Bias Correction of ML and QML Estimators in the EGARCH(1,1) Model

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Abstract

In this paper we derive the bias approximations of the Maximum Likelihood (ML) and Quasi-Maximum Likelihood (QML) Estimators of the EGARCH(1,1) parameters and we check our theoretical results through simulations. With the approximate bias expressions up to $O\left(\frac{1}{T}\right)$, we are then able to correct the bias of all estimators. To this end, a Monte Carlo exercise is conducted and the results are presented and discussed. We conclude that, for given sets of parameters values, the bias correction works satisfactory for all parameters. The results for the bias expressions can be used in order to formulate the approximate Edgeworth distribution of the estimators.

Keywords: Quasi Maximum Likelihood, EGARCH, bias approximations, bias correction.

JEL classification: C13, C22

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1 Introduction

The last years there has been a substantial interest in deriving the asymptotic properties of econometric estimators in time series models. Although there is an important and growing literature that deals with the asymptotics of the Generalized Autoregressive Conditional Heteroskedastic (GARCH) models, either in terms of consistency and asymptotic normality of the estimators or in terms of the finite-sample theory, the asymptotic properties of the estimators in the Exponential GARCH (EGARCH) process of Nelson (1991) have not been fully explored. Comparing to the GARCH process, the advantages of the EGARCH model are well-known, with the main one being the fact that the model captures the negative dynamic asymmetries noticed in many financial series, i.e. the so-called leverage effects.

The asymptotic aspects of the conditionally heteroskedastic models have been discussed under many different considerations, in order to analyze the statistical properties of these estimators. Since the important work of Engle (1982) and that of Bollerslev (1986), who introduced the Autoregressive Conditional Heteroskedasticity (ARCH) and Generalized ARCH models, respectively, a huge amount of literature on the asymptotics has appeared in short time. Weiss (1986) proved Consistency and Asymptotic Normality (CAN) of the maximum likelihood estimators in ARCH models, assuming normal distribution of the errors and imposing a rather restrictive condition that the data have bounded fourth moments, excluding in that way from the proof many other interesting conditionally heteroskedastic models. Quite parallel, Lee and Hansen (1994) and Lumsdaine (1996) relaxed the condition which Weiss imposed and they looked at the consequences of the possible failure of the normality assumption on the errors, providing conditions under which CAN exist in the GARCH$(1,1)$ specification (for multivariate frameworks see e.g. Jeantheau, 1998; Comte and Lieberman, 2003).

The finite sample properties of the QML estimators in the first order GARCH model are investigated through an asymptotic expansion of the Edgeworth type, as Linton (1997) developed[1] in which he also provided the higher-order bias of the estimators. Furthermore, Iglesias and Linton (2007) derive the second-order asymptotic theory of the quasi-maximum likelihood estimator in stationary and nonstationary GARCH models, when constaints are imposed and they correct the first- and second-order bias of the estimator. Nowadays, many researchers work

[1] The validity of the Edgeworth expansions in the GARCH model is established in the paper of Corradi and Iglesias (2008).

on the asymptotic behaviour of these estimators, with unceasing interest.

Until the influential work of Nelson (1991), the conditional heteroskedastic models that had been developed could not explain the asymmetry effects, indicating that alternative models might be suitable for financial applications. Turning our attention to asymmetric GARCH models, and more specifically to the EGARCH model which has become a popular model in applied work, very little is known about its statistical properties. Although we are endowed with the moment structure investigated by He, Terasvirta and Malmsten (2002), the limiting properties of the maximum likelihood estimators in the EGARCH models do not exist in the literature. The interest in consistency and asymptotic normality results of EGARCH has been growing and the problem of the theoretical properties not yet been explored await for an answer; see, for example, Straumann and Mikosch (2006)[2]. The finite sample properties of the maximum likelihood and quasi-maximum likelihood estimators of the EGARCH$(1,1)$ process using Monte Carlo methods have been examined in the paper of Deb (1996)[3]. He used, however, response surface methodology in order to examine the finite sample bias and other properties in interest, by summarizing the results of a wide array of experiments.

In this paper we derive the bias approximations of the Maximum Likelihood (ML) and Quasi-Maximum Likelihood (QML) Estimators of the EGARCH$(1,1)$ parameters and we check our theoretical results through simulations. With the approximate bias expressions, we are then able to correct the bias of all estimators. To this end, a Monte Carlo exercise is conducted and the results are presented and discussed. We provide two types of bias correction mechanisms in order to decide for the bias reduction in practice for the popular model of Nelson, the EGARCH. It is the first time that analytically the higher order biases appear in this literature for a nonlinear model like the EGARCH one and these results can now be used to be incorporated into the relative analysis of other similar specifications, see e.g. Iglesias and Linton (2007). We conclude that, for given sets of parameters values, the bias correction works satisfactory for all parameters. The results for the bias expressions can be used in order to formulate the approximate Edgeworth distribution of the estimators.

The organisation of the paper is as follows: Section 2 presents the model and estimators.

[2] In a recent paper, Zaffaroni (2009) estimates the EGARCH parameters with Whittle methods and the asymptotic distribution theory of these estimators is established.

[3] Perez and Zaffaroni (2008) compare the finite sample properties of the MLE and Whittle estimators, in terms of bias and efficiency, in the EGARCH model and its long-memory version.

Section 3 deals with the main results of our analysis. First, analytic derivatives and their expected values are presented. Second, conditions for stationarity of the log-variance derivatives are investigated. In the sequel, the theoretical bias approximations of the Maximum Likelihood and Quasi Maximum Likelihood Estimators are calculated and the simulation results for the bias correction of the estimators are presented. Finally, Section 4 concludes. All proofs, rather lengthy, are collected in the Appendix. Let us now turn our attention to the definition of the EGARCH$(1,1)$ model and the estimators.

2 The Model and Estimators

Let us consider the following model, where the observed data $\{y_t\}_{t=1}^{T}$ are generated by the EGARCH$(1,1)$ process, see Nelson (1991), in which the conditional variance, h_t, depends on both the size and the sign of the lagged residuals:

$$
\begin{aligned}
y_t &= \mu + u_t, \qquad t = 1, ...T, \qquad \textit{where} && (1)\\
u_t &= z_t\sqrt{h_t}, \quad z_t \sim iidD\,(0,1) &&\\
\ln(h_t) &= \alpha + \theta z_{t-1} + \gamma g\,(z_{t-1}) + \beta \ln(h_{t-1})\,, \qquad \textit{where} && (2)\\
g\,(z_t) &= |z_t| - E\,|z_t|\,. &&
\end{aligned}
$$

The process $\{u_t\}$ is a real-valued discrete time stochastic process (the error process) and h_t is a positive with probability one $\mathcal{A}_{t-1}$-measurable function (the conditional variance), where $\mathcal{A}_{t-1}$ is the sigma-algebra generated by the past values of z_t, i.e. $\{z_{t-1}, z_{t-2}, z_{t-3}, ...\}$. The function $g\,(z_t)$ is a well-defined function of z_t. The process h_t is not observed and thus is constructed via recursion using the estimating values of the parameters and a proper initial value for the conditional variance. The only distributional assumption made about the innovations $z_t^{\prime}s$ is that they are independently and identically distributed (iid) with zero mean and unit variance. We do not impose any symmetric distributional property, however the proofs automatically become very tedious. The conditional variance is constrained to be non-negative by the assumption that the logarithm of h_t is a function of past $z_t^{\prime}s$. Comparing to the relative analysis, Nelson's paper was the first which models the conditional variance as a function of variables which are not solely squares of the observations.

Note from (2) that $\ln(h_t)$ constitutes a causal AR(1) process with mean $\alpha/(1-\beta)$ and error sequence $[\theta z_{t-1} + \gamma(|z_{t-1}| - E|z_{t-1}|)]$. The unique stationary solution to (2), provided that $|\beta| < 1$, is given by its almost sure (a.s.) representation:

$$\begin{aligned} \ln(h_t) &= \alpha(1-\beta)^{-1} + \sum_{k=0}^{\infty} \beta^k \left(\theta z_{t-1-k} + \gamma g(z_{t-1-k})\right) \Rightarrow \\ \ln(h_t) &\geq (\alpha - \gamma E|z_t|)(1-\beta)^{-1} \qquad a.s. \end{aligned}$$

The conditional variance responds asymmetrically to rises and falls in stock price, which is believed to be important for example in modelling the behaviour of stock returns. It is an important stylized fact for many assets. The coefficients $(\theta+\gamma)$ and $(\theta-\gamma)$ (if $z_t \geq 0$ and $z_t < 0$, respectively) show the asymmetry in response to positive and negative y_t. The parameter θ is referred to as the leverage parameter, which shows the effect of the sign of y_t. The term $\gamma[|z_t| - E|z_t|]$ represents a magnitude effect. Formulae for the higher order moments of u_t are given in Nelson (1991). The parameter α can be made a function of time (α_t) to accommodate the effect of any non-trading periods of forecastable effects.

The unconditional mean and variance of y_t is:

$$E(y_t) = \mu,$$

and

$$Var(y_t) = \exp\left(\frac{\alpha}{1-\beta}\right) \prod_{i=0}^{\infty} E\left[\exp\left[\beta^i\left(\theta z_0 + \gamma g(z_0)\right)\right]\right],$$

which, under normality of the errors, becomes the following result:

$$Var(y_t) = \exp\left(\frac{\alpha - \gamma\sqrt{\frac{2}{\pi}}}{1-\beta}\right) \prod_{i=0}^{\infty} \left[\exp\left(\frac{\beta^{2i}(\gamma^*)^2}{2}\right)\Phi\left(\beta^i\gamma^*\right) + \exp\left(\frac{\beta^{2i}\delta^2}{2}\right)\Phi\left(\beta^i\delta\right)\right],$$

where $\gamma^* = \gamma + \theta$, $\delta = \gamma - \theta$ and $\Phi(k)$ is the value of the cumulative standard Normal evaluated at k, i.e. $\Phi(k) = \int_{-\infty}^{k} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$.

Proof. The proof of the unconditional variance is given in the Appendix. ■

To estimate the parameters of the model in (1) and (2), we employ the quasi-maximum likelihood estimation. Maximum likelihood is the procedure which is most often used in estimating the parameters in time series models, but for most applications it is very difficult to justify the conditional normality assumption. Therefore, the log-likelihood function may be misspecified. However, we can still obtain estimates by maximizing a Gaussian quasi-log-likelihood function and under the auxiliary assumption of an *iid* distribution for the standardized innovations $z_t^{\prime}s$. The estimators which are derived by this maximization problem are the so-called Quasi Maximum Likelihood Estimators (QMLEs). The fact that we maximize a quasi-log-likelihood is justified by the evidence that distributions of asset returns are often thick tailed and as a consequence the normality assumption is violated.

An important and really interesting feature of our model is that the assumption of the block diagonality of the information matrix no longer holds. This is also the case for the ARCH-M model and the asymmetric model of the Augmented ARCH (see Bera and Higgins, 1993, p. 349; also Bollerslev, Engle and Nelson, 1994, p. 2981). This implies that the off-diagonal blocks involving partial derivatives with respect to both mean and variance parameters are not null matrices, while this is the case in other GARCH-type models. Below we present analytic proofs of this argument in the context of the EGARCH$(1,1)$ model and these results disaccord with Malmsten (2004), even if the distribution of the innovations is symmetric, which implies that $Ez^3 = 0$.

In the EGARCH$(1,1)$ model, there is no explicit expression of the probability density of the vector $(y_1, ..., y_T)'$ since the distribution of $(h_1, ..., h_T)'$ is not known. To overcome this difficulty, we consider an approximate conditional log-likelihood instead. Some assumptions are also required for the initial values of the conditional variance h_t, which should be drawn from the stationary distribution, and the squared standardized residuals z_t^2. Assuming that $z_0 = 0$ and $\ln(h_0) = \frac{\alpha}{1-\beta}$, we obtain a good approximation to the conditional Gaussian log-likelihood, as follows:

$$\begin{aligned} \ell\left(\mu, \alpha, \theta, \beta, \gamma \middle| z_0, h_0\right) &= -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln(h_t) - \sum_{t=1}^{T}\frac{(y_t-\mu)^2}{2h_t} = \\ &= -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln(h_t) - \frac{1}{2}\sum_{t=1}^{T} z_t^2. \qquad (3) \end{aligned}$$

Notice that h_t and z_t are both functions of $\boldsymbol{\omega}$ and μ, where $\boldsymbol{\omega} = (\alpha, \theta, \beta, \gamma)^{/}$, i.e. the vector of unknown log-variance parameters, so that both are functions of $\boldsymbol{\varphi} = \left(\boldsymbol{\omega}^{/}, \mu\right)^{/}$, which represents the vector of all unknown parameters. The first order conditions are recursive and consequently do not have explicit solutions.

