Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

ELSEVIER

Fuzzy Sets and Systems 146 (2004) 271–284

FUZZY sets and systems

www.elsevier.com/locate/fss

Fuzzy clan games and bi-monotonic allocation rules☆

S.H. Tijs[a], M.R. Brânzei[b,*], S. Muto[c], S. Ishihara[c], E. Fukuda[c]

[a]*CentER and Department of Econometrics and Operations Research, Tilburg University, The Netherlands*

[b]*Faculty of Computer Science, Alexandru Ioan Cuza University, 11, Carol I Bd., 6600 Iasi, Romania*

[c]*Department of Value and Decision Science, Graduate School of Decision Science and Technology, Tokyo Institute of Technology, Tokyo, Japan*

Received 8 November 2002; received in revised form 15 May 2003; accepted 20 May 2003

Abstract

In this paper the class of fuzzy clan games is introduced. The cores of such games have an interesting shape which inspires to define a class of compensation-sharing rules that are additive and stable on the cone of fuzzy clan games. Further, the notion of bi-monotonic participation allocation scheme (bi-pamas) is introduced and it turns out that each core element of a fuzzy clan game is extendable to a bi-pamas.

MSC: 90D12; 03E72

Keywords: Clan game; Big boss game; Core; Decision making; Fuzzy coalition; Fuzzy game; Monotonic allocation rule

1. Introduction

Cooperative games with fuzzy coalitions are introduced in [1]. Such games are helpful for approaching sharing problems arising from economic situations, where agents have the possibility to cooperate with different participation levels, varying from non-cooperation to full cooperation, and where the obtained reward depends on the levels of participation. A fuzzy coalition describes the participation levels to which each player is involved in cooperation. Classical cooperative games, which model situations, where agents are either fully involved or not involved at all in cooperation with some other agents, can then be seen as a simplified version of games with fuzzy coalitions. In the pioneering work of Aubin the focus was on the core of cooperative fuzzy games and on

☆ This paper was written while the first and the second authors were visiting Tokyo Institute of Technology, September–December 2002. The authors wish to thank D. Dimitrov who read carefully the previous version of the paper. Valuable comments given by two anonymous referees are gratefully appreciated.

* Corresponding author. Tel.: +4032201534; fax: +4032201490.
E-mail address: branzeir@infoiasi.ro (M.R. Brânzei).

0165-0114/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/S0165-0114(03)00302-6

the Shapley value. Since 1974, much research has been done in the field of cooperative games with fuzzy coalitions. For a survey, the reader is referred to Nishizaki and Sakawa [10]. As in the classical cooperative game theory, some classes of games with fuzzy coalitions deserve special attention. The class of convex fuzzy games is introduced in [2] together with the notion of participation monotonic scheme (pamas), the existence of which is assured by the convexity of the game. Some additive and monotonic rules on the convex cone of convex fuzzy games are also considered (Proposition 8, in [2]). In the classical cooperative game theory extensive attention is paid to monotonicity properties of solution concepts (see [4,8,13,18–20]). Additivity of rules on specific cones of cooperative TU-games is also an important research topic (see [3,6,12,15]).

In this paper, we introduce a new class of games partly with fuzzy coalitions, namely, the class of fuzzy clan games (and its subclass of fuzzy big boss games), and focus on the core and bi-monotonic allocation schemes and rules. Inspired by Branzei et al. [4] and Voorneveld et al. [19] who consider the notion of bi-monotonic allocation scheme (bi-mas) for classical total big boss and total clan games, respectively, we introduce here for fuzzy clan games, the notion of bi-monotonic participation allocation scheme (bi-pamas). Big boss games and clan games are introduced in classical cooperative game theory by Muto et al. [9] and Potters et al. [11], respectively; see also [14].

The outline of the rest of the paper is as follows: In Section 2 we briefly recall some notions and facts from the theory of games with fuzzy coalitions. The notion of fuzzy clan game is introduced and exemplified in Section 3. Further, in Section 4, the cores of a fuzzy clan game (and a fuzzy big boss game, respectively) and its restricted games are explicitly described and the geometrical shape of the core is discussed. Compensation-sharing rules and the notion of bi-participation monotonic allocation scheme (bi-pamas) are introduced in Section 5. It turns out that each compensation-sharing rule is additive, stable and generates for each game a bi-pamas, and each core element of a fuzzy clan game is bi-pamas extendable. We conclude with some final remarks in Section 6.

2. Preliminaries on games with fuzzy coalitions

Let $N=\{1,\dots,n\}$ be a finite set of players. A fuzzy coalition on N is a vector $s=(s_1,\dots,s_n)$ in $[0,1]^N$, where the ith coordinate s_i is referred to as the participation level of player i in the fuzzy coalition s. The set of fuzzy coalitions on N is denoted by $[0,1]^N$ and also by $\mathscr{F}^N$. A crisp coalition $S\in 2^N$ corresponds in a canonical way to the fuzzy coalition e^S, where $e^S\in\mathscr{F}^N$ is the vector with $(e^S)_i=1$ if $i\in S$, and $(e^S)_i=0$ if $i\in N\backslash S$. The fuzzy coalition e^S corresponds to the situation where the players in S fully cooperate (i.e. they have participation level 1) and the players outside S are not involved in cooperation at all (i.e. they have participation level 0). Instead of $e^{\{i\}}$ we often write e^i. The fuzzy coalition $0=(0,\dots,0)$ is called the"empty" fuzzy coalition and the fuzzy coalition $e^N=(1,\dots,1)$ is called the "grand" coalition. For $s,t\in\mathscr{F}^N$ we use the notation $s\leqslant t$ iff $s_i\leqslant t_i$ for each $i\in N$; we say that s is (weakly) smaller than t, or, equivalently, that t is larger than s with respect to participation levels.

A cooperative fuzzy game with player set N is a function $v:\mathscr{F}^N\to\mathbb{R}$, with $v(0)=0$, assigning to each fuzzy coalition a real number telling what such a coalition can achieve in cooperation. The set of games with fuzzy coalitions on N is an infinite dimensional linear space that we denote by FG^N.

The core [1] of a fuzzy game v is defined by

$$Core(v)=\left\{x\in\mathbb{R}^n \;\middle|\; \sum_{i\in N} x_i = v(e^N), \sum_{i\in N} s_i x_i \geqslant v(s) \text{ for each } s\in\mathscr{F}^N\right\}.$$

Here $\sum_{i\in N} s_i x_i$ is the inner product of s and x, denoted by sx in the following.

