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Appointment Games in Fixed-Route Traveling Salesman Problems and the Shapley Value

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Appointment Games in Fixed-Route Traveling Salesman Problems and The Shapley Value*

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September 14, 2009

Abstract

Starting from her home, a service provider visits several customers, following a predetermined route, and returns home after all customers are visited. The problem is to find a fair allocation of the total cost of this tour among the customers served. A transferable-utility cooperative game can be associated with this cost allocation problem. We introduce a new class of games, which we refer as the *fixed-route traveling salesman games with appointments.* We study the Shapley Value in this class and show that it is in the core. Our first characterization of the Shapley value involves a property which requires that sponsors do not benefit from mergers, or splitting into a set of sponsors. Our second theorem involves a property which requires that the cost shares of two sponsors who get connected are equally effected. We also show that except for our second theorem, none of our results for appointment games extend to the class of routing games (Potters et al, 1992).

Keywords : fixed-route traveling salesman games, routing games, appointment games, the Shapley value, the core, transferable-utility games, merging and splitting proofness, equal impact, networks, cost allocation.

1 Introduction

Finding the least-costly route that visits a given set of locations and returns to the starting location, the so called "traveling salesman problem (TSP)" is one of the most well-known combinatorial optimization problems in operations research. A vide variety of problems can be modelled as a TSP or one of its extensions.[1] In several of these problems, the cost of the tour has to be allocated among the customers visited (sponsors). This kind of a cost allocation problem in a TSP was first investigated by Fishburn and Pollack (1983). Some examples where a cost allocation problem arises include a salesman (repairman, cable guy, parcel delivery guy etc.) visiting his customers, a professor invited by several universities for

*The first draft of this paper was written when I was a Ph.D student in the University of Rochester. I am grateful to William Thomson for his guidance and advice.

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[1]For instance, location routing, closed-loop material flow system design in production settings, sequencing jobs in a flexible manufacturing environmet, post box collection, stochastic vehicle routing, grocery shopping, scheduling of home deliveries of online shopping, robotic travel problems like soldering or drilling operations on printed circuit boards, sequencing local genome maps to produce a global map, planning the order in which a satellite interferometer studies a sequence of stars, seriation problems in archeology, etc.

seminars, passengers using shuttle buses or car-pooling, and distribution planning situations such as delivery of supplies to grocery stores by a manufacturer.[2]

In some of the above examples, the traveler may need to follow a route that is not necessarily the least costly one. We study the so called "*fixed-route traveling salesman problems*" where the route is fixed according to the restrictions in the agenda of the traveler. Here, starting from her home (main office, factory, or depot), a service provider visits several customers, following a predetermined route, and returns home after all customers are visited. Each customer is to be visited exactly once but home can be visited more than once, which may be necessary, for instance, when the service provider needs to replenish her supplies after visiting a group of customers and before visiting the rest. Another reason may be that the traveler has appointments to meet with the customers and there is a considerable waiting time between two consecutive appointments. Then, in between those appointments, she would go home and wait there.

Various factors other than the cost may affect the route. Some of the sponsors may need to be visited before the others due to the urgency of their needs, their higher priority status, or the availability of their free times for a visit. For example, a professor may have to visit several universities in the order specified by their available seminar dates or a service provider has to visit her customers according to their appointments. In some cases, it is not possible to visit a location before visiting certain others. For instance, an employer may need to pick up some files from some offices and submit them to other offices to get them signed and there is an authority structure according to which signatures must be collected. Other examples include a communication network where the flow of information has to follow the specified network structure[3] or a product which has to be processed in several departments in a firm according to the stage of its development (e.g. it can not be sent to the marketing department before quality control).

Our goal is to find a fair distribution of the total cost generated in a fixed-route TSP among the sponsors. One way to solve this distribution problem is to associate a *cooperative game with transferable utilities* (TU-game) with the cost allocation problem. A TU-game is a pair (N, v) where N is a finite set of agents and $v : 2^N \to \mathbb{R}$ is a characteristic function which assigns to each coalition $S \subseteq N$, a value $v(S)$ such that $v(\emptyset) = 0$. In the current context, $v(S)$ represents the cost of the tour in which only the members of S are served by the service provider. Potters, Curiel, and Tijs (1992) formulate a TU-game associated with a fixed-route TSP as follows: for each coalition $S \subset N$, $v(S)$ is defined as the cost of the original route restricted to S, where the salesman visits the members of S in the same order as they were visited in the original route over N, skipping all agents in $N \backslash S$.[4] They refer to these games as *routing games.* Note that if the salesman and all sponsors live along the coast of an island and the travel costs are proportional to the Euclidean distances, then the least costly tour for

[2]For a case study of the cost allocation problem concerning the transportation of gas and gas oil to the customers of Norsk Hydro, see Engevall et al (1998).

[3]For example, consider a network in which a central office sends information (or papers/products to be processed) to several satellite offices which need to send back the processed information. There is a fixed order of satellite offices that must be respected: office i's information has to be obtained by the central office before office $i+1$. Each satellite office i can only communicate with the central office and office $i+1$. Hence, she can send the information to the central office via the office next to her.

[4]Potters et al (1992) also studied TSPs where the route is not fixed. They introduced the *traveling salesman games,* where the value of a coalition is the cost of a least costly tour over the members of that coalition. The salesman is allowed to visit any agent more than once and he is free to visit the agents in a coalition in any order he wishes as long as the cost of the trip is minimized. See [3], [8], [12], [14].

a subset of sponsors is the one specified in a routing game (Derks and Kuipers, 1997).

We introduce a new class of games which we refer as the class of *fixed-route traveling salesman games with appointments* (here after, *appointment games*). Consider the case in which each sponsor in N makes an appointment to meet the traveler at a specified time. After all the appointments are made, suppose the members of $S \subseteq N$ decide to hire the traveler without cooperating with the sponsors in $N \backslash S$. That is, the members of S together will pay $v(S)$ to the traveler. This can be thought as if all the sponsors in $N \backslash S$ cancel their appointments. The permissible route over S is the one where the traveler still visits the sponsors in S according to their original appointments. So, the traveler follows the original route, skipping the sponsors who are not in S, and *when she skips a sponsor, she goes home from where she goes to the next unvisited sponsor in S.*[5] The value of a coalition S, is the cost of this permissible route over S.

Our formulation of permissible routes over coalitions makes sense in several TSPs where the service provider makes appointments with the customers which can not be changed in a short notice of time. Hence, if some appointments are cancelled, the remaining ones can not be rescheduled. Also, suppose that when a traveler visits a sponsor, she has to spend a considerable period of time to complete her service for that sponsor. In that case, if an appointment is cancelled, then the traveler has to wait a lengthy period of time till the next appointment. Hence, it is not feasible for the traveler to go to the next sponsor immediately. Hence, when the traveler skips a sponsor, she goes home where she waits till it is time for the next appointment. For instance, consider a professor who wants to visit universities in different cities at specific dates as a visiting professor. When she visits a university, suppose she has to stay there for a few weeks. If a university cancels its appointment, instead of visiting the next university in the route right away, the professor goes back to her home and waits there until the appointed date for the next university arrives.

Several papers discuss the "core" in traveling salesman games and routing games (see [2], [4], [19], [20], [21]). Here, we study another well-known solution, the "Shapley value" (Shapley, 1971). In general, the Shapley value is computationally complex. However, in appointment games, we show that this is not the case. We also show that under a mild condition on the costs, the class of appointment games is convex, hence, in this class, the Shapley value is in the core. Moreover, the Shapley value may be an appealing alternative to core since it is always non-empty, single-valued, and is the unique solution satisfying certain desirable properties. Characterizations of the Shapley value in general networks are provided by Myerson (1997) and Jackson and Wolinsky (1996). Kar (2002) characterizes the Shapley value in minimum cost spanning tree games. Shapley value has also been characterized in sequencing and queuing problems (Maniquet, 2003; Chun, 2004, 2006; Moulin, 2007, 2008).

For the TSP games with appointments, we present two sets of characterizations of the Shapley value. The first set of results involves several variations of a strategic property called *merging and splitting proofness* which requires that a set of sponsors who follow each other on a route should not gain by merging or a sponsor should not gain by splitting into several sponsors located next to each other. Our second set of results involves a property which requires that when two sponsors get connected, they are effected equally. This characterization is in the same spirit of Kar's characterization of the Shapley value in minimum cost spanning tree games and Myerson's characterization in general networks.

We also analyze the Shapley value in the class of routing games. Not all results for

[5] Note that an appointment game would coincide with a routing game if for each pair of sponsors $\{i, j\}$, the cost of traveling between i and j is equal to cost of traveling from i to home and from home to j.

appointment games carry over to routing games. For instance, in the class of routing games, the calculation of the Shapley value is complex unlike in appointment games. We also show that our first set of results doesn't extend to the routing games. However, we extend our second set of results to this class. Potters et al (1992) specified the conditions which ensure that the core of routing games is non-empty (these conditions are stronger than our condition that ensures the convexity of appointment games). However, we show that these conditions do not guarantee the convexity of the routing games. Hence, we can not guarantee that Shapley value is in the core whenever it is non-empty.

In Section 2, the model is described. The results for the appointment games are presented in Section 3. Section 4 presents the results for the routing games. All proofs are in the Appendix.

2 The Model

2.1 The Economy

Let $N = \{1, ..., n\}$ with $|N| = n \geq 2$ be an ordered list of sponsors and 0 be home. Without loss of generality, we assume that the sponsors are visited in the same order as they appear in N. Let $N^0 \equiv N \cup \{0\}$ and for each $S \subseteq N$, let $S^0 \equiv S \cup \{0\}$. A route $r = (i_1, i_2, ..., i_M)$ is an ordered list of the agents (sponsors and home) to be visited by a "*traveler*" such that

(i) the route starts from home and ends at home (i.e. $i_1 = i_M = 0$),

(ii) each sponsor is visited exactly once,

(iii) home can be visited more than once,

(iv) after sponsor $i \in N$ is visited, either home or sponsor $i+1$ is to be visited (i.e. the relative order of the sponsors in r respect their order in N).

For each pair $\{i, j\} \subseteq N^0$, i is *connected* to j on a route r (denoted as $i \succ_r j$), if after i, the next agent visited is $j : r = (0, ..., i, j, ..., 0)$.

For each $\{i, j\} \subseteq N^0$, let $c_{i,j} \geq 0$ be the cost of traveling between agents i and j. Let $c_i \equiv c_{0,i}$ be the cost of traveling between home and sponsor i. The cost of a route r is $c(r) = \sum_{\{i,j\} \subseteq N^0 :\, i \succ_r j} c_{i,j}$.

Let $\mathbf{c} = \{c_{i,j} : \{i, j\} \subseteq N^0\}$. An *economy* is given by $\mathbf{e} = \langle N, \mathbf{c}, r \rangle$. Let the domain of all economies be $\mathcal{E}$.

A sponsor set $S = \{l, l+1, ..., m-1, m\} \subseteq N$ is a *connected set on* r if and only if $0 \succ_r l \succ_r l+1 \succ_r ... \succ_r m-1 \succ_r m \succ_r 0$. Let $\mathcal{S}_{\mathbf{e}}$ be the set of all connected sets in economy $\mathbf{e}$.

In order to visualize the problem, we can associate a graph with each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$. The elements of N^0 are called *nodes,* 0 being the *source.* A *link* between nodes i and j (denoted as l_{ij}) is a direct path between them. Let $l_i \equiv l_{0i}$ be the link between home and i. Let $L = \{l_{ij} : \{i, j\} \subseteq N^0\}$ be the set of all links between all agents. A graph g over N^0 is a subset of L. The graph associated with $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ is $g(\mathbf{e}) = \{l_{ij} : \{i, j\} \subseteq N^0$ and

$i \succ_r j\}$ where each link l_{ij} in $g(\mathbf{e})$ is associated with weight $c_{i,j}$.

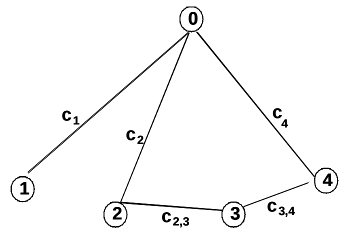

Figure 1

Example 1. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ with $r = (i_1, i_2, i_3, ..., i_7) = (0, 1, 0, 2, 3, 4, 0)$. The route r describes a trip where starting from 0 (home), the traveler visits sponsor 1, then goes back home. From home, she visits sponsors 2, 3, and 4, in that order, and returns home and completes the tour.
Here, the connected sets are $S = \{1\}$ and $S' = \{2, 3, 4\}$. Hence, $\mathcal{S}_\mathbf{e} = \{S, S'\}$. The cost of the route is $c(r) = 2c_1 + c_2 + c_{2,3} + c_{3,4} + c_4$. The associated graph $g(\mathbf{e})$ is as in Figure 1.

2.2 Appointment Games

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $S \subseteq N$. Let the *permissible route over* S (denoted as r_S) be as follows:

Starting from home, the traveler first visits the smallest numbered sponsor in S, let us call this sponsor j_1. Suppose, in the original route r, after visiting sponsor j_1, the traveler visits agent (home or a sponsor) $i \in N^0$ (i.e. $j_1 \succ_r i$). If $i \in S^0$, then in route r_s, the traveler goes to i right after visiting j_1 (i.e. $j_1 \succ_{r_S} j_2 \equiv i$). If $i \notin S$, then it is as if i has cancelled her appointment. In this case, in r_S, after visiting j_1, the traveler goes home and she waits there till it is time to attend the next outstanding appointment with the sponsors in $S \backslash \{j_1\}$. That is, if $j_1 \succ_r i$ and $i \notin S$, then $j_1 \succ_{r_S} 0 \succ_{r_S} l$ where $l = \min\{k : k \in S \text{ and } k > j_1\}$). A similar procedure is followed until all the sponsors in S are visited, then the traveler returns home. Note that each time after the traveler visits home, the next agent she visits is the smallest numbered agent in S that has not been visited so far.

Formally, for some $T \geq |S|$, let $r_S = (0, j_1, j_2, ..., j_T, 0)$ be such that:

(i) for each $t \in \{1, .., T\}$, $j_t \in S^0$, and for each $i \in S$, there is a unique $t \in \{1, .., T\}$ such that $i \equiv j_t$ on r_S,

(ii) $j_1 = \min\limits_{i \in S} i$ and $j_T = \max\limits_{i \in S} i$,

(iii) for each $j_t \in S$ with $t \in \{1, 2, .., T\}$ and each $i \in N$ such that $j_t \succ_r i$, if $i \in S^0$, then $j_t \succ_{r_S} j_{t+1} \equiv i$, otherwise $j_t \succ_{r_S} j_{t+1} \equiv 0$, and

(iv) for each $j_t \equiv 0$ with $t \in \{2, .., T-1\}$, we have $j_t \succ_{r_S} \min\{k : k \in S \text{ and } k > j_{t-1}\}$.

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$. For each $S \subseteq N$, let $\mathbf{c}_S = \{c_{i,j} \geq 0 : \{i, j\} \subseteq S^0\}$. The economy restricted to S with respect to r_S is $\mathbf{e}_S = \langle S, \mathbf{c}_S, r_S\rangle \in \mathcal{E}$.

For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, the *fixed-route traveling salesman game with appointments* (in short, *appointment game*) *associated with* $\mathbf{e}$ is $V_\mathbf{e} = (N, v_\mathbf{e})$ where $v_\mathbf{e} : 2^N \to \mathbb{R}_+$ is such that for each $S \subseteq N$, $v_\mathbf{e}(S) = c(r_S)$. Let $\mathcal{V}_\mathcal{E} = \{V_\mathbf{e} : \mathbf{e} \in \mathcal{E}\}$ be the class of appointment games.

Note that $v_{\mathbf{e}}(N) = c(r)$ and for each $S \in \mathcal{S}_{\mathbf{e}}$, $v_{\mathbf{e}}(S) = c(r_S)$. Since $c(r) = \sum_{S \in \mathcal{S}_{\mathbf{e}}} c(r_S)$, $v_{\mathbf{e}}(N) = \sum_{S \in \mathcal{S}_{\mathbf{e}}} v_{\mathbf{e}}(S)$.

Example 2. Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle$ where $r = (0, 1, 0, 2, 3, 4, 5, 0, 6, 0, 7, 8, 9, 0)$. Let $S = \{1, 4, 5, 6, 7, 9\}$. Then, $r_S = (0, 1, 0, 4, 5, 0, 6, 0, 7, 0, 9, 0)$. Here, $7 \succ_r 8$ but $8 \notin S$ (i.e. 8 cancelled her appointment). Thus, after visiting 7, the traveler goes home from where she goes to sponsor 9.
The graphs $g(\mathbf{e})$ and $g(\mathbf{e}_S) = \{l_{ij} : \{i, j\} \subseteq S^0 \text{ and } i \succ_{r_S} j\}$ are as in Figures 2a and 2b.