The likelihood function is derived as though the errors are conditionally normal and is still maximized at the true parameters. Having specified the log-likelihood function, the quasi maximum likelihood estimator is then defined as

$$\widehat{\varphi_T} = \arg\max_{\varphi \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \ell(\varphi) . \tag{4}$$

The parameter space is of the form

$$\Theta = \mathbb{R} \times [0,1) \times D,$$

where

$$D = \left\{ (\theta, \gamma)' \in \mathbb{R}^2 \mid \theta \in \mathbb{R}, \gamma \geq |\theta| \right\} .$$

Let us proceed with the main results of our analysis, beginning with the analytic derivatives of the log-likelihood function and their expected values.

3 The Main Results

3.1 Analytic derivatives and their expected values

In this section we present analytic derivatives[4] of the log-likelihood function and their expected values, which are needed in the sequel to evaluate the asymptotic bias of the QMLEs and to calculate the cumulants of the Edgeworth distribution. It is of great importance to mention that there are no such analytic results in the related literature of the finite sample theory, and it is especially this feature that makes this analysis to differ from the previous one, that of Linton (1997), who studied the case of the GARCH$(1,1)$ model. Let us first proceed with the derivatives of the log-likelihood function and their analytic representation.

Following henceforth the notation employed in Linton (1997), i.e. $h_{t;\circ} = \frac{\partial \ln(h_t)}{\partial \circ}$ and so on,

[4]Fiorentini, Calzolari and Panattoni (1996) argue that the computation of analytic derivatives of the log-likelihood is essential, as the computational benefit of their use is really substantial for estimation purposes.

the derivatives of the log-likelihood function with respect to all the parameters are:

$$\begin{aligned}
\mathcal{L}_{\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu}+\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}},\\
\mathcal{L}_{\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu}-\sum_{t=1}^{T}\left(\frac{1}{h_t}+2\frac{z_t}{\sqrt{h_t}}h_{t;\mu}+\frac{1}{2}z_t^2h_{t;\mu}^2\right),\\
\mathcal{L}_{\mu\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu,\mu}+3\sum_{t=1}^{T}\frac{1}{h_t}h_{t;\mu}\\
&\quad -3\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\left(h_{t;\mu,\mu}-h_{t;\mu}^2\right)-\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(3h_{t;\mu}h_{t;\mu,\mu}-h_{t;\mu}^3\right)
\end{aligned}$$

while for $i,j,k\in\{\alpha,\theta,\gamma,\beta\}$ the derivatives are:

$$\begin{aligned}
\mathcal{L}_{i} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i},\\
\mathcal{L}_{ij} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;i}^2,\\
\mathcal{L}_{ijk} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j,k}-\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(3h_{t;i}h_{t;j,k}-h_{t;i}^3\right).
\end{aligned}$$

The cross derivatives are given by the following expressions:

$$\begin{aligned}
\mathcal{L}_{i\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,\mu}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;i}h_{t;\mu}-\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}h_{t;i},\\
\mathcal{L}_{i\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,\mu,\mu}-2\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\left(h_{t;i,\mu}-h_{t;i}h_{t;\mu}\right)\\
&\quad -\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(2h_{t;\mu}h_{t;i,\mu}-h_{t;i}h_{t;\mu}^2+h_{t;i}h_{t;\mu,\mu}\right)+\sum_{t=1}^{T}\frac{1}{h_t}h_{t;i},\\
\mathcal{L}_{ij\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j,\mu}-\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\left(h_{t;i,j}-h_{t;i}h_{t;j}\right)\\
&\quad -\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(h_{t;j}h_{t;i,\mu}-h_{t;j}h_{t;i}h_{t;\mu}+h_{t;i,j}h_{t;\mu}+h_{t;i}h_{t;j,\mu}\right).
\end{aligned}$$

Note that the log-likelihood derivatives are expressions of the log-variance derivatives, $h_{t;\circ}$, where the latter are given in the Appendix. The expected values of the log-likelihood derivatives are also given in the Appendix.

The cross-products of the log-likelihood derivatives are:

for $i, j \in \{\alpha, \theta, \gamma, \beta\}$,

$$\begin{aligned}
\mathcal{L}_i\mathcal{L}_{ij} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i}\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;i}^2\right),\\
\mathcal{L}_i\mathcal{L}_{j\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i}\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;j,\mu}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;j}h_{t;\mu}-\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}h_{t;i}\right),\\
\mathcal{L}_i\mathcal{L}_{\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i}\left[\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu}-\sum_{t=1}^{T}\left(\frac{1}{h_t}+2\frac{z_t}{\sqrt{h_t}}h_{t;\mu}+\frac{1}{2}z_t^2h_{t;\mu}^2\right)\right],\\
\mathcal{L}_\mu\mathcal{L}_{ij} &= \left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu}+\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\right)\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;i}^2\right),\\
\mathcal{L}_\mu\mathcal{L}_{j\mu} &= \left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu}+\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\right)\left(\begin{array}{c}\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;j,\mu}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;j}h_{t;\mu}\\ -\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}h_{t;i}\end{array}\right),\\
\mathcal{L}_\mu\mathcal{L}_{\mu\mu} &= \left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu}+\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\right)\left[\begin{array}{c}\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu}\\ -\sum_{t=1}^{T}\left(\frac{1}{h_t}+2\frac{z_t}{\sqrt{h_t}}h_{t;\mu}+\frac{1}{2}z_t^2h_{t;\mu}^2\right)\end{array}\right].
\end{aligned}$$

The expectations of the cross-products are given in the Appendix.

Let us turn our attention to the conditions for stationarity of the log-variance derivatives.

3.2 Conditions for stationarity of the log-variance derivatives

In this section we investigate under which conditions there is a second-order stationary solution to the log-variance derivatives, needed for the existence and the evaluation of the log-likelihood derivatives, and hence in order to calculate the bias expressions of the QMLEs. The existence, stationarity and ergodicity of the second order derivatives of the conditional variance are necessary so that the Taylor expansion of the first order derivatives of the log-likelihood is validated.

We consider the following example:

$$\begin{aligned}
h_{t;\alpha}h_{t;\alpha\alpha} &= \frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^2+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^3\\
&+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\alpha}\\
&+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2h_{t-1;\alpha}h_{t-1;\alpha,\alpha}. \qquad (5)
\end{aligned}$$

In order to calculate the expected value of the above expression, we first assume that $E\left(h_{t;\alpha}^2\right), E\left(h_{t;\alpha}^3\right)$

and $E\left(h_{t;\alpha,\alpha}\right)$ exist. Next, define:

$$\begin{aligned} A\left(z_{t-1}\right) &= \frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha}^{2} \\ &+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha}^{3} \\ &+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha,\alpha}, \end{aligned}$$

and

$$B^{2}\left(z_{t-1}\right)=\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^{2}.$$

Then,

$$\begin{aligned} h_{t;\alpha}h_{t;\alpha\alpha} &= A\left(z_{t-1}\right)+B^{2}\left(z_{t-1}\right) h_{t-1;\alpha}h_{t-1;\alpha,\alpha}= \\ &= A\left(z_{t-1}\right)+\sum_{k=1}^{\infty}\prod_{i=0}^{k-1} B^{2}\left(z_{t-1-i}\right) A\left(z_{t-1-k}\right). \end{aligned}$$

The infinite sum converges almost surely. To see this, let:

$$S_{n}=A\left(z_{t-1}\right)+\sum_{k=1}^{n}\prod_{i=0}^{k-1} B^{2}\left(z_{t-1-i}\right) A\left(z_{t-1-k}\right).$$

Then we have:

$$\begin{aligned} E\left(S_{n}\right) &= E\left[A\left(z_{t-1}\right)\right]+\sum_{k=1}^{n} E\left[\prod_{i=0}^{k-1} B^{2}\left(z_{t-1-i}\right)\right] E\left[A\left(z_{t-1-k}\right)\right]= \\ &= E\left[A\left(z_{t-1}\right)\right]\left[\sum_{k=0}^{n}\left\{E\left[B^{2}\left(z_{t-1-i}\right)\right]\right\}^{k}\right]. \end{aligned}$$

Thus, $E\left(\lim_{n\to\infty} S_{n}\right)=E\left[A\left(z_{t-1}\right)\right]\left\{1-E\left[B^{2}\left(z_{t-1-i}\right)\right]\right\}^{-1}<\infty$, providing that $E\left[A\left(z_{t-1}\right)\right]<\infty$. In order to ensure the existence of a stationary solution to the (5), we should impose the condition that

$$\left|E\left[B^{2}\left(z_{t-1-i}\right)\right]\right|<1.$$

In a similar manner, the rest stationarity conditions of all log-variance derivatives and the products of them follow.

Proposition 1 *Given*

a) $\left|\beta_0 - \frac{1}{2}\gamma_0 E\,|z|\right| < 1$

b) $\left|\beta_0^2 + \frac{1}{4}\theta_0^2 + \frac{1}{4}\gamma_0^2 - \gamma_0\beta_0 E\,|z| + \frac{1}{2}\gamma_0\theta_0 E\,(z\,|z|)\right| < 1$

and

c) $$\left|\begin{array}{c} \beta_0^3 + \frac{3}{4}\beta_0\theta_0^2 + \frac{3}{4}\beta_0\gamma_0^2 - \frac{1}{8}\theta_0\left(\theta_0^2 + 3\gamma_0^2\right)E\left(z^3\right) - \frac{3}{2}\beta_0^2\gamma_0 E\,|z| \\ +\frac{3}{2}\beta_0\theta_0\gamma_0 E\,(z\,|z|) - \frac{1}{8}\gamma_0\left(\gamma_0^2 + 3\theta_0^2\right)E\,|z|^3 \end{array}\right| < 1,$$

then

the second-order stationarity of all log-variance derivatives follows.

Proof. The proof comes immediately from the results in the *Appendices C* and *G*. ■

Let us now proceed with the bias approximations of the QMLEs.

3.3 Bias Approximations

In this section we develop the bias approximations for the ML and QML estimators in the EGARCH$(1,1)$[5]. One of the main advantages of developing the bias expressions is to use them as a bias correction mechanism. This is one of the practical applications of the bias approximations. Moreover, these results help to analyse the consequences of introducing restrictions in the log-variance parameters. With these expressions, one can compute the Edgeworth approximate distribution. It is also important to explore the theoretical properties of the estimators so that the statistical inference is possible.

We use a McCullagh (1986) result for the standardized estimator having a stochastic expansion, see in p.209, and taking expectations we end up with the asymptotic bias of the QML estimator. Our next step is to check our bias approximations through simulations. Note that McCullagh's expansion has already been applied in the literature to retrieve the bias in many nonlinear models, such as Linton (1997). When dealing with nonlinear models, it is very common to have the bias expressions in terms of expectations and applying these expressions for bias correction. At this point, it is important to state briefly the main differences between our analysis and that of Linton. First of all, we generalize the finite-sample analysis of heteroskedastic time series models considering a non-symmetric distribution of the errors. Furthermore, we show that the block-diagonality of the information matrix does not hold in our case, which

[5] Iglesias and Phillips (2002) developed theoretical bias approximations for the MLEs of the parameters in an ARCH(1) model.

implies that there are new terms in the bias expressions of the estimators. This means that we cannot use the results that appear in the literature from the analysis of the GARCH model.

Assumption 3.3.1 *We assume that the errors have bounded J^{th} moments, for some $J > 6$, and we denote by κ_3 and κ_4 their third and fourth order cumulants, where the latter is given by:*

$$\kappa_4 = E\left(z^4 - 3\right).$$

Under the above assumptions, we are now able to present our Theorem which is useful for the evaluation of the bias approximations of all estimators and also to construct the Edgeworth expansions in this setting.