For each fuzzy game v we define its corresponding crisp game $w:2^N\to\mathbb{R}$ by $w(S)=v(e^S)$ for each $S\in 2^N$.

In [2] the notion of t-restricted game of v is introduced, which plays a role, similar to that of a subgame of a crisp game (see Remark 4 in [2]).

Let $v\in FG^N$ and $t\in\mathscr{F}^N$. In what follows for each $t\in\mathscr{F}^N$ we denote the set $\{i\in N\,|\,t_i>0\}$ by $car(t)$. The t-restricted game of v with player set N is the game v_t with $v_t:\mathscr{F}^N\to\mathbb{R}$ given by $v_t(s)=v(t*s)$ for all $s\in\mathscr{F}^N$, where $t*s=(t_1s_1,\dots,t_ns_n)$ is the coordinate-wise product of t and s. Note that $s\mapsto t*s$ maps $\mathscr{F}^N$ in the set of fuzzy coalitions not bigger than t. When $t=e^T$ and $s=e^S$ we obtain $v_{e^T}(e^S)=v(e^{S\cap T})$. This implies that the restriction of $cr(v_{e^T}):2^N\to\mathbb{R}$ to 2^T is the subgame of $cr(v)$ with player set T. Moreover, in v_{e^T} for each player $i\in N\backslash T$ the null-player property $v_{e^T}(s+\varepsilon e^i)=v_{e^T}(s)$ holds for all $s\in\mathscr{F}^N$ and for all $\varepsilon\in[0,1-s_i]$. For each core element $x\in Core(v_t)$ we have $x_k=0$ for each $k\notin car(t)$ (see Remark 5 in [2]).

By means of restricted games Branzei et al. [2] have extended the notion of population monotonic allocation scheme (pmas) for cooperative crisp games [13] to that of participation monotonic allocation scheme (pamas) in the context of cooperative fuzzy games. Convexity of the fuzzy game (and its restricted games) is a sufficient condition for the existence of a pamas.

3. The cone of fuzzy clan games

There are various economic situations, where the group of agents involved consists of two subgroups with different status: a "clan" whose members can "manage" the situation, and a set of available agents willing to join the clan. However, the non-clan members are completely dependent on the collective of clan members, in the sense that, a coalition never can obtain a positive reward if not all clan members are present in the coalition. Such situations are modeled in the classical theory of cooperative games with transferable utility by means of (total) clan games, where only the full cooperation and non-cooperation at all of non-clan members with the clan are taken into account. Here, we take over this simplifying assumption and allow non-clan members to cooperate with all clan members and some other non-clan members to a certain extent. As a result the notion of fuzzy clan game is introduced.

Let $N=\{1,\dots,n\}$ be a finite set of players. We denote the non-empty set of clan members by C, and treat clan members as crisp players. In the following we denote the set of crisp subcoalitions of C by $\{0,1\}^C$, the set of fuzzy coalitions on $N\backslash C$ by $[0,1]^{N\backslash C}$ (equivalent to $\mathscr{F}^{N\backslash C}$), and denote $[0,1]^{N\backslash C}\times\{0,1\}^C$ by $\mathscr{F}^N_C$. For each $s\in\mathscr{F}^N_C$, $s_{N\backslash C}$ and s_C will denote its restriction to $N\backslash C$ and C, respectively. We denote the vector $(e^N)_C$ by 1_C in the following. Further, we denote by $F^N_{1_C}$ the set $[0,1]^{N\backslash C}\times\{1_C\}$ of fuzzy coalitions on N, where all clan members have full participation level, and where the participation level of non-clan members may vary between 0 and 1.

In this section fuzzy clan games are defined using veto power of clan members, monotonicity, and a so-called decreasing average marginal return property (DAMR-property). The DAMR-property takes over for fuzzy clan games, the role which the total concavity (TC) property has for crisp clan games. Specifically, the DAMR-property reflects the fact that an increase in the participation level of a non-clan member, in growing coalitions containing at least all clan members with full participation level, results in a decrease of the average marginal return of that player.

Formally, a game $v:\mathscr{F}_C^N\to\mathbb{R}$ is a *fuzzy clan game* if v satisfies the following three properties:

(i) (*veto-power of clan members*) $v(s)=0$ if $s_C\neq 1_C$;
(ii) (*Monotonicity*) $v(s)\leqslant v(t)$ for all $s,t\in\mathscr{F}_C^N$ with $s\leqslant t$;
(iii) (*DAMR-property*) for each $i\in N\backslash C$, all $s^1,s^2\in\mathscr{F}_{1_C}^N$ and all $\varepsilon_1,\varepsilon_2>0$ such that $s^1\leqslant s^2$ and $0\leqslant s^1-\varepsilon_1 e^i\leqslant s^2-\varepsilon_2 e^i$ we have

$$\varepsilon_1^{-1}(v(s^1)-v(s^1-\varepsilon_1 e^i))\geqslant\varepsilon_2^{-1}(v(s^2)-v(s^2-\varepsilon_2 e^i)).$$

Property (i) expresses the fact that the full participation level of all clan members is a necessary condition for generating a positive reward for coalitions.

Fuzzy clan games for which the clan consists of a single player are called *fuzzy big boss games*, with the single clan member as the big boss.

As an introduction we give two examples of interactive situations, one of them leading to a fuzzy clan game, but the other one not.

Example 1 (A production situation with owners and gradually available workers). Let $N\backslash C=\{1,\dots,m\}$, $C=\{m+1,\dots,n\}$. Let $f:[0,1]^{N\backslash C}\to\mathbb{R}$ be a monotonic non-decreasing function with $f(0)=0$ and with the decreasing average marginal return property. Then $v:[0,1]^{N\backslash C}\times\{0,1\}^C\to\mathbb{R}$ defined by $v(s)=0$ if $s_C\neq 1_C$ and $v(s)=f(s_1,s_2,\dots,s_m)$ otherwise, is a fuzzy clan game with clan C.

One can think of a production situation, where the clan members are providers of different (complementary) essential tools needed for the production and the production function measures the gains if all clan members are cooperating with the set of workers $N\backslash C$ (cf. [5,11]), where each worker i can participate at level s_i which may vary from lack of participation to full participation.