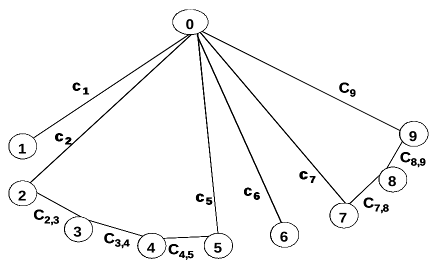

Figure 2a : $g(\mathbf{e})$

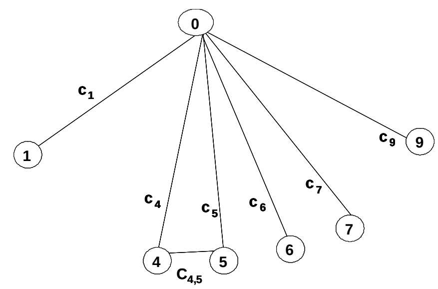

Figure 2b : $g(\mathbf{e}_S)$

2.3 The Shapley Value

For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$, to determine a cost allocation vector $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ with $\sum_{i \in N} x_i = c(r)$, we have two options. First option is to define a rule that selects an allocation for each economy directly. Second one is to associate a TU-game with each economy, and define a rule that selects an allocation for the TU-game. In this paper, we follow the later approach.[6]

Let $V = (N, v)$ be a TU-game where $v : 2^N \to \mathbb{R}_+$ is a characteristic function such that $v(\emptyset) = 0$. A *solution* F is a mapping that associates with each $V = (N, v)$, an allocation vector $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ where $\sum_{i \in N} x_i = v(N)$.[7] An example of a solution is the *Shapley value, SV:* for each $V = (N, v)$ and each $i \in N$,

$$SV_i(V) = \sum_{S \subseteq N \backslash \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)].$$

In general, the Shapley value is computationally complex since we need to calculate the marginal contribution of each agent to each possible coalition. But, for appointment games, it turns out that the Shapley value has a simple form (see the Appendix for the derivation of the Shapley Value). Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$, $i \in N$, and $S_i \in \mathcal{S}_{\mathbf{e}}$ be the connected set such that $i \in S_i$.
• If $S_i = \{i\}$, then

$$SV_i(V_{\mathbf{e}}) = 2c_i.$$

[6] Examples of other cost allocation problems where cooperative game theory is used include airplane landing fees [15], water resource planning [18], telephone billing rates [1], investment in electric power [6], minimum cost spanning trees [11].

[7] Note that we define solutions to apply for general TU-games and not for only TU-games associated with TSPs.

- If $S_i \cap \{i-1, i+1\} = j$, then

$$SV_i(V_{\mathbf{e}}) = \frac{3c_i + c_{i,j} - c_j}{2}.$$

- If $\{i-1, i+1\} \subseteq S_i$, then

$$SV_i(V_{\mathbf{e}}) = \frac{1}{2}(2c_i + c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}).$$

That is, for each sponsor $i \in S_i$, her Shapley value in an appointment game is the average of her marginal contribution to the coalition of sponsors that are in the same connected set as her and precede her in the route $(v_{\mathbf{e}}(\{j \in S_i : j \leq i\}) - v_{\mathbf{e}}(\{j \in S_i : j < i\}))$ and her marginal contribution to the coalition of sponsors that are in the same connected set as her and come after her in the route $(v_{\mathbf{e}}(\{j \in S_i : j \geq i\}) - v_{\mathbf{e}}(\{j \in S_i : j > i\}))$.

Note that in the appointment games, the Shapley value of a sponsor only depends on the cost of traveling from herself to home, and to the sponsors that are connected to her. A change in the cost of a sponsor to connect home affects only herself and the sponsors who are connected to her. Also, a change in the cost of traveling between two sponsors only effect those sponsors and effect them equally.

3 Characterizations of the Shapley Value in Appointment Games

3.1 The Core and the Shapley value

In a cost allocation problem, the *core* of a TU-game $V = (N, v)$ is the set of vectors $\mathbf{x} \in \mathbb{R}^n$ such that for each $S \subseteq N$, $\sum_{i \in S} x_i \leq v(S)$ and $\sum_{i \in N} x_i = v(N)$. If an allocation $\mathbf{x} \in \mathbb{R}^n_+$ is in the core of a game V, then no coalition of sponsors has an incentive to leave the grand coalition N. In general, the core can be empty. Potters et al (1992) state that in the class of routing games, if the route r chosen for the grand coalition is a least-costly tour and triangle inequalities hold for all the agents (i.e. for each triple $\{i, j, k\} \subseteq N^0$, $c_{i,j} + c_{j,k} \geq c_{i,k}$), then the core is non-empty.

In appointment games, a much weaker condition is sufficient for the core to be non-empty. First of all, we do not need that r be a least costly tour for N. Second, we only need that given a route, for each pair of connected sponsors, the sum of their costs of connecting to home is greater than the cost of connecting to each other. Formally, for each r and each pair $\{i, j\} \subseteq N$ such that $i \succ_r j$, $c_i + c_j \geq c_{i,j}$. Let $\mathcal{E}_T$ be the set of economies in which this condition holds. Let $\mathcal{V}_{\mathcal{E}_T}$ be the class of appointment games associated with economies in $\mathcal{E}_T$. Actually, on $\mathcal{E}_T$, we achieve more than the non-emptiness of the core. Here, we also have the convexity of the appointment games[8] and hence, by Theorem 7 of Shapley (1971), the Shapley value is an element of the core.

Proposition 1. *On the domain $\mathcal{E}_T$, appointment games are convex and the Shapley value is in the core.*

[8]In a cost allocation problem, a TU-game $V = (N, v)$ is convex if for each $i \in N$ and $S \subseteq T \subseteq N \backslash \{i\}$, $v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$.

In the rest of the paper, unless stated otherwise, the results hold on both of the domains $\mathcal{E}$ and $\mathcal{E}_T$.

Let us present other axioms that compare the cost shares of sponsors with the values of coalitions in different situations.

Although, the core compares, for each coalition, the sum of the cost shares of the sponsors in the coalition with the value of that coalition, the following two axioms are concerned with only the grand coalition N and singleton coalitions, respectively.

Efficiency: For each $V{=}(N, v)$, $\sum_{i \in N} F_i(V) = v(N)$.

Individual Rationality: For each $V{=}(N, v)$, $F_i(V) \leq v(\{i\})$.

Note that the Shapley value satisfies *Individual Rationality* only on $\mathcal{V}_{\mathcal{E}_T}$. To see this, let $\mathbf{e} = \langle \{i, j\}, \mathbf{c}, r \rangle$ with $i \succ_r j$ and suppose that $c_i + c_j < c_{i,j}$ (i.e. $\mathbf{e} \notin \mathcal{E}_T$). Then, since $SV_i(V_\mathbf{e}) = 1/2(3c_i + c_{i,j} - c_j)$ and $v(\{i\}) = 2c_i$, $SV_i(V_\mathbf{e}) > v(\{i\})$.

The following axiom states that in each connected set, the sponsors should together pay the value of that set. Hence, connected sets should not cross-subsidize each other.[9]

Respect of Connected Sets: For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and each connected set $S \in \mathcal{S}_\mathbf{e}$,

$$\sum_{i \in S} F_i(V_\mathbf{e}) = v_\mathbf{e}(S).$$

We also consider the following weakening of *Respect of Connected Sets* where for each connected set S, the sum of the cost shares of the sponsors in S sum up to an amount that depends on the value of S. Hence, instead of the cost of visiting all the sponsors in S, the traveler collects an amount from S which is a function of this cost. For instance, the service provider may charge a flat fee to each connected set, regardless of the cost of visiting them or may use markup pricing. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$.

Weak Respect of Connected Sets with respect to ϕ : For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and each connected set $S \in \mathcal{S}_\mathbf{e}$,

$$\sum_{i \in S} F_i(V_\mathbf{e}) = \phi(v_\mathbf{e}(S)).$$

Consider the difference between the value of a coalition consisting of only one sponsor and the cost share of this sponsor in the grand coalition. This difference measures how much a sponsor benefits from cooperating with the other sponsors rather than being alone. The following fairness axiom requires that in a two-sponsor TU-game, the sponsors should equally benefit from cooperation. In a sense, in two-sponsor games, we require the sponsors to have equal bargaining powers when it comes to sharing the benefits from cooperation.

Equal Benefit: For each $V{=}(\{i, j\}, v)$,

$$v(\{i\}) - F_i(V) {=} v(\{j\}) - F_j(V).$$

Hart and Mas-Colell (1989) call a solution F "standard for two-person games" if it satisfies *Efficiency* and *Equal Benefit.* For each $V{=}(\{i, j\}, v)$, such a solution divides the surplus

[9] Note that we define some of the axioms (such as *Efficiency*) for any TU-game where as some (such as *Respect of Connected Sets*) are defined only for those TU-games associated with TSPs.

$v(\{i,j\}){-}v(\{i\}){-}v(\{j\})$ equally among the sponsors.[10] Most solutions satisfy this requirement, one of them being the Shapley value.

Remark 1. A solution F satisfies *Efficiency* in two-sponsor TU-games and *Equal Benefit* if and only if for each $V = (N, v)$ with $n = 2$, and each $i \in N$,

$$F_i(V){=}v(\{i\}) + \frac{1}{2}\left[v(N){-}v(\{i\}){-}v(N\backslash\{i\})\right] = SV_i(V).$$

3.2 Mergers and the Shapley value

Manipulation of solutions by collusion has been analyzed in several different contexts including problems of bargaining (Harsanyi, 1977), rationing and bankruptcy (Banker 1981; Moulin, 1987; Ju, 2003), cost sharing (Sprumont, 2005), quasi-linear social choice (Moulin, 1985; Chun, 2000), queuing and scheduling (Maniquet, 2003; Chun, 2004, 2006; Moulin, 2007, 2008) Merging and splitting proofness leads to characterizations of the Proportional rule in rationing problems, the egalitarian division of surplus in the quasi-linear social choice model, and the Aumann-Shapley rule in the cost sharing problem with variable demands (for a survey, see Ju et al, 2005). In the deterministic scheduling model, Moulin (2007) introduces two rules which correspond to the Shapley value of two different games that can be associated to the scheduling problem. He analyzes these rules with respect to their merge proofness and split proofness properties.

There are several ways in which agents can collude. In the context of TU-games, two approaches can be noticed: either a new game with the same player set evolves when agents make binding agreements (Haller, 1994) or a group of agents merge into one player so that the set of players for the new game is reduced (Lehrer, 1988; Derks and Tijs, 2000, Knudsen and Østerdal, 2005). In some of the papers, only bilateral agreements/amalgamations are studied (e.g. Lehrer, 1988; Haller, 1994) or there is a given partition of the agent set that dictates which coalitions can merge (Derks and Tijs, 2000).

In our context, we consider mergers which result in a reduced player set. Also, instead of any group of sponsors, we allow only those sponsors who follow each other on a route to merge. This requirement is intuitive especially when we think that sponsors can only effectively communicate and merge with their neighbors in the network (for instance, when the sponsors do not follow each other on a route, they may not be able to collude due to the geographical distance between them or the traveler may easily detect such mergers and prohibit them).

Suppose a group of consecutive sponsors $K = \{k, k+1, k+2, ..., l\}$ for some $\{k, l\} \subseteq N$, form a coalition and act as a single sponsor $k \in K$ (i.e. K merge into k).[11] Note that K does not have to be a connected set, the traveler may visit home in between visiting any two sponsors in K. However, the traveler does not visit any sponsor outside K in between visiting any two sponsors in K.

If K merges and acts like a single sponsor $k \in K$, then we assume that as a group, K is willing to pay the traveler up to $v(K)$. Also, after the merger, no subset of K can behave on its own and form coalitions with sponsors outside K, however all the sponsors in K, as a group, can cooperate with other sponsors. Hence, in effect, by requiring K to act as a single

[10] Hart and Mas-Colell (1989) introduce the concept of "preservation of differences" which can be regarded as a generalization of the "equal division of the surplus" idea for two-person problems.

[11] Note that the choice of k as the representative of K is arbitrary, K can merge into any $i \in K$.

entity, we are imposing restrictions on which coalitions can form. The resulting TU-game can be defined as follows.

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $K \subseteq N$ be such that $K = \{k, k+1, k+2, ..., l\}$ for some $1 \leq k < l \leq n$. Let $\widehat{v} : 2^{(N\backslash K)\cup\{k\}} \to \mathbb{R}_+^{n-|K|+1}$ be such that

- $\widehat{v}(\{k\}) = v_{\mathbf{e}}(K)$,
- for each $S \subseteq N\backslash K$; $\widehat{v}(S) = v_{\mathbf{e}}(S)$, and
- for each $S \subseteq N\backslash K$; $\widehat{v}(S \cup \{k\}) = v_{\mathbf{e}}(S \cup K)$.

We refer $\widehat{V} = ((N\backslash K) \cup \{k\}, \widehat{v})$ as the TU-game obtained from $V_{\mathbf{e}} = (N, v_{\mathbf{e}})$ when K merges into a single sponsor k.

The following axiom states that no group of consecutive sponsors can change the total cost its members pay by a merger.

Merging and Splitting Proofness: For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, each $K = \{k, k+1, k+2, ..., l\}$ with $1 \leq k < l \leq n$, and each $\widehat{V}$ as described above,

$$F_k(\widehat{V}) = \sum_{i\in K} F_i(V_{\mathbf{e}}).$$

We can strengthen *Merging and Splitting Proofness* by allowing for the possibility that there may be more than one merger at the same time. In this case, for each of the merging groups, the total cost its members pay should remain unchanged. Although, the Shapley value would satisfy this stronger requirement as well[12], for our results, we only need a much weaker (but less intuitive) version of this requirement: Suppose the grand coalition is partitioned into two groups and each of these groups merge into a single sponsor. Then, none of these two groups should change their total cost share by these mergers. Formally, let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $\{K, K'\} \subseteq 2^N$ be such that $K = \{1, 2, ..., k\}$ and $K' = \{k+1, k+2, ..., n\}$ for some $1 \leq k < n$. Let $\widetilde{v} : 2^{\{k,k'\}} \to \mathbb{R}_+^2$ be such that $\widetilde{v}(\{k\}) = v_{\mathbf{e}}(K)$, $\widetilde{v}(\{k'\}) = v_{\mathbf{e}}(K')$, and $\widetilde{v}(\{k, k'\}) = v_{\mathbf{e}}(N)$. Let $\widetilde{V} = (\{k, k'\}, \widetilde{v})$ be the TU-game obtained from $V_{\mathbf{e}} = (N, v_{\mathbf{e}})$ when K merges into a single sponsor k and K' merges into a single sponsor $k' \in K'$.

Merging and Splitting Proofness-2: For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, each $\{K, K'\} \subseteq 2^N$, and each $\widetilde{V}$ as described above,

$$F_k(\widetilde{V}) = \sum_{i\in K} F_i(V_{\mathbf{e}}) \text{ and } F_{k'}(\widetilde{V}) = \sum_{i\in K'} F_i(V_{\mathbf{e}}).$$

Another variable population property is concerned with departures from the original economy in the following way. Let S be a connected set in $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$. Now, suppose all the sponsors which do not belong to S leave after paying their cost shares. Note that since S is a connected set, the cost of visiting the sponsors in S is same both before and after the departure of sponsors in $N\backslash S$. Hence, fairness may require that whether the sponsors in S cooperate with the grand coalition or not should not effect their cost shares. In other words, the sponsors in S should not be affected when the other sponsors leave the economy. Let $\mathbf{e}_S = \langle S, \mathbf{c}_S, r_S\rangle \in \mathcal{E}$ be the reduced economy after the departure of $N\backslash S$.

Consistency over Connected Sets: For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, each $S \in \mathcal{S}_{\mathbf{e}}$, and each $i \in S$,

$$F_i(V_{\mathbf{e}}) = F_i(V_{\mathbf{e}_S}).$$

[12] Our proofs will also work with this stronger requirement.

Between the axioms stated so far, certain (sometimes rather obvious) logical relations hold as the following remark presents.

Remark 2. a) If a solution satisfies *Weak Respect of Connected Sets with respect to* ϕ for some $\phi : \mathbb{R}_+ \to \mathbb{R}$, *Merging and Splitting Proofness-2*, and *Equal Benefit*, then ϕ is an identity function (i.e. the solution satisfies *Respect of Connected Sets*).

b) *Efficiency* and *Merging and Splitting Proofness* together imply *Merging and Splitting Proofness-2.*

c) *Efficiency, Individual Rationality,* and *Merging and Splitting Proofness* together imply *Respect of Connected Sets.*

d) *Efficiency* in two-sponsor TU-games, *Merging and Splitting Proofness-2,* and *Equal Benefit* together imply *Respect of Connected Sets*.

e) *Core* implies *Respect of Connected Sets* which in turn implies *Efficiency* in two-sponsor TU-games.[13]

f) *Efficiency* and *Consistency over Connected Sets* together imply *Respect of Connected Sets.*

By Remark 1, *Efficiency* in two-sponsor TU-games and *Equal Benefit* characterize the Shapley value for two sponsor TU-games. By Remark 2d, we also have *Respect of Connected Sets*. This axiom and *Merging and Splitting Proofness-2* lifts the characterization from two sponsor games to larger economies as stated in our main theorem, next.

