Top (2007) 15: 78–88
DOI 10.1007/s11750-007-0002-7

ORIGINAL PAPER

Type monotonic allocation schemes for a class of market games

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Received: 5 September 2005 / Accepted: 12 July 2006 /
Published online: 10 March 2007

Abstract We consider an extension of glove markets, called T-markets, characterize a family of weighted allocation rules, and define related cooperative games. For the class of T-market games we introduce a new solution concept called the type monotonic allocation scheme. It turns out that the nucleolus and the τ-value generate the same type monotonic allocation scheme with nice extra properties.

Keywords Market game · Monotonic allocation scheme · τ-value · Nucleolus

Mathematics Subject Classification (2000) 91A12

1 Introduction

The main purpose of this paper is to introduce a new kind of monotonic allocation schemes for special cooperative games with players of different types. Other sort of

Initial research for this paper was supported by the Hungarian Scientific Research Fund grant OTKA T030945. The authors thank Ruud Hendrickx for his valuable comments. Research of T. Solymosi was also supported in part by OTKA grant T046194.

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monotonic allocation rules for various special types of (totally) balanced cooperative games have been introduced earlier by Sprumont (1990), Brânzei et al. (2001), Voorneveld et al. (2003), and Fernández et al. (2005).

In this paper we consider markets whose corners are characterized by different types of resources and where each player belongs to one corner and possesses one unit of the corner-specific resource. We refer to such a market as a T(ype-based)-market and associate with it a cooperative game which we call a T-market game. For T-market games we introduce the notion of type monotonic allocation scheme (tmas) and tackle the question of existence of a tmas and of the possibility of extensions of core elements to a tmas.

The outline of the paper is as follows. In Sect. 2, we introduce T-markets and consider allocation rules for them. Special attention is paid to an interesting class of weighted allocation rules, which are based on hierarchical weight systems on the type space. In Sect. 3, T-market games corresponding to T-markets are studied. For this kind of cooperative games we describe the imputation set, the core, the τ-value, and the nucleolus. The τ-value and the nucleolus coincide, and these single-valued solutions turn out to be in the barycenter of the core of such a game. In Sect. 4, the notion of type monotonic allocation scheme for T-market games is introduced. Each weighted allocation rule for T-markets generates, for each T-market game, a tmas, and all core elements of a T-market game can be extended to a tmas. The τ-value (or the nucleolus) applied to a T-market game and to each of its subgames (all T-market games) turns out to generate a tmas with nice extra properties. Section 5 contains some concluding remarks.

2 T-markets and allocation rules

In this section, we consider special markets in which t different types of complementary goods $1, 2, \ldots, t$ $(t \geq 2)$ are available in t corners of the market and where each agent in the market belongs to one corner and possesses precisely one unit of the corner-specific good; moreover, for each type of goods there is at least one owner. Furthermore, a profit of one unit can be obtained only if one unit of each of the t types of goods are combined. The agents in the market with one unit of good s are called agents of type s, and this set of agents is denoted by N_s. Then $N_1, N_2, \ldots, N_t$ is a partition of the set N of all agents in the market. Let us denote the set of types $\{1, 2, \ldots, t\}$ by T, and let us denote by n_s the number of agents in N_s. Then the agents in N can make a profit of $\min_{s\in T} n_s$ units. We will call such a market a *T-market* and denote it by $m_T := \langle T, N_1, N_2, \ldots, N_t\rangle$, and, in situations where only the number of agents of the different types matters, we also denote the market by $n = (n_1, n_2, \ldots, n_t)$. If there are only two types $(t = 2)$, such a market is called a glove market, where the agents of type 1 (type 2) possess a left (right) glove and where a left–right pair of gloves has value 1. For relevant related literature we refer to Shapley (1959), Shapley and Shubik (1969), Owen (1975), Rosenmüller and Sudhölter (2002, 2004), and Apartsin and Holzman (2003).

We are interested in efficient and symmetric allocation rules for T-markets which divide for each market $(n_1, n_2, \ldots, n_t)$ the total profit $p(n) := \min_{s\in T} n_s$ among the agents in such a way that agents of the same type get the same share of the profit. Such allocation rules can be described by a function $F : \mathbb{N}^t \to \mathbb{R}^t_+$, where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers, and where, for each market $n = (n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$ and for each $s \in T$, the share of the profit of each agent of type s is $F_s(n)$. Then $\sum_{s \in T} n_s F_s(n) = \min_{u \in T} n_u = p(n)$.