Theorem 3.3.1 *Given that $z_t \sim iidD\left(0,1\right)$ and non-symmetric, and for $i,j,k \in \{\mu,\alpha,\theta,\gamma,\beta\}$ unless the parameter μ is used seperately to underline the difference, the following moments of the log-likelihood derivatives converge to finit limits as $T \to \infty$:*

$$c_{ij} = \frac{1}{T}E\left(\mathcal{L}_{ij}\right) = -\frac{1}{2}\tau_{i,j},$$

$$c_{ijk} = \frac{1}{T}E\left(\mathcal{L}_{ijk}\right) = -\frac{1}{2}\left(\tau_{ij,k} + \tau_{ik,j} + \tau_{jk,i} - \tau_{i,j,k}\right),$$

$$c_{ij,k} = \frac{1}{T}E\left(\mathcal{L}_{ij}\mathcal{L}_{k}\right) = -\frac{1}{4}\left[\tau^{zz}_{k;i,j} - \left(\kappa_4 + 2\right)\left(\tau_{ij,k} - \tau_{i,j,k}\right)\right],$$

$$c_{\mu\mu} = \frac{1}{T}E\left(\mathcal{L}_{\mu\mu}\right) = -\left(\overline{\pi} + \frac{\tau_{\mu,\mu}}{2}\right),$$

$$c_{i\mu\mu} = \frac{1}{T}E\left(\mathcal{L}_{i\mu\mu}\right) = \overline{\pi_i} - \frac{1}{2}\left(\tau_{i,\mu\mu} + 2\tau_{\mu i,\mu} - \tau_{\mu,i,\mu}\right),$$

$$c_{\mu\mu\mu} = \frac{1}{T}E\left(\mathcal{L}_{\mu\mu\mu}\right) = -\frac{1}{2}\left(3\tau_{\mu\mu,\mu} - \tau^3_{\mu}\right) + 3\overline{\pi_\mu},$$

$$c_{i\mu,\mu} = \frac{1}{T}E\left(\mathcal{L}_{i\mu}\mathcal{L}_{\mu}\right) = -\frac{1}{4}\left\{\begin{array}{c} 4\overline{\pi_i} - \left(\kappa_4 + 2\right)\left(\tau_{i\mu,\mu} - \tau_{i,\mu,\mu}\right) \\ +\tau^{zz}_{\mu;i\mu} + 2\tau^{zh}_{i,\mu} + 2\kappa_3\left(2\tau^{h}_{i,\mu} - \tau^{h}_{i\mu}\right) \end{array}\right\},$$

$$c_{i\mu,j} = \frac{1}{T}E\left(\mathcal{L}_{i\mu}\mathcal{L}_{j}\right) = -\frac{1}{4}\left\{-\left(\kappa_4 + 2\right)\left(\tau_{i\mu,j} - \tau_{i,j,\mu}\right) + \tau^{zz}_{\mu;i\mu} + 2\kappa_3\tau^{h}_{ij}\right\},$$

$$c_{\mu\mu,i} = \frac{1}{T}E\left(\mathcal{L}_{\mu\mu}\mathcal{L}_{i}\right) = -\frac{1}{4}\left\{-\left(\kappa_4 + 2\right)\left(\tau_{\mu\mu,i} - \tau_{i,\mu,\mu}\right) + \tau^{zz}_{i;\mu\mu} + 4\kappa_3\tau^{h}_{i,\mu}\right\},$$

$$c_{ij,\mu} = \frac{1}{T}E\left(\mathcal{L}_{ij}\mathcal{L}_{\mu}\right) = -\frac{1}{4}\left\{\begin{array}{c} -\left(\kappa_4 + 2\right)\left(\tau_{ij,\mu} - \tau_{i,j,\mu}\right) + \tau^{zz}_{\mu;ij} + 2\tau^{zh}_{i,j} \\ +2\kappa_3\left(2\tau^{h}_{i,j} - \tau^{h}_{ij}\right) \end{array}\right\},$$

$$c_{\mu\mu,\mu} = \frac{1}{T}E\left(\mathcal{L}_{\mu\mu}\mathcal{L}_{\mu}\right) = -\frac{1}{4}\left\{\begin{array}{c} 8\overline{\pi_{\mu}} - (\kappa_4+2)\left(\tau_{\mu\mu,\mu} - \tau_{\mu,\mu,\mu}\right) \\ +\tau^{zz}_{\mu;\mu\mu} + 2\tau^{zh}_{\mu,\mu} + 2\kappa_3\left(3\tau^{h}_{\mu,\mu} - \tau^{h}_{\mu\mu}\right) \end{array}\right\},$$

where $\tau_i = \frac{1}{T}\sum_{t=1}^{T} E\left(h_{t;i}\right)$, $\tau_{i,j} = \frac{1}{T}\sum_{t=1}^{T} E\left(h_{t;i}h_{t;j}\right)$, $\tau_{ij,k} = \frac{1}{T}\sum_{t=1}^{T} E\left(h_{t;ij}h_{t;k}\right)$

and $\tau_{i,j,k} = \frac{1}{T}\sum_{t=1}^{T} E\left(h_{t;i}h_{t;j}h_{t;k}\right)$.

Also, $\overline{\pi} = \frac{1}{T}\sum_{t=1}^{T} E\left(\frac{1}{h_t}\right)$, *and* $\overline{\pi_i} = \frac{1}{T}\sum_{t=1}^{T} E\left(\frac{1}{h_t}h_{t;i}\right)$,

while $\tau^{zz}_{k;i,j} = \frac{1}{T}\sum\sum_{s<t} E\left[\left(z_s^2 - 1\right)h_{s;k}h_{t;i}h_{t;j}\right]$, $\tau^{zh}_{i,j} = \frac{1}{T}\sum\sum_{s<t} E\left(z_s\frac{1}{\sqrt{h_t}}h_{t;i}h_{t;j}\right)$,

$\tau^{h}_{i,\mu} = \frac{1}{T}\sum_{t=1}^{T} E\left(\frac{1}{\sqrt{h_t}}h_{t;i}h_{t;\mu}\right)$ *and* $\tau^{h}_{i\mu} = \frac{1}{T}\sum_{t=1}^{T} E\left(\frac{1}{\sqrt{h_t}}h_{t;i,\mu}\right)$.

Proof. Given in the Appendix. ■

The basic approach to finding the bias approximations requires that we find expressions for the c^{ij}, c_{ijk} and $c_{jk,l}$. Let us first condider the case when the mean parameter is supposed to be equal to zero and not estimated. With techniques of McCullagh (1986), the standardized estimators, derived from choosing θ to solve $\mathcal{L}_i(\omega,\mu) = 0$, for $i \in \{\alpha, \theta, \gamma, \beta\}$, have the following stochastic expansions[6]:

$$\sqrt{T}\left\{\widehat{\varphi}_i - \varphi_i\right\} \approx -c^{ij}Z_j + \frac{1}{\sqrt{T}}\left\{c^{ij}c^{kl}Z_{jk}Z_l - c^{ij}c^{kl}c^{mn}c_{j\ln}Z_kZ_m/2\right\} + O_P\left(\frac{1}{T}\right), \qquad (6)$$

where

$$Z_j = T^{-1/2}\mathcal{L}_j$$

and

$$Z_{jk} = T^{-1/2}\left\{\mathcal{L}_{jk} - E\left(\mathcal{L}_{jk}\right)\right\}$$

are evaluated at the true parameters and are jointly asymptotically normal. Raising pairs of indices signifies components from the matrix inversion.

Taking expectations of the right-hand side in (6), we get:

$$E\left[\sqrt{T}\upsilon'\left\{\widehat{\varphi}(\mu) - \varphi\right\}\right] \approx \frac{1}{\sqrt{T}}\upsilon_i c^{ij}c^{kl}\left\{c_{jk,l} + c_{jkl}\left(\kappa_4+2\right)/4\right\},$$

[6] We make use of the summation convention, that is: $c^{ij}Z_j = \sum_j c^{ij}Z_j$, in which repeated indices in an expression are to be summed over.

where υ is the 5×1 parameter vector. If $\kappa_4 = 0$, QML equals ML and then the above formula equals the one of Cox and Snell (1968), i.e.:

$$E\left[\sqrt{T}\upsilon'\left\{\widehat{\varphi}\left(\mu\right) - \varphi\right\}\right] \approx \frac{1}{\sqrt{T}}\upsilon_i c^{ij} c^{kl}\left\{c_{jk,l} + \frac{1}{2}c_{jkl}\right\}.$$

Let us now consider the other case, when the mean parameter is unknown and estimated. Hence, if we incorporate the effects of estimating μ, the stochastic expansions take the following form:

$$\sqrt{T}\left\{\widehat{\varphi}_i\left(\widehat{\mu}\right) - \varphi_i\right\} - \sqrt{T}\left\{\widehat{\varphi}_i\left(\mu\right) - \varphi_i\right\} \approx \frac{1}{\sqrt{T}}\left\{c^{ij}c^{kl}Z_{jk}Z_l - c^{ij}c^{kl}c^{mn}c_{j\ln}Z_kZ_m/2\right\},$$

where now $i, j, k, l \in \{\alpha, \theta, \gamma, \beta, \mu\}$. Taking expectations of the right-hand side, we find the asymptotic bias of the estimators in this case.

In terms of the mean squared error, from (6) we have up to $O_P\left(\frac{1}{T}\right)$:

$$E\left[\sqrt{T}\upsilon'\left\{\widehat{\varphi}\left(\mu\right) - \varphi\right\}\right]^2 \approx -\upsilon_i c^{ij}\left(\kappa_4 + 2\right)/2, \tag{7}$$

which is the asymptotic variance. If we let the remainder to be of $O\left(T^{-3/2}\right)$, then the mean squared error is again evaluated by (7), with the difference now that there would be added terms of $O\left(T^{-1}\right)$. Of course, as $T \to \infty$, the mean squared error approaches the asymptotic variance. In what follows, we present the simulation results and discuss the bias correction of all estimators.

3.4 Simulations

In this section we make a simulation exercise in order to check the adequacy of our theoretical results and be able to proceed with the bias correction of the estimators. We draw a random sample of $T = \{750, 1500, 3000, 5000, 10000, 25000, 50000\}$ observations and 500 observations for initialization, under the assumption of normality. We make 50000 replications for sample sizes up to 10000 and 300000 replications for 25000 and more observations, in order to decrease the Monte Carlo error. The mean parameter is supposed to be equal to zero and hence is not estimated, so the parameter vector is $(\alpha, \beta, \gamma, \theta)'$. We check the performance of the bias correction mechanism for different sets of parameter values and we will present the results for three sets, i.e. $(0.1, 0.9, 0.7, -0.4)$, $(-0.1, 0.9, 0.6, -0.2)$ and $(0.5, 0.5, 0.8, -0.5)$. The first two sets include values for the parameters that are close to what is observed from the financial data. We multiply the bias by T and not $\sqrt{T}$, i.e. $E\left(T\left(\widehat{\varphi}-\varphi\right)\right)$, as in this way we keep a constant term in the bias expressions that is important to distinguish what happens when we increase the sample size, as the next terms in the expressions will tend to zero, as $T \to \infty$.

The bias correction mechanism is constructed under the specification of two methods. The first one, called first-step correction, is the classical one, in which we estimate the model and we retrieve the estimated parameters. Next, we compute the bias expressions by using the estimates and we are then able to correct the bias of the estimators with the corresponding values of the bias, i.e.

$$\widetilde{\varphi} = \widehat{\varphi} - \frac{1}{T} bias\left(\widehat{\varphi}\right).$$

Notice that there is nothing to prevent the case of $\widetilde{\varphi}$ being outside the admissible area (see also Linton, 1997 as well as Iglesias and Linton, 2007). In such a case we throw away the random sample and draw a new one.