Theorem 1. *The Shapley value is the only solution which satisfies Efficiency in two-sponsor TU-games, Merging and Splitting Proofness-2, and Equal Benefit.*

Several alternative combinations of axioms still characterize the Shapley value in appointment games due to the logical relations stated in Remark 2. For instance, by Remark 2e and Theorem 1, the Shapley value is also the only solution that satisfies *Respect of Connected Sets, Merging and Splitting Proofness-2,* and *Equal Benefit.* Moreover, by Remark 2a, we can weaken *Respect of Connected Sets* and still obtain the Shapley value. The other result in the next corollary is due to Remark 2b.

Corollary to Theorem 1

a) *A solution F satisfies Weak Respect of Connected Sets with respect to ϕ for some $\phi :\ \mathbb{R}_+ \to \mathbb{R}$, Merging and Splitting Proofness-2, and Equal Benefit if and only if $F = SV$.*

b) *The Shapley value is the only solution which satisfies Efficiency, Merging and Splitting Proofness, and Equal Benefit.*

One may argue that only those sponsors that belong to the same connected set can effectively communicate and hence can merge into a single sponsor. That is, the network structure does not permit sponsors in different connected sets to merge. We can weaken *Merging and*

[13] In general, *Respect of Connected Sets* in appointment games does not imply *Efficiency* in general TU-games with more than 2 agents. For instance, consider $V = (\{1,2,3\}, v)$ such that $v(\{1\}) = 1$, $v(\{2\}) = 5$, $v(\{3\}) = 3$, $v(\{1,2\}) = 4$, $v(\{2,3\}) = 6$, $v(\{1,3\}) = 3$. Here, there is no route r such that for $\mathbf{e} = \langle\{1,2,3\}, \mathbf{c}, r\rangle$, we can have $v_{\mathbf{e}} = v$. To see this, if $r = (0, i, j, k, 0)$ (we can add 0 in between $i\&j$ and/or $j\&k$), then $v_{\mathbf{e}}(\{i,k\}) = v_{\mathbf{e}}(\{i\}) + v_{\mathbf{e}}(\{k\})$ which is not satisfied by v for any $\{i,k\} \subset \{1,2,3\}$.

Splitting Proofness-2 to take into account this argument by requiring that we can only apply the axiom when the agent set is a connected set.[14]

Weak Merging and Splitting Proofness-2: For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$, if $|\mathcal{S}_{\mathbf{e}}| = 1$, then for each $\{K', K''\} \subseteq 2^N$ such that $N = K' \cup K''$ and $K' \cap K'' = \emptyset$, if $\widetilde{V} = (\{k', k''\}, \widetilde{v})$ is the TU-game obtained from $V_{\mathbf{e}} = (N, v_{\mathbf{e}})$ when K' merges into a single sponsor $k' \in K'$ and K'' merges into a single sponsor $k'' \in K''$, then

$$F_{k'}(\widetilde{V}) = \sum_{i \in K'} F_i(V_{\mathbf{e}}) \text{ and } F_{k''}(\widetilde{V}) = \sum_{i \in K''} F_i(V_{\mathbf{e}}).$$

If we use this weaker axiom in Theorem 1, to obtain a characterization result, we also need *Consistency over Connected Sets.*

Proposition 2. *The Shapley value is the only solution which satisfies Efficiency in two-sponsor TU-games, Weak Merging and Splitting Proofness-2, Equal Benefit, and Consistency over Connected Sets.*[15]

Hart and Mas-Colell (1989) characterize the Shapley value in general TU-games using *Efficiency* in two-sponsor TU-games, *Equal Benefit,* and a consistency property which, if adapted to our setting, is stronger than *Consistency over Connected Sets* since it allows the departure of any set of agents (not only the sponsors that are outside a given connected set) after paying their cost shares. Theorem 1 indicates that in appointment games, instead of consistency, we can use *Merging and Splitting Proofness-2* and still obtain the Shapley value. Also, comparison of Proposition 2 to Hart and Mas-Colell's result shows that we can weaken their consistency idea to *Consistency over Connected Sets* while using a supplementary axiom *Weak Merging and Splitting Proofness-2* and still obtain the Shapley value.

In almost all characterizations of the Shapley value, *Efficiency* (or *Respect of Connected Sets*) is used. In Theorem 1, we weakened *Efficiency* so that it is only required to hold in two-sponsor TU-games. To see how far we would move away from the Shapley value when we drop the requirement that the traveler collects the cost of visiting sponsors, let us consider *Weak Respect of Connected Sets with respect to* ϕ for some $\phi : \mathbb{R}_+ \to \mathbb{R}$. Remark 2a states that if $\phi : \mathbb{R}_+ \to \mathbb{R}$ is an arbitrary function and not necessarily the identity function, then *Weak Respect of Connected Sets with respect to* ϕ, *Merging and Splitting Proofness-2*, and *Equal Benefit* are incompatible. For instance, if the traveler wants to use markup pricing or a flat fee, she can not use a solution which satisfies *Merging and Splitting Proofness-2* and *Equal Benefit.* The proof of this incompatibility result requires using economies with at least three connected sets. Let $\mathcal{V}_{\mathcal{E}^2}$ be the class of appointment games associated with economies with at most two connected sets. If F is defined on $\mathcal{V}_{\mathcal{E}^2}$, then F can satisfy the aforementioned three axioms and yet ϕ does not have to be an identity function. The interesting point is that F would still be closely related to the Shapley value: in economies where it is defined,

[14]Note that the Shapley value also satisfies the following stronger requirement: For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$, $S \in \mathcal{S}_{\mathbf{e}}$, and $\{S', S''\} \subseteq 2^S$ such that $S = S' \cup S''$ and $S' \cap S'' = \emptyset$, if $\widetilde{V} = ((N \backslash S) \cup \{s', s''\}, \widetilde{v})$ is the TU-game obtained from $V_{\mathbf{e}} = (N, v_{\mathbf{e}})$ when S' merges into a single sponsor $s' \in S'$ and S'' merges into a single sponsor $s'' \in S''$, then $F_{s'}(\widetilde{V}) = \sum_{i \in S'} F_i(V_{\mathbf{e}})$ and $F_{s''}(\widetilde{V}) = \sum_{i \in S''} F_i(V_{\mathbf{e}})$. For the proof, see Appendix 8 in the working version of our paper.

[15]Note that the same result can be obtained if we replaced *Weak Merging and Splitting Proofness-2* with the following stronger but more intuitive requirement: there may be more than one merger at the same time but only the sponsors that belong to the same connected set are allowed to merge, then, for each of the merging groups, the total cost its members pay should remain unchanged.

F coincides with the Shapley value for each sponsor $i \in N$ that is connected to both $i-1$ and $i+1$. Hence, weakening *Respect of Connected Sets* only effects the cost shares of the first and the last sponsors to be visited in any connected set. This result shows that, on $\mathcal{V}_{\mathcal{E}^2}$, the traveler can be flexible in her pricing strategy and still not move too far away from the Shapley value in appointment games.

Proposition 3. *Let F be solution defined on $\mathcal{V}_{\mathcal{E}^2}$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$.*
a) *Solution F satisfies Weak Respect of Connected Sets with respect to ϕ, (Weak) Merging and Splitting Proofness-2, and Equal Benefit if and only if for each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ such that $|\mathcal{S}_{\mathbf{e}}| = 1$ and each $i \in N$,*

$$\begin{aligned} F_i(V_{\mathbf{e}}) &= \tfrac{1}{2}(\phi(v_{\mathbf{e}}(N)) + v_{\mathbf{e}}(\{i\}) - v_{\mathbf{e}}(N\backslash\{i\})), & \text{if } i \in \{1, n\}, \\ &= SV_i(V_{\mathbf{e}}), & \text{if } i \notin \{1, n\}. \end{aligned}$$

b) *Solution F satisfies Weak Respect of Connected Sets with respect to ϕ, (Weak) Merging and Splitting Proofness-2, Equal Benefit, and Consistency over Connected Sets if and only if for each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ such that $|\mathcal{S}_{\mathbf{e}}| = 2$, each $i \in N$ and $S_i \in \mathcal{S}_{\mathbf{e}}$ with $i \in S_i$,*

$$\begin{aligned} F_i(V_{\mathbf{e}}) &= \phi(v_{\mathbf{e}}(\{i\})), & \text{if } |\{i-1, i+1\} \cap S_i| = 0, \\ &= \tfrac{1}{2}(\phi(v_{\mathbf{e}}(\{S_i\})) + v_{\mathbf{e}}(\{i\}) - v_{\mathbf{e}}(S_i\backslash\{i\})), & \text{if } |\{i-1, i+1\} \cap S_i| = 1, \\ &= SV_i(V_{\mathbf{e}}), & \text{if } |\{i-1, i+1\} \cap S_i| = 2. \end{aligned}$$

3.3 Changes in the route and the Shapley value

Consider the following two economies where the only difference between them is that there are two sponsors i and $i+1$ such that the service provider visits them consecutively in one economy, and via home in the other. In such a situation, one may require that other things being equal, when two sponsors become connected, their cost shares should be affected equally. This requirement is similar to the "equal-gains principle" of Myerson (1977) and "equal bargaining power" of Jackson and Wolinsky (1996).

Formally, let $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ where $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ and $\mathbf{e}' = \langle N, \mathbf{c}, r'\rangle$ be such that there exists $i \in N\backslash\{n\}$ where
(i) $i \succ_r 0 \succ_r i+1$ and $i \succ_{r'} i+1$, and
(ii) for each $j \in N^0\backslash\{i\}$ and $k \in N^0\backslash\{i+1\}$, $j \succ_r k$ if and only if $j \succ_{r'} k$.

Equal Impact : For each pair $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ and each $\{i, i+1\} \subseteq N$ as described above,

$$F_i(V_{\mathbf{e}}) - F_i(V_{\mathbf{e}'}) = F_{i+1}(V_{\mathbf{e}}) - F_{i+1}(V_{\mathbf{e}'}).$$

The next result states that requiring no-cross subsidization between connected sets and equal treatment of sponsors when they get connected is also enough to characterize the Shapley value.

Theorem 2. *The Shapley value is the only solution which satisfies Respect of Connected Sets and Equal Impact.*

Theorem 2 shows that Myerson's characterization of the Shapley value in general networks can be adapted to TSPs. Note that in a general network, each node on the graph is an agent and the total value generated in the graph is distributed to all the agents. However, in TSPs, home, although being a node in the graph, does not get a share of the total cost generated by the TSP network.

Theorem 2 also resembles Kar's (2002) characterization of the Shapley value in minimum cost spanning tree games. However, Kar uses more axioms and his "Equal treatment" axiom differs from our *Equal Impact* in the sense that while we consider a change on the route keeping the cost structure same, he considers a change in the cost of the link between two sponsors which will lead to a change in the graph.

Note that, in the class of appointment games, the Shapley value actually also satisfies a stronger version of *Equal Impact* which allows for the possibility that, besides the link between i and $i+1$, there may be other changes in the respective routes in the two economies. However, there should be no change on the links between $i-1$ and i; and $i+1$ and $i+2$. For instance, either $i-1$ and i are connected in both economies, or they are unconnected in both economies.[16]

Formally, if $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ with $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ and $\mathbf{e}' = \langle N, \mathbf{c}, r'\rangle$ are such that there exists $i \in N\backslash\{n\}$ where $i \succ_r i+1$ and $i \succ_{r'} 0 \succ_{r'} i+1$; $i-1 \succ_r i$ if and only if $i-1 \succ_{r'} i$; and $i+1 \succ_r i+2$ if and only if $i+1 \succ_{r'} i+2$, then $SV_i(V_{\mathbf{e}}) - SV_i(V_{\mathbf{e}'}) = SV_{i+1}(V_{\mathbf{e}}) - SV_{i+1}(V_{\mathbf{e}'})$.

The following corollary to Theorem 2 is obtained by using Remark 2.

Corollary to Theorem 2

a) *On $\mathcal{V}_{\mathcal{E}_T}$, the Shapley value is the only solution which satisfies Efficiency, Individual Rationality, Merging and Splitting Proofness, and Equal Impact.*

b) *The Shapley value is the only solution which satisfies Efficiency, Consistency over Connected Sets, and Equal Impact.*

3.4 Summary of results for appointment games

Characterizing one solution as the only one which satisfies a set of axioms is not always good news since adding other axioms to this set may lead to impossibility, as Thomson (2001) states "...more often than we would like, impossibilities are precipitated by relatively short lists of properties". Fortunately, in our setting, the Shapley value satisfies a great variety of requirements one may expect from a solution. Indeed, our results show that several combinations of axioms characterize the Shapley value. The fact that the Shapley value is also in the core of appointment games (on $\mathcal{V}_{\mathcal{E}_T}$) adds to its desirability. Table 1 below summarizes the different ways that the Shapley value is characterized.

	Eff	*Eff-2*	*WRCS*	*RCS*	*MSP*	*MSP-2*	*CSMSP-2*	*CCS*	*EB*	*IR*	*EI*
Thm1		✓				✓			✓		
Cor1a			✓			✓			✓		
Cor1b	✓				✓				✓		
Prop 2		✓					✓	✓	✓		
Thm2				✓							✓
Cor2a (on $\mathcal{V}_{\mathcal{E}_T}$)	✓				✓					✓	✓
Cor2b	✓							✓			✓

Table 1

[16]Note that the Shapley value would not satisfy this stronger requirement in general networks.

Eff: Efficiency, Eff-2: Efficiency in 2-sponsor TU-games, WRCS: Weak Respect of Connected Sets, RCS: Respect of Connected Sets, MSP: Merging and Splitting Proofness, MSP-2: Merging and Splitting Proofness-2, CSMSP-2: Connected-Set Merging and Splitting Proofness-2, CCS: Consistency over Connected Sets, EB: Equal Benefit, IR: Individual Rationality, EI: Equal Impact.

4 Characterization of the Shapley Value in Routing Games

Potters et al (1992) introduced the *routing games* to analyze the cost allocation problem in fixed-route TSPs, and they studied the *Core* in these games. Here, we analyze the Shapley value in routing games.

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $S \subset N$. The permissible route over S in a routing game is the one where the traveler follows the original route r, skipping all the sponsors who are absent in S. Let r^*_S be the resulting route over S. The *routing game* associated with $\mathbf{e}$ is $V^*_{\mathbf{e}} = (N, v^*_{\mathbf{e}})$ where $v^*_{\mathbf{e}} : 2^N \to \mathbb{R}_+$ is such that for each $S \subseteq N$, $v^*_{\mathbf{e}}(S) = c(r^*_S)$.

Example 3. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ where $r = (0,1,0,2,3,4,5,0,6,0,7,8,9,0)$. Let $S = \{1,4,5,6,7,9\}$. Then, $r^*_S = (0,1,0,4,5,0,6,0,7,9,0)$ and $v^*_{\mathbf{e}}(S) = 2c_1 + c_4 + c_{4,5} + c_5 + 2c_6 + c_7 + c_{7,9} + c_9$.

The axioms in Section 3 can be stated for routing games just by replacing all $V_{\mathbf{e}}$ with $V^*_{\mathbf{e}}$, $v_{\mathbf{e}}(.)$ with $v^*_{\mathbf{e}}(.)$, r_S with r^*_S, etc.

Not all of the results we derived in Section 3 carry over to the class of routing games. First of all, in the class of routing games, the Shapley value doesn't reduce into a simple formula as it does in the class of appointment games. Moreover, Theorem 1 no longer holds in the class of routing games since the Shapley value violates *Merging and Splitting Proofness.*

Proposition 4. *In the class of routing games, the Shapley value violates Merging and Splitting Proofness and Merging and Splitting Proofness-2.*

The good news is that Theorem 2 extends to the class of routing games.

Theorem 3. *In the class of routing games, the Shapley value is the only solution which satisfies Respect of Connected Sets and Equal Impact.*

In routing games, the Shapley value satisfies a stronger version of *Equal Impact* as well: let $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ where $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ and $\mathbf{e}' = \langle N, \mathbf{c}, r'\rangle$ be such that there exists $i \in N\backslash\{n\}$ where

(i) $i \succ_r i+1$ and $i \succ_{r'} 0 \succ_{r'} i+1$, and

(ii) for $S_i \in \mathcal{S}_{\mathbf{e}}$ with $i \in S_i$, for each $k \in S_i\backslash\{i\}$ and $l \in S_i\backslash\{i+1\}$, $k \succ_r l$ if and only if $k \succ_{r'} l$.

Hence, besides the change on the link between i and $i+1$, there may be other changes between the routes r and r', as long as those changes only concern the sponsors that do not belong to S_i where S_i is the connected set that includes i and $i+1$ in economy $\mathbf{e}$. That is, the links between the sponsors in other connected sets are allowed to change. In this case, we have $SV_i(V^*_{\mathbf{e}}) - SV_i(V^*_{\mathbf{e}'}) = SV_{i+1}(V^*_{\mathbf{e}}) - SV_{i+1}(V^*_{\mathbf{e}'})$.