In the following we denote the set of scarce types of the market n by $sc(n)$, so $sc(n) = \{s \in T \mid n_s = p(n)\} = \arg\min_{u \in T} n_u$.

Example 1 The allocation rule $F : \mathbb{N}^t \to \mathbb{R}^t_+$ defined by $F_s(n) = t^{-1} n_s^{-1} p(n)$, for each $s \in T$, divides the profit in such a way that the total share of each of the types is the same amount $t^{-1} p(n)$. Here F assigns also to non-scarce agents a positive payoff.

Example 2 Let $E : \mathbb{N}^t \to \mathbb{R}^t_+$ be such that, for each T-market n, $E_s(n) = 0$ if $s \notin sc(n)$, and $E_s(n) = |sc(n)|^{-1}$ otherwise. Then E divides the profit equally among the scarce members.

We introduce now a special class of efficient and symmetric allocation rules induced by hierarchical weight systems. A (*hierarchical*) *weight system* for T-markets with type space $T = \{1, 2, \ldots, t\}$ is determined by a partition $\{C^1, C^2, \ldots, C^k\}$ of T in classes of types and corresponding vectors $w^1, w^2, \ldots, w^k$ in $\mathbb{R}^t_+$, where, for each $r \in \{1, 2, \ldots, k\}$, the carrier $\mathrm{carr}(w^r) = \{s \in T \mid w^r_s > 0\}$ of w^r is equal to C^r and where $\sum_{s \in C^r} w^r_s = 1$. We will denote such a weight system by $(w^1, w^2, \ldots, w^k)$ or shortly by w. The class C^r is called the *class of types of rank r*, and the vector w^r is called the *weight vector* for this class.

Each given $(w^1, w^2, \ldots, w^k)$ induces an allocation rule $F^w = F^{(w^1, w^2, \ldots, w^k)}$ which divides the total profit in a T-market $n \in \mathbb{N}^t$ among the agents that belong to those scarce types which have the lowest rank, and the distribution of the profit for these types is determined by the corresponding weight vector. Formally, if $\ell(n) = \min\{r \in \{1, 2, \ldots, k\} \mid sc(n) \cap C^r \neq \emptyset\}$, then

$$F_s^{(w^1, w^2, \ldots, w^k)}(n) = \left(\sum \left\{w_r^{\ell(n)} \mid r \in sc(n) \cap C^{\ell(n)}\right\}\right)^{-1} w_s^{\ell(n)}$$

if $s \in sc(n) \cap C^{\ell(n)}$, and $F_s^{(w^1, w^2, \ldots, w^k)}(n) = 0$ otherwise.

Example 3 Let $T = \{1, 2, 3, 4, 5\}$, $k = 2$, $C^1 = \{1, 2, 3\}$, $C^2 = \{4, 5\}$, $w^1 = (\frac{1}{3}, \frac{1}{2}, \frac{1}{6}, 0, 0)$, and $w^2 = (0, 0, 0, \frac{1}{4}, \frac{3}{4})$. For the T-market $(6, 4, 4, 4, 4)$ we have $sc(6, 4, 4, 4, 4) = \{2, 3, 4, 5\}$ and $\ell(6, 4, 4, 4, 4) = 1$ implying that

$$F^{(w^1, w^2)}(6, 4, 4, 4, 4) = \left(0, \frac{3}{4}, \frac{1}{4}, 0, 0\right).$$

Further, $F^{(w^1, w^2)}(6, 4, 4, 3, 4) = (0, 0, 0, 1, 0)$.

Note that the rule $E : \mathbb{N}^t \to \mathbb{R}^t_+$ from Example 2 is also a rule based on a hierarchical weight system with only one class $C^1 = T$ and $w^1 = |T|^{-1}(1, 1, \ldots, 1)$; so $E = F^{w^1}$.

Let us now introduce for efficient and symmetric allocation rules two properties: the *scarcety property* SCARCE and the *ratio consistency property* CONS.

We say that $F : \mathbb{N}^t \to \mathbb{R}^t_+$ satisfies SCARCE if $F_s(n) = 0$ for each T-market $n \in \mathbb{N}^t$ and each $s \notin sc(n)$.