The second method that we employ, called full-step correction, is a method proposed by Arvanitis and Demos (2010), in which we solve an optimization problem of the form

$$\min_{\varphi} \left\{ \widehat{\varphi} - \varphi - \frac{1}{T} bias\left(\varphi\right) \right\}^2 .$$

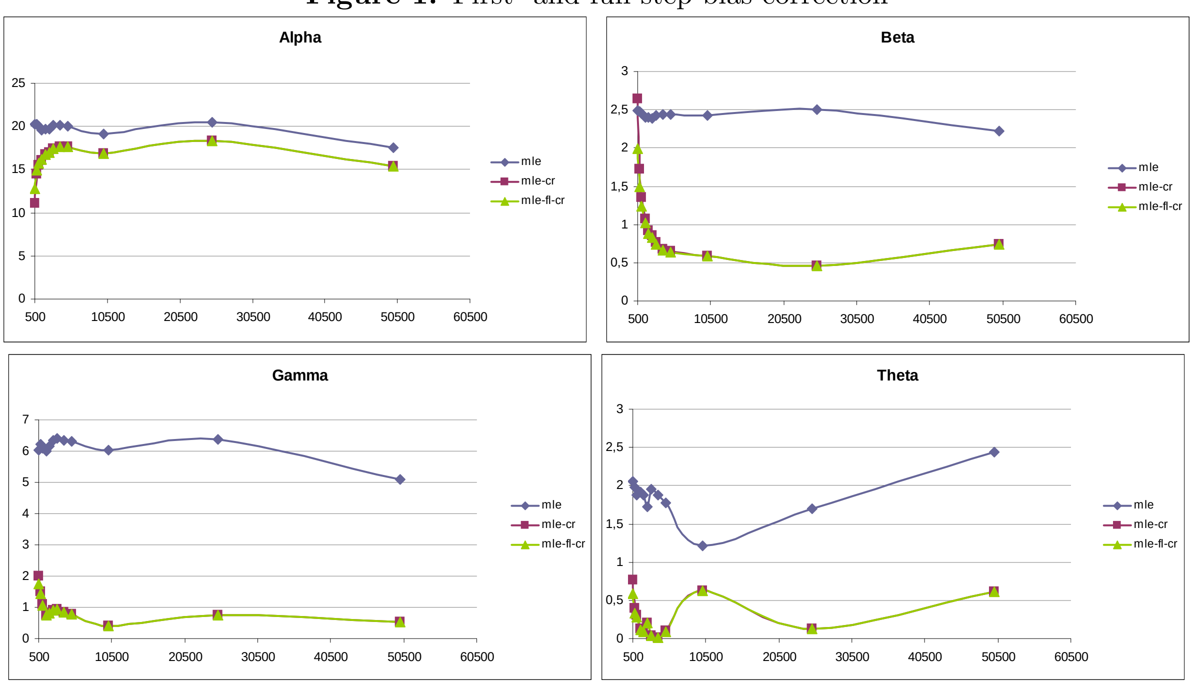
In this respect, this method is a multi-step maximization procedure, using numerical derivatives. This justifies the name of the first method, which is the first step of the multi-step optimization problem. In this way, the second method incorporates the constraints that are imposed on the coefficients and as a consequence the corrected estimate of the EGARCH parameter cannot lie outside the admissible region, i.e. the corrected beta will be less than one in absolute value.

Figures 1 and 2 represent the bias correction performance under the normality assumption. For the first set of parameter values (Figure 1) we see that the bias correction works in all cases and the corrected bias of the MLEs tend to zero, as the sample size increases. For Figure 2, the bias correction represents some intervals in which it behaves well, especially for small sample sizes. The case of the beta coefficient is the most ideal in the sense that the bias of the MLE is stabilized in the constant term of its expression, as T increases.

When dropping the normality assumption, we run the simulations under the hypothesis of mixture of normals for standardized random variables (see Figure 3 and Figure 4). In fact, the errors are drawn from a normal distribution with mean 0.01 and variance 9, with probability 0.1, and with probability 0.9 they are drawn from a normal distribution with mean -0.001 and variance 0.111. In this way, the theoretical mean and variance of the distribution are 0 and 1, respectively. Notice that with these hyperparameter values the theoretical skewness and kurtosis of the random errors are 0.0266 and 24.334 respectively, approximately matching the sample counterparts of most financial data.

Figures 3 and 4 represent two sets of parameter values, in which we have selected different values of the beta coefficient, i.e. low (0.5) and high (0.9). Figure 1 (under normality) and Figure 4 (under mixture of normals) are constructed under the same set of parameter values and it is interesting to compare between the two cases. As in the case of normality, we see that in Figure 4 the bias correction of the estimators works in most cases and the results are satisfactory. In Figure 3, the corrected bias is again under the bias of the MLEs, indicating that the theoretical results correct the bias, under the assumptions made.

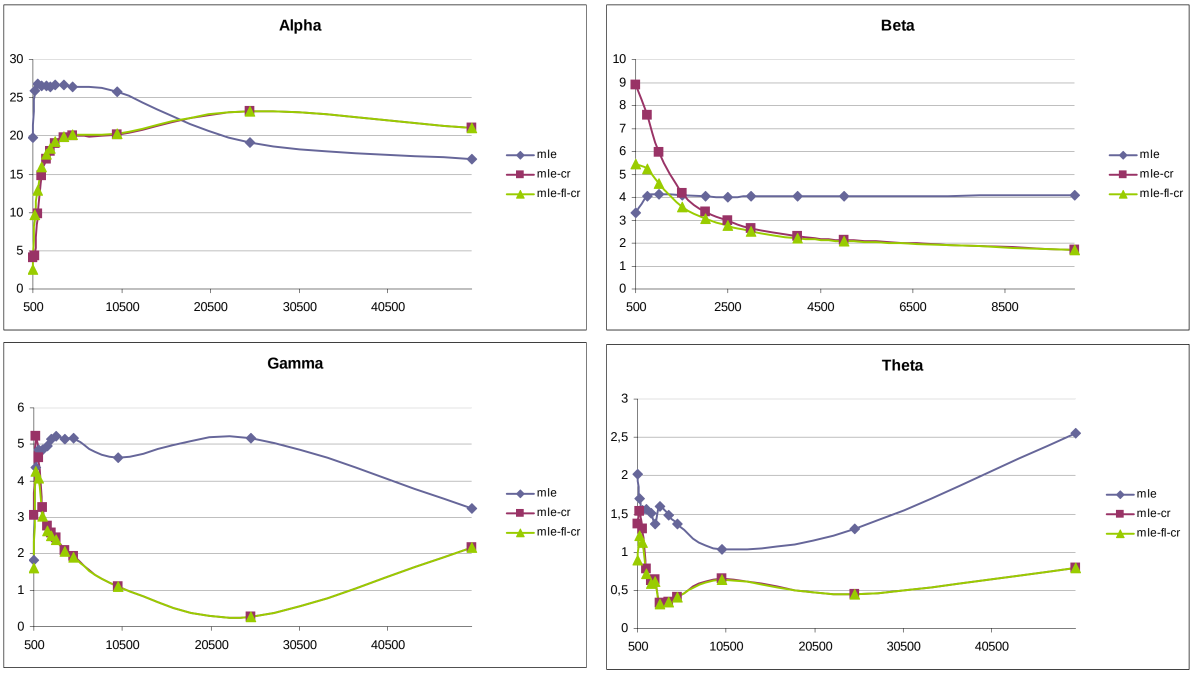
Figure 1: First- and full-step bias-correction



Note: $\alpha_0 = 0.1, \beta_0 = 0.9, \gamma_0 = 0.7, \theta_0 = -0.4,$ under normality

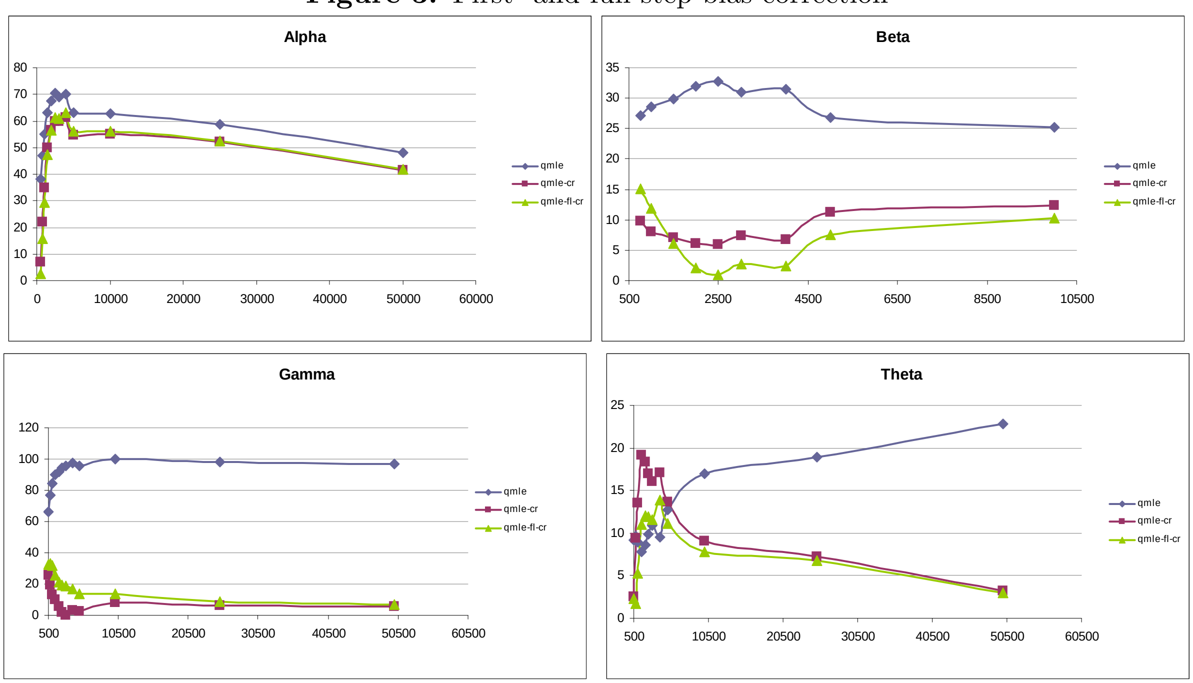
1-step correction denoted by "mle-cr", full-step correction by "mle-fl-cr" (the same applies to all graphs)

Figure 2: First- and full-step bias-correction



Note: $\alpha_0 = -0.1, \beta_0 = 0.9, \gamma_0 = 0.6, \theta_0 = -0.2,$ under normality

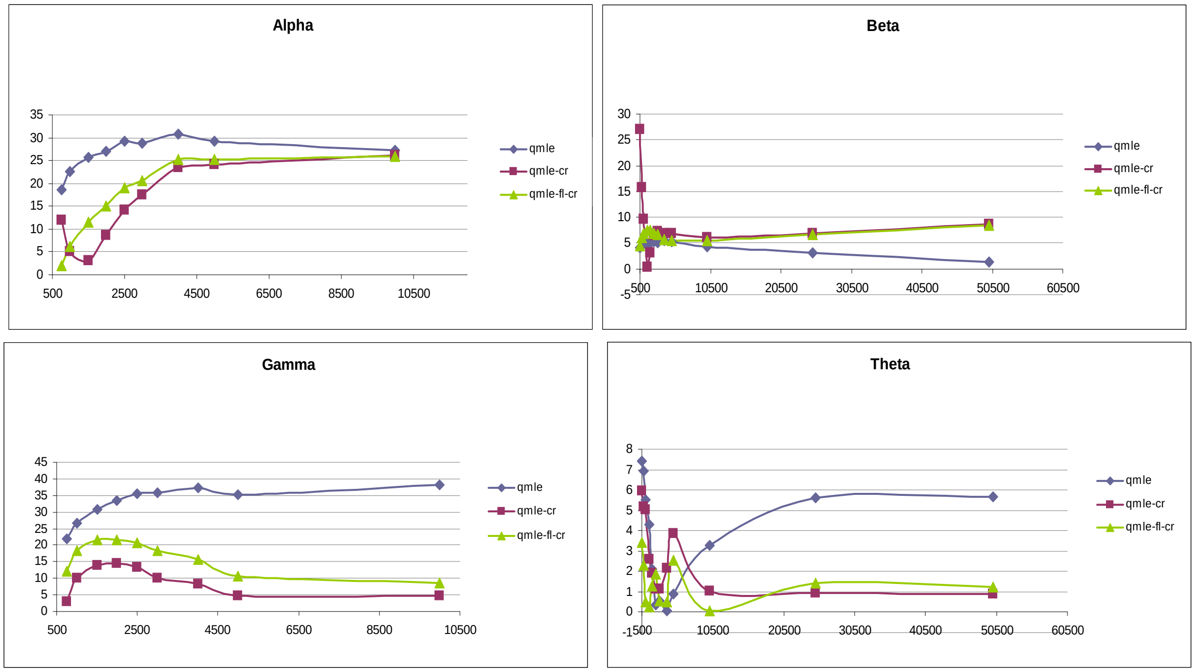
Figure 3: First- and full-step bias-correction



Note: $\alpha_0 = 0.5, \beta_0 = 0.5, \gamma_0 = 0.8, \theta_0 = -0.5,$ under mixture of normals

with $p = 0.1$: $N(0.01, 9)$: $E\left(z^3\right) = 0.0266$, $E\left(z^4\right) = 24.334$

Figure 4: First- and full-step bias-correction



Note: $\alpha_0 = 0.1, \beta_0 = 0.9, \gamma_0 = 0.7, \theta_0 = -0.4,$ under mixture of normals with $p = 0.1$: $N(0.01, 9)$: $E\left(z^3\right) = 0.0266$, $E\left(z^4\right) = 24.334$

4 Conclusions

In this paper we study the asymptotic properties of the MLEs and QMLEs in the EGARCH$(1,1)$ model of Nelson (1991). In the current context, we present analytic derivatives both of the log-likelihood and the log-variance functions and also their expected values. We further develop theoretical bias approximations for the estimators of the model parameters and we find conditions for the second-order stationarity of the log-variance derivatives. The theoretical results in this paper can be used to bias-correct the QMLEs in practice directly. In small or moderate-sized samples, a bias correction could be appreciable and it is helpful to have a rough estimate of its size.