It is easy to see that, in the class of routing games, the Shapley value still satisfies *Efficiency* and *Consistency over Connected Sets*[17]. Hence, Corollary to Theorem 2b extends to the routing games:

[17] This is because for each $i \in N$, each $S_i \in \mathcal{S}_{\mathbf{e}}$ such that $i \in S_i$, and each $S \subseteq N\backslash\{i\}$, the marginal contribution of i to the value of a coalition S is equal to the marginal contribution of i to the value of $S \cap (S_i\backslash\{i\})$. That is, $v^*_{\mathbf{e}}(S \cup \{i\}) - v^*_{\mathbf{e}}(S) = v^*_{\mathbf{e}}(S \cap (S_i\backslash\{i\}) \cup \{i\}) - v^*_{\mathbf{e}}(S \cap (S_i\backslash\{i\}))$.

Corollary to Theorem 3 *In the class of routing games, the Shapley value is the only solution which satisfies Efficiency, Consistency over Connected Sets, and Equal Impact*

Potters et al (1992) state that in the class of routing games, if the route r chosen for the grand coalition is a least-costly tour and triangle inequalities hold for all the agents (i.e. for each triple $\{i,j,k\} \subseteq N^0$, $c_{i,j} + c_{j,k} \geq c_{i,k}$), then the core is non-empty. Let $\mathcal{E}_T^*$ be the set of economies in which these conditions hold. Note that to ensure the convexity of appointment games (which also implies the non-emptiness of the core), we only needed the triangle inequalities to hold for those sponsors who are connected rather than all sponsors and we did not need the route r be a least-costly route for the economy. Hence, $\mathcal{E}_T$ is a larger set of economies than $\mathcal{E}_T^*$.

We know that if a class of TU-games is convex, then the Shapley value is an element of the core. In general, routing games are not convex. Here, we show that even under the conditions Potters et al (1992) specify for the non-emptiness of the core, the routing games are still not convex. Hence, we do not know for sure that Shapley value is in the core whenever it is non-empty. It is an open question to characterize the conditions under which the Shapley value is in the core of routing games.

Proposition 5. *On the domain $\mathcal{E}_T^*$, the routing games are not convex.*

5 Appendix

5.1 Derivation of the Shapley Value in Appointment Games

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, $i \in N$, and $S_i \subseteq N$ be the connected set such that $i \in S_i$. For each $S \subseteq N$, let $|S| = s$ and $f(s) = \frac{s!(n-s-1)!}{n!}$.

• If $S_i = \{i\}$, then since for each $S \subseteq N\backslash\{i\}$, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = v_{\mathbf{e}}(\{i\}) = 2c_i$, we have

$$SV_i(V_{\mathbf{e}}) = 2c_i.$$

• If $S_i \cap \{i-1, i+1\} = j$, then since

for each $S \subseteq N\backslash\{i\}$ such that $j \in S$, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = c_i + c_{i,j} - c_j$, and

for each $S \subseteq N\backslash\{i\}$ such that $j \notin S$, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = 2c_i$, we have

$$\begin{aligned} SV_i(V_{\mathbf{e}}) &= \sum_{S \subseteq N\backslash\{i\}} f(s)\left(v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S)\right) \\ &= \sum_{S \subseteq N\backslash\{i\}: j \in S} f(s)\left(c_i + c_{i,j} - c_j\right) + \sum_{S \subseteq N\backslash\{i,j\}} f(s)\left(2c_i\right) \\ &= (c_i + c_{i,j} - c_j)\sum_{s=1}^{n-1}\binom{n-2}{s-1} f(s) + 2c_i \sum_{s=0}^{n-2}\binom{n-2}{s} f(s) \\ &= (c_i + c_{i,j} - c_j)\frac{1}{2} + (2c_i)\frac{1}{2} \\ &= \frac{3c_i + c_{i,j} - c_j}{2}. \end{aligned}$$

Here, $\binom{n-2}{s-1}$ is the number of $(s-1)$-combinations from the set $N\backslash\{i,j\}$. It gives us the number of subsets of $N\backslash\{i\}$ that contains j and has s number of sponsors: to find such subsets, we need to pick $s-1$ sponsors from the set $N\backslash\{i,j\}$. Similar interpretation applies to $\binom{n-2}{s}$ and all other binomial coefficients from now on.

• If $\{i-1,i+1\}\subseteq S_i$, then since

for each $S\subseteq N\backslash\{i\}$ such that $\{i-1,i+1\}\subseteq S$, $v_{\mathbf{e}}(S\cup\{i\})-v_{\mathbf{e}}(S)=c_{i-1,i}+c_{i,i+1}-c_{i-1}-c_{i+1}$,

for each $S\subseteq N\backslash\{i\}$ such that $S\cap\{i-1,i+1\}=\{j\}$, $v_{\mathbf{e}}(S\cup\{i\})-v_{\mathbf{e}}(S)=c_i+c_{i,j}-c_j$, and

for each $S\subseteq N\backslash\{i\}$ such that $S\cap\{i-1,i+1\}=\emptyset$, $v_{\mathbf{e}}(S\cup\{i\})-v_{\mathbf{e}}(S)=2c_i$, we have

$$\begin{aligned}
SV_i(V_{\mathbf{e}}) = & \sum_{S\subseteq N\backslash\{i\}:\{i-1,i+1\}\subseteq S} f(s)\,(c_{i-1,i}+c_{i,i+1}-c_{i-1}-c_{i+1}) \\
& + \sum_{S\subseteq N\backslash\{i\}:\{i-1,i+1\}\cap S=\{i-1\}} f(s)\,(c_i+c_{i-1,i}-c_{i-1}) \\
& + \sum_{S\subseteq N\backslash\{i\}:\{i-1,i+1\}\cap S=\{i+1\}} f(s)\,(c_i+c_{i,i+1}-c_{i+1}) + \sum_{S\subseteq N\backslash\{i\}:\{i-1,i+1\}\cap S=\emptyset} f(s)\,(2c_i) \\
& = (c_{i-1,i}+c_{i,i+1}-c_{i-1}-c_{i+1}) \sum_{s=2}^{n-1}\binom{n-3}{s-2}f(s) + (c_i+c_{i-1,i}-c_{i-1}) \sum_{s=1}^{n-2}\binom{n-3}{s-1}f(s) \\
& + (c_i+c_{i,i+1}-c_{i+1}) \sum_{s=1}^{n-2}\binom{n-3}{s-1}f(s) + 2c_i \sum_{s=0}^{n-3}\binom{n-3}{s}f(s) \\
& = (c_{i-1,i}+c_{i,i+1}-c_{i-1}-c_{i+1})\,\tfrac{1}{3}+(c_i+c_{i-1,i}-c_{i-1})\,\tfrac{1}{6}+(c_i+c_{i,i+1}-c_{i+1})\,\tfrac{1}{6}+(2c_i)\tfrac{1}{3} \\
& = \tfrac{1}{2}(2c_i+c_{i-1,i}+c_{i,i+1}-c_{i-1}-c_{i+1}).
\end{aligned}$$

5.2 Proofs of the Results in Section 3

Proof of Proposition 1: Let $\mathbf{e}=\langle N,\mathbf{c},r\rangle\in\mathcal{E}_T$ and $i\in N$. Let $K=\{j\in N\backslash\{i\}$: either $i\succ_r j$ or $j\succ_r i\}$.[18] Note that on $\mathcal{E}_T$, for each $j\in K$,

$$c_i+c_j\geq c_{i,j}. \tag{1}$$

We need to show that[19] for each $S\subseteq T\subseteq N\backslash\{i\}$,

$$v_{\mathbf{e}}(S\cup\{i\})-v_{\mathbf{e}}(S)\geq v_{\mathbf{e}}(T\cup\{i\})-v_{\mathbf{e}}(T). \tag{2}$$

There are 6 possible cases. We will show that in each case, (2) holds.

1. $K\cap S=\emptyset$. Then, $v_{\mathbf{e}}(S\cup\{i\})-v_{\mathbf{e}}(S)=2c_i$.
a) $K\cap T=\emptyset$. Then, $v_{\mathbf{e}}(T\cup\{i\})-v_{\mathbf{e}}(T)=2c_i$. Hence, (2) holds.
b) $K\cap T=\{j\}$. Then, $v_{\mathbf{e}}(T\cup\{i\})-v_{\mathbf{e}}(T)=c_{j,i}+c_i-c_j$. Hence, by (1), (2) holds.
c) $K\cap T=\{i-1,i+1\}$. Then, $v_{\mathbf{e}}(T\cup\{i\})-v_{\mathbf{e}}(T)=c_{i-1,i}+c_{i,i+1}-c_{i-1}-c_{i+1}$. Hence, by (1), (2) holds.

2. $K\cap S=\{j\}$. Then, $v_{\mathbf{e}}(S\cup\{i\})-v_{\mathbf{e}}(S)=c_{j,i}+c_i-c_j$.

[18] If $0\succ_r i\succ_r 0$, then $K=\emptyset$. If $i-1\succ_r i\succ_r 0$, then $K=\{i-1\}$. If $0\succ_r i\succ_r i+1$, then $K=\{i+1\}$. If $i-1\succ_r i\succ_r i+1$, then $K=\{i-1,i+1\}$.

[19] Since $v_{\mathbf{e}}(S)$ measures the cost that coalition S generates as opposed to the benefit it can ensure, the convexity of the game requires inequality (2) to hold.

a) $K \cap T = \{j\}$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{j,i} + c_i - c_j$. Hence, (2) holds.
b) $K \cap T = \{i-1, i+1\}$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}$. Hence, by (1), (2) holds.

3. $K \cap S = K \cap T = \{i-1, i+1\}$. Then, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}$. Hence, (2) holds. □

Proof of Remark 1: Let F satisfy *Efficiency* in two-sponsor TU-games and *Equal Benefit*. Let $V = (N, v)$ be such that $n = 2$. Then, for each $i \in N$ and $j = N \backslash \{i\}$, by *Equal Benefit*, (I) $F_i(V) - F_j(V) = v(\{i\}) - v(\{j\})$, and by *Efficiency*, (II) $v(N) = F_i(V) + F_j(V)$. By (I) and (II), for each $i \in N$, $F_i(V) = \frac{1}{2}\left[v(N) + v(\{i\}) - v(\{j\})\right] = SV_i(V)$. □

Proof of Remark 2:

a) Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ and F satisfy the first 3 axioms listed in Remark 2a. We need to show that for each $a \in \mathbb{R}_+$, $\phi(a) = a$. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ be such that $|\mathcal{S}_{\mathbf{e}}| \geq 3$ and there is $S \in \mathcal{S}_{\mathbf{e}}$ with $S = \{l, l+1, ..., m\}$ for some $1 < l \leq m < n$ and $v_{\mathbf{e}}(S) = a$. Let $K_1 = \{i \in N : i < l\}$ and $K_2 = \{i \in N : i > m\}$.

Let K_1 and S merge into a single sponsor denoted by $k_1 \in K_1$ and K_2 merge into a single sponsor denoted by n. Let $V^1 = (\{k_1, n\}, v^1)$ be the TU-game obtained from $V_{\mathbf{e}}$ by these mergers. Thus, $v^1(\{k_1\}) = v_{\mathbf{e}}(K_1 \cup S) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S)$, $v^1(\{n\}) = v_{\mathbf{e}}(K_2)$, and $v^1(\{k_1, n\}) = v_{\mathbf{e}}(N) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) + v_{\mathbf{e}}(K_2)$.

Since V^1 is a two-sponsor TU-game, by *Equal Benefit*,

$$F_{k_1}(V^1) - F_n(V^1) = v^1(\{k_1\}) - v^1(\{n\}) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) - v_{\mathbf{e}}(K_2). \tag{3}$$

By *Merging and Splitting Proofness-2*, $F_{k_1}(V^1) = \sum\limits_{i \in K_1} F_i(V_{\mathbf{e}}) + \sum\limits_{i \in S} F_i(V_{\mathbf{e}})$ and $F_n(V^1) = \sum\limits_{i \in K_2} F_i(V_{\mathbf{e}})$. These equalities and (3) together imply

$$\sum_{i \in K_1} F_i(V_{\mathbf{e}}) + \sum_{i \in S} F_i(V_{\mathbf{e}}) - \sum_{i \in K_2} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) - v_{\mathbf{e}}(K_2). \tag{4}$$

Note that K_1 is a union of connected sets and so is K_2. Let $\Phi(K_1) = \sum\limits_{S' \subseteq K_1 : S' \in \mathcal{S}_{\mathbf{e}}} \phi(v_{\mathbf{e}}(S'))$ and $\Phi(K_2) = \sum\limits_{S' \subseteq K_2 : S' \in \mathcal{S}_{\mathbf{e}}} \phi(v_{\mathbf{e}}(S'))$. By *Weak Respect of Connected Sets* and (4),

$$\phi(v_{\mathbf{e}}(S)) = \Phi(K_2) - \Phi(K_1) + v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) - v_{\mathbf{e}}(K_2). \tag{5}$$

Now, let K_2 and S merge into a single sponsor denoted by $k_2 \in K_2$ and K_1 merge into a single sponsor denoted by 1. Let $V^2 = (\{1, k_2\}, v^2)$ be the TU-game obtained from $V_{\mathbf{e}}$ by these mergers. Again, by *Equal Benefit*, (I) $F_{k_2}(V^2) - F_1(V^2) = v_{\mathbf{e}}(K_2) + v_{\mathbf{e}}(S) - v_{\mathbf{e}}(K_1)$. By *Merging and Splitting Proofness-2*, (II) $F_{k_2}(V^2) = \sum\limits_{i \in K_2} F_i(V_{\mathbf{e}}) + \sum\limits_{i \in S} F_i(V_{\mathbf{e}})$ and $F_1(V^2) = \sum\limits_{i \in K_1} F_i(V_{\mathbf{e}})$. By (I), (II), and *Weak Respect of Connected Sets*,

$$\phi(v_{\mathbf{e}}(S)) = \Phi(K_1) - \Phi(K_2) + v_{\mathbf{e}}(K_2) + v_{\mathbf{e}}(S) - v_{\mathbf{e}}(K_1). \tag{6}$$

By (5) and (6),

$$\Phi(K_2) - v_{\mathbf{e}}(K_2) = \Phi(K_1) - v_{\mathbf{e}}(K_1). \tag{7}$$

Since $v_{\mathbf{e}}(S) = a$, substituting (7) into (5), $\phi(a) = a$. Since we can repeat this procedure for any $a \in \mathbb{R}_+$, ϕ is the identity function. Therefore, F satisfies *Respect of Connected Sets.*

b) Let F satisfy the first two axioms in Remark 1b. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $\{K, K'\} \subseteq 2^N$ be such that $K = \{1, 2, ..., k\}$ and $K' = \{k+1, k+2, ..., n\}$ for some $1 \leq k < n$. Let $\widehat{V} = (\{k\} \cup K', \widehat{v})$ be the TU-game obtained from $V_{\mathbf{e}} = (N, v_{\mathbf{e}})$ when K merges into a single sponsor k. By *Efficiency*, $F_k(\widehat{V}) = \widehat{v}(\{k\} \cup K') - \sum_{i \in K'} F_i(\widehat{V})$ and $\sum_{i \in K} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(N) - \sum_{i \in K'} F_i(V_{\mathbf{e}})$. Since $\widehat{v}(\{k\} \cup K') = v_{\mathbf{e}}(N)$ and by *Merging and Splitting Proofness*, $F_k(\widehat{V}) = \sum_{i \in K} F_i(V_{\mathbf{e}})$, we have (I) $\sum_{i \in K'} F_i(\widehat{V}) = \sum_{i \in K'} F_i(V_{\mathbf{e}})$.

Now, let $\widetilde{V} = (\{k, k'\}, \widetilde{v})$ be the TU-game obtained from $\widehat{V}$ when K' merges into a single sponsor $k' \in K'$. By *Merging and Splitting Proofness*, $F_{k'}(\widetilde{V}) = \sum_{i \in K'} F_i(\widehat{V})$. This equality and (I) together imply (II) $F_{k'}(\widetilde{V}) = \sum_{i \in K'} F_i(V_{\mathbf{e}})$. Also, by *Efficiency*, $F_{k'}(\widetilde{V}) = \widetilde{v}(\{k, k'\}) - F_k(\widetilde{V})$ and $\sum_{i \in K'} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(N) - \sum_{i \in K} F_i(V_{\mathbf{e}})$. Since $\widetilde{v}(\{k, k'\}) = v_{\mathbf{e}}(N)$, by (II), $F_k(\widetilde{V}) = \sum_{i \in K} F_i(V_{\mathbf{e}})$. This equality and (II) together imply that F satisfies *Merging and Splitting Proofness-2.*

c) Let F satisfy the first 3 axioms listed in Remark 2c. Suppose, by contradiction, that F does not satisfy *Respect of Connected Sets*. Then, there are $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}_T$ and $\{S', S''\} \subseteq \mathcal{S}_{\mathbf{e}}$ such that $\sum_{i \in S''} F_i(V_{\mathbf{e}}) < v_{\mathbf{e}}(S'')$ and (I) $\sum_{i \in S'} F_i(V_{\mathbf{e}}) > v_{\mathbf{e}}(S')$. Such S' and S'' exist since by *Efficiency*, $\sum_{S \in \mathcal{S}_{\mathbf{e}}} \left(\sum_{i \in S} F_i(V_{\mathbf{e}}) \right) = v_{\mathbf{e}}(N) = \sum_{S \in \mathcal{S}_{\mathbf{e}}} v_{\mathbf{e}}(S)$.