We say that $F : \mathbb{N}^t \to \mathbb{R}^t_+$ satisfies CONS if for all pairs $n, m \in \mathbb{N}^t$, for which there is an $s \in T$ such that $F_s(n) > 0$ and $F_s(m) > 0$, we have

$$\frac{F_u(m)}{F_s(m)} = \frac{F_u(n)}{F_s(n)} \quad \text{for all } u \in T. \tag{1}$$

This implies, for example, that for two types $s, u \in T$ with positive shares in both T-markets n and m w.r.t. the rule F, if s gets α times the amount which u gets in T-market n, then this is also (consistently) the case in T-market m.

These properties hold for rules based on hierarchical weight systems. Clearly, each F^w satisfies SCARCE. Now, take $F^w = F^{(w^1, w^2, \ldots, w^k)}$ and T-markets n, m for which there is an $s \in T$ such that $F^w_s(n) > 0$ and $F^w_s(m) > 0$. Then s is scarce in n as well as in m, and if $s \in C^r$ then $\ell(n) = \ell(m) = r$. So, $F^w_u(m) = F^w_u(n) = 0$ if $u \notin C^r$, and for such u equality (1) holds. For $u \in C^r$, we have

$$\frac{F^w_u(n)}{F^w_s(n)} = \frac{F^w_u(m)}{F^w_s(m)} = \frac{w^r_u}{w^r_s}.$$

Theorem 1 (Characterization of rules based on weight systems) *An allocation rule $F : \mathbb{N}^t \to \mathbb{R}^t_+$ satisfies* SCARCE *and* CONS *if and only if there are $k \in \mathbb{N}$ and a weight system $w = (w^1, w^2, \ldots, w^k)$ such that $F = F^w$.*

Proof We have already shown that a rule of the form F^w satisfies SCARCE and CONS. Conversely, let F be a rule with the SCARCE and CONS properties. We construct a weight system w as follows. Take the T-market $n^1 = (1, 1, \ldots, 1)$, where all types are scarce, and take $w^1 = F(n^1)$. If $\text{carr}(w^1) = T$, take $w = (w^1)$. If $\text{carr}(w^1) \neq T$, take a T-market n^2 with $n^2_s = 2$ if $s \in \text{carr}(w^1)$ and $n^2_s = 1$ otherwise. Then $sc(n^2) = T \setminus \text{carr}(w^1)$, and take $w^2 = F(n^2)$. If $\text{carr}(w^1) \cup \text{carr}(w^2) = T$, then take $w = (w^1, w^2)$. Otherwise take $w^3 = F(n^3)$, where $n^3_s = 2$ for $s \in \text{carr}(w^1) \cup \text{carr}(w^2)$ and $n^3_s = 1$ otherwise. And so we go on.

This leads to a weight system, say $w = (w^1, w^2, \ldots, w^k)$. Note that

$$F\left(n^h\right) = w^h = F^w\left(n^h\right) \quad \text{for all } h \in \{1, 2, \ldots, k\}. \tag{2}$$

It remains to prove that $F(n) = F^w(n)$ for all $n \in \mathbb{N}^t$.

Take $n \in \mathbb{N}^t$. Take among the scarce types in n a type s with $F_s(n) > 0$ and such that there is no s' with $F_{s'}(n) > 0$, which is in a lower class w.r.t. the hierarchy corresponding to w. Let $s \in C^h$. Then the T-markets n and n^h have a positive payoff for s. So, by CONS, for all $u \in T$,

$$\frac{F_u(n)}{F_s(n)} = \frac{F_u(n^h)}{F_s(n^h)}.$$

Now, from (2) it follows

$$\frac{F_u(n^h)}{F_s(n^h)} = \frac{w_u^h}{w_s^h} = \frac{F_u^w(n)}{F_s^w(n)}.$$

This implies that $F_u(n) = 0$ if $u \notin \mathrm{carr}(w)$ and $F(n) = \alpha F^w(n)$ with $\alpha = \frac{F_s(n)}{F_s^w(n)}$. Since $\sum_{s\in T} F_s(n) = \sum_{s\in T} F_s^w(n) = 1$, we obtain $\alpha = 1$. Hence $F = F^w$. □

The next theorem, whose proof is left to the reader, will be useful in Sect. 4.