Example 2 (A fuzzy voting situation with a fixed group with veto-power). Let N and C be as in Example 1, and $0<k<|N\backslash C|$. Let $v:[0,1]^{N\backslash C}\times\{0,1\}^C\to\mathbb{R}$ with

$$v(s)=\begin{cases}1 \text{ if } s_C=1_C \text{ and } \displaystyle\sum_{i=1}^{m}s_i\geqslant k,\\ 0 \text{ otherwise.}\end{cases}$$

Then v has the veto-power property for members in C and the monotonicity property, but not the DAMR-property with respect to members of $N\backslash C$, hence it is not a fuzzy clan game. This game can be seen as arising from a voting situation, where to pass the bill all members of C have to (fully) agree and the sum of the support levels $\sum_{i\in N\backslash C}s_i$ of $N\backslash C$ should exceed a fixed threshold k, where $s_i=1$ $(s_i=0)$ correspond to full support (no support) of the bill, but also partial supports count.

In the following the set of all fuzzy clan games with a fixed non-empty set of players N and a fixed clan C is denoted by FCG_C^N. We notice that FCG_C^N is a convex cone in FG^N, that is for all $v, w \in FCG_C^N$ and $p, q \in \mathbb{R}_+$, $pv + qw \in FCG_C^N$, where $\mathbb{R}_+$ denotes the set of non-negative real numbers.

Now, we show that for each game $v \in FCG_C^N$ the corresponding crisp game w is a total clan game if $|C| \geqslant 2$, and a total big boss game if $|C| = 1$.

Let $v \in FCG_C^N$. The corresponding crisp game w has the following properties which follow straightforwardly from the properties of v:

(V) $w(S) = 0$ if $C \not\subset S$;
(M) $w(S) \leqslant w(T)$ for all S, T with $S \subset T \subset N$;
(TC) for all S, T with $C \subset S \subset T$ and each $i \in S \backslash C$,

$$w(S) - w(S \backslash \{i\}) \geqslant w(T) - w(T \backslash \{i\}).$$

Here (TC) stands for the total concavity property expressing the fact that the marginal contribution of each non-clan member is (weakly) greater in smaller coalitions which contain all clan members.

So, w is a total clan game in the terminology of Voorneveld et al. [19] if $|C| \geqslant 2$, and a total big boss game in the terminology of Branzei et al. [4] if $|C| = 1$.

Fuzzy clan games can be seen as an extension of crisp clan games, in what concerns the possibilities of cooperation available for non-clan members. Specifically, in a fuzzy clan game each non-clan member can be involved in cooperation at each extent between 0 and 1, whereas in a crisp clan game a non-clan member can only be or not a member of a (crisp) coalition containing all clan members.

In the following we consider t-restricted games corresponding to a fuzzy clan game and prove, in Proposition 1, that these games are also fuzzy clan games.

Let $v \in FCG_C^N$ and $t \in \mathcal{F}_{1_C}^N$. Recall that the t-restricted game v_t of v with respect to t is given by $v_t(s) = v(t * s)$ for each $s \in \mathcal{F}_C^N$.

Proposition 1. *Let v_t be the t-restricted game of $v \in FCG_C^N$, with $t \in \mathcal{F}_{1_C}^N$. Then $v_t \in FCG_C^N$.*

Proof. First, note that for each $s \in \mathcal{F}_C^N$ with $s_C \neq 1_C$ we have $(t * s)_C \neq 1_C$, and then the veto-power property of v implies $v_t(s) = v(t * s) = 0$. To prove the monotonicity property, let $s^1, s^2 \in \mathcal{F}_C^N$ with $s^1 \leqslant s^2$. Then $v_t(s^1) = v(t * s^1) \leqslant v(t * s^2) = v_t(s^2)$, where the inequality follows from the monotonicity of v. Now, we focus on the DAMR-property. Let $i \in N \backslash C$, $s^1, s^2 \in \mathcal{F}_{1_C}^N$, and let $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that $s^1 \leqslant s^2$ and $0 \leqslant s^1 - \varepsilon_1 e^i \leqslant s^2 - \varepsilon_2 e^i$. Then

$$\begin{aligned} \varepsilon_2^{-1}(v_t(s^2) - v_t(s^2 - \varepsilon_2 e^i)) &= \varepsilon_2^{-1}(v(t * s^2) - v(t * s^2 - t_i \varepsilon_2 e^i)) \\ &\leqslant \varepsilon_1^{-1}(v(t * s^1) - v(t * s^1 - t_i \varepsilon_1 e^i)) \\ &= \varepsilon_1^{-1}(v_t(s^1) - v_t(s^1 - \varepsilon_1 e^i), \end{aligned}$$

where the inequality follows from the DAMR-property of v. □

For each $i \in N \backslash C$, $x \in [0,1]$ and $t \in \mathcal{F}_C^N$, let $(t^{-i} \| x)$ be the element in $\mathcal{F}_C^N$ such that $(t^{-i} \| x)_j = t_j$ for each $j \in N \backslash \{i\}$ and $(t^{-i} \| x)_i = x$. The function $v : [0,1]^{N \backslash C} \times \{0,1\}^C \to \mathbb{R}$ is called *coordinate-wise* *concave regarding non-clan members* if for each $i \in N \backslash C$ the function $g_{t^{-i}}:[0,1] \to \mathbb{R}$ with $g_{t^{-i}}(x) = v(t^{-i} \| x)$ for each $x \in [0,1]$ is a concave function. Intuitively, this means that the function v is concave in each coordinate corresponding to (the participation level of) a non-clan member when all the other coordinates are kept fixed. Recall that a function $f:[0,1] \to \mathbb{R}$ is concave if for all $a, b, \alpha \in [0,1]$ it holds $f(\alpha a + (1-\alpha)b) \geqslant \alpha f(a) + (1-\alpha)f(b)$.

The function $v:[0,1]^{N\backslash C} \times \{0,1\}^C \to \mathbb{R}$ is said to have the *submodularity property on* $[0,1]^{N\backslash C}$ if $v(s \vee t) + v(s \wedge t) \leqslant v(s) + v(t)$ for all $s, t \in \mathscr{F}^N_{1_C}$, where $s \vee t$ and $s \wedge t$ are those elements of $[0,1]^{N\backslash C} \times \{1_C\}$ with the ith coordinate equal, for each $i \in N \backslash C$, to $\max\{s_i, t_i\}$ and $\min\{s_i, t_i\}$, respectively.

(The operations $\vee$ and $\wedge$ play a similar role for fuzzy coalitions as the union and intersection for crisp coalitions.)