Now, let S' merge into a single sponsor denoted by $s' \in S'$. Let $\widehat{V} = ((N \backslash S') \cup \{s'\}, \widehat{v})$ be the TU-game obtained from $V_{\mathbf{e}}$ by this merger. Thus, (II) $\widehat{v}(\{s'\}) = v_{\mathbf{e}}(S')$. By *Merging and Splitting Proofness*, (III) $F_{s'}(\widehat{V}) = \sum_{i \in S'} F_i(V_{\mathbf{e}})$. By *Individual Rationality*, (IV) $F_{s'}(\widehat{V}) \leq \widehat{v}(\{s'\})$. By (II), (III), and (IV), $\sum_{i \in S'} F_i(V_{\mathbf{e}}) \leq v_{\mathbf{e}}(S')$ which contradicts (I).

d) Let F satisfy the first 3 axioms listed in Remark 2d. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $\mathcal{S}_{\mathbf{e}} = \{S_1, S_2, ..., S_T\}$ for some $T \leq n$. The proof is by induction.

$*$*Base Step:* Let S_1 merge into a single sponsor denoted by 1 and $N \backslash S_1$ merge into a single sponsor denoted by n. Let $V^1 = (\{1, n\}, v^1)$ be the TU-game obtained from $V_{\mathbf{e}}$ by these mergers. Thus, $v^1(\{1\}) = v_{\mathbf{e}}(S_1)$, $v^1(\{n\}) = v_{\mathbf{e}}(N \backslash S_1)$, and $v^1(\{1, n\}) = v_{\mathbf{e}}(N)$. Note that since S_1 is a connected set, $v_{\mathbf{e}}(N) = v_{\mathbf{e}}(S_1) + v_{\mathbf{e}}(N \backslash S_1)$. These equalities and Remark 1 together imply

$$\begin{aligned} F_1(V^1) &= \tfrac{1}{2}\left[v^1(\{1, n\}) + v^1(\{1\}) - v^1(\{n\})\right], \\ &= v_{\mathbf{e}}(S_1). \end{aligned} \tag{8}$$

By *Merging and Splitting Proofness-2*, $F_1(V^1) = \sum_{i \in S_1} F_i(V_{\mathbf{e}})$. This equality and (8) together imply $\sum_{i \in S_1} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S_1)$.

$*$*Induction Step:* Let $k < T$. Assume that for each $t < k$, $\sum_{i \in S_t} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S_t)$. We will prove that $\sum_{i \in S_k} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S_k)$.

Let $\{S_1, S_2, ..., S_k\}$ merge into a single sponsor denoted by k, and $\{S_{k+1}, .., S_T\}$ merge into a single sponsor denoted by n. Let $V^k = (\{k, n\}, v^k)$ be the TU-game obtained from $V_{\mathbf{e}}$ by these mergers. Thus, $v^k(\{k\}) = v_{\mathbf{e}}(\bigcup_{t=1}^{k} S_t) = \sum_{t=1}^{k} v_{\mathbf{e}}(S_t)$, $v^k(\{n\}) = v_{\mathbf{e}}(\bigcup_{t=k+1}^{T} S_t) = \sum_{t=k+1}^{T} v_{\mathbf{e}}(S_t)$, and $v^k(\{k, n\}) = v_{\mathbf{e}}(N) = \sum_{t=1}^{k} v_{\mathbf{e}}(S_t) + \sum_{t=k+1}^{T} v_{\mathbf{e}}(S_t)$.

These equalities and Remark 1 together imply

$$F_k(V^k) \quad = \quad v_{\mathbf{e}}(S_k) + \sum_{t=1}^{k-1} v_{\mathbf{e}}(S_t). \tag{9}$$

By *Merging and Splitting Proofness-2*, $F_k(V^k) = \sum_{i \in S_k} F_i(V_{\mathbf{e}}) + \sum_{t=1}^{k-1} \sum_{i \in S_t} F_i(V_{\mathbf{e}})$. This equality, (9), and the induction hypothesis together imply $\sum_{i \in S_k} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S_k)$.

$*$*Conclusion Step:* By the Base and the Induction steps, for each $t < T$, $\sum_{i \in S_t} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S_t)$. Now, consider $V^{T-1} = (\{T-1, n\}, v^{T-1})$ obtained from $V_{\mathbf{e}}$ when S_T merges into a single sponsor denoted by n and $N \backslash S_T$ merge into a single sponsor denoted by $T-1$. Similar to the argument in the Base step, we have $\sum_{i \in S_T} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S_T)$. Therefore, F satisfies *Respect of Connected Sets.*

e) For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$, let $F(V_{\mathbf{e}})$ be in the *Core* of $V_{\mathbf{e}}$. Then, $\sum_{S \in \mathcal{S}_{\mathbf{e}}} \sum_{i \in S} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(N)$ and for each $S \in \mathcal{S}_{\mathbf{e}}$, $\sum_{i \in S} F_i(V_{\mathbf{e}}) \leq v_{\mathbf{e}}(S)$. Since $v_{\mathbf{e}}(N) = \sum_{S \in \mathcal{S}_{\mathbf{e}}} v_{\mathbf{e}}(S)$, for each $S \in \mathcal{S}_{\mathbf{e}}$, $\sum_{i \in S} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S)$ and F satisfies *Respect of Connected Sets.*

Now, let F satisfy *Respect of Connected Sets.* Let $V = (\{i, j\}, v)$ be a two-sponsor TU-game. Let $\mathbf{e} = \langle \{i, j\}, \mathbf{c}, r \rangle$ be such that $c_i = v(\{i\})/2$, $c_j = v(\{j\})/2$, $c_i + c_{i,j} + c_j = v(\{i, j\})$, and $r = (0, i, j, 0)$. By *Respect of Connected Sets,* (I) $F_i(V_{\mathbf{e}}) + F_j(V_{\mathbf{e}}) = v_{\mathbf{e}}(\{i, j\})$. Since for each $S \subseteq \{i, j\}$, $v_{\mathbf{e}}(S) = v(S)$, we have $V \equiv V_{\mathbf{e}}$. This equivalency and (I) together imply that F satisfies *Efficiency* in two-sponsor TU-games.

f) Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and $S \in \mathcal{S}_{\mathbf{e}}$. Consider $\mathbf{e}_S = \langle S, \mathbf{c}_S, r_S \rangle \in \mathcal{E}$. By *Efficiency,* (I) $\sum_{i \in S} F_i(V_{\mathbf{e}_S}) = v_{\mathbf{e}_S}(S)$. By *Consistency over Connected Sets,* for each $i \in S$, (II) $F_i(V_{\mathbf{e}}) = F_i(V_{\mathbf{e}_S})$. Note that by definition, $v_{\mathbf{e}}(S) = c(r_S) = v_{\mathbf{e}_S}(S)$. Hence, by (I) and (II), $\sum_{i \in S} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S)$. That is, F satisfies *Respect of Connected Sets.* ■

Lemma 1: The Shapley value satisfies *Efficiency,* (on $\mathcal{V}_{\mathcal{E}_T}$) *Individual Rationality, Respect of Connected Sets, Equal Benefit, Consistency over Connected Sets, Merging and Splitting Proofness,* and *Equal Impact.*

Proof of Lemma 1:

It is easy to see that the Shapley value satisfies the first five axioms listed in Lemma 1. Now, we will show that it satisfies the rest of the axioms in Lemma 1.

- *Merging and Splitting Proofness:*

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $K \subseteq N$ be such that $K = \{k, k+1, k+2, ..., l\}$ for some $1 \leq k < l \leq n$. Let $\widehat{V} = ((N\backslash K) \cup \{k\}, \widehat{v})$ be the TU-game obtained from $V_{\mathbf{e}}$ when K merges into k.

Note that K may involve some connected sets. For some $1 \leq M \leq |K|$, let $P_K = \{K_1, K_2, ..., K_M\}$ be the partitioning of K such that

∗ for each $m \in \{1, M-1\}$, each $i \in K_m$, and each $j \in K_{m+1}$, we have $i < j$,
∗ for each $m \in \{1, M\}$, $K_m \subseteq S$ for some $S \in \mathcal{S}_{\mathbf{e}}$, and
∗ for each $m \notin \{1, M\}$, $K_m \in \mathcal{S}_{\mathbf{e}}$.

For example, if $r = (0, 1, 2, 3, 4, 0, 5, 6, 0, 7, 0, 8, 9, 0)$ and $K = \{3, 4, .., 8\}$, then $P_K = \{\{3, 4\}, \{5, 6\}, \{7\}, \{8\}\}$.

Note that there are $n - |K| + 1$ agents in the game $\widehat{V}$. For each $S \subseteq N$, let $|S| = s$ and $g(s) = \frac{s!(n-|K|-s)!}{(n-|K|+1)!}$. For each $1 \leq m \leq M$, let $K_m = \{k_m, k_m + 1, ..., l_m\}$. The following four cases are possible.

1) $P_K \subseteq \mathcal{S}_{\mathbf{e}}$: That is, for each $1 \leq m \leq M$, K_m is a connected set. Then, for each $S \subseteq N\backslash K$,

$$\widehat{v}(S \cup \{k\}) - \widehat{v}(S) = \widehat{v}(\{k\}) = v_{\mathbf{e}}(K) = \sum_{m=1}^{M} v_{\mathbf{e}}(K_m) = \sum_{m=1}^{M} (c_{k_m} + \sum_{t=k_m}^{l_m-1} c_{t,t+1} + c_{l_m}). \quad (10)$$

Hence, $SV_k(\widehat{V}) = \widehat{v}(\{k\})$. By *Respect of Connected Sets*, for each $K_m \in P_K$, $\sum\limits_{i\in K_m} SV_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(K_m)$. These equalities and (10) together imply that $SV_k(\widehat{V}) = \sum\limits_{m=1}^{M} \sum\limits_{i\in K_m} SV_i(V_{\mathbf{e}}) = \sum\limits_{i\in K} SV_i(V_{\mathbf{e}})$.

2) $P_K\backslash\mathcal{S}_{\mathbf{e}} = \{K_1\}$ and either $M \geq 2$ or $l = n$: That is, except for K_1, each $K_m \in P_K$ is a connected set. Then, for each $S \subseteq N\backslash K$ such that $k_1 - 1 \notin S$, (10) holds. For each $S \subseteq N\backslash K$ such that $k_1 - 1 \in S$,

$$\widehat{v}(S \cup \{k\}) - \widehat{v}(S) = c_{k_1-1,k_1} + (v_{\mathbf{e}}(K) - c_{k_1}) - c_{k_1-1}. \quad (11)$$

Then,

$$SV_k(\widehat{V}) = \sum_{S\subseteq N\backslash K: k_1-1\in S} g(s)(\widehat{v}(S \cup \{k\}) - \widehat{v}(S)) + \sum_{S\subseteq N\backslash K: k_1-1\notin S} g(s)(\widehat{v}(S \cup \{k\}) - \widehat{v}(S))$$

$$= \sum_{s=1}^{n-|K|} \binom{n-|K|-1}{s-1} g(s)[c_{k_1-1,k_1} + (v_{\mathbf{e}}(K) - c_{k_1}) - c_{k_1-1}] + \sum_{s=0}^{n-|K|-1} \binom{n-|K|-1}{s} g(s) v_{\mathbf{e}}(K).$$

Note that $\sum\limits_{s=1}^{n-|K|} \binom{n-|K|-1}{s-1} g(s) = \sum\limits_{s=0}^{n-|K|-1} \binom{n-|K|-1}{s} g(s) = \frac{1}{2}$.[20] Hence,

$$SV_k(\widehat{V}) = v_{\mathbf{e}}(K) + \tfrac{1}{2}(c_{k_1-1,k_1} - c_{k_1} - c_{k_1-1})$$

$$= \tfrac{1}{2}(2c_{l_1} + 2\sum_{t=k_1}^{l_1-1} c_{t,t+1} + c_{k_1-1,k_1} + c_{k_1} - c_{k_1-1}) + \sum_{m=2}^{M} v_{\mathbf{e}}(K_m)$$

$$= \sum_{i\in K_1} SV_i(V_{\mathbf{e}}) + \sum_{m=2}^{M} \sum_{i\in K_m} SV_i(V_{\mathbf{e}})$$

[20] For the calculation of these values, see Appendix 7 in the working version of our paper on http://www.adelaide.edu.au/directory/duygu.yengin.

$= \sum_{i\in K} SV_i(V_{\mathbf{e}}).$

3) $P_K \backslash \mathcal{S}_{\mathbf{e}} = \{K_M\}$ and either $M \geq 2$ or $k = 1$: That is, except for K_M, each $K_m \in P_K$ is a connected set. Then, for each $S \subseteq N\backslash K$ such that $l_M + 1 \notin S$, (10) holds. For each $S \subseteq N\backslash K$ such that $l_M + 1 \in S$,

$$\widehat{v}(S \cup \{k\}) - \widehat{v}(S) = (v_{\mathbf{e}}(K) - c_{l_M}) + c_{l_M,l_M+1} - c_{l_M+1}. \tag{12}$$

Then,

$SV_k(\widehat{V}) = \sum_{s=1}^{n-|K|} \binom{n-|K|-1}{s-1} g(s)[(v_{\mathbf{e}}(K) - c_{l_M}) + c_{l_M,l_M+1} - c_{l_M+1}] + \sum_{s=0}^{n-|K|-1} \binom{n-|K|-1}{s} g(s) v_{\mathbf{e}}(K)$

$= v_{\mathbf{e}}(K) + \frac{1}{2}(c_{l_M,l_M+1} - c_{l_M} - c_{l_M+1})$

$= \frac{1}{2}(2c_{k_M} + 2\sum_{t=k_M}^{l_M-1} c_{t,t+1} + c_{l_M,l_M+1} + c_{l_M} - c_{l_M+1}) + \sum_{m=1}^{M-1} v_{\mathbf{e}}(K_m)$

$= \sum_{i\in K_M} SV_i(V_{\mathbf{e}}) + \sum_{m=1}^{M-1} \sum_{i\in K_m} SV_i(V_{\mathbf{e}})$

$= \sum_{i\in K} SV_i(V_{\mathbf{e}}).$

4) $P_K \backslash \mathcal{S}_{\mathbf{e}} = \{K_1, K_M\}$: That is, except for K_1 and K_M, each $K_m \in P_K$ is a connected set. Note that this case covers the possibility that $K = K_1 = K_M$ and $K \notin \mathcal{S}_{\mathbf{e}}$.

Then, for each $S \subseteq N\backslash K$ such that $\{k_1 - 1, l_M + 1\} \cap S = \emptyset$, (10) holds. For each $S \subseteq N\backslash K$ such that $k_1 - 1 \in S$ and $l_M + 1 \notin S$, (11) holds. For each $S \subseteq N\backslash K$ such that $l_M + 1 \in S$ and $k_1 - 1 \notin S$, (12) holds. For each $S \subseteq N\backslash K$ such that $\{k_1 - 1, l_M + 1\} \subseteq S$,

$$\widehat{v}(S \cup \{k\}) - \widehat{v}(S) = c_{k_1-1,k_1} + (v_{\mathbf{e}}(K) - c_{k_1} - c_{l_M}) + c_{l_M,l_M+1} - c_{k_1-1} - c_{l_M+1}.$$

Then,

$SV_k(\widehat{V}) = \sum_{s=0}^{n-|K|-2} \binom{n-|K|-2}{s} g(s) v_{\mathbf{e}}(K) + \sum_{s=1}^{n-|K|-1} \binom{n-|K|-2}{s-1} g(s)[c_{k_1-1,k_1} + (v_{\mathbf{e}}(K) - c_{k_1}) - c_{k_1-1}] + \sum_{s=1}^{n-|K|-1} \binom{n-|K|-2}{s-1} g(s)[(v_{\mathbf{e}}(K) - c_{l_M}) + c_{l_M,l_M+1} - c_{l_M+1}] + \sum_{s=2}^{n-|K|} \binom{n-|K|-2}{s-2} g(s)[c_{k_1-1,k_1} + (v_{\mathbf{e}}(K) - c_{k_1} - c_{l_M}) + c_{l_M,l_M+1} - c_{k_1-1} - c_{l_M+1}]$

Note that $\sum_{s=0}^{n-|K|-2} \binom{n-|K|-2}{s} g(s) = \sum_{s=2}^{n-|K|} \binom{n-|K|-2}{s-2} g(s) = \frac{1}{3}$ and $\sum_{s=1}^{n-|K|-1} \binom{n-|K|-2}{s-1} g(s) = \frac{1}{6}$.
Hence,

$= v_{\mathbf{e}}(K) + \frac{1}{2}(c_{k_1-1,k_1} - c_{k_1-1} - c_{k_1} + c_{l_M,l_M+1} - c_{l_M} - c_{l_M+1})$

$= \frac{1}{2}(2c_{l_1} + 2\sum_{t=k_1}^{l_1-1} c_{t,t+1} + c_{k_1-1,k_1} + c_{k_1} - c_{k_1-1}) + \sum_{m=2}^{M-1} v_{\mathbf{e}}(K_m) + \frac{1}{2}(2c_{k_M} + 2\sum_{t=k_M}^{l_M-1} c_{t,t+1} + c_{l_M,l_M+1} + c_{l_M} - c_{l_M+1})$

$= \sum_{i\in K_1} SV_i(V_{\mathbf{e}}) + \sum_{m=2}^{M-1} \sum_{i\in K_m} SV_i(V_{\mathbf{e}}) + \sum_{i\in K_M} SV_i(V_{\mathbf{e}})$

$= \sum_{i\in K} SV_i(V_{\mathbf{e}}).$

In all the possible cases, we showed that $SV_k(\widehat{V}) = \sum_{i \in K} SV_i(V_{\mathbf{e}})$. Therefore, the Shapley value satisfies *Merging and Splitting Proofness*.