Theorem 2 (Type monotonicity for allocation rules based on weight systems) *Let $w = (w^1, w^2, \ldots, w^k)$ be a hierarchical weight system. Let $n \in \mathbb{N}^t$ and $u \in T$ with $n_u \geq 2$. Then*

$$F_u^w\big(n - e^u\big) \geq F_u^w(n), \qquad F_s^w\big(n - e^u\big) \leq F_s^w(n) \quad \text{for } s \in T \setminus \{u\}, \tag{3}$$

where $(n - e^u) = (n_1, n_2, \ldots, n_u - 1, \ldots, n_t)$, e^u is the u-th basis vector in $\mathbb{R}^t$.

Interpretation. If gains in T-markets are distributed using an allocation rule based on weight systems, and one player of certain type leaves a T-market, then in the new T-market the remaining agents of the same type are not worse off, while agents of other types are not better off.

3 T-market games

In this section, we assign to a T-market a cooperative game and discuss how some classical solution concepts look like for such T-market games.

First, we recall that a cooperative game is a pair $\langle N, v\rangle$, where $N = \{1, 2, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is the worth function assigning to each coalition $S \in 2^N$ the worth $v(S)$ which satisfies the assumption $v(\emptyset) = 0$.

Now, let $(n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$ be a T-market, where $N_1 = \{1, 2, \ldots, n_1\}$ is the set of agents of type 1, $N_2 = \{n_1 + 1, n_1 + 2, \ldots, n_1 + n_2\}$ the set of agents of type 2, ..., and $N_t = \{\sum_{s=1}^{t-1} n_s + 1, \sum_{s=1}^{t-1} n_s + 2, \ldots, \sum_{s=1}^{t} n_s\}$. The corresponding T-market game $\langle N, v\rangle$ is given by $N = \bigcup_{s=1}^{t} N_s$ and, for each $S \subset N$,

$$v(S) = \min\big\{|S \cap N_1|, |S \cap N_2|, \ldots, |S \cap N_t|\big\} \tag{4}$$

where $|R|$ is the number of elements in set R. Note that $v(N) = p(n)$. Non-trivial coalitions, i.e. coalitions $S \in 2^N \setminus \{\emptyset\}$ with $v(S) \neq 0$ and their (positive) type distribution vectors $t(S) = (|N_1 \cap S|, \ldots, |N_t \cap S|)$ will play a key role in Sect. 4. Note that $n = t(N)$. Let us call a coalition $S \in 2^N$ a *simple coalition* if it consists of t players, all of a different type.

The imputation set $I(v)$ is the set of individual rational allocations of the worth of the grand coalition, i.e.

$$I(v) = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = v(N),\ x_i \geq v\big(\{i\}\big) \text{ for each } i \in N \right\}.$$

The core $C(v)$ (cf. Gillies 1953) is the subset of $I(v)$ of split-off stable allocations, i.e.

$$C(v) = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \right\}.$$

Theorem 3 *Let $(n_1, n_2, \ldots, n_t) \in \mathbb{N}^t$ be a T-market with corresponding T-market game $\langle N, v \rangle$. Then*:

(i) $I(v) = \text{conv}\{p(n)e^i \mid i \in N\}$, *where e^i is the i-th basis vector in $\mathbb{R}^n$.*
(ii) $C(v) \neq \emptyset$, *and it consists of all vectors of the form $[\alpha_1, \alpha_2, \ldots, \alpha_t]$, where $\sum_{s=1}^{t} \alpha_s = 1$, $\alpha_s \geq 0$ for each s, and $\alpha_s = 0$ if $s \notin sc(n)$, and where $[\alpha_1, \alpha_2, \ldots, \alpha_t]$ denotes the vector x in $\mathbb{R}^n$ with $x_i = \alpha_s$ for each $s \in T$ and $i \in N_s$.*

Proof (i) Since $t \geq 2$, the worth $v(\{i\}) = 0$ for each $i \in N$. So $I(v) = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i = v(N)\} = \text{conv}\{v(N)e^i \mid i \in N\}$.