The next steps in our research are to compute the approximate skewness of the estimators and hence the Edgeworth-type distributions. An interesting topic would be the investigation of necessary and sufficient conditions for the existence and validity of the Edgeworth approximations in this context. These issues are an ongoing research.

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5 Appendix

A. Proof of the unconditional variance

We write the variance equation as follows:

$$\ln(h_t) = \alpha^* + \theta \sum_{i=0}^{\infty} \beta^i z_{t-1-i} + \gamma \sum_{i=0}^{\infty} \beta^i \left(|z_{t-1-i}| - E\,|z_{t-1-i}| \right),$$

where $\alpha^* = \frac{\alpha}{1-\beta}$. Taking the expectation of the exponential of $\ln(h_t)$ we have:

$E\exp(\ln h_t) = \exp(\alpha^*)\, E\exp\left[\sum_{i=0}^{\infty} \left(\theta\beta^i z_{t-1-i} + \gamma\beta^i \left(|z_{t-1-i}| - E\,|z_{t-1-i}|\right)\right)\right] =$

$= \exp(\alpha^*)\, E\prod_{i=0}^{\infty} \exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i \left(|z_{t-1-i}| - E\,|z_{t-1-i}|\right)\right] =$

$= \exp(\alpha^*) \prod_{i=0}^{\infty} E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i \left(|z_{t-1-i}| - E\,|z_{t-1-i}|\right)\right]$

Now,

$\prod_{i=0}^{\infty} E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i \left(|z_{t-1-i}| - E\,|z_{t-1-i}|\right)\right] =$

$= \prod_{i=0}^{\infty} \exp\left(-\gamma E\,|z_{t-1-i}|\,\beta^i\right) E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\,|z_{t-1-i}|\right] =$

$= \exp\left(-\frac{\gamma E|z|}{1-\beta}\right) \prod_{i=0}^{\infty} E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\,|z_{t-1-i}|\right]$

$$\kappa_1 = \theta\beta^i$$

$$\kappa_2 = \gamma\beta^i$$

$E\exp\left[\theta\beta^i z + \gamma\beta^i |z|\right] \quad = \quad E\exp\left[\kappa_1 z + \kappa_2 |z|\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\kappa_1 z + \kappa_2 |z| - \frac{1}{2} z^2\right) dz =$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{1}{2}\left(-2(\kappa_1-\kappa_2)z + z^2 \pm (\kappa_1-\kappa_2)^2\right)\right) dz$

$+ \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \exp\left(-\frac{1}{2}\left(-2(\kappa_1+\kappa_2)z + z^2 \pm (\kappa_1+\kappa_2)^2\right)\right) dz =$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{1}{2}\left(-2(\kappa_1-\kappa_2)z + z^2 \pm (\kappa_1-\kappa_2)^2\right)\right) dz$

$+ \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \exp\left(-\frac{1}{2}\left(-2(\kappa_1+\kappa_2)z + z^2 \pm (\kappa_1+\kappa_2)^2\right)\right) dz =$

$= \exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{1}{2}\left(z-(\kappa_1-\kappa_2)\right)^2\right) dz$

$+ \exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{1}{2}\left(z-(\kappa_1+\kappa_2)\right)^2\right) dz =$

$= \exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-(\kappa_1-\kappa_2)} \exp\left(-\frac{1}{2}u^2\right) du + \exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-(\kappa_1+\kappa_2)} \exp\left(-\frac{1}{2}u^2\right) du =$

$= \exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right) \Phi\left(-(\kappa_1-\kappa_2)\right) + \exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right) \left(1 - \Phi\left(-(\kappa_1+\kappa_2)\right)\right) =$

$= \exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right) \Phi\left(-(\kappa_1-\kappa_2)\right) + \exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right) \Phi(\kappa_1+\kappa_2) =$

$= \exp\left(\frac{\beta^{2i}(\gamma-\theta)^2}{2}\right) \Phi\left(\beta^i(\gamma-\theta)\right) + \exp\left(\frac{\beta^{2i}(\gamma+\theta)^2}{2}\right) \Phi\left(\beta^i(\gamma+\theta)\right)$

$= \exp(\Delta)\,\Phi(-B) + \exp(\Gamma)\,\Phi(A),$

where $\Gamma = \frac{\beta^{2i}(\gamma+\theta)^2}{2}$, $\Delta = \frac{\beta^{2i}(\gamma-\theta)^2}{2}$, $A = \beta^i(\gamma+\theta)$ and $B = \beta^i(\gamma-\theta)$, and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Therefore,

$$\begin{aligned} E\exp(q\ln h_t) &= \exp\left(\frac{\alpha-\gamma E|z|}{1-\beta}\right)\prod_{i=0}^{\infty}\left(\begin{array}{c}\exp\left(\frac{\beta^{2i}(\gamma-\theta)^2}{2}\right)\Phi\left(\beta^i(\gamma-\theta)\right)\\ +\exp\left(\frac{\beta^{2i}(\gamma+\theta)^2}{2}\right)\Phi\left(\beta^i(\gamma+\theta)\right)\end{array}\right) \\ &= \exp(\Psi)\prod_{i=0}^{\infty}\left(\exp(\Delta)\Phi(-B)+\exp(\Gamma)\Phi(A)\right) \\ &= b^*, \end{aligned}$$

where $\Psi = \frac{\alpha-\gamma E|z|}{1-\beta}$.■

B. Expected values of the log-likelihood derivatives

The expected values of all first order derivatives are equal to zero.

Second order derivatives:

For $i, j \in \{\alpha, \theta, \gamma, \beta\}$,

$$\begin{aligned} E(\mathcal{L}_{ij}) &= -\frac{T}{2}E(h_{t;i}h_{t;j}), \\ E(\mathcal{L}_{\mu j}) &= -\frac{T}{2}E(h_{t;\mu}h_{t;j}), \\ E(\mathcal{L}_{\mu\mu}) &= -TE\left(\frac{1}{h_t}\right) - \frac{T}{2}E\left(h_{t;\mu}^2\right). \end{aligned}$$

Third order derivatives:

For $i \in \{\alpha, \theta, \gamma, \beta\}$,

$$E(\mathcal{L}_{iii}) = -\frac{T}{2}E\left(3h_{t;i}h_{t;i,i} - h_{t;i}^3\right),$$

for $i \in \{\alpha, \theta, \gamma, \beta\}, j \in \{\alpha, \theta, \gamma, \beta, \mu\}$,

$$E(\mathcal{L}_{iij}) = -\frac{T}{2}E\left(h_{t;j}h_{t;i,i} - h_{t;i}^2h_{t;j} + 2h_{t;i}h_{t;i,j}\right),$$

for $i, j \in \{\alpha, \theta, \gamma, \beta\}, k \in \{\alpha, \theta, \gamma, \beta, \mu\}$,

$$E(\mathcal{L}_{ijk}) = -\frac{T}{2}E\left(h_{t;j}h_{t;i,k} + h_{t;k}h_{t;i,j} + h_{t;i}h_{t;j,k} - h_{t;j}h_{t;i}h_{t;k},\right),$$

for $i \in \{\alpha, \theta, \gamma, \beta\}, j \in \{\mu\}$,

$$E(\mathcal{L}_{ijj}) = -\frac{T}{2} E\left(h_{t;i} h_{t;j,j} + 2h_{t;j} h_{t;i,j} - h_{t;i}(h_{t;j})^2\right) + TE\left(\frac{1}{h_t} h_{t;i}\right),$$

for $j \in \{\mu\}$,

$$E(\mathcal{L}_{jjj}) = -\frac{T}{2} E\left(3h_{t;j} h_{t;j,j} - h_{t;j}^3\right) + TE\left(3\frac{1}{h_t} h_{t;j}\right).$$

In this Appendix we make a list of the results that are needed for the bias approximations. Please note that the last Appendix should be studied first in order to be familiarized with the symbols used.

First, provided that $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\theta\gamma E(z|z|)\right| < 1$, the expected values of all second order derivatives are:

1. $E(\mathcal{L}_{\alpha\alpha}) = -\frac{T}{2} \frac{1+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\theta\gamma E(z|z|)\right)}$

2. $E(\mathcal{L}_{\alpha\beta}) = -\frac{T}{2} \frac{E(\ln(h_{t-1}))+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\theta\gamma E(z|z|)\right)}$

3. $E(\mathcal{L}_{\alpha\gamma}) = -\frac{T}{2} \frac{-\frac{1}{2}\left[E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

4. $E(\mathcal{L}_{\alpha\theta}) = -\frac{T}{2} \frac{-\frac{1}{2}[\theta+\gamma E(z|z|)]E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

5. $E(\mathcal{L}_{\mu\alpha}) = -\frac{T}{2} \frac{-(\theta+\gamma EI)E\left(\frac{1}{\sqrt{h}}\right)+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\mu}+[\theta(\beta-\gamma E|z|)+\gamma\beta EI]E_{-\frac{1}{2}}E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

6. $E(\mathcal{L}_{\beta\beta}) = -\frac{T}{2} \frac{E\left(\ln^2(h_{t-1})\right)+2\left(\beta-\frac{1}{2}\gamma|z_{t-1}|\right)LE_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

7. $E(\mathcal{L}_{\beta\gamma}) = -\frac{T}{2} \frac{\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\gamma}-\frac{1}{2}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

8. $E(\mathcal{L}_{\beta\theta}) = -\frac{T}{2} \frac{-\frac{1}{2}[\theta+\gamma E(z|z|)]E_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

9. $E(\mathcal{L}_{\beta\mu}) = -\frac{T}{2} \frac{-(\theta+\gamma EI)LE_{-\frac{1}{2}}+[\theta(\gamma E|z|-\beta)-\beta\gamma EI]E_{-\frac{1}{2}}E_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\mu}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

10. $E(\mathcal{L}_{\gamma\gamma}) = -\frac{T}{2} \frac{1-E^2|z|}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

11. $E(\mathcal{L}_{\theta\gamma}) = -\frac{T}{2} \frac{E(z|z|)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

12. $E(\mathcal{L}_{\mu\gamma}) = -\frac{T}{2} \frac{-\gamma EI_{g(z)}E\left(\frac{1}{\sqrt{h}}\right)-\frac{1}{2}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\mu}-(\theta(\beta-\gamma E|z|)+\gamma\beta EI)E_{-\frac{1}{2}}E_{;\gamma}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

13. $E(\mathcal{L}_{\theta\theta}) = -\frac{T}{2} \frac{1}{1-\left(\beta_2^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

14. $E\left(\mathcal{L}_{\mu\theta}\right) = -\frac{T}{2}\frac{-\frac{1}{2}(\theta+\gamma E(z|z|))E(h_{t;\mu})-\gamma E|z|E\left(\frac{1}{\sqrt{h}}\right)+[\theta(\gamma E|z|-\beta)-\beta\gamma EI]E_{-\frac{1}{2}}E_{;\theta}}{1-\left(\beta_2^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

15. $E\left(\mathcal{L}_{\mu\mu}\right) = -TE\left(\frac{1}{h_t}\right) - \frac{T}{2}\frac{\left(\theta^2+\gamma^2+2\gamma\theta EI\right)E\left(\frac{1}{\sqrt{h_{t-1}}}\right)-2(\theta(\beta-\gamma E|z|)+\gamma\beta EI)E_{-\frac{1}{2}}E_{;\mu}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$. ■

Second, the expected values of the third order derivatives are:

1. $E\left(\mathcal{L}_{\alpha\alpha\alpha}\right) = -\frac{T}{2}E\left(3\left(h_{t;\alpha}h_{t;\alpha,\alpha}\right) - h_{t;\alpha}^3\right)$

2. $E\left(\mathcal{L}_{\alpha\alpha\beta}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\alpha} - h_{t;\alpha}^2h_{t;\beta} + 2h_{t;\alpha}h_{t;\alpha,\beta}\right)$

3. $E\left(\mathcal{L}_{\alpha\alpha\gamma}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\alpha,\alpha} - h_{t;\alpha}^2h_{t;\gamma} + 2h_{t;\alpha}h_{t;\alpha,\gamma}\right)$

4. $E\left(\mathcal{L}_{\alpha\alpha\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\alpha,\alpha} - h_{t;\alpha}^2h_{t;\theta} + 2h_{t;\alpha}h_{t;\alpha,\theta}\right)$

5. $E\left(\mathcal{L}_{\mu\alpha\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha,\alpha}h_{t;\mu} + 2\left(h_{t;\alpha}h_{t;\mu,\alpha}\right) - h_{t;\alpha}^2h_{t;\mu}\right)$