• *Equal Impact:*

Let $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ where $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ and $\mathbf{e}' = \langle N, \mathbf{c}, r'\rangle$ be such that there exists $i \in N \backslash \{n\}$ where
(i) $i \succ_r 0 \succ_r i+1$ and $i \succ_{r'} i+1$, and
(ii) $i-1 \succ_{r'} i$ if and only if $i-1 \succ_r i$, and
(ii) $i+1 \succ_{r'} i+2$ if and only if $i+1 \succ_r i+2$.

(Note that the proof would not change, if we assumed the stronger requirement stated in the definition of *Equal Impact*, namely, for each $j \in N^0 \backslash \{i\}$ and $k \in N^0 \backslash \{i+1\}$, $j \succ_r k$ if and only if $j \succ_{r'} k$.)

There are four cases to consider:
Case 1: $i-1 \succ_r i \succ_r 0 \succ_r i+1 \succ_r i+2$. Then, $SV_i(V_{\mathbf{e}}) = \frac{1}{2}(3c_i + c_{i-1,i} - c_{i-1})$, $SV_i(V_{\mathbf{e}'}) = \frac{1}{2}(2c_i + c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1})$, $SV_{i+1}(V_{\mathbf{e}}) = \frac{1}{2}(3c_{i+1} + c_{i+1,i+2} - c_{i+2})$, and $SV_{i+1}(V_{\mathbf{e}'}) = \frac{1}{2}(2c_{i+1} + c_{i,i+1} + c_{i+1,i+2} - c_i - c_{i+2})$. Hence,

$$SV_i(V_{\mathbf{e}}) - SV_i(V_{\mathbf{e}'}) = \frac{1}{2}(c_i + c_{i+1} - c_{i,i+1}) = SV_{i+1}(V_{\mathbf{e}}) - SV_{i+1}(V_{\mathbf{e}'}). \quad (13)$$

Case 2: $0 \succ_r i \succ_r 0 \succ_r i+1 \succ_r i+2$. Then, $SV_i(V_{\mathbf{e}}) = 2c_i$, $SV_i(V_{\mathbf{e}'}) = \frac{1}{2}(3c_i + c_{i,i+1} - c_{i+1})$; and $SV_{i+1}(V_{\mathbf{e}})$ and $SV_{i+1}(V_{\mathbf{e}'})$ are as in Case 1. It is easy to check that equality 13 still holds in Case 2.

Case 3: $0 \succ_r i \succ_r 0 \succ_r i+1 \succ_r 0$. Then, $SV_i(V_{\mathbf{e}})$ and $SV_i(V_{\mathbf{e}'})$ are as in Case 2; and $SV_{i+1}(V_{\mathbf{e}}) = 2c_{i+1}$, $SV_{i+1}(V_{\mathbf{e}'}) = \frac{1}{2}(3c_{i+1} + c_{i,i+1} - c_i)$. Hence, again equality 13 holds.

Case 4: $i-1 \succ_r i \succ_r 0 \succ_r i+1 \succ_r 0$. Then, $SV_i(V_{\mathbf{e}})$ and $SV_i(V_{\mathbf{e}'})$ are as in Case 1; and $SV_{i+1}(V_{\mathbf{e}})$ and $SV_{i+1}(V_{\mathbf{e}'})$ are as in Case 3. Hence, equality 13 still holds. ■

Proof of Theorem 1:

By Lemma 1 the Shapley value satisfies *Efficiency, Merging and Splitting Proofness*, and *Equal Benefit*. By Remark 2b, it satisfies *Merging and Splitting Proofness-2.*

Now, we show that the Shapley value is the only solution that satisfies the axioms listed in Theorem 1.
Let F satisfy those axioms and $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$. We will show that for each $S \in \mathcal{S}_{\mathbf{e}}$ and each $i \in S$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$.

If $n = 2$, by Remark 1, $F = SV$. Let $n > 2$. By Remark 2d, F satisfies *Respect of Connected Sets*. Hence, for each $\{i\} \in \mathcal{S}_{\mathbf{e}}$, $F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(\{i\}) = SV_i(V_{\mathbf{e}})$.

Now, let $S \in \mathcal{S}_{\mathbf{e}}$ be such that $|S| \geq 2$ and $S = \{l, l+1, ..., m\}$ for some $\{l, m\} \subseteq N$. Let $K_1 = \{i \in N : i < l\}$ and $K_2 = \{i \in N : i > m\}$.[21]

The proof is by induction.

[21] If $l = 1$, then $K_1 = \emptyset$ and if $m = n$, then $K_2 = \emptyset$. Note that K_1 is a union of connected sets and so is K_2.

$*$*Base Step:* Let K_1 and $\{l\}$ merge into a single sponsor denoted by l. Let K_2 and $\{l+1, l+2, ..., m\}$ merge into a single sponsor denoted by n. Let $V^l = (\{l,n\}, v^l)$ be the TU-game obtained from $V_{\mathbf{e}}$ by these mergers. Thus,

$v^l(\{l\}) = v_{\mathbf{e}}(\{1,...,l\}) = v_{\mathbf{e}}(K_1) + 2c_l$,

$v^l(\{n\}) = v_{\mathbf{e}}(\{l+1,...,n\}) = c_{l+1} + \sum_{t=l+1}^{m-1} c_{t,t+1} + c_m + v_{\mathbf{e}}(K_2) = (v_{\mathbf{e}}(S) - c_l - c_{l,l+1} + c_{l+1}) + v_{\mathbf{e}}(K_2)$, and

$v^l(\{l,n\}) = v_{\mathbf{e}}(N) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) + v_{\mathbf{e}}(K_2)$.

These equalities and Remark 1 together imply

$$\begin{aligned} F_l(V^l) &= \tfrac{1}{2}\left[v^l(\{l,n\}) + v^l(\{l\}) - v^l(\{n\})\right], \\ &= v_{\mathbf{e}}(K_1) + \tfrac{1}{2}(3c_l + c_{l,l+1} - c_{l+1}). \end{aligned} \tag{14}$$

By *Respect of Connected Sets,*

$$\sum_{i\in K_1} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(K_1),\ \sum_{i\in S} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S),\ \text{and} \sum_{i\in K_2} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(K_2). \tag{15}$$

By *Merging and Splitting Proofness-2*,

$$F_l(V^l) = \sum_{i\in K_1} F_i(V_{\mathbf{e}}) + F_l(V_{\mathbf{e}}). \tag{16}$$

By equalities (14), (15), and (16),

$$F_l(V_{\mathbf{e}}) = \frac{1}{2}(3c_l + c_{l,l+1} - c_{l+1}) = SV_l(V_{\mathbf{e}}). \tag{17}$$

$*$*Induction Step:* Let $l < k \leq m$. Assume that, for each $l < i < k$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$. We will prove that $F_k(V_{\mathbf{e}}) = SV_k(V_{\mathbf{e}})$.

Let K_1 and $S_1 = \{l, l+1, .., k\}$ merge into a single sponsor denoted by k. Let K_2 and $S\backslash S_1$ merge into a single sponsor denoted by n. Let $V^k = (\{k,n\}, v^k)$ be the TU-game obtained from $V_{\mathbf{e}}$ by these mergers.

If $k < m$, then

$v^k(\{k\}) = v_{\mathbf{e}}(\{1,...,k\}) = v_{\mathbf{e}}(K_1) + c_l + \sum_{t=l}^{k-1} c_{t,t+1} + c_k$,

$v^k(\{n\}) = v_{\mathbf{e}}(\{k+1,...,n\}) = (v_{\mathbf{e}}(S) - c_l - \sum_{t=l}^{k} c_{t,t+1} + c_{k+1}) + v_{\mathbf{e}}(K_2)$, and

$v^k(\{k,n\}) = v_{\mathbf{e}}(N) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) + v_{\mathbf{e}}(K_2)$.

These equalities and Remark 1 together imply

$$\begin{aligned} F_k(V^k) &= \tfrac{1}{2}\left[v^k(\{k,n\}) + v^k(\{k\}) - v^k(\{n\})\right], \\ &= v_{\mathbf{e}}(K_1) + \tfrac{1}{2}(2c_l + 2\sum_{t=l}^{k-1} c_{t,t+1} + c_k + c_{k,k+1} - c_{k+1}). \end{aligned} \tag{18}$$

By *Merging and Splitting Proofness-2*,

$$F_k(V^k) = \sum_{i\in K_1} F_i(V_{\mathbf{e}}) + \sum_{i=l}^{k-1} F_i(V_{\mathbf{e}}) + F_k(V_{\mathbf{e}}). \tag{19}$$

Note that $\sum_{i=l}^{k-1} SV_i(V_{\mathbf{e}}) = \frac{1}{2}(2c_l + 2\sum_{t=l}^{k-2} c_{t,t+1} + c_{k-1} + c_{k-1,k} - c_k)$. Hence, by the induction hypothesis and equalities (15), (18), and (19),

$$F_k(V_{\mathbf{e}}) = (2c_k + c_{k-1,k} + c_{k,k+1} - c_{k-1} - c_{k+1})/2 = SV_k(V_{\mathbf{e}}).$$

If $k = m$, then
$v^m(\{m\}) = v_{\mathbf{e}}(\{1, ..., m\}) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S)$,
$v^m(\{n\}) = v_{\mathbf{e}}(\{m+1, ..., n\}) = v_{\mathbf{e}}(K_2)$, and
$v^m(\{m, n\}) = v_{\mathbf{e}}(N) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) + v_{\mathbf{e}}(K_2)$.

These equalities and Remark 1 together imply

$$\begin{aligned} F_m(V^m) &= \tfrac{1}{2}\left[v^m(\{m,n\}) + v^m(\{m\}) - v^m(\{n\})\right], \\ &= v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S). \end{aligned} \tag{20}$$

By *Merging and Splitting Proofness-2*,

$$F_m(V^m) = \sum_{i \in K_1} F_i(V_{\mathbf{e}}) + \sum_{i=l}^{m-1} F_i(V_{\mathbf{e}}) + F_m(V_{\mathbf{e}}). \tag{21}$$

Hence, by the induction hypothesis and equalities (15), (20), and (21),

$$F_m(V_{\mathbf{e}}) = (3c_m + c_{m-1,m} - c_{m-1})/2 = SV_m(V_{\mathbf{e}}).$$

This concludes the induction step.

$*$*Conclusion Step:* By the Base and the Induction steps, for each $l \leq k \leq m$, we have $F_k(V_{\mathbf{e}}) = SV_k(V_{\mathbf{e}})$.

By repeating the induction proof for each $S \in \mathcal{S}_{\mathbf{e}}$, we obtain that for each $i \in S$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$. This completes the proof. ■

Proof of Proposition 2: Let F satisfy the axioms listed in Proposition 2. Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$. For each $S \in \mathcal{S}_{\mathbf{e}}$, consider $\mathbf{e}_S = \langle S, \mathbf{c}_S, r_S \rangle$. Since $|\mathcal{S}_{\mathbf{e}_S}| = 1$, we can use *Weak Merging and Splitting Proofness-2* instead of *Merging and Splitting Proofness-2* in the induction proof of Theorem 1 (taking $K_1 = \emptyset$ and $K_2 = \emptyset$ in that proof), and show that for each $i \in S$, $F_i(V_{\mathbf{e}_S}) = SV_i(V_{\mathbf{e}_S})$. Since both F and SV are *Consistent over Connected Sets*, for each $i \in S$, $F_i(V_{\mathbf{e}_S}) = F_i(V_{\mathbf{e}})$ and $SV_i(V_{\mathbf{e}_S}) = SV_i(V_{\mathbf{e}})$. Hence, for each $i \in S$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$. Repeating this procedure for each $S \in \mathcal{S}_{\mathbf{e}}$, we have for each $i \in N$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$. ■

Proof of Proposition 3:

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$. Let $\mathcal{E}^2 = \{\mathbf{e} \in \mathcal{E} : |\mathcal{S}_{\mathbf{e}}| \leq 2\}$ and $\mathcal{V}_{\mathcal{E}^2}$ be the class of appointment games associated with economies in $\mathcal{E}^2$. Let F be defined on $\mathcal{V}_{\mathcal{E}^2}$.

a) Let F satisfy *Weak Respect of Connected Sets with respect to* ϕ, *Weak Merging and Splitting Proofness-2,* and *Equal Benefit.* Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}^2$ be such that $|\mathcal{S}_{\mathbf{e}}| = 1$.

First, suppose that $i \in \{1, n\}$. Let $N \backslash \{i\}$ merge into a single sponsor denoted by $j \in N \backslash \{i\}$. Let $V^i = (\{i, j\}, v^i)$ be the TU-game obtained from $V_{\mathbf{e}}$ by this merger. Thus, $v^i(\{i\}) = v_{\mathbf{e}}(\{i\})$, $v^i(\{j\}) = v_{\mathbf{e}}(N \backslash \{i\})$, and $v^i(\{i, j\}) = v_{\mathbf{e}}(N)$.

Since V^i is a two-sponsor TU-game, by *Equal Benefit*, (I) $F_i(V^i){-}F_j(V^i){=}v_{\mathbf{e}}(\{i\}){-}v_{\mathbf{e}}(N\backslash\{i\})$.

By *Weak Merging and Splitting Proofness-2*, (II) $F_i(V^i) = F_i(V_{\mathbf{e}})$ and $F_j(V^i) = \sum_{l \in N\backslash\{i\}} F_l(V_{\mathbf{e}})$.

By (I) and (II), we have (III) $F_i(V_{\mathbf{e}}) - \sum_{l \in N\backslash\{i\}} F_l(V_{\mathbf{e}}) = v_{\mathbf{e}}(\{i\}) - v_{\mathbf{e}}(N\backslash\{i\})$.

By *Weak Respect of Connected Sets,* (IV) $F_i(V_{\mathbf{e}}) + \sum_{l \in N\backslash\{i\}} F_l(V_{\mathbf{e}}) = \phi(v_{\mathbf{e}}(N))$. By (III) and (IV),

$$F_i(V_{\mathbf{e}}){=}\frac{1}{2}\left[\phi(v_{\mathbf{e}}(N)){+}v_{\mathbf{e}}(\{i\}) - v_{\mathbf{e}}(N\backslash\{i\})\right]. \tag{22}$$

Now, we will show that for each $i \in \{2,..,n-1\}$, $F_i(V_{\mathbf{e}}){=}SV_i(V_{\mathbf{e}})$. The proof is by induction.

$*$*Base Step:* Let $\{1,2\}$ merge into a single sponsor denoted by 2 and $N\backslash\{1,2\}$ merge into a single sponsor denoted by n. Let $V^2 = (\{2,n\}, v^2)$ be the TU-game obtained from $V_{\mathbf{e}}$ by these mergers. Similarly, V^2 is a two-sponsor TU-game. Hence, by *Equal Benefit*,

$$F_2(V^2) - F_n(V^2){=}v_{\mathbf{e}}(\{1,2\}) - v_{\mathbf{e}}(N\backslash\{1,2\}). \tag{23}$$

By *Weak Merging and Splitting Proofness-2*, (I) $F_2(V^2) = \sum_{i \in \{1,2\}} F_i(V_{\mathbf{e}})$ and $F_n(V^2) = \sum_{i \in N\backslash\{1,2\}} F_i(V_{\mathbf{e}})$.