(ii) Take first $x = [\alpha_1, \alpha_2, \ldots, \alpha_t] \in \mathbb{R}^n_+$ with $\sum_{s \in T} \alpha_s = 1$ and $\alpha_s = 0$ if $s \notin sc(n)$. Then $\sum_{i \in N} x_i = \sum_{s \in T} \alpha_s n_s = \sum_{s \in sc(n)} \alpha_s n_s = \sum_{s \in sc(n)} \alpha_s v(N) = v(N)$. For $R \in 2^N$, we have $\sum_{i \in R} x_i = \sum_{s \in T} \alpha_s |R \cap N_s| \geq \sum_{s \in T} \alpha_s \min_{u \in T} |R \cap N_u| = \sum_{s \in T} \alpha_s v(R) = v(R)$. So $x \in C(v)$. Conversely, let $z \in C(v)$. We have to prove that there are $\beta_1, \ldots, \beta_t$ such that $z = [\beta_1, \beta_2, \ldots, \beta_t] \in \mathbb{R}^n_+$, where $\sum_{s \in T} \beta_s = 1$ and $\beta_s = 0$ if $s \notin sc(n)$.

For each simple coalition S, we have $\sum_{i \in S} z_i \geq v(S) = 1$. Take one of the many systems consisting of $p(n)$ disjoint simple coalitions, say $S_1, \ldots, S_{p(n)}$. We have $p(n) = v(N) = \sum_{i \in N} z_i \geq \sum_{j=1}^{p(n)} \sum_{i \in S_j} z_i \geq \sum_{j=1}^{p(n)} v(S_j) = p(n)$. This implies that $\sum_{i \in S_j} z_i = 1$ for each $j = 1, \ldots, p(n)$, and that $z_i = 0$ for each $i \in N \setminus (\bigcup_{j=1}^{p(n)} S_j)$. Since each simple coalition is a member of such a family, we have $\sum_{i \in S} z_i = 1$ for each simple coalition S. Similarly, since each non-scarce player is left out from at least one such family, we have $z_i = 0$ for each non-scarce player i. Finally, let $s \in sc(n)$ and let $k, \ell \in N_s$. We claim that $z_k = z_\ell$. Take a simple coalition S with $k \in S$. Then $S' = (S \cup \{\ell\}) \setminus \{k\}$ is also simple. From $\sum_{i \in S} z_i = \sum_{i \in S'} z_i = 1$ it follows that $z_k = z_\ell$.

Now let $\beta_s = z_i$ for each $s \in T$ and some $i \in N_s$. Then z corresponds to $[\beta_1, \beta_2, \ldots, \beta_t]$. □

Remark 1 The imputation set of a T-market game is a simplex in $\mathbb{R}^n$ whose extreme points are those allocations where one player obtains the whole worth of the grand coalition N. Core elements are imputations where players of the same scarce type get the same share and players of a non-scarce type get nothing.

Theorem 4 *Let $\langle N, v \rangle$ be the T-market game corresponding to the T-market $n \in \mathbb{N}^t$. Then for the τ-value $\tau(v)$ and nucleolus $\eta(v)$ the following assertions hold*:

(i) $\tau_i(v) = |sc(n)|^{-1}$ *for $i \in N_r$ and $r \in sc(n)$, and $\tau_i(v) = 0$ otherwise.*
(ii) $\tau(v) = \eta(v)$.
(iii) *The τ-value and the nucleolus, restricted to T-market games, are the only rules which assign to each such a game a core element with coordinates the same for*

all players of a scarce type and coordinates zero for the players of a non-scarce type.

Proof First, we note that in a T-market game non-scarce players are not entitled to a positive value according to the τ-value (Tijs 1981) and the nucleolus (Schmeidler 1969). Further, from the anonymity property that both the nucleolus and the τ-value satisfy, we obtain equal share for scarce players of the same type. □

Remark 2 Let us make some geometric observations following from the two preceding theorems, where we deal with a game $\langle N, v\rangle$ corresponding to the T-market $n \in \mathbb{N}^t$. For each $r \in sc(n)$, we can consider the simplex $I^r(v) = \text{conv}\{v(N)e^i \mid i \in N^r\}$, which is a face of the simplex $I(v)$ with the barycenter $b^r = \sum_{i\in N^r} e^i$. The core $C(v)$ of the game $\langle N, v\rangle$ equals the convex hull of these barycenters, i.e. $C(v) = \text{conv}\{b^r \mid r \in sc(n)\}$, so the core is also a simplex. The barycenter of the core equals the τ-value and the nucleolus: $\tau(v) = \eta(v) = |sc(n)|^{-1} \sum_{r\in sc(n)} b^r$. The following example may be illustrative.