6. $E\left(\mathcal{L}_{\beta\beta\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\beta,\beta} + 2h_{t;\beta}h_{t;\beta,\alpha} - h_{t;\alpha}h_{t;\beta}^2\right)$

7. $E\left(\mathcal{L}_{\alpha\beta\gamma}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\gamma} + h_{t;\gamma}h_{t;\alpha,\beta} + h_{t;\alpha}h_{t;\beta,\gamma} - h_{t;\beta}h_{t;\alpha}h_{t;\gamma}\right)$

8. $E\left(\mathcal{L}_{\alpha\beta\theta}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\theta} + h_{t;\alpha}h_{t;\beta,\theta} + h_{t;\theta}h_{t;\alpha,\beta} - h_{t;\alpha}h_{t;\beta}h_{t;\theta}\right)$

9. $E\left(\mathcal{L}_{\mu\beta\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\beta,\mu} + h_{t;\beta,\alpha}h_{t;\mu} + h_{t;\beta}h_{t;\mu,\alpha} - h_{t;\alpha}h_{t;\beta}h_{t;\mu}\right)$

10. $E\left(\mathcal{L}_{\mu\beta\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\beta,\mu} + h_{t;\beta,\alpha}h_{t;\mu} + h_{t;\beta}h_{t;\mu,\alpha} - h_{t;\alpha}h_{t;\beta}h_{t;\mu}\right)$

11. $E\left(\mathcal{L}_{\alpha\gamma\gamma}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\gamma,\gamma} + 2h_{t;\gamma}h_{t;\alpha,\gamma} - h_{t;\alpha}h_{t;\gamma}^2\right)$

12. $E\left(\mathcal{L}_{\alpha\gamma\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\alpha,\gamma} + h_{t;\gamma}h_{t;\alpha,\theta} + h_{t;\alpha}h_{t;\gamma,\theta} - h_{t;\alpha}h_{t;\gamma}h_{t;\theta}\right)$

13. $E\left(\mathcal{L}_{\alpha\gamma\mu}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\gamma,\mu} - h_{t;\alpha}h_{t;\gamma}h_{t;\mu} + h_{t;\gamma,\alpha}h_{t;\mu} + h_{t;\gamma}h_{t;\alpha,\mu}\right)$

14. $E\left(\mathcal{L}_{\alpha\theta\theta}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\theta,\theta} + 2h_{t;\theta}h_{t;\alpha,\theta} - h_{t;\alpha}h_{t;\theta}^2\right)$

15. $E\left(\mathcal{L}_{\alpha\theta\mu}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\theta,\mu} - h_{t;\alpha}h_{t;\theta}h_{t;\mu} + h_{t;\theta,\alpha}h_{t;\mu} + h_{t;\theta}h_{t;\alpha,\mu}\right)$

16. $E\left(\mathcal{L}_{\alpha\mu\mu}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\alpha,\mu} - h_{t;\alpha}\left(h_{t;\mu}\right)^2\right) + TE\left(\frac{1}{h_t}h_{t;\alpha}\right)$

17. $E\left(\mathcal{L}_{\beta\beta\beta}\right) = -\frac{T}{2}E\left(3h_{t;\beta}h_{t;\beta,\beta} - Eh_{t;\beta}^3\right)$

18. $E\left(\mathcal{L}_{\beta\beta\gamma}\right) = -\frac{T}{2}E\left(2h_{t;\beta}h_{t;\gamma,\beta} + h_{t;\gamma}h_{t;\beta,\beta} - h_{t;\beta}^2h_{t;\gamma}\right)$

19. $E\left(\mathcal{L}_{\beta\beta\theta}\right) = -\frac{T}{2}E\left(2h_{t;\beta}h_{t;\theta,\beta} + h_{t;\theta}h_{t;\beta,\beta} - h^2_{t;\beta}h_{t;\theta}\right)$

20. $E\left(\mathcal{L}_{\beta\beta\mu}\right) = -\frac{T}{2}\left(2E\left(h_{t;\beta}h_{t;\beta,\mu}\right) + E\left(h_{t;\beta;\beta}h_{t;\mu}\right) - E\left(h^2_{t;\beta}h_{t;\mu}\right)\right)$

21. $E\left(\mathcal{L}_{\beta\gamma\gamma}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\gamma,\gamma} + 2h_{t;\gamma}h_{t;\beta,\gamma} - h_{t;\beta}h^2_{t;\gamma}\right)$

22. $E\left(\mathcal{L}_{\beta\gamma\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\beta,\gamma} + h_{t;\gamma}h_{t;\beta,\theta} + h_{t;\beta}h_{t;\gamma,\theta} - h_{t;\beta}h_{t;\gamma}h_{t;\theta}\right)$

23. $E\left(\mathcal{L}_{\beta\gamma\mu}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\gamma,\mu} - h_{t;\beta}h_{t;\gamma}h_{t;\mu} + h_{t;\gamma,\beta}h_{t;\mu} + h_{t;\gamma}h_{t;\beta,\mu}\right)$

24. $E\left(\mathcal{L}_{\beta\theta\theta}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\theta,\theta} + 2h_{t;\theta}h_{t;\beta,\theta} - h_{t;\beta}h^2_{t;\theta}\right)$

25. $E\left(\mathcal{L}_{\beta\theta\mu}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\theta,\mu} - h_{t;\beta}h_{t;\theta}h_{t;\mu} + h_{t;\theta,\beta}h_{t;\mu} + h_{t;\theta}h_{t;\beta,\mu}\right)$

26. $E\left(\mathcal{L}_{\mu\mu_\beta}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\mu,\beta} - h_{t;\beta}\left(h_{t;\mu}\right)^2\right) + TE\left(\frac{1}{h_t}h_{t;\beta}\right)$

27. $E\left(\mathcal{L}_{\gamma\gamma\gamma}\right) = -\frac{T}{2}E\left(3\left(h_{t;\gamma}h_{t;\gamma,\gamma}\right) - \left(h^3_{t;\gamma}\right)\right)$

28. $E\left(\mathcal{L}_{\gamma\gamma\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\gamma,\gamma} + 2h_{t;\gamma}h_{t;\gamma,\theta} - h^2_{t;\gamma}h_{t;\theta}\right)$

29. $E\left(\mathcal{L}_{\gamma\gamma\mu}\right) = -\frac{T}{2}E\left(2h_{t;\gamma}h_{t;\gamma,\mu} - h^2_{t;\gamma}h_{t;\mu} + h_{t;\gamma,\gamma}h_{t;\mu}\right)$

30. $E\left(\mathcal{L}_{\beta\theta\theta}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\theta,\theta} + 2h_{t;\theta}h_{t;\gamma,\theta} - h_{t;\gamma}h^2_{t;\theta}\right)$

31. $E\left(\mathcal{L}_{\gamma\theta\mu}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\theta,\mu} - h_{t;\gamma}h_{t;\theta}h_{t;\mu} + h_{t;\theta,\gamma}h_{t;\mu} + h_{t;\theta}h_{t;\gamma,\mu}\right)$

32. $E\left(\mathcal{L}_{\gamma\mu\mu}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\gamma,\mu} - h_{t;\gamma}h^2_{t;\mu}\right) + TE\left(\frac{1}{h_t}h_{t;\gamma}\right)$

33. $E\left(\mathcal{L}_{\theta\theta\mu}\right) = -\frac{T}{2}E\left(2h_{t;\theta}h_{t;\theta,\mu} - h^2_{t;\theta}h_{t;\mu} + h_{t;\theta,\theta}h_{t;\mu}\right)$

34. $E\left(\mathcal{L}_{\theta\mu\mu}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\theta,\mu} - h_{t;\theta}h^2_{t;\mu}\right) + TE\left(\frac{1}{h_t}h_{t;\theta}\right)$

35. $E\left(\mathcal{L}_{\mu\mu\mu}\right) = -\frac{T}{2}E\left(3h_{t;\mu}h_{t;\mu,\mu} - h^3_{t;\mu}\right) + TE\left(3\frac{1}{h_t}h_{t;\mu}\right)$. ■

C. Expected values of the log-variance derivatives

In the current Appendix, we present some of the results for the expected values of the log-variance derivatives and more specifically those that are needed for the evaluation of some of the expected values of the third order log-likelihood derivatives of the previous Appendix, that is:

Assuming first $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1$ and $\left|\beta^3 + \frac{3}{4}\beta\theta^2 - \frac{1}{8}\theta^3 Ez^3 - \frac{3}{2}\gamma\left(\beta^2 E\left|z\right| - \beta\theta E\left(z\left|z\right|\right) + \frac{1}{4}\theta^2 E\left|z\right|^3\right) + \frac{3}{4}\gamma^2\left(\beta - \frac{1}{2}\theta Ez^3\right) - \frac{1}{8}\gamma^3 E\left|z\right|^3\right| < 1$, we have:

1. $E\left(h_{t;\alpha}h_{t;\alpha,\alpha}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\alpha)^2}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^3}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha,\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

2. $E\left(h_{t;\alpha}^3\right)=\frac{1+3\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha}+3\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)E_{(;\alpha)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3 Ez^3-\frac{3}{2}\gamma\left(\beta^2 E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2 E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3 E|z|^3\right]}$

3. $E\left(h_{t;\beta}h_{t;\alpha,\alpha}\right)=\frac{\frac{1}{4}\gamma E|z|LE_{(;\alpha)^2}+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)E_{(;\alpha)^2;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

4. $E\left(h_{t;\alpha}h_{t;\alpha,\beta}\right)=\frac{E_{;\alpha}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{(;\alpha)^2}+\frac{1}{4}\gamma E|z|E_{;\alpha;\beta}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha,\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

5. $E\left(h_{t;\gamma}h_{t;\alpha,\alpha}\right)=\frac{\frac{1}{4}[\theta E(z|z|)+\gamma(1-E|z|)]E_{(;\alpha)^2}-\frac{1}{2}[\theta E(z|z|)+\gamma(1-E|z|)]E_{;\alpha,\alpha}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\gamma}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

6. $E\left(h_{t;\theta}h_{t;\alpha,\alpha}\right)=\frac{\frac{1}{4}[\theta+\gamma E(z|z|)]E_{(;\alpha)^2}-\frac{1}{2}[\theta+\gamma E(z|z|)]E_{;\alpha,\alpha}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

7. $E\left(h_{t;\alpha}^2 h_{t;\theta}\right)=\frac{-[\theta+\gamma E(z|z|)]E_{;\alpha}+\left(\frac{1}{2}\gamma\theta E|z|^3-\theta\beta+\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3-\beta\gamma E(z|z|)\right)E_{(;\alpha)^2}+2\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)E_{;\alpha;\theta}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3 Ez^3-\frac{3}{2}\gamma\left(\beta^2 E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2 E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3 E|z|^3\right]}$

8. $E\left(h_{t;\alpha}h_{t;\alpha,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{;\alpha;\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha,\theta}+\frac{1}{4}[\theta+\gamma E(z|z|)]E_{(;\alpha)^2}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

9. $E\left(h_{t;\alpha}h_{t;\beta,\beta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\beta)^2}+2E_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta,\beta}+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\beta}+\left(\frac{1}{4}\beta\gamma|z_{t-1}|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{;\alpha(;\beta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

10. $E\left(h_{t;\beta}h_{t;\alpha,\beta}\right)=\frac{LE_{;\alpha}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\beta}+\frac{1}{4}\gamma E|z|LE_{;\alpha;\beta}+\left(\frac{1}{4}\beta\gamma|z_{t-1}|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{;\alpha(;\beta)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

11. $E\left(h_{t;\beta}h_{t;\alpha,\gamma}\right)=\frac{-\frac{1}{2}E|z|LE_{;\alpha}+\frac{1}{4}\gamma E|z|LE_{;\alpha;\gamma}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\gamma}-\frac{1}{2}\left(\beta E|z|-\frac{1}{2}\theta E(z|z|)-\frac{1}{2}\gamma\right)E_{;\alpha}E_{;\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\theta\gamma E(z|z|)\right)E_{;\alpha;\beta;\gamma}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

12. $E\left(h_{t;\beta}h_{t;\alpha,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|LE_{;\alpha;\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

13. $E\left(h_{t;\alpha}h_{t;\beta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{;\beta;\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta,\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

14. $E\left(h_{t;\theta}h_{t;\alpha,\beta}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\beta}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\alpha,\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\theta}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\theta^2-\frac{1}{8}\gamma^2-\frac{1}{4}\gamma\theta E(z|z|)\right)E_{;\alpha;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