By *Weak Respect of Connected Sets,* (II) $\sum_{i \in \{1,2\}} F_i(V_{\mathbf{e}}) + \sum_{i \in N\backslash\{1,2\}} F_i(V_{\mathbf{e}}) = \phi(v_{\mathbf{e}}(N))$. By (I), (II), and (23)

$$\sum_{i \in \{1,2\}} F_i(V_{\mathbf{e}}){=}\frac{1}{2}\left[\phi(v_{\mathbf{e}}(N)){+}v_{\mathbf{e}}(\{1,2\}){-}v_{\mathbf{e}}(N\backslash\{1,2\})\right].$$

This equality and (22) together imply

$$F_2(V_{\mathbf{e}}){=}\frac{1}{2}\left[v_{\mathbf{e}}(\{1,2\}){-}v_{\mathbf{e}}(N\backslash\{1,2\}) - v_{\mathbf{e}}(\{1\}){+}v_{\mathbf{e}}(N\backslash\{1\})\right].$$

Note that $v_{\mathbf{e}}(\{1,2\}){-}v_{\mathbf{e}}(\{1\}){=}c_2 + c_{1,2} - c_1$ and $v_{\mathbf{e}}(N\backslash\{1\}){-}v_{\mathbf{e}}(N\backslash\{1,2\}) = c_2 + c_{2,3} - c_3$. Hence, $F_2(V_{\mathbf{e}}){=}\frac{1}{2}\left[2c_2 + c_{1,2} + c_{2,3} - c_1 - c_3\right] = SV_2(V_{\mathbf{e}})$.

$*$*Induction Step:* Let $1 < k < n$. Assume that for each $1 < j < k$, $F_j(V_{\mathbf{e}}){=}SV_j(V_{\mathbf{e}})$. That is, for each $j < k$, $\sum_{i \in \{1,..,j\}} F_i(V_{\mathbf{e}}){=}\frac{1}{2}\left[\phi(v_{\mathbf{e}}(N)){+}v_{\mathbf{e}}(\{1,...,j\}){-}v_{\mathbf{e}}(N\backslash\{1,...,j\})\right]$.

Let $\{1,..,k\}$ merge into k and $\{k{+}1,..,n\}$ merge into n. Let $V^k = (\{k,n\}, v^k)$ be the TU-game obtained from $V_{\mathbf{e}}$ by these mergers. Since V^k is a two-sponsor TU-game, by *Equal Benefit*, (I) $F_k(V^k) - F_n(V^k){=}v_{\mathbf{e}}(\{1,..,k\}) - v_{\mathbf{e}}(N\backslash\{1,..,k\})$.

By *Weak Merging and Splitting Proofness-2*, (II) $F_k(V^k) = \sum_{i \in \{1,..,k\}} F_i(V_{\mathbf{e}})$ and $F_n(V^k) = \sum_{i \in N\backslash\{1,..,k\}} F_i(V_{\mathbf{e}})$.

By *Weak Respect of Connected Sets*, (III) $\sum_{i\in\{1,..,k\}} F_i(V_{\mathbf{e}}) + \sum_{i\in N\backslash\{1,..,k\}} F_i(V_{\mathbf{e}}) = \phi(v_{\mathbf{e}}(N))$. By (I), (II), and (III),

$$\sum_{i\in\{1,..,k\}} F_i(V_{\mathbf{e}}) = \frac{1}{2}\left[\phi(v_{\mathbf{e}}(N)) + v_{\mathbf{e}}(\{1,..,k\}) - v_{\mathbf{e}}(N\backslash\{1,...,k\})\right].$$

This equality and the induction hypothesis together imply

$$F_k(V_{\mathbf{e}}) = \frac{1}{2}\left[v_{\mathbf{e}}(\{1,..,k\}) - v_{\mathbf{e}}(N\backslash\{1,...,k\}) - v_{\mathbf{e}}(\{1,..,k-1\}) + v_{\mathbf{e}}(N\backslash\{1,...,k-1\})\right].$$

Since $v_{\mathbf{e}}(\{1,..,k\}) - v_{\mathbf{e}}(\{1,..,k-1\}) = c_k + c_{k-1,k} - c_{k-1}$ and $v_{\mathbf{e}}(N\backslash\{1,...,k-1\}) - v_{\mathbf{e}}(N\backslash\{1,...,k\}) = c_k + c_{k,k+1} - c_{k+1}$, we have

$$F_k(V_{\mathbf{e}}) = = \frac{1}{2}\left[2c_k + c_{k-1,k} + c_{k,k+1} - c_{k-1} - c_{k+1}\right] = SV_k(V_{\mathbf{e}}).$$

$*$*Conclusion Step:* By the Base and the Induction steps, for each $i \in \{2,..,n-1\}$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$.

b) Let F satisfy *Weak Respect of Connected Sets with respect to* ϕ, *Weak Merging and Splitting Proofness-2, Equal Benefit,* and *Consistency over Connected Sets.* Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}^2$ be such that $|\mathcal{S}_{\mathbf{e}}| = 2$, $i \in N$, and $S_i \in \mathcal{S}_{\mathbf{e}}$ with $i \in S_i$.

If $|\{i-1, i+1\} \cap S_i| = 0$, that is $S_i = \{i\}$, then by *Weak Respect of Connected Sets*, $F_i(V_{\mathbf{e}}) = \phi(v_{\mathbf{e}}(\{i\}))$.

Now, suppose that $|\{i-1, i+1\} \cap S_i| \geq 1$. Let $S_i = \{l, l+1, ..., m\}$ for some $\{l, m\} \subseteq N$. Consider $\mathbf{e}_{S_i} = \langle S_i, \mathbf{c}_{S_i}, r_{S_i}\rangle \in \mathcal{E}^2$.

If $|\{i-1, i+1\} \cap S_i| = 1$, that is $i \in \{l, m\}$, then by part (a), $F_i(V_{\mathbf{e}_{S_i}}) = \frac{1}{2}\left[\phi(v_{\mathbf{e}}(S_i)) + v_{\mathbf{e}}(\{i\}) - v_{\mathbf{e}}(S_i\backslash\{i\})\right]$ and by *Consistency over Connected Sets* $F_i(V_{\mathbf{e}_{S_i}}) = F_i(V_{\mathbf{e}})$.

If $|\{i-1, i+1\} \cap S_i| = 2$, that is $i \in \{l+1, .., m-1\}$, then by part (a), $F_i(V_{\mathbf{e}_{S_i}}) = SV_i(V_{\mathbf{e}_{S_i}})$ and by *Consistency over Connected Sets* $F_i(V_{\mathbf{e}_{S_i}}) = F_i(V_{\mathbf{e}})$ and $SV_i(V_{\mathbf{e}_{S_i}}) = SV_i(V_{\mathbf{e}})$. This concludes the proof. ■

Proof of Theorem 2:

By Lemma 1, the Shapley value satisfies the axioms listed in Theorem 2.

Now, we show that the Shapley value is the only solution which satisfies the axioms in Theorem 2.

Let F satisfy those axioms and $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$. We will show, by induction on t, that for each connected set $S^t \in \mathcal{S}_{\mathbf{e}}$ such that $|S^t| = t$, $1 \leq t \leq n$, and each $i \in S^t$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$.

For each $i < n$ such that $i \succ_r i+1$, let $\mathbf{e}_i = \langle N, \mathbf{c}, r_i\rangle$ be such that $i \succ_{r_i} 0 \succ_{r_i} i+1$, and for each $k \in N\backslash\{i\}$ and $l \in N\backslash\{i+1\}$, $k \succ_{r_i} l$ if and only if $k \succ_r l$. That is, the only change between $\mathbf{e}$ and $\mathbf{e}_i$ is that i and $i+1$ are connected in $\mathbf{e}$ but not in $\mathbf{e}_i$.

Step 1: For each $S^1 \in \mathcal{S}_{\mathbf{e}}$ such that $S^1 = \{i\}$ for some $i \in N$, by *Respect of Connected Sets*, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}}) = 2c_i$.

Step 2: Suppose $S^2 \in \mathcal{S}_{\mathbf{e}}$ is such that $S^2 = \{i, i+1\}$ for some $i \in N \backslash \{n\}$.

By *Equal Impact,*

$$F_i(V_{\mathbf{e}}) - F_{i+1}(V_{\mathbf{e}}) = F_i(V_{\mathbf{e}_i}) - F_{i+1}(V_{\mathbf{e}_i}).$$

This equality and Step 1 together imply

$$F_i(V_{\mathbf{e}}) - F_{i+1}(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}_i}) - SV_{i+1}(V_{\mathbf{e}_i}). \tag{24}$$

Since SV satisfies *Equal Impact,*

$$SV_i(V_{\mathbf{e}}) - SV_{i+1}(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}_i}) - SV_{i+1}(V_{\mathbf{e}_i}). \tag{25}$$

Equalities (24) and (25) together imply that there is a real number $\gamma_S(\mathbf{e})$ such that

$$F_i(V_{\mathbf{e}}) - SV_i(V_{\mathbf{e}}) = F_{i+1}(V_{\mathbf{e}}) - SV_{i+1}(V_{\mathbf{e}}) = \gamma_{S^2}(\mathbf{e}).$$

By *Respect of Connected Sets,* $F_i(V_{\mathbf{e}}) + F_{i+1}(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}}) + SV_{i+1}(V_{\mathbf{e}}) = v_{\mathbf{e}}(\{i, i+1\})$. Hence, $[F_i(V_{\mathbf{e}}) - SV_i(V_{\mathbf{e}})] + [F_{i+1}(V_{\mathbf{e}}) - SV_{i+1}(V_{\mathbf{e}})] = 2\gamma_{S^2}(\mathbf{e}) = 0$. Thus, $\gamma_{S^2}(\mathbf{e}) = 0$ and $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$.

Let $2 < T \leq n$.

Step T*:* Assume that for each $t < T$, each $S^t \in \mathcal{S}_{\mathbf{e}}$, and each $i \in S^t$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$. We will show that for each $S^T \in \mathcal{S}_{\mathbf{e}}$ and each $i \in S^T$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$.

Let $S^T \in \mathcal{S}_{\mathbf{e}}$. For each $\{i, i+1\} \subseteq S^T$, by *Equal Impact,*

$$F_i(V_{\mathbf{e}}) - F_{i+1}(V_{\mathbf{e}}) = F_i(V_{\mathbf{e}_i}) - F_{i+1}(V_{\mathbf{e}_i}). \tag{26}$$

Note that in $\mathbf{e}_i$, i and $i+1$ belong to connected sets with less than T sponsors. That is, there are $\{S, S'\} \subseteq \mathcal{S}_{\mathbf{e}_i}$ such that $S \cup S' = S^T$, $\max(|S|, |S'|) < T$, and $i = \arg\max\{j \in S\}$ and $i+1 = \arg\min\{j \in S'\}$. Hence, by the induction hypothesis, $F_i(V_{\mathbf{e}_i}) = SV_i(V_{\mathbf{e}_i})$ and $F_{i+1}(V_{\mathbf{e}_i}) = SV_{i+1}(V_{\mathbf{e}_i})$. These equalities and (26) together imply

$$F_i(V_{\mathbf{e}}) - F_{i+1}(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}_i}) - SV_{i+1}(V_{\mathbf{e}_i}). \tag{27}$$

Since SV satisfies *Equal Impact,*

$$SV_i(V_{\mathbf{e}}) - SV_{i+1}(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}_i}) - SV_{i+1}(V_{\mathbf{e}_i}). \tag{28}$$

By (27) and (28),

$$F_i(V_{\mathbf{e}}) - SV_i(V_{\mathbf{e}}) = F_{i+1}(V_{\mathbf{e}}) - SV_{i+1}(V_{\mathbf{e}}). \tag{29}$$

Since (29) is true for each $\{i, i+1\} \subseteq S^T$, there is $\gamma_{_{S^T}}(\mathbf{e}) \in \mathbb{R}$ such that for each $i \in S^T$, $F_i(V_{\mathbf{e}}) - SV_i(V_{\mathbf{e}}) = \gamma_{_{S^T}}(\mathbf{e})$. By *Respect of Connected Sets,* $\sum\limits_{i \in S^T} F_i(V_{\mathbf{e}}) = \sum\limits_{i \in S^T} SV_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S^T)$. Hence,

$$\sum_{i \in S^T} [F_i(V_{\mathbf{e}}) - SV_i(V_{\mathbf{e}})] = T\ \gamma_{_{S^T}}(\mathbf{e}) = 0.$$

Thus, $\gamma_{_{S^T}}(\mathbf{e}) = 0$ and for each $i \in S^T$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$. This concludes the proof. ■

5.3 Independence of Axioms

Independence of the axioms in Theorem 1

• The following Dictatorial solution satisfies *Efficiency* in two-sponsor TU-games and *Merging and Splitting Proofness-2*, but not *Equal Benefit*.
For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and each $i \in N$, if $\mathcal{S}_{\mathbf{e}} = \{N\}$, then

$$D_i(V_{\mathbf{e}}) = \begin{cases} v_{\mathbf{e}}(N) & \text{if } i = \arg\min\{j \in N\}, \\ 0 & \text{otherwise.} \end{cases}$$

if $|\mathcal{S}_{\mathbf{e}}| > 1$, then $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$.

• Solution F^{ϕ} characterized in Proposition 2 satisfies *Merging and Splitting Proofness-2* and *Equal Benefit*, but not *Efficiency* in two-sponsor TU-games.

• The following solution satisfies *Efficiency* in two-sponsor TU-games and *Equal Benefit*, but not *Merging and Splitting Proofness-2*.
For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, each $i \in N$, and each $S_i \in \mathcal{S}_{\mathbf{e}}$ with $i \in S_i$,

$$F_i(V_{\mathbf{e}}) = \begin{cases} SV_i(V_{\mathbf{e}}) & \text{if } n \leq 2, \\ \frac{c_i}{\sum\limits_{j \in S_i} c_j} v_{\mathbf{e}}(S_i) & \text{if } n > 2. \end{cases}$$

Independence of the axioms in Theorem 2

• The following solution satisfies *Respect of Connected Sets* but not *Equal Impact*.
For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, each $i \in N$, and each $S_i \in \mathcal{S}_{\mathbf{e}}$ with $i \in S_i$,

$$P_i(V_{\mathbf{e}}) = \frac{c_i}{\sum\limits_{j \in S_i} c_j} v_{\mathbf{e}}(S_i).$$

• The following solution satisfies *Equal Impact* but not *Respect of Connected Sets*.
Let $\lambda \geq 0$. For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and each $i \in N$,

$$F_i^{\lambda}(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}}) - \lambda.$$

5.4 Proof of the Result in Section 4

Proof of Proposition 4: First, we demonstrate the calculation of the Shapley value for a 3-sponsor economy. Suppose $N = \{1, 2, 3\}$ and $r = (0, 1, 2, 3, 0)$. For each $S \subseteq N$, let $|S| = s$ and $f(s) = \frac{s!(n-s-1)!}{n!}$. Then,

$f(s)$	$S : 1 \notin S$	$v^*_{\mathbf{e}}(S \cup \{1\}) - v^*_{\mathbf{e}}(S)$	$S : 2 \notin S$	$v^*_{\mathbf{e}}(S \cup \{2\}) - v^*_{\mathbf{e}}(S)$	$S : 3 \notin S$	$v^*_{\mathbf{e}}(S \cup \{3\}) - v^*_{\mathbf{e}}(S)$
2/6	$\emptyset$	$2c_1$	$\emptyset$	$2c_2$	$\emptyset$	$2c_3$
1/6	$\{2\}$	$c_1 + c_{1,2} - c_2$	$\{1\}$	$c_2 + c_{1,2} - c_1$	$\{1\}$	$c_3 + c_{1,3} - c_1$
1/6	$\{3\}$	$c_1 + c_{1,3} - c_3$	$\{3\}$	$c_2 + c_{2,3} - c_3$	$\{2\}$	$c_3 + c_{2,3} - c_2$
2/6	$\{2,3\}$	$c_1 + c_{1,2} - c_2$	$\{1,3\}$	$c_{1,2} + c_{2,3} - c_{1,3}$	$\{1,2\}$	$c_3 + c_{2,3} - c_2$

Since $SV_i(V^*_{\mathbf{e}}) = \sum\limits_{S \subseteq N\setminus\{i\}} f(s)[v^*_{\mathbf{e}}(S \cup \{i\}) - v^*_{\mathbf{e}}(S)]$, we have

$$\begin{aligned} SV_1(V^*_{\mathbf{e}}) &= \frac{4}{3}c_1 - \frac{1}{2}c_2 - \frac{1}{6}c_3 + \frac{1}{2}c_{1,2} + \frac{1}{6}c_{1,3} \\ SV_2(V^*_{\mathbf{e}}) &= c_2 - \frac{1}{6}c_1 - \frac{1}{6}c_3 + \frac{1}{2}c_{1,2} - \frac{1}{3}c_{1,3} + \frac{1}{2}c_{2,3} \\ SV_3(V^*_{\mathbf{e}}) &= \frac{4}{3}c_3 - \frac{1}{2}c_2 - \frac{1}{6}c_1 + \frac{1}{6}c_{1,3} + \frac{1}{2}c_{2,3}. \end{aligned}$$

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ where $c_1 = 30$, $c_2 = 6$, $c_3 = 15$, $c_{1,2} = 25$, $c_{1,3} = 16$, $c_{2,3} = 20$, and $r = (0, 1, 2, 3, 0)$. Note that $\mathbf{e} \in \mathcal{E}_T$. We have $SV_1(V^*_{\mathbf{e}}) = \frac{149}{3}$, $SV_2(V^*_{\mathbf{e}}) = \frac{47}{3}$, and $SV_3(V^*_{\mathbf{e}}) = \frac{74}{3}$.