Example 4 Let $\langle N, v\rangle$ be the game corresponding to the T-market $n = (2, 2, 3)$. Then $N = \{1, 2, \ldots, 7\}$ and $I(v) = \text{conv}\{2e^i \mid i \in N\}$. Further, $sc(n) = \{1, 2\}$, $I^1(v) = \text{conv}\{2e^1, 2e^2\} \subset I(v)$, $I^2(v) = \text{conv}\{2e^3, 2e^4\} \subset I(v)$, $b^1(v) = e^1 + e^2$, $b^2(v) = e^3 + e^4$, and $C(v) = \{(\alpha_1, \alpha_1, \alpha_2, \alpha_2, 0, 0, 0) \in \mathbb{R}^7_+ \mid \alpha_1 + \alpha_2 = 1\} = \text{conv}\{(1, 1, 0, 0, 0, 0, 0), (0, 0, 1, 1, 0, 0, 0)\} = \text{conv}\{b^1, b^2\}$, $\tau(v) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0) = \frac{1}{2}(b^1(v) + b^2(v))$.

4 Type monotonic allocation schemes

In the following for a game $\langle N, v\rangle$ also its subgames $\langle S, v\rangle$ play a role. Here, $S \in 2^N \setminus \{\emptyset\}$ and $v : 2^S \to \mathbb{R}$ is the restriction of $v : 2^N \to \mathbb{R}$ to the set 2^S of subcoalitions of S. Special attention is paid to so-called allocation schemes, which assign to a game and its subgames a collection of payoff vectors, one for each subgame. Such an allocation scheme can be denoted by $[x_i^S]_{S\in 2^N\setminus\{\emptyset\}, i\in S}$, where $x^S = (x_i^S)_{i\in S} \in \mathbb{R}^S$, and can be seen as a (defect) matrix with n columns corresponding to the players and $2^n - 1$ rows corresponding to the nonempty coalitions. An allocation scheme for a game $\langle N, v\rangle$ is called stable if for each $S \in 2^N \setminus \{\emptyset\}$ the vector x^S is an element of the core $C(S, v)$ of the subgame $\langle S, v\rangle$. So, $C(v) = C(N, v)$.

In the literature one finds stable allocation schemes with special monotonicity properties. Sprumont (1990) has studied for totally balanced games $\langle N, v\rangle$ population monotonic allocation schemes (pmas) $[x_i^S]_{S\in 2^N\setminus\{\emptyset\}, i\in S}$, where $x^S \in C(S, v)$ for each $S \in 2^N \setminus \{\emptyset\}$ and where, for each $S \in 2^N \setminus \{\emptyset\}$, $i \in S$, $j \in N \setminus S$, we have $x_i^{S\cup\{j\}} \geq x_i^S$. So, a player i gets a better payoff in the larger coalition $S \cup \{j\}$ than in S. In Brânzei et al. (2001), for total big boss games (cf. Muto et al. 1988) $\langle N, v\rangle$ with player 1 as the big boss, bi-monotonic allocation schemes (bi-mas) $[x_i^S]_{S\in B^N, i\in S}$ were introduced, where $B^N = \{S \in 2^N \mid 1 \in S\}$, $x^S \in C(S, v)$ for each $S \in B^N$, and for all $S \in B^N$ with $i \in S \setminus \{1\}$, $j \in N \setminus S$ we have $x_i^{S\cup\{j\}} \leq x_i^S$ and $x_1^{S\cup\{j\}} \geq x_1^S$. So, in a bi-mas the big boss is not worse off in a larger coalition and the other players are not better off. Y. Sprumont proved that for convex games each core element $x \in C(N, v)$ is extendable to a pmas $[x_i^S]_{S \in 2^N \setminus \{\emptyset\}, i \in S}$, where $x^N = x$. Further, if x^S is the Shapley value (Shapley 1953) of $\langle S, v \rangle$, then $[x_i^S]_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ is a pmas for the convex game $\langle N, v \rangle$. For total big boss games also each core element is bi-mas extendable and the τ-value applied to the subgames containing the big boss generates a bi-mas (cf. Brânzei et al. 2001).

The objective of this section is to introduce a new sort of stable monotonic allocation schemes for the class of T-market games with the type set T.