Remark 1. The DAMR-property implies two important properties of v, namely coordinate-wise concavity and submodularity. Note that the coordinate-wise concavity follows straightforwardly from the DAMR-property of v. The proof of the submodularity follows the same line as in the proof of Theorem 6 in [2], where it is shown that the IAMR-property implies supermodularity.

Let $\varepsilon > 0$ and let $s \in \mathscr{F}^N_C$. For each $i \in N \backslash C$ we denote by $D_i v(s)$ the ith left derivative of v in s if $s_i > 0$, and the ith right derivative of v in s if $s_i = 0$, i.e.

$$D_i v(s) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(v(s) - v(s - \varepsilon e^i)) \quad \text{if } s_i > 0$$

and

$$D_i v(s) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(v(s + \varepsilon e^i) - v(s)) \quad \text{if } s_i = 0.$$

It is well known that for a concave real-valued function each tangent line to the graph lies above the graph of the function. Based on this property we state

Lemma 1. *Let* $v \in FCG^N_C$, $t \in \mathscr{F}^N_{1_C}$, *and* $i \in N \backslash C$. *Then for* $s_i \in [0, t_i]$

$$v(t^{-i} \| t_i) - v(t^{-i} \| s_i) \geqslant (t_i - s_i) D_i v(t).$$

Proof. Applying the coordinate-wise concavity of v and the property of tangent lines to the graph of g_{-i} in $(t_i, g_{-i}(t_i))$ one obtains

$$v(t^{-i} \| t_i) - (t_i - s_i) D_i v(t) \geqslant v(t^{-i} \| s_i). \qquad \square$$

4. The core of fuzzy clan games

The main aim of this section is to provide an explicit description of the core of a fuzzy clan game and give some insight into its geometrical shape. We start with a lemma.

Lemma 2. *Let* $v \in FCG_C^N$ *and let* $s \in \mathscr{F}_{1_C}^N$. *Then*

$$v(e^N) - v(s) \geqslant \sum_{i\in N\setminus C} (1-s_i) D_i v(e^N).$$

Proof. Suppose that $|N\setminus C| = m$ and denote $N\setminus C = \{1,2,\dots,m\}$, $C = \{m+1, m+2, \dots, n\}$. Let $a^0, a^1, \dots, a^m$ and $b^1, b^2, \dots, b^m$ be the sequences of fuzzy coalitions on N given by $a^0 = e^N$, $a^r = e^N - \sum_{k=1}^r (1-s_k)e^k$, $b^r = e^N - (1-s_r)e^r$ for each $r \in \{1,2,\dots,m\}$. Note that $a^m = s \in \mathscr{F}_{1_C}^N$, and $a^{r-1} \vee b^r = e^N$, $a^{r-1} \wedge b^r = a^r$ for each $r \in \{1,2,\dots,m\}$. Then

$$v(e^N) - v(s) = \sum_{r=1}^m (v(a^{r-1}) - v(a^r)) \geqslant \sum_{r=1}^m (v(e^N) - v(b^r)), \tag{1}$$

where the inequality follows from the submodularity property of v applied for each $r \in \{1,2,\dots,m\}$. Now for each $r \in \{1,2,\dots,m\}$ we have by Lemma 1

$$D_r v(e^N) \leqslant (1-s_r)^{-1}(v(e^N) - v(e^N - (1-s_r)e^r)),$$

thus obtaining

$$v(e^N) - v(b^r) = v(e^N) - v(e^N - (1-s_r)e^r) \geqslant (1-s_r) D_r v(e^N). \tag{2}$$

Now we combine (1) and (2). □

Theorem 1. *Let* v *be an element of* FCG_C^N. *Then*

(i) $Core(v) = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = v(e^N),\ 0 \leqslant x_i \leqslant D_i v(e^N)$ *for each* $i \in N\setminus C$, $0 \leqslant x_i$ *for each* $i \in C\}$, *if* $|C| > 1$;

(ii) $Core(v) = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = v(e^N),\ 0 \leqslant x_i \leqslant D_i v(e^N)$ *for each* $i \in N\setminus\{n\}$, $v(e^n) \leqslant x_n\}$, *if* $C = \{n\}$.

Proof. We only prove (i).

(a) Let $x \in Core(v)$. Then $x_i = e^i x \geqslant v(e^i) = 0$ for each $i \in N$ and $\sum_{i=1}^n x_i = v(e^N)$. Further, for each $i \in N\setminus C$ and each $\varepsilon \in (0,1)$ we have

$$x_i = \varepsilon^{-1}(e^N x - (e^N - \varepsilon e^i)x) \leqslant \varepsilon^{-1}(v(e^N) - v(e^N - \varepsilon e^i)).$$

We use now the monotonicity property and the coordinate-wise concavity property of v obtaining that $\lim_{\varepsilon\to 0} \varepsilon^{-1}(v(e^N) - v(e^N - \varepsilon e^i))$ exists and this limit is equal to $D_i v(e^N)$. Hence, $x_i \leqslant D_i v(e^N)$, thus implying that $Core(v)$ is a subset of the set on the right side of the equality in (i).

(b) To prove the converse inclusion, let $x \in \mathbb{R}^n$ with $\sum_{i=1}^n x_i = v(e^N)$, $0 \leqslant x_i \leqslant D_i v(e^N)$ for each $i \in N\setminus C$, and $0 \leqslant x_i$ for each $i \in C$. We have to show that the inequality $sx \geqslant v(s)$ holds for each $s \in [0,1]^N$. First, if $s \in [0,1]^N$ is such that $s_C \neq 1_C$, then $v(s) = 0 \leqslant sx$. Now, let $s \in [0,1]^N$, with

$s_C = 1_C$. Then

$$\begin{aligned} sx &= \sum_{i\in C} x_i + \sum_{i\in N\setminus C} s_i x_i \\ &= v(e^N) - \sum_{i\in N\setminus C} (1-s_i)x_i \geqslant v(e^N) - \sum_{i\in N\setminus C} (1-s_i) D_i v(e^N). \end{aligned}$$

The inequality $sx \geqslant v(s)$ follows then from Lemma 2. □

The core of a fuzzy clan game has an interesting geometric shape. It is the intersection of a simplex with 'hyperbands' corresponding to the non-clan members. To be more precise, for fuzzy clan games (and fuzzy big boss games with $v(e^n)=0$), we have $Core(v) = \Delta(v(e^N)) \cap B_1(v) \cap \cdots \cap B_m(v)$, where $\Delta(v(e^N))$ is the simplex $\{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = v(e^N)\}$, and for each player $i \in \{1,2,\ldots,m\}$, $B_i(v) = \{x \in \mathbb{R}^n \mid 0 \leqslant x_i \leqslant D_i v(e^N)\}$ is the region between the two parallel hyperplanes in $\mathbb{R}^n$, $\{x \in \mathbb{R}^n \mid x_i = 0\}$ and $\{x \in \mathbb{R}^n \mid x_i = D_i v(e^N)\}$, which we call the 'hyperband' corresponding to i.