15. $E\left(h_{t;\alpha}h_{t;\gamma,\gamma}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\gamma)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\gamma,\gamma}-\left(\beta E|z|-\frac{1}{2}\gamma-\frac{1}{2}\theta E(z|z|)\right)E_{;\alpha;\gamma}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha(;\gamma)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

16. $E\left(h_{t;\alpha}h_{t;\gamma,\theta}\right)=\frac{-\frac{1}{2}\left(\beta E|z|-\frac{1}{2}\theta E(z|z|)-\frac{1}{2}\gamma\right)E_{;\alpha;\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\gamma}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha;\gamma;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

17. $E\left(h_{t;\alpha}h_{t;\theta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\theta)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\theta,\theta}+\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\alpha;\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

18. $E\left(h_{t;\theta}h_{t;\alpha,\theta}\right)=\frac{-\frac{1}{2}E_{;\alpha}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\theta}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\alpha,\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

19. $E\left(h_{t;\beta}h_{t;\beta,\beta}\right)=\frac{\frac{1}{4}\gamma E|z|LE_{(;\beta)^2}+2LE_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\beta,\beta}+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{(;\beta)^2}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\beta)^3}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

20. $E\left(h_{t;\beta}^3\right)=\frac{L^3+3\left(\beta-\frac{1}{2}\gamma E|z|\right)L^2 E_{;\beta}+3\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)LE_{(;\beta)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3 Ez^3-\frac{3}{2}\gamma\left(\beta^2 E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2 E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3 E|z|^3\right]}$

21. $E\left(h_{t;\theta}h_{t;\beta,\beta}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{(;\beta)^2}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\beta,\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\beta)^2;\theta}+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

22. $E\left(h_{t;\beta}h_{t;\theta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|LE_{(;\theta)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\theta,\theta}+\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\beta;\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\beta(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

23. $E\left(h_{t;\gamma}^3\right)=\frac{\left(E|z|^3-3E|z|+2E^3|z|\right)+3\left[\left(\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2\right)\left(E|z|^3-E|z|\right)-\gamma\beta\left(1-E^2|z|\right)-\beta\theta E(z|z|)+\frac{1}{2}\gamma\theta\left(Ez^3-E|z|E(z|z|)\right)\right]E_{(;\gamma)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}$

24. $E\left(h_{t;\theta}h_{t;\gamma,\gamma}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{(;\gamma)^2}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\gamma,\gamma}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\gamma)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

25. $E\left(h_{t;\gamma}h_{t;\gamma,\theta}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{(;\gamma)^2}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\gamma)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

26. $E\left(h_{t;\gamma}^2h_{t;\theta}\right)=\frac{Ez^3-2E|z|E(z|z|)+\left[\theta\left(\frac{1}{2}\gamma E|z|^3-\beta\right)+\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3-\beta\gamma E(z|z|)\right]E_{(;\gamma)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}$

27. $E\left(h_{t;\gamma}h_{t;\theta,\theta}\right)=\frac{\frac{1}{4}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{(;\theta)^2}-\frac{1}{2}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\theta,\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\gamma(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

28. $E\left(h_{t;\theta}h_{t;\gamma,\theta}\right)=\frac{-\frac{1}{2}\left(\beta E|z|-\frac{1}{2}\theta E(z|z|)-\frac{1}{2}\gamma\right)E_{(;\theta)^2}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\gamma(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

29. $E\left(h_{t;\gamma}h_{t;\theta}^2\right)=\frac{E|z|^3-E|z|+\left[\left(\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2\right)\left(E|z|^3-E|z|\right)-\gamma\beta\left(1-E^2|z|\right)-\beta\theta E(z|z|)+\frac{1}{2}\gamma\theta\left(Ez^3-E|z|E(z|z|)\right)\right]E_{(;\theta)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}$. ■

The whole results are available on demand from the corresponding author.

D. Expected values of cross products of the log-likelihood derivatives

In this Appendix, we present the expected values of cross-products of the log-likelihood derivatives. To conserve space, we present only some indicative. That is,

1. $\frac{1}{T}E\left(\mathcal{L}_\alpha\mathcal{L}_{\alpha\alpha}\right)=-\frac{1}{4}\left[\sum_{s<t}\sum E\left[\left(z_s^2-1\right)h_{s;\alpha}h_{t;\alpha}h_{t;\alpha}\right]-(\kappa_4+2)\sum_{t=1}^{T}E\left(h_{t;\alpha}h_{t;\alpha,\alpha}-h_{t;\alpha}^3\right)\right]$

2. $\frac{1}{T}E\left(\mathcal{L}_\alpha\mathcal{L}_{\alpha\mu}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum_{s<t}\sum E\left[\left(z_s^2-1\right)h_{s;\mu}h_{t;\alpha}h_{t;\mu}\right]-(\kappa_4+2)\sum_{t=1}^{T}E\left(h_{t;\alpha}h_{t;\alpha,\mu}-h_{t;\mu}h_{t;\alpha}^2\right)\\+2\kappa_3\sum_{t=1}^{T}E\left(\frac{1}{\sqrt{h_t}}h_{t;\alpha}^2\right)\end{array}\right]$

3. $\frac{1}{T}E\left(\mathcal{L}_\alpha\mathcal{L}_{\mu\mu}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum_{s<t}\sum E\left[\left(z_s^2-1\right)h_{s;\alpha}h_{t;\mu}^2\right]-(\kappa_4+2)\sum_{t=1}^{T}E\left(h_{t;\alpha}h_{t;\mu,\mu}-h_{t;\alpha}h_{t;\mu}^2\right)\\+4\kappa_3\sum_{t=1}^{T}E\left(\frac{1}{\sqrt{h_t}}h_{t;\alpha}h_{t;\mu}\right)\end{array}\right]$

4. $\frac{1}{T}E\left(\mathcal{L}_\mu\mathcal{L}_{\alpha\alpha}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum_{s<t}\sum E\left[\left(z_s^2-1\right)h_{s;\mu}h_{t;\alpha}^2\right]-(\kappa_4+2)\sum_{t=1}^{T}E\left(h_{t;\mu}h_{t;\alpha,\alpha}-h_{t;\mu}h_{t;\alpha}^2\right)\\+2\sum_{s<t}\sum E\left(z_s\frac{1}{\sqrt{h_t}}h_{t;\alpha}^2\right)+2\kappa_3\sum_{t=1}^{T}E\left(\frac{1}{\sqrt{h_t}}h_{t;\alpha}^2-\frac{1}{\sqrt{h_t}}h_{t;\alpha,\alpha}\right)\end{array}\right]$

5. $\frac{1}{T}E\left(\mathcal{L}_\mu\mathcal{L}_{\alpha\mu}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum\sum_{s<t}E\left[\left(z_s^2-1\right)h_{s;\mu}h_{t;\alpha}h_{t;\mu}\right]-\left(\kappa_4+2\right)\sum_{t=1}^T E\left(h_{t;\mu}h_{t;\alpha,\mu}-h_{t;\alpha}h_{t;\mu}^2\right)\\ +2\sum\sum_{s<t}E\left(z_s\frac{1}{\sqrt{h_t}}h_{t;\alpha}h_{t;\mu}\right)+2\kappa_3\sum_{t=1}^T E\left(\frac{2}{\sqrt{h_t}}h_{t;\alpha}h_{t;\mu}-\frac{1}{\sqrt{h_t}}h_{t;\alpha,\mu}\right)\\ +4\sum_{t=1}^T E\left(\frac{1}{h_t}h_{t;\alpha}\right)\end{array}\right]$

6. $\frac{1}{T}E\left(\mathcal{L}_\mu\mathcal{L}_{\mu\mu}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum\sum_{s<t}E\left[\left(z_s^2-1\right)h_{s;\mu}h_{t;\mu}^2\right]-\left(\kappa_4+2\right)\sum_{t=1}^T E\left(h_{t;\mu}h_{t;\mu,\mu}-h_{t;\mu}^3\right)\\ +2\sum\sum_{s<t}E\left(z_s\frac{1}{\sqrt{h_t}}h_{t;\mu}^2\right)+2\kappa_3\sum_{t=1}^T E\left(\frac{3}{\sqrt{h_t}}h_{t;\mu}^2-\frac{1}{\sqrt{h_t}}h_{t;\mu,\mu}\right)\\ +8\sum_{t=1}^T E\left(\frac{1}{h_t}h_{t;\mu}\right)\end{array}\right]$

At this point, we should note that these results differ from those in the paper of Linton [15], due to the fact that we assume non-symmetric distribution of the errors and also none of these expressions are zero, since the block-diagonality of the information matrix in our case that we study the EGARCH$(1,1)$ model does not hold.

Analytic proof of the first result is given as follows:

$$\begin{aligned}
\mathcal{L}_\alpha\mathcal{L}_{\alpha\alpha} &= \frac{1}{2}\sum_{t=1}^T\left(z_t^2-1\right)h_{t;\alpha}\left(\frac{1}{2}\sum_{t=1}^T\left(z_t^2-1\right)h_{t;\alpha,\alpha}-\frac{1}{2}\sum_{t=1}^T z_t^2\left(h_{t;\alpha}\right)^2\right)\\
&= \frac{1}{4}\sum_{t=1}^T\left(z_t^2-1\right)h_{t;\alpha}\sum_{t=1}^T\left(z_t^2-1\right)h_{t;\alpha,\alpha}-\frac{1}{4}\sum_{t=1}^T\left(z_t^2-1\right)h_{t;\alpha}\sum_{t=1}^T z_t^2h_{t;\alpha}^2\\
&= \frac{1}{4}\sum_{t=1}^T\left(z_t^2-1\right)^2h_{t;\alpha}h_{t;\alpha,\alpha}+\frac{1}{4}\sum_{t=1}^T\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=t+1}^T\left(z_s^2-1\right)h_{s;\alpha,\alpha}\\
&\quad+\frac{1}{4}\sum_{t=1}^T\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=1}^{t-1}\left(z_s^2-1\right)h_{s;\alpha,\alpha}\\
&\quad-\frac{1}{4}\sum_{t=1}^T\left(z_t^2-1\right)z_t^2h_{t;\alpha}^3-\frac{1}{4}\sum_{t=1}^T\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=t+1}^T z_s^2h_{s;\alpha}^2-\frac{1}{4}\sum_{t=1}^T\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=1}^{t-1} z_s^2h_{s;\alpha}^2
\end{aligned}$$

Hence

$$E\left(\mathcal{L}_\alpha\mathcal{L}_{\alpha\alpha}\right)=\frac{T\left(\kappa_4+2\right)}{4}\left[E\left(h_{t;\alpha}h_{t;\alpha,\alpha}\right)-E\left(h_{t;\alpha}^3\right)\right]-\frac{1}{4}E\sum\sum_{s<t}\left(z_s^2-1\right)z_t^2h_{t;\alpha}^2h_{s;\alpha},$$

where

$h_{t;\alpha}=1+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}$ and $h_{t;\alpha}^2=1+2\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2h_{t-1;\alpha}^2$.

Let

$h_{t+k;\alpha} = 1+\left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right) h_{t+k-1;\alpha}$ and $h^2_{t+k;\alpha} = 1+2\left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right) h_{t+k-1;\alpha}+ \left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right)^2 h^2_{t+k-1;\alpha}$.

Hence,

$E\left[\left(z_t^2-1\right) h^2_{t+k;\alpha} h_{t;\alpha}\right] =$

$= E\left[\left(z_t^2-1\right)\left[h_{t;\alpha} + 2\left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right) h_{t+k-1;\alpha}h_{t;\alpha} + \left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right)^2 h^2_{t+k-1;\alpha}h_{t;\alpha}\right]\right] =$

$\overset{k\geq 1}{=} 2\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) E\left(z_t^2-1\right) h_{t+k-1;\alpha}h_{t;\alpha} + \left[\beta^2 + \frac{1}{4}\left(\theta^2+\gamma^2\right) - \beta\gamma E\left|z\right| + \frac{1}{2}\theta\gamma E\left(z\left|z\right|\right)\right] E\left(z_t^2-1\right) h^2_{t+k-1;\alpha}h_{t;\alpha}$.

$k=1: E\left[\left(z_t^2-1\right) h^2_{t+1;\alpha}h_{t;\alpha}\right] = E\left[\left(z_t^2-1\right)\left[h_{t;\alpha} + 2\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma\left|z_t\right|\right) h^2_{t;\alpha} + \left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma\left|z_t\right|\right)^2 h^3_{t;\alpha}\right]\right] =$

$= 2E\left[\left(z_t^2-1\right)\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma\left|z_t\right|\right)\right] Eh^2_{t;\alpha} + E\left[\left(z_t^2-1\right)\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma\left|z_t\right|\right)^2\right] Eh^3_{t;\alpha}$.