Let sponsors 1 and 2 merge into a single sponsor denoted by k. Let $\widehat{V} = ((N\setminus\{1, 2\}) \cup \{k\}, \widehat{v})$ be the TU-game obtained from $V^*_{\mathbf{e}}$ by this merger. Thus, $\widehat{v}(\{k\}) = c_1 + c_{1,2} + c_2$, $\widehat{v}(\{k, 3\}) - \widehat{v}(\{3\}) = c_1 + c_{1,2} + c_{2,3} - c_3$. Then, $SV_k(\widehat{V}) = \frac{1}{2}(2c_1 + c_2 - c_3 + 2c_{1,2} + c_{2,3}) = \frac{121}{2}$. Since, $SV_k(\widehat{V}) \neq SV_1(V^*_{\mathbf{e}}) + SV_2(V^*_{\mathbf{e}})$, SV is not *Merging and Splitting Proof* or *Merging and Splitting Proof-2.* ■

Proof of Theorem 3:

• Let F satisfy the axioms in Theorem 3. Then, by the same argument in the proof of Theorem 2, $F = SV$.

• Now, we show that the Shapley value satisfies the axioms listed in Theorem 3.

Respect of Connected Sets:

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$. Note that for each $i \in N$, each $S_i \in \mathcal{S}_{\mathbf{e}}$ such that $i \in S_i$, and each $S \subseteq N\setminus\{i\}$, the marginal contribution of i to the value of a coalition S is equal to the marginal contribution of i to the value of $S \cap (S_i\setminus\{i\})$. That is,

$$v^*_{\mathbf{e}}(S \cup \{i\}) - v^*_{\mathbf{e}}(S) = v^*_{\mathbf{e}}(S \cap (S_i\setminus\{i\}) \cup \{i\}) - v^*_{\mathbf{e}}(S \cap (S_i\setminus\{i\})). \tag{30}$$

Let $\widehat{S} \in \mathcal{S}_{\mathbf{e}}$ and $\widehat{\mathbf{e}} = \langle N, \widehat{\mathbf{c}}, r\rangle$ be such that for each $j \in N\setminus\widehat{S}$ and each $l \in N\setminus\{j\}$, $\widehat{c}_j = 0$ and $\widehat{c}_{j,l} = 0$.

By 30, for each $i \in \widehat{S}$ and each $S \subseteq N\setminus\{i\}$, $v^*_{\mathbf{e}}(S \cup \{i\}) - v^*_{\mathbf{e}}(S) = v^*_{\widehat{\mathbf{e}}}(S \cup \{i\}) - v^*_{\widehat{\mathbf{e}}}(S)$ and hence, $SV_i(V^*_{\mathbf{e}}) = SV_i(V^*_{\widehat{\mathbf{e}}})$. Also, by 30, for each $j \in N\setminus\widehat{S}$ and each $S \subseteq N\setminus\{j\}$, $v^*_{\widehat{\mathbf{e}}}(S \cup \{j\}) - v^*_{\widehat{\mathbf{e}}}(S) = 0$ and hence, $SV_j(V^*_{\widehat{\mathbf{e}}}) = 0$. By *Efficiency*, $\sum\limits_{i \in N} SV_i(V^*_{\widehat{\mathbf{e}}}) = \sum\limits_{i \in \widehat{S}} SV_i(V^*_{\widehat{\mathbf{e}}}) = \widehat{c}(r) = v^*_{\mathbf{e}}(\widehat{S})$. All together, $\sum\limits_{i \in \widehat{S}} SV_i(V^*_{\mathbf{e}}) = v^*_{\widehat{\mathbf{e}}}(\widehat{S})$. Hence, the Shapley value satisfies *Respect of Connected Sets.*

Equal Impact:

Let $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ where $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ and $\mathbf{e}' = \langle N, \mathbf{c}, r'\rangle$ be such that there exists $i \in N\setminus\{n\}$ where

(i) $i \succ_r i + 1$ and $i \succ_{r'} 0 \succ_{r'} i + 1$, and

(ii) for $S_i \in \mathcal{S}_{\mathbf{e}}$ with $i \in S_i$, for each $k \in S_i\setminus\{i\}$ and $l \in S_i\setminus\{i+1\}$, $k \succ_r l$ if and only if $k \succ_{r'} l$.

Let $\overline{S} = \{i, i + 1\}$. For each $S \subseteq N\setminus\overline{S}$ and each $j \in \overline{S}$,

$$v^*_{\mathbf{e}}(S \cup \{j\}) = v^*_{\mathbf{e}'}(S \cup \{j\}) \text{ and } v^*_{\mathbf{e}}(S) = v^*_{\mathbf{e}'}(S). \tag{31}$$

By (31), $\sum_{S \subseteq N \setminus \overline{S}} f(s)[(v^*_{\mathbf{e}}(S \cup \{j\}) - v^*_{\mathbf{e}}(S)) - (v^*_{\mathbf{e}'}(S \cup \{j\}) - v^*_{\mathbf{e}'}(S))] = 0$. Hence,

$$\begin{aligned} SV_j(V^*_{\mathbf{e}}) - SV_j(V^*_{\mathbf{e}'}) &= \sum_{S \subseteq N \setminus \overline{S}} f(s)[(v^*_{\mathbf{e}}(S \cup \{j,l\}) - v^*_{\mathbf{e}}(S \cup \{l\})) - (v^*_{\mathbf{e}'}(S \cup \{j,l\}) - v^*_{\mathbf{e}'}(S \cup \{l\}))] \\ &= \sum_{S \subseteq N \setminus \overline{S}} f(s)[v^*_{\mathbf{e}}(S \cup \overline{S}) - v^*_{\mathbf{e}'}(S \cup \overline{S})]. \end{aligned}$$

Therefore, $SV_i(V^*_{\mathbf{e}}) - SV_i(V^*_{\mathbf{e}'}) = \sum_{S \subseteq N \setminus \overline{S}} f(s)[v^*_{\mathbf{e}}(S \cup \overline{S}) - v^*_{\mathbf{e}'}(S \cup \overline{S})] = SV_{i+1}(V^*_{\mathbf{e}}) - SV_{i+1}(V^*_{\mathbf{e}'})$ and the Shapley value satisfies *Equal Impact*. ■

Proof of Proposition 5:
Let $\mathbf{e} = \langle \{1,2,3,4\}, \mathbf{c}, r \rangle \in \mathcal{E}^*_T$ be such that $r^* = (0,1,2,3,4,0)$ and $c_1 = c_{1,2} = c_{3,4} = c_4 = 5$, $c_2 = c_3 = c_{2,3} = 3$, $c_{1,4} = 10$, and $c_{1,3} = c_{2,4} = 6$. Note that triangle inequalities hold among all agents and r is a least costly route for $\mathbf{e}$. Let $i = 2$, $S = \{1,4\}$, and $T = \{1,3,4\}$. Since, $v^*_{\mathbf{e}}(S \cup \{i\}) - v^*_{\mathbf{e}}(S) = 1$ and $v^*_{\mathbf{e}}(T \cup \{i\}) - v^*_{\mathbf{e}}(T) = 2$, and $1 < 2$, $V^*_{\mathbf{e}}$ is not convex.

In general, let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}^*_T$ be such that there is $S_i \in \mathcal{S}_{\mathbf{e}}$ with $|S_i| \geq 4$ and $\{i-1, i, i+1, i+2\} \subseteq S_i$ for some $i \in N$ where $c_{i,i+2} + c_{i-1,i+1} < c_{i-1,i+2} + c_{i,i+1}$. Note that this inequality is compatible with a cost vector satisfying triangle inequalities (as in the example in the previous paragraph). Let $S = \{i-1, i+2\}$ and $T = \{i-1, i+1, i+2\}$. Then, $v^*_{\mathbf{e}}(S \cup \{i\}) - v^*_{\mathbf{e}}(S) < v^*_{\mathbf{e}}(T \cup \{i\}) - v^*_{\mathbf{e}}(T)$ and $V^*_{\mathbf{e}}$ is not convex. ■

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7 Appendix for referees

Here, we depict the calculation of Shapley values we used in the proof of Theorem 1 where we showed the Shapley value is merging and splitting proof.

In part (b) and (c), we used the following expressions whose simplifications are as follows:

- $\sum_{s=1}^{n-|K|} \binom{n-|K|-1}{s-1} g(s) = \sum_{s=1}^{n-|K|} \frac{(n-|K|-1)!}{(s-1)!(n-|K|-s)!} \frac{s!(n-|K|+1-s-1)!}{(n-|K|+1)!} = \sum_{s=1}^{n-|K|} \frac{s}{(n-|K|+1)(n-|K|)}$

  $= \frac{1}{(n-|K|+1)(n-|K|)} \frac{(n-|K|)(n-|K|+1)}{2} = 1/2$

- $\sum_{s=0}^{n-|K|-1} \binom{n-|K|-1}{s} g(s) = \sum_{s=0}^{n-|K|-1} \frac{(n-|K|-1)!}{(s)!(n-|K|-s-1)!} \frac{s!(n-|K|-s)!}{(n-|K|+1)!}$

$$= \sum_{s=0}^{n-|K|-1} \frac{(n-|K|-s)}{(n-|K|+1)(n-|K|)} = \sum_{s=0}^{n-|K|-1} \frac{1}{(n-|K|+1)} - \frac{1}{(n-|K|+1)(n-|K|)} \sum_{s=0}^{n-|K|-1} s$$

$$= \frac{(n-|K|)}{(n-|K|+1)} - \frac{1}{(n-|K|+1)(n-|K|)} \frac{(n-|K|-1)(n-|K|)}{2} = \frac{2(n-|K|)-(n-|K|-1)}{2(n-|K|+1)} = 1/2$$

In part (d), we used the following expressions:

- $\sum_{s=0}^{n-|K|-2} \binom{n-|K|-2}{s} g(s) = \sum_{s=0}^{n-|K|-2} \frac{(n-|K|-2)!}{(s)!(n-|K|-s-2)!} \frac{s!(n-|K|-s)!}{(n-|K|+1)!}$

  $= \frac{(n-|K|-2)!}{(n-|K|+1)!} \sum_{s=0}^{n-|K|-2} (n-|K|-s)(n-|K|-s-1)$

  Let $x = n - |K| + 1$

  Then,

  $\frac{(n-|K|-2)!}{(n-|K|+1)!} \sum_{s=0}^{n-|K|-2} (n-|K|-s)(n-|K|-s-1) = \frac{(x-3)!}{x!} \sum_{s=0}^{x-3} \left[(x^2 - 3x + 2) + s(3 - 2x) + s^2\right]$

  $= \frac{(x-3)!}{x!} \left[\sum_{s=0}^{x-3} (x^2 - 3x + 2) + (3 - 2x) \sum_{s=0}^{x-3} s(3 - 2x) + (3 - 2x)s^2\right]$

  $= \frac{(x-3)!}{x!} \left[(x^2 - 3x + 2)(x - 2) + \frac{(x-3)(x-2)}{2}(3 - 2x) + \frac{(x-3)(x-2)(2x-5)}{6}\right]$

  $= \frac{(x-3)!}{x!} \frac{(x-2)(x-1)(3x-6-2x+6)}{3} = \frac{1}{3}$

- $\sum_{s=1}^{n-|K|-1} \binom{n-|K|-2}{s-1} g(s) = \sum_{s=1}^{n-|K|-1} \frac{(n-|K|-2)!}{(s-1)!(n-|K|-s-1)!} \frac{s!(n-|K|-s)!}{(n-|K|+1)!}$

  $= \sum_{s=1}^{n-|K|-1} \frac{s(n-|K|-s)}{(n-|K|+1)(n-|K|)(n-|K|-1)}$

  $= \sum_{s=1}^{n-|K|-1} \frac{s}{(n-|K|+1)(n-|K|-1)} - \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)} \sum_{s=1}^{n-|K|-1} s^2$

$= \frac{(n-|K|-1)(n-|K|)}{2(n-|K|+1)(n-|K|-1)} - \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)} \frac{(n-|K|-1)(n-|K|)(2n-2|K|-1)}{6}$

$= \frac{(n-|K|)}{2(n-|K|+1)} - \frac{1}{(n-|K|+1)} \frac{(2n-2|K|-1)}{6} = \frac{3(n-|K|)}{6(n-|K|+1)} - \frac{1}{(n-|K|+1)} \frac{(2n-2|K|-1)}{6}$

$= \frac{3n-3|K|-2n+2|K|+1}{6(n-|K|+1)} = \frac{n-|K||+1}{6(n-|K|+1)} = 1/6$

- $\sum_{s=2}^{n-|K|} \binom{n-|K|-2}{s-2} g(s) = \sum_{s=2}^{n-|K|-2} \frac{(n-|K|-2)!}{(s-2)!(n-|K|-s)!} \frac{s!(n-|K|+1-s-1)!}{(n-|K|+1)!}$

  $= \sum_{s=2}^{n-|K|} \frac{s(s-1)}{(n-|K|+1)(n-|K|)(n-|K|-1)} = \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)} (\sum_{s=2}^{n-|K|} s^2 - \sum_{s=2}^{n-|K|} s)$

$= \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)}\{[\frac{(n-|K|)(n-|K|+1)(2n-2|K|+1)}{6} - 1] - [\frac{(n-|K|)(n-|K|+1)}{2} - 1]\}$

$= \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)}\{\frac{(n-|K|)(n-|K|+1)(2n-2|K|+1)}{6} - \frac{3(n-|K|)(n-|K|+1)}{6}\}$

$= \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)}\frac{(n-|K|)(n-|K|+1)(2n-2|K|+1-3)}{6}$

$= \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)}\frac{(n-|K|)(n-|K|+1)2(n-|K|-1)}{6} = 1/3$

8 Appendix 8

Connected-Set Merging and Splitting Proofness-2: For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, $S \in \mathcal{S}_{\mathbf{e}}$, and $\{S', S''\} \subseteq 2^S$ such that $S = S' \cup S''$ and $S' \cap S'' = \emptyset$, if $\widetilde{V} = ((N\backslash S) \cup \{s', s''\}, \widetilde{v})$ is the TU-game obtained from $V_{\mathbf{e}} = (N, v_{\mathbf{e}})$ when S' merges into a single sponsor $s' \in S'$ and S'' merges into a single sponsor $s'' \in S''$, then

$$F_{s'}(\widetilde{V}) = \sum_{i \in S'} F_i(V_{\mathbf{e}}) \text{ and } F_{s''}(\widetilde{V}) = \sum_{i \in S''} F_i(V_{\mathbf{e}}).$$

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, $\overline{S} \in \mathcal{S}_{\mathbf{e}}$, and $\{S', S''\} \subseteq 2^{\overline{S}}$ be such that $\overline{S} = S' \cup S''$ and $S' \cap S'' = \emptyset$. Let $\widetilde{V} = ((N\backslash\overline{S}) \cup \{s', s''\}, \widetilde{v})$ be the TU-game obtained from $V_{\mathbf{e}} = (N, v_{\mathbf{e}})$ when S' merges into a single sponsor $s' \in S'$ and S'' merges into a single sponsor $s'' \in S''$.

For each $S \subseteq N$, let $|S| = s$ and $g(s) = \frac{s!(n-|\overline{S}|+2-s-1)!}{(n-|\overline{S}|+2)!}$. Note that for each $S \subseteq \{s''\} \cup N\backslash\overline{S}$, if $s'' \notin S$, then $\widetilde{v}(S \cup \{s'\}) - \widetilde{v}(S) = v_{\mathbf{e}}(S')$; and if $s'' \in S$, then $\widetilde{v}(S \cup \{s'\}) - \widetilde{v}(S) = v_{\mathbf{e}}(\overline{S}) - v_{\mathbf{e}}(S'')$. Then,

$SV_{s'}(\widetilde{V}) = \sum_{S \subseteq N\backslash\overline{S}} g(s)(\widetilde{v}(S \cup \{s'\}) - \widetilde{v}(S)) + \sum_{S \subseteq \{s''\} \cup N\backslash\overline{S}: s'' \in S} g(s)(\widetilde{v}(S \cup \{s'\}) - \widetilde{v}(S))$

$= v_{\mathbf{e}}(S') \sum_{s=0}^{n-|\overline{S}|} \binom{n-|\overline{S}|}{s} g(s) + (v_{\mathbf{e}}(\overline{S}) - v_{\mathbf{e}}(S'')) \sum_{s=1}^{n-|\overline{S}|+1} \binom{n-|\overline{S}|}{s-1} g(s)$

$= \frac{1}{2}(v_{\mathbf{e}}(S') + v_{\mathbf{e}}(\overline{S}) - v_{\mathbf{e}}(S'')) = \sum_{i \in S'} SV_i(V_{\mathbf{e}})$.

Similarly, we can show that $SV_{s''}(\widetilde{V}) = \sum_{i \in S''} SV_i(V_{\mathbf{e}})$. Therefore, the Shapley value satisfies *Connected-Set Merging and Splitting Proofness-2*.