An interesting core element is $b(v) = (D_1 v(e^N)/2, \ldots, D_m v(e^N)/2, t, \ldots, t)$, with $t = |C|^{-1}(v(e^N) - \sum_{i=1}^m D_i v(e^N)/2)$, which corresponds to the point with a central location in this geometric structure. Note that $b(v)$ is in the intersection of middle-hyperplanes of all hyperbands $B_i(v)$, $i = 1, \ldots, m$, and it has the property that the coordinates corresponding to clan members are equal.

Example 3. For a three-person fuzzy big boss game with player 3 as the big boss and $v(e^3) = 0$ the core has the shape of a parallelogram (in the imputation set) with vertices:

$$\begin{aligned} &(0,0,v(e^N)),\ (D_1 v(e^N), 0, v(e^N) - D_1 v(e^N)),\ (0, D_2 v(e^N), v(e^N) - D_2 v(e^N)), \\ &(D_1 v(e^N), D_2 v(e^N), v(e^N) - D_1 v(e^N) - D_2 v(e^N)). \end{aligned}$$

Note that $b(v) = (D_1 v(e^N)/2, D_2 v(e^N)/2, v(e^N) - (D_1 v(e^N) + D_2 v(e^N))/2)$ is the middle point of this parallelogram.

For a convex fuzzy game the core of v and the core of the corresponding crisp game w coincide (see Theorem 7(iii) in [2]). This is not the case in general for fuzzy clan games as the next example shows.

Example 4. Let $N = \{1,2\}$ and let $v : [0,1] \times \{0,1\} \to \mathbb{R}$ be given by $v(s_1, 1) = \sqrt{s_1}$, $v(s_1, 0) = 0$ for each $s_1 \in [0,1]$. Then v is a fuzzy big boss game with player 2 as the big boss, and $Core(v) = \{(\alpha, 1-\alpha) \mid \alpha \in [0, 1/2]\}$, $Core(w) = \{(\alpha, 1-\alpha) \mid \alpha \in [0,1]\}$. Hence $Core(v) \neq Core(w)$.

The next lemma plays a role in the rest of the paper.

Lemma 3. *Let $v \in FCG_C^N$. Let $t \in \mathscr{F}^N_{1_C}$ and v_t be the t-restricted game of v. Then for each non-clan member $i \in car(t)$: $D_i v_t(e^N) = t_i D_i v(t)$.*

Proof. $D_i v_t(e^N) = \lim_{\varepsilon\to 0} \varepsilon^{-1}(v_t(e^N) - v_t(e^N - \varepsilon e^i)) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(v(t) - v(t - \varepsilon t_i e^i)) = t_i D_i v(t)$. □

Theorem 2. *Let* $v \in FCG_C^N$. *Then for each* $t \in \mathscr{F}_{1_C}^N$ *the core* $Core(v_t)$ *of the t-restricted game* v_t *is described by*

(i) $Core(v_t) = \{x \in \mathbb{R}^n \mid \sum_{i\in N} x_i = v(t),\ 0 \leqslant x_i \leqslant t_i D_i v(t)$ *for each* $i \in N\setminus C,\ 0 \leqslant x_i$ *for each* $i \in C\}$, *if* $|C| > 1$;

(ii) $Core(v_t) = \{x \in \mathbb{R}^n \mid \sum_{i\in N} x_i = v(t),\ 0 \leqslant x_i \leqslant t_i D_i v(t)$ *for each* $i \in N\setminus\{n\},\ v(t_n e^n) \leqslant x_n\}$, *if* $C = \{n\}$.

Proof. We only prove (i). Let $t \in \mathscr{F}_{1_C}^N$, with $|C| > 1$. Then, by the definition of the core of a fuzzy game, $Core(v_t) = \{x \in \mathbb{R}^n \mid \sum_{i\in N} x_i = v_t(e^N), \sum_{i\in N} s_i x_i \geqslant v_t(s)$ for each $s \in \mathscr{F}_C^N\}$. Since $v_t(e^N) = v(t)$ and since, by Proposition 1, v_t is itself a fuzzy clan game, we can apply Theorem 1(i), thus obtaining $Core(v_t) = \{x \in \mathbb{R}^n \mid \sum_{i\in N} x_i = v(t),\ 0 \leqslant x_i \leqslant D_i v_t(e^N)$ for each $i \in N\setminus C$, $0 \leqslant x_i$ for each $i \in C\}$.

Now we apply Lemma 3. □

5. Monotonic allocation rules and bi-pamas

Let $N\setminus C = \{1,\ldots,m\}$ and $C = \{m+1,\ldots,n\}$. We introduce for each $\alpha \in [0,1]^m$ and $\beta \in \Delta(C) = \Delta(\{m+1,\ldots,n\}) = \{z \in \mathbb{R}_+^{n-m}, \sum_{i=m+1}^n z_i = 1\}$ an allocation rule $\psi^{\alpha,\beta} : FCG_C^N \to \mathbb{R}^n$ given by

$$\psi_i^{\alpha,\beta}(v) = \begin{cases} \alpha_i D_i v(e^N) & \text{if } i \in \{1,\ldots,m\}, \\ \beta_i \left(v(e^N) - \sum_{k=1}^m \alpha_k D_k v(e^N) \right) & \text{if } i \in \{m+1,\ldots,n\}. \end{cases}$$

We call this rule the *compensation-sharing rule with compensation vector* α *and sharing vector* β. The ith coordinate α_i of the compensation vector α indicates that player $i \in \{1,\ldots,m\}$ obtains the part $\alpha_i D_i v(e^N)$ of his marginal contribution $D_i v(e^N)$ to e^N. Then for each $i \in \{m+1,\ldots,n\}$, the ith coordinate β_i of the sharing vector β determines the share $\beta_i(v(e^N) - \sum_{j=1}^m \alpha_j D_j v(e^N))$ for the clan member i from what is left for the group of clan members in e^N.