Note first that for a T-market game $\langle N, v \rangle$ corresponding to the T-market $\langle T, N_1, N_2, \ldots, N_t \rangle$, the subgame $\langle S, v \rangle$ with $S \in 2^N \setminus \{\emptyset\}$ corresponds to the sub-market $\langle T, N_1 \cap S, N_2 \cap S, \ldots, N_t \cap S \rangle$ if $v(S) \neq 0$. So, non-trivial subgames of a T-market game are also T-market games.

We are only interested in allocation schemes for (totally balanced) T-market games with special properties. First, we want that such an allocation scheme assign to the non-trivial subcoalitions S core elements of the subgame $\langle S, v \rangle$. This implies, according to Theorem 3, that in the subgame players of the same type obtain the same payoff, while non-scarce players obtain zero. Secondly, we want that the allocation scheme be type distribution consistent, i.e. if for two subgames $\langle S_1, v \rangle$ and $\langle S_2, v \rangle$ we have $t(S_1) = t(S_2)$ for each $s \in T$, then the allocation scheme should assign the same payoff to the players of type s in both subgames. This implies that such an allocation scheme for $\langle N, v \rangle$ can be represented by a map $\alpha^n : \{m \in \mathbb{N}^t | m \leq n\} \to \Delta$, where $\Delta = \{\beta \in \mathbb{R}_+^t \mid \sum_{s \in T} \beta_s = 1\}$ and $n = t(N)$. For a non-trivial coalition $S \in 2^N$ with positive type distribution vector $t(S)$, $\alpha_u^n(t(S))$ is the payoff assigned to players of type u in S.

Now we come to our main notion.

Definition 1 A *type monotonic allocation scheme* for a T-market game $\langle N, v \rangle$ with type set T and type distribution vector $n = t(N)$ is a map $\alpha^n : \{m \in \mathbb{N}^t \mid m \leq n\} \to \Delta$ with the following two properties:

- (*Stability*) $\alpha_s^n(m) = 0$ for each $m \leq n$ and $s \in T \setminus sc(m)$.
- (*Type monotonicity*) For all $m \in \mathbb{N}^t$ and $s \in T$ such that $m - e^s \in \mathbb{N}^t$, we have $\alpha_s^n(m - e^s) \geq \alpha_s^n(m)$ and $\alpha_u^n(m - e^s) \leq \alpha_u^n(m)$ for each $u \in T \setminus \{s\}$.

So, a type monotonic allocation scheme has the following property: if a player leaves a coalition, the remaining players of the same type are not worse off and the other types of players are not better off in the new core allocation (for the remaining coalition) than in the old core element.

We illustrate this notion with two examples.

Example 5 Let $\langle N, v \rangle$ be the 5-person game with $N_1 = \{1\}$, $N_2 = \{2, 3\}$, and $N_3 = \{4, 5\}$. Then assigning to each non-trivial subgame the τ-value leads to a type monotonic allocation scheme which can be represented by the matrix

$$\begin{array}{c} \\ (1,2,2) \\ (1,1,2) \\ (1,2,1) \\ (1,1,1) \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{array}$$

where the columns correspond to the types 1, 2, 3 and the rows to the type distribution vectors of the non-trivial subcoalitions. In $\langle N \setminus \{2\}, v\rangle$ the remaining player of type 2 is, in comparison to $\langle N, v\rangle$, not worse off ($\frac{1}{2} > 0$) and the players of types 1 and 3 are not better off ($\frac{1}{2} < 1, 0 = 0$).

It turns out (Theorem 5) that the τ-value and the nucleolus generate for each T-market game a type monotonic allocation scheme. Also, other weighted allocation schemes for T-markets generate type monotonic allocation schemes for T-market games, as we see in Example 6 and Theorem 5.

Example 6 Consider the game in Example 5 and apply the weighted allocation scheme $F^{((\frac{1}{3},\frac{2}{3},0),(0,0,1))}$ to the type distribution vectors of the game and the subgames. This leads to the type monotonic allocation scheme represented by the matrix

$$\begin{array}{c} \\ (1,2,2) \\ (1,1,2) \\ (1,2,1) \\ (1,1,1) \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \end{array}$$

where in the third row we find $(1, 0, 0) = F^{((\frac{1}{3},\frac{2}{3}),(1))}(1, 2, 1)$.

