric Kalai-Smorodinsky solution. Furthermore, Kalai-Smorodinsky solution is a *itself dual* function, yielding the same properties if we focus on disagreement or utopia point, this implies that, as well as PL^{α} solution, KS could considered as mutual concessions among the agents starting on the utopia point in the case that all agents have the same bargaining power (a particular case of PL^{α} when $\alpha = 1 - \alpha$). We propose and characterize the weighted proportional losses solution using a straightforward transformation of Concavity axiom that is a weaker condition than usual version. This weighted proportional losses solution is a dual version of *weighted Kalai -Smorodinsky* solution proposed by Thomson (1994) [21] and can resolve the problem of not Pareto optimality in $\mathbb{R}^2$ maintaining restricted set concavity property, that it is also satisfied by KS solution. In this sense, the asymmetric KS solution (lKS^{α}) proposed by Dubra (2001) [4] can yield an strong Pareto optimal in $\mathbb{R}^2$, but renouncing to restricted concavity property. Hence that PL^{α} should be implemented by the agents when they want individual monotonic solutions and they are bargaining under restricted probabilistic uncertainty. As the literature point, does not exist any solution that can satisfy strong Pareto optimality, monotonicity and set concavity for 2-person bargaining problem but our solution is the only that satisfies strong Pareto optimality, restricted monotonicity and restricted concavity and scale invariance for 2-person bargaining problem.

We also characterize PL^{α} solution for $n \geq 3$ agents, replacing strong Pareto optimality axiom by weak Pareto optimality version and adding an axiom that we named *utopia continuity*, yielding a *losses* solution that has not rationality problems, as it is the case of the equal losses or weighted losses solution. Utopia continuity means that if a sequence of sets tend to utopia point, the solution it should also tend to this free conflict point, since to achieve it is desirable for all agents. Hence, before to know the set, the agents will prefer that the solution will be as near to the utopia point as be possible. The motivation of this axiom is that if the utopia point is a desirable solution for all agents, we have certainty that if the utopia point is attainable, the agents will reach an agreement. In this line, we hypothesize that the willingness to reach an agreement of each agent will be higher the closer the solution is to her ideal payoff (her utopia point coordinate). When more agents achieve a payoff close to their ideal payoff, more likely be an agreement among the agents.

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