Theorem 3. *Let* $v \in FCG_C^N$. *Then*

(i) $\psi^{\alpha,\beta} : FCG_C^N \to \mathbb{R}^n$ *is stable (i.e.* $\psi^{\alpha,\beta}(v) \in Core(v)$ *for each* $v \in FCG_C^N$*) and additive for each* $\alpha \in [0,1]^m$ *and each* $\beta \in \Delta(C)$;

(ii) $Core(v) = \{\psi^{\alpha,\beta}(v) \mid \alpha \in [0,1]^{N\setminus C}, \beta \in \Delta(C)\}$;

(iii) *The multi-function* $Core : FCG_C^N \to\to \mathbb{R}^n$ *which assigns to each* $v \in FCG_C^N$ *the subset* $Core(v)$ *of* $\mathbb{R}^n$ *is additive.*

Proof. (i) $\psi^{\alpha,\beta}(pv + qw) = p\psi^{\alpha,\beta}(v) + q\psi^{\alpha,\beta}(w)$ for all $v, w \in FCG_C^N$ and all $p, q \in \mathbb{R}_+$, so $\psi^{\alpha,\beta}$ is additive on the cone of fuzzy clan games. The stability follows from Theorem 1.

(ii) Clearly, each $\psi^{\alpha,\beta}(v) \in Core(v)$. Conversely, let $x \in Core(v)$. Then, according to Theorem 1, $x_i \in [0, D_i v(e^N)]$ for each $i \in N\setminus C$. Hence, for each $i \in \{1,\ldots,m\}$ there is $\alpha_i \in [0,1]$ such that $x_i = \alpha_i D_i v(e^N)$.

Now we show that

$$v(e^N) - \sum_{i=1}^{m} \alpha_i D_i v(e^N) \geqslant 0. \tag{3}$$

Note that $e^C \in \mathscr{F}_{1_C}^N$ is the fuzzy coalition where each non-clan member has participation level 0 and each clan-member has participation level 1. We have

$$\begin{aligned} v(e^N) - v(e^C) &= \sum_{i=1}^{m} \left(v\left(\sum_{k=1}^{i} e^k + e^C \right) - v\left(\sum_{k=1}^{i-1} e^k + e^C \right) \right) \\ &\geqslant \sum_{i=1}^{m} (v(e^N) - v(e^N - e^i)) \geqslant \sum_{i=1}^{m} D_i v(e^N) \\ &\geqslant \sum_{i=1}^{m} \alpha_i D_i v(e^N), \end{aligned}$$

where the first inequality follows from the DAMR-property of v by taking $s^1 = \sum_{k=1}^{i} e^k + e^C$, $s^2 = e^N$, $\varepsilon_1 = \varepsilon_2 = 1$, the second inequality follows from Lemma 1 with $t = e^N$ and $s_i = 0$, and the third inequality since $D_i v(e^N) \geqslant 0$ in view of the monotonicity property of v. Hence (3) holds.

Inequality (3) expresses the fact that the group of clan members is left a non-negative amount in the grand coalition.

The fact that $x_i \geqslant v(e^i)$ for each $i \in C$ implies that $x_i \geqslant 0$ for each $i \in \{m+1, \ldots, n\}$. But then there is a vector $\beta \in \Delta(C)$ such that $x_i = \beta_i(v(e^N) - \sum_{j=1}^{m} \alpha_j D_j v(e^N))$. (Take $\beta \in \Delta(C)$ arbitrarily if $v(e^N) - \sum_{j=1}^{m} D_j v(e^N) = 0$, and $\beta_i = x_i(v(e^N) - \sum_{j=1}^{m} \alpha_j D_j v(e^N))^{-1}$, for each $i \in C$, otherwise.) Hence $x = \psi^{\alpha,\beta}(v)$.

(iii) Trivially, $Core(v+w) \supset Core(v) + Core(w)$ for all $v, w \in FCG_C^N$. Conversely, let $v, w \in FCG_C^N$. Then

$$\begin{aligned} Core(v+w) &= \{\psi^{\alpha,\beta}(v+w) \mid \alpha \in [0,1]^{N\setminus C}, \beta \in \Delta(C)\} \\ &= \{\psi^{\alpha,\beta}(v) + \psi^{\alpha,\beta}(w) \mid \alpha \in [0,1]^{N\setminus C}, \beta \in \Delta(C)\} \\ &\subset \{\psi^{\alpha,\beta}(v) \mid \alpha \in [0,1]^{N\setminus C}, \beta \in \Delta(C)\} + \{\psi^{\alpha,\beta}(w) \mid \alpha \in [0,1]^{N\setminus C}, \beta \in \Delta(C)\} \\ &= Core(v) + Core(w), \end{aligned}$$

where the equalities follow from (ii). □

For fuzzy clan games the notion of bi-monotonic allocation scheme which we introduce now plays a similar role as pamas for convex fuzzy games in [2]. A bi-monotonic allocation scheme is an $\infty \times n$-matrix, where the columns correspond to the players (clan and non-clan members) and the rows to the fuzzy coalitions containing all clan members with full participation levels. In each row t there is a core element of the game v_t. Each two different rows s, t with $s \leqslant t$ are related via a bi-monotonicity condition with respect to the participation levels. To be more precise, if the scheme is used as a regulator for the payoff distributions in the restricted (fuzzy) games, non-clan members are paid per unit of participation less in larger coalitions than in smaller coalitions, while clan members are better off in larger coalitions.

Let $v \in FCG_C^N$. A scheme $[b_{t,i}]_{t\in\mathscr{F}_{1_C}^N, i\in N}$ is called a *bi-monotonic participation allocation scheme* (*bi-pamas*) for v if the following conditions hold:

(i) (*Stability*) $(b_{t,i})_{i\in N} \in Core(v_t)$ for each $t \in \mathscr{F}_{1_C}^N$;
(ii) (*Bi-monotonicity w.r.t. participation levels*). For all $s,t \in \mathscr{F}_{1_C}^N$ with $s \leqslant t$ we have,

- $s_i^{-1} b_{s,i} \geqslant t_i^{-1} b_{t,i}$ for each $i \in (N\backslash C) \cap car(s)$;
- $b_{s,i} \leqslant b_{t,i}$ for each $i \in C$.

Remark 2. The restriction of $[b_{t,i}]_{t\in\mathscr{F}_{1_C}^N, i\in N}$ to a crisp environment (where only the crisp coalitions are considered) is a bi-monotonic allocation scheme according to Branzei et al. [4] for the case $|C| = 1$, and Voorneveld et al. [19] for the case $|C| \geqslant 2$; see also [7]. A bi-pmas of a crisp game is a bunch of core elements, one for each subgame v_T with $T \supset C$ and the game itself, which are related via a bi-monotonicity condition guaranteeing that in coalitions containing all clan members each non-clan member is worst off when more other non-clan members join him, while clan members are better off in larger coalition than in smaller ones.