Take an arbitrary weighted allocation rule F^w for T-markets, where $w = (w_1, w_2, \ldots)$. Then, for each T-market game $\langle N, v\rangle$, the vector $F^w(t(N))$ is a core element for $\langle N, v\rangle$. This together with Theorem 2 implies that $\alpha^w : \{m \in \mathbb{N}^n \mid m \le t(N)\} \to \Delta$ with $\alpha^w(m) = F^w(m)$ for each m is a type monotonic allocation scheme for the T-market game $\langle N, v\rangle$. The weight system (w^1) with $w^1 = t^{-1}(e^1 + e^2 + \cdots + e^t)$ corresponds to $\alpha^{(w^1)}$, and $\alpha^{(w^1)}$ assigns to each subgame its τ-value (or nucleolus). So we have proved the following theorem.

Theorem 5 *Let $\langle N, v\rangle$ be a T-market game with type set T. Then*:

(i) *For each hierarchical weight system w, $\alpha^w : \{m \in \mathbb{N}^t \mid m \le t(N)\} \to \mathbb{R}^t_+$ is a type monotonic allocation scheme for $\langle N, v\rangle$.*
(ii) *An allocation scheme for $\langle N, v\rangle$ which assigns to each subgame the τ-value* (*or the nucleolus*) *is a type monotonic allocation scheme.*

The following theorem implies that each core element z of a T-market game $\langle N, v\rangle$ is extendable to a type monotonic allocation scheme, where the element assigned to $\langle N, v\rangle$ corresponds to z.

Theorem 6 *Let $z \in C(N, v)$ and $t(N) \in \mathbb{N}^t$. Then there is a hierarchical weight system w such that $\alpha^w(t(N)) = z$.*

Proof Construct the vector $w^1 \in \mathbb{R}^t_+$ as follows. For each type $s \in T$ take an $i \in N_s$ and put $w^1_s = z_i$. If $\text{carr}(w^1) = T$, then take $w = (w^1)$. If $\text{carr}(w^1) \neq T$, take as weight system $w = (w^1, w^2)$, where w^2 is an arbitrarily chosen vector $w^2 \in \mathbb{R}^T_+$ with the properties that $\text{carr}(w^2) = T \setminus \text{carr}(w^1)$ and that the sum of the coordinates is 1. Then $\alpha^w_s(t(N)) = z_i$ for each $i \in N_s$. So $\alpha^w(t(N))$ corresponds to the core element z of $\langle N, v\rangle$, and α^w is a type monotonic allocation scheme which 'extends' z. $\square$

Example 7 Let $\langle N, v\rangle$ be the 4-person T-market game with $N_1 = \{1, 2\}$ and $N_2 = \{3, 4\}$. Then $z = (1, 1, 0, 0) \in C(N, v)$, which corresponds to $\alpha^w(2, 2)$, where $w = (w^1, w^2)$ with $w^1 = (1, 0)$ and $w^2 = (0, 1)$. Note that $\alpha^w(2, 2) = \alpha^w(1, 2) = \alpha^w(1, 1) = (1, 0)$, and $\alpha^w(2, 1) = (0, 1)$.

5 Concluding remarks

This paper considers a special class of markets, called T-markets, whose corners are characterized by different types of resources and where each player belongs to one corner and possesses one unit of the corner-specific resource. We have introduced allocation rules for T-markets and analyzed some of the properties and solutions of the related cooperative games, called T-market games. Our main contribution is the introduction of a new monotonicity property, which we have called type monotonicity. We have shown that for T-market games the nucleolus and the τ-value coincide and that they yield a type monotonic allocation scheme.

We note here that a T-market can be seen as a linear production situation with one product, t resources, a technology matrix containing only ones, and an endowment matrix in which each player possesses only one resource. Such linear production (LP-) situations arise from simple max flow situations (cf. Kalai and Zemel 1982, Reijnierse et al. 1996) where each player owns one arc with capacity one, and there are $t + 1$ nodes $p_0, p_1, \ldots, p_t$, where p_0 is the source, p_t is the sink, and players whose arcs connect p_{k-1} to p_k, for $k = 1, \ldots, t$, are all of the same type k. It is well known that for these simple flow situations the minimum cut solutions correspond to the extreme points of the core of the corresponding flow game. So, this core corresponds to the set of Owen vectors of the related LP-situation. It looks intriguing but difficult to extend the existence of tmas to a larger subclass of LP-situations.

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