Lemma 4. *Let $v \in FCG_C^N$. Let $s,t \in \mathscr{F}_{1_C}^N$ with $s \leqslant t$ and let $i \in car(s)$ be a non-clan member. Then $D_i v(s) \geqslant D_i v(t)$.*

Proof. $D_i v(s) = \lim_{\varepsilon\to 0} \varepsilon^{-1}(v(s) - v(s - \varepsilon e^i)) \geqslant \lim_{\varepsilon\to 0} \varepsilon^{-1}(v(t) - v(t - \varepsilon e^i)) = D_i v(t)$, where the inequality follows from the DAMR-property of v, with $\varepsilon_1 = \varepsilon_2 = \varepsilon$. □

Theorem 4. *Let $v \in FCG_C^N$, with $N\backslash C = \{1,\ldots,m\}$. Then for each $\alpha \in [0,1]^m$ and $\beta \in \Delta(C) = \Delta(\{m+1,\ldots,n\})$ the compensation-sharing rule $\psi^{\alpha,\beta}$ generates a bi-monotonic participation allocation scheme for v, namely $[\psi_i^{\alpha,\beta}(v_t)]_{t\in\mathscr{F}_{1_C}^N, i\in N}$.*

Proof. We treat only the case $|C| > 1$. In Theorem 2(i) we have proved that for each $t \in \mathscr{F}_{1_C}^N$ the core $Core(v_t)$ of the t-restricted game v_t is given by $Core(v_t) = \{x \in \mathbb{R}^n \mid \sum_{i\in N} x_i = v(t),\ 0 \leqslant x_i \leqslant t_i D_i v(t)$ for each $i \in N\backslash C$, $0 \leqslant x_i$ for each $i \in C\}$.

Then, for each non-clan member i the α-based compensation (regardless of β) in the "grand coalition" t of the t-restricted game v_t is $\psi_i^{\alpha,\beta} = \alpha_i t_i D_i v(t)$, $i \in \{1,\ldots,m\}$. Hence, $\psi_i^{\alpha,\beta} = \beta_i(v(t) - \sum_{j=1}^m \alpha_j t_j D_j v(t))$ for each $i \in \{m+1,\ldots,n\}$.

First we prove that for each non-clan member i the compensation per unit of participation level is weakly decreasing when the coalition containing all clan members with full participation level and in which player i is active (i.e. $s_i > 0$) becomes larger.

Let $s,t \in \mathscr{F}_{1_C}^N$ with $s \leqslant t$ and $i \in car(s) \cap (N\backslash C)$. We have

$$\begin{aligned}\psi_i^{\alpha,\beta}(v_s) &= \alpha_i D_i v_s(e^N) = \alpha_i s_i D_i(v(s))\\ &\geqslant \alpha_i s_i D_i(v(t)) = \alpha_i s_i (t_i)^{-1} D_i v_t(e^N) = s_i (t_i)^{-1} \psi_i^{\alpha,\beta}(v_t),\end{aligned}$$

where the second and third equalities by Lemma 3, and the inequality follows from Lemma 4. Hence, for each $s,t \in \mathscr{F}_{1_C}^N$ with $s \leqslant t$ and each non-clan member $i \in car(s)$

$$s_i^{-1}\psi_i^{\alpha,\beta}(v_s) \geqslant t_i^{-1}\psi_i^{\alpha,\beta}(v_t).$$

Now, denote by $R_\alpha(v_t)$ the α-based remainder for the clan members in the "grand coalition" t of the t-restricted game v_t. Formally,

$$R_\alpha(v_t) = v_t(e^N) - \sum_{i \in N \setminus C} \alpha_i D_i v_t(e^N) = v(t) - \sum_{i \in N \setminus C} \alpha_i D_i v(t).$$

First we prove that for each $s,t \in \mathscr{F}_{1_C}^N$ with $s \leqslant t$

$$R_\alpha(v_t) \geqslant R_\alpha(v_s). \tag{4}$$

Inequality (4) expresses the fact that the remainder for the clan members is weakly larger in larger coalitions (when non-clan members increase their participation level).

Let $s,t \in \mathscr{F}_{1_C}^N$ with $s \leqslant t$. Then

$$\begin{aligned} v(t) - v(s) &= \sum_{k=1}^{m}\left(v\left(s + \sum_{i=1}^{k}(t_i - s_i)e^i\right) - v\left(s + \sum_{i=1}^{k-1}(t_i - s_i)e^i\right)\right) \\ &\geqslant \sum_{k=1}^{m}(t_k - s_k)D_k v\left(s + \sum_{i=1}^{k}(t_i - s_i)e^i\right) \\ &\geqslant \sum_{k=1}^{m}(t_k - s_k)D_k v(t) \geqslant \sum_{k=1}^{m}(t_k - s_k)\alpha_k D_k v(t), \end{aligned}$$

where the first inequality follows from Lemma 1 and the second inequality from Lemma 4. This implies $v(t) - \sum_{k=1}^m t_k\alpha_k D_k v(t) \geqslant v(s) - \sum_{k=1}^m s_k\alpha_k D_k v(t) \geqslant v(s) - \sum_{k=1}^m s_k\alpha_k D_k v(s)$, where the last inequality follows from Lemma 4. So, we proved that $R_\alpha(v_t) \geqslant R_\alpha(v_s)$ for all $s,t \in \mathscr{F}_{1_C}^N$ with $s \leqslant t$.

Now, note that inequality (4) implies that for each clan member the individual share (of the remainder for the whole group of clan members) in v_t, that is $\beta_i R_\alpha(v_t)$, is weakly increasing when non-clan members increase their participation level. □

In crisp game theory a prominent class of total big boss games is the class of holding games (cf. [17]). In the next example we consider a fuzzy approach to holding situations leading to a fuzzy big boss game.

Example 5. Let agents 1 and 2 have goods to be stored and let agent 3 possess a holding house with capacity 1. Agents 1 and 2 at activity level s_1 and s_2, respectively, want to store s_1 and s_2 units, respectively, with corresponding benefit $10s_1$ and $4s_2$. This economic situation leads to a fuzzy game with $N = \{1,2,3\}$, $v(s_1,s_2,0) = 0$ for all $s_1,s_2 \in [0,1]$, $v(s_1,s_2,1) = 10s_1 + 4s_2$ if $s_1 + s_2 \leqslant 1$, and $v(s_1,s_2,1) = 10s_1 + 4(1 - s_1) = 6s_1 + 4$ if $s_1 + s_2 > 1$.

One can easily check that this is a fuzzy big boss game with player 3 as a big boss. The bi-pamas $[b_{t,i}]_{t \in \mathscr{F}_{1_{\{3\}}}^{\{1,2,3\}},\, i \in \{1,2,3\}}$ corresponding to the compensation-sharing rule where players 1 and 2 obtain half of their marginal contribution is given by: $b_{t,1}=5t_1$; $b_{t,2}=2t_2$; $b_{t,3}=5t_1+2t_2$, if $t_1+t_2\leqslant 1$, and $b_{t,1}=3t_1$; $b_{t,2}=0$; $b_{t,3}=3t_1+4$, if $t_1+t_2>1$.

Let $v\in FCG_C^N$ and $x\in Core(v)$. Then we call x *bi-pamas extendable* if there exists a bi-pamas $[b_{t,i}]_{t\in\mathscr{F}_{1_C}^N, i\in N}$ such that $b_{e^N,i}=x_i$ for each $i\in N$. In the next theorem we show that each core element of a fuzzy clan game is bi-pamas extendable.

Theorem 5. *Let $v\in FCG_C^N$ and $x\in Core(v)$. Then x is bi-pamas extendable.*

Proof. Let $x\in Core(v)$. Then, according to Theorem 3(ii), x is of the form $\psi^{\alpha,\beta}(v_{e^N})$. Take now $[\psi_i^{\alpha,\beta}(v_t)]_{t\in\mathscr{F}_{1_C}^N, i\in N}$, which is a bi-pamas by Theorem 4. □

6. Concluding remarks

In this paper games of the form $v:[0,1]^{N_1}\times\{0,1\}^{N_2}\to\mathbb{R}$ are considered, where the players in N_1 have participation levels which may vary between 0 and 1, while the players in N_2 are crisp players in the sense that they can fully cooperate or not at all. Special attention is given to a subclass of such games, which we call fuzzy clan games, where the clan members are the crisp players. For the class of fuzzy clan games we have focused on the core [1] and on bi-monotonic participation allocation rules and schemes that are introduced in this paper. In [16] we have paid attention to other cores (the proper core, the dominance core and the crisp core) and stable sets for fuzzy clan games. For crisp clan and big boss games properties of other solutions and their relations are studied, namely: the bargaining set, the kernel, subsolutions, the Shapley value, the nucleolus, and the τ-value [9,11]. A topic for further research could be to introduce for games with (partly) fuzzy coalitions solutions corresponding to the kernel, the bargaining set and subsolutions, and to study for fuzzy clan games their properties, as well as their relations with other solution concepts.

References

[1] J.P. Aubin, Coeur et valeur des jeux flous a paiement lateraux, C. R. Acad. Sci. Paris 279 A (1974) 891–894.

[2] R. Branzei, D. Dimitrov, S. Tijs, Convex fuzzy games and participation monotonic allocation schemes, CentER DP 2002-13, Tilburg University, The Netherlands, 2002, Fuzzy Sets and Systems, to appear.

[3] R. Branzei, S. Tijs, Additivity regions in cooperative game theory, Libertas Math. 21 (2001) 155–167.

[4] R. Branzei, S. Tijs, J. Timmer, Information collecting situations and bi-monotonic allocation schemes, Math. Methods Oper. Res. 54 (2001) 303–313.

[5] V.K. Chetty, D. Dasgupta, T.E.S. Raghavan, Power and distribution of profit, Discussion Paper No. 139, Indian Statistical Institute, Delhi Center, New Delhi, 1976.

[6] I. Dragan, J. Potters, S. Tijs, Superadditivity for solutions on coalitional games, Libertas Math. 9 (1989) 101–110.

[7] S. Grahn, Topics in cooperative game theory, Ph.D. Thesis, Department of Economics, Upsala University, Economic Studies 58, 2002.

[8] H. Moulin, Axioms of Cooperative Decision Making, Cambridge University Press, Cambridge, NY, 1988.

[9] S. Muto, M. Nakayama, J. Potters, S. Tijs, On big boss games, Econom. Stud. Quart. 39 (1988) 303–321.

[10] I. Nishizaki, M. Sakawa, Fuzzy and Multiobjective Games for Conflict Resolution, Physica-Verlag, Wurzberg (Wien), Heidelberg, 2001.

[11] J. Potters, R. Poos, S. Tijs, S. Muto, Clan games, Games Econom. Behavior 1 (1989) 275–293.
[12] L.S. Shapley, A value for *n*-person games, Ann. Math. Stud. 28 (1953) 307–317.
[13] Y. Sprumont, Population monotonic allocation schemes for cooperative games with transferable utility, Games Econom. Behavior 2 (1990) 378–394.
[14] S.H. Tijs, Big boss games, clan games and information market games, in: T. Ichiishi, A. Neyman, Y. Tauman (Eds.), Game Theory and Applications, Academic Press, Inc., San Diego, 1990, pp. 410–412.
[15] S. Tijs, R. Branzei, Additive stable solutions on perfect cones of cooperative games, CentER DP 2001-105, Tilburg University, Tilburg, The Netherlands, Internat. J. Game Theory 31 (2002) 469–474.
[16] S. Tijs, R. Branzei, S. Ishihara, S. Muto, On Cores and Stable Sets for Fuzzy Games, CentER DP 2002-116, Tilburg University, Tilburg, The Netherlands, and VALDES RP Series E, 03-02 (2003), Tokyo Institute of Technology, Tokyo, Japan, 2002.
[17] S. Tijs, A. Meca, M.A. Lopez, Benefit Sharing in Holding Situations, DP-I-2000-01, OR Center, Universitas Miguel Hernandez, Elche, Spain, 2000.
[18] S. Tijs, J. Timmer, R. Branzei, Compensations in Information Collecting Situations, CentER DP 2001-02, Tilburg University, The Netherlands, 2001.
[19] M. Voorneveld, S. Tijs, S. Grahn, Monotonic allocation schemes in clan games, Math. Methods Oper. Res. 56 (2002) 439–449.
[20] H.P. Young, Monotonic solutions of cooperative games, Internat. J. Game Theory 14 (1985) 65–72.