Hence,

$$E\left[\left(z_t^2-1\right) h^2_{t+k;\alpha}h_{t;\alpha}\right] \overset{k\geq 1}{=} -\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)\left[\theta E z^3 + \gamma\left(E\left|z\right|^3 - E\left|z\right|\right)\right]\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-2} Eh^2_{t;\alpha}$$
$$+\left[\beta^2 + \frac{1}{4}\left(\theta^2+\gamma^2\right) - \beta\gamma E\left|z\right| + \frac{1}{2}\theta\gamma E\left(z\left|z\right|\right)\right] E\left(z_t^2-1\right) h^2_{t+k-1;\alpha}h_{t;\alpha}.$$

Set: $A = -\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)\left[\theta E z^3 + \gamma\left(E\left|z\right|^3 - E\left|z\right|\right)\right] Eh^2_{t;\alpha}$ and $C = \beta^2 + \frac{1}{4}\left(\theta^2+\gamma^2\right) - \beta\gamma E\left|z\right| + \frac{1}{2}\theta\gamma E\left(z\left|z\right|\right)$.

We have that: $E\left[\left(z_t^2-1\right) h^2_{t+k;\alpha}h_{t;\alpha}\right] \overset{k\geq 1}{=} A\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-2} + CE\left(z_t^2-1\right) h^2_{t+k-1;\alpha}h_{t;\alpha}$.

By repeating substitution, $E\left[\left(z_t^2-1\right) h^2_{t+k;\alpha}h_{t;\alpha}\right] \overset{k\geq 1}{=} A\left[\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-2} + C\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-3} + ... + C^{k-2}\right]$
$+C^{k-1}E\left(z_t^2-1\right) h^2_{t+1;\alpha}h_{t;\alpha}$.

This formula can be written as:

$E\left[\left(z_t^2-1\right) h^2_{t+k;\alpha}h_{t;\alpha}\right] \overset{k\geq 1}{=} A\frac{C^{k-1} - \left(\beta - \frac{1}{2}\gamma E|z|\right)^{k-1}}{C - \left(\beta - \frac{1}{2}\gamma E|z|\right)} + C^{k-1}E\left(z_t^2-1\right) h^2_{t+1;\alpha}h_{t;\alpha}$.

Consequently,

$E\left[\left(z_t^2-1\right) h^2_{t+k;\alpha}h_{t;\alpha}\right] \overset{k\geq 1}{=} A\frac{C^{k-1} - \left(\beta - \frac{1}{2}\gamma E|z|\right)^{k-1}}{C - \left(\beta - \frac{1}{2}\gamma E|z|\right)}$
$+C^{k-1}\left[2E\left[\left(z_t^2-1\right)\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma\left|z_t\right|\right)\right] Eh^2_{t;\alpha} + E\left[\left(z_t^2-1\right)\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma\left|z_t\right|\right)^2\right] Eh^3_{t;\alpha}\right]$,

where

$2E\left[\left(z_t^2-1\right)\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma\left|z_t\right|\right)\right] Eh^2_{t;\alpha} + E\left[\left(z_t^2-1\right)\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma\left|z_t\right|\right)^2\right] Eh^3_{t;\alpha} =$

$$= -\left(\theta E z^3 + \gamma\left(E\left(\left|z\right|^3\right) - E\left|z\right|\right)\right) Eh^2_{t;\alpha} + \begin{bmatrix} \frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right) - \beta\theta E z^3 \\ +\beta\gamma\left(E\left|z\right| - E\left|z\right|^3\right) + \frac{1}{2}\theta\gamma\left(E\left(z^3\left|z\right|\right) - E\left(z\left|z\right|\right)\right) \end{bmatrix} Eh^3_{t;\alpha}$$

Hence we have $E\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(z_s^2-1\right) z_t^2 h^2_{t;\alpha}h_{s;\alpha} = E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right) h^2_{t+k;\alpha}h_{t;\alpha} =$

$= \sum_{t=1}^{T-1}\sum_{k=1}^{T-t} A\frac{C^{k-1} - \left(\beta - \frac{1}{2}\gamma E|z|\right)^{k-1}}{C - \left(\beta - \frac{1}{2}\gamma E|z|\right)} + C^{k-1}\Lambda$

where $\Lambda = -\left(\theta E z^3 + \gamma\left(E\left(\left|z\right|^3\right) - E\left|z\right|\right)\right) Eh^2_{t;\alpha} + \begin{bmatrix} \frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right) - \beta\theta E z^3 \\ +\beta\gamma\left(E\left|z\right| - E\left|z\right|^3\right) + \frac{1}{2}\theta\gamma\left(E\left(z^3\left|z\right|\right) - E\left(z\left|z\right|\right)\right) \end{bmatrix} Eh^3_{t;\alpha}$.

Hence, $E\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(z_s^2-1\right)z_t^2h_{t;\alpha}^2h_{s;\alpha}=\left(\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E|z|\right)}+\Lambda\right)\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}C^{k-1}$
$-\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)^{k-1}=$
...(keeping only terms of order T)= $\left(\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E|z|\right)}+\Lambda\right)\frac{T}{1-C}-\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\frac{T}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$, provided that $|C|<1$ and $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$. Hence

$$
\begin{aligned}
E\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(z_s^2-1\right)z_t^2h_{t;\alpha}^2h_{s;\alpha} &= T\left(\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\frac{1}{1-C}+\Lambda\frac{1}{1-C}-\frac{A}{C-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\frac{1}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\right)+O\left(\right.\\
&= T\frac{A}{\left(1-C\right)\left(1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right)}+T\frac{\Lambda}{1-C}+O\left(1\right)
\end{aligned}
$$

where $A=-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\left[\theta Ez^3+\gamma\left(E\left|z\right|^3-E\left|z\right|\right)\right]Eh_{t;\alpha}^2$ and $C=\beta^2+\frac{1}{4}\left(\theta^2+\gamma^2\right)-\beta\gamma E\left|z\right|+\frac{1}{2}\theta\gamma E\left(z\left|z\right|\right)$, $\Lambda=-\left(\theta Ez^3+\gamma\left(E\left(\left|z\right|^3\right)-E\left|z\right|\right)\right)Eh_{t;\alpha}^2$

$$
+\left[\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta Ez^3\\ +\beta\gamma\left(E\left|z\right|-E\left|z\right|^3\right)+\frac{1}{2}\theta\gamma\left(E\left(z^3\left|z\right|\right)-E\left(z\left|z\right|\right)\right)\end{array}\right]Eh_{t;\alpha}^3.\ \blacksquare
$$

E. Proof of the Theorem

The proof comes immediately from the results of *Appendix B* and *Appendix D*. ■

F. The log-variance derivatives

In this Appendix we present the expressions of the log-variance derivatives, in a form useful to explore their properties.

$$h_{t;\alpha}=1+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}$$

$$h_{t;\alpha,\alpha}=\left(\frac{1}{4}\theta z_{t-1}+\frac{1}{4}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^2+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\alpha}$$

$$h_{t;\alpha,\beta}=h_{t-1;\alpha}+\left(\frac{1}{4}\theta z_{t-1}+\frac{1}{4}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\beta}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\beta}$$

$$h_{t;\alpha,\gamma}=-\frac{1}{2}\left|z_{t-1}\right|h_{t-1;\alpha}+\left(\frac{1}{4}\theta z_{t-1}+\frac{1}{4}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\gamma}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\gamma}$$

$$h_{t;\alpha,\theta}=-\frac{1}{2}z_{t-1}h_{t-1;\alpha}+\left(\frac{1}{4}\theta z_{t-1}+\frac{1}{4}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\theta}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\theta}$$

$$h_{t;\alpha,\mu} = \frac{1}{2}\left(\theta+\gamma\left[I\left(z_{t-1\geq 0}\right)-I\left(z_{t-1<0}\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\alpha}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\mu} + \left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\mu}$$

$$h_{t;\beta} = \ln\left(h_{t-1}\right)+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}$$

$$h_{t;\beta,\beta} = \left(\frac{1}{4}\theta z_{t-1}+\frac{1}{4}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}^{2}+2h_{t-1;\beta}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta,\beta}$$

$$h_{t;\beta,\gamma} = h_{t-1;\gamma}-\frac{1}{2}\left|z_{t-1}\right|h_{t-1;\beta}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}h_{t-1;\gamma}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta,\gamma}$$

$$h_{t;\beta,\theta} = h_{t-1;\theta}-\frac{1}{2}z_{t-1}h_{t-1;\beta}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}h_{t-1;\theta}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta,\theta}$$

$$h_{t;\beta,\mu} = h_{t-1;\mu}+\frac{1}{2}\left(\theta+\gamma\left[I\left(z_{t-1\geq 0}\right)-I\left(z_{t-1<0}\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\beta}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}h_{t-1;\mu} + \left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta,\mu}$$

$$h_{t;\gamma} = g\left(z_{t-1}\right)+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\gamma}$$

$$h_{t;\gamma,\gamma} = -\left|z_{t-1}\right|h_{t-1;\gamma}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\gamma}^{2}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\gamma,\gamma}$$

$$h_{t;\gamma,\theta} = -\frac{1}{2}\left|z_{t-1}\right|h_{t-1;\theta}-\frac{1}{2}z_{t-1}h_{t-1;\gamma}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\gamma}h_{t-1;\theta}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\gamma,\theta}$$

$$h_{t;\gamma,\mu} = -\left[I\left(z_{t-1\geq 0}\right)-I\left(z_{t-1<0}\right)\right]\frac{1}{\sqrt{h_{t-1}}}+\frac{1}{2}\left(\theta+\gamma\left[I\left(z_{t-1\geq 0}\right)-I\left(z_{t-1<0}\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\gamma} -\frac{1}{2}\left|z_{t-1}\right|h_{t-1;\mu}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\gamma}h_{t-1;\mu}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\gamma,\mu}$$

$$h_{t;\theta} = z_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta}$$

$$h_{t;\theta,\theta} = -z_{t-1}h_{t-1;\theta}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta}^{2}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta,\theta}$$

$$h_{t;\theta,\mu} = -\frac{1}{\sqrt{h_{t-1}}} + \frac{1}{2}\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\theta} - \frac{1}{2}z_{t-1}h_{t-1;\mu}$$
$$+\frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\theta}h_{t-1;\mu} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta,\mu}$$

$$h_{t;\mu} = -\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\mu}$$

$$h_{t;\mu,\mu} = \left(\theta + \gamma\left[I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\mu} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\mu}^2$$
$$+\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\mu,\mu}$$

G. Expected values of the first & second order log-variance derivatives

We assume $\left|\beta - \frac{1}{2}\gamma E\left|z\right|\right| < 1$.

First order derivatives:

1. $E\left(h_{t;\alpha}\right) = \frac{1}{1-\beta+\frac{1}{2}\gamma E(|z|)}$
2. $E\left(h_{t;\beta}\right) = \frac{\alpha}{\left(1-\beta+\frac{\gamma}{2}E|z|\right)(1-\beta)}$
3. $E\left(h_{t;\gamma}\right) = 0$
4. $E\left(h_{t;\theta}\right) = 0$
5. $E\left(h_{t;\mu}\right) = -\frac{\theta E_{-\frac{1}{2}}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$. ■

Second order derivatives:

1. $E\left(h_{t;\alpha,\alpha}\right) = \frac{\frac{1}{4}\gamma E|z|E_{(;\alpha)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
2. $E\left(h_{t;\alpha,\beta}\right) = \frac{E_{;\alpha}+\frac{1}{4}\gamma E|z|E_{;\alpha;\beta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
3. $E\left(h_{t;\alpha,\gamma}\right) = \frac{-\frac{1}{2}E|z|E_{;\alpha}+\frac{1}{4}\gamma E|z|E_{;\alpha;\gamma}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
4. $E\left(h_{t;\alpha,\theta}\right) = \frac{\frac{1}{4}\gamma E|z|E_{;\alpha;\theta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
5. $E\left(h_{t;\alpha,\mu}\right) = \frac{\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\alpha}+\frac{1}{4}\gamma E|z|E_{;\alpha;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

6. $E\left(h_{t;\beta,\beta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\beta)^2}+2E_{;\beta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

7. $E\left(h_{t;\beta,\gamma}\right)=\frac{-\frac{1}{2}E|z|E_{;\beta}+\frac{1}{4}\gamma E|z|E_{;\beta;\gamma}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

8. $E\left(h_{t;\beta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{;\beta;\theta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

9. $E\left(h_{t;\beta,\mu}\right)=\frac{E_{;\mu}+\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\beta}+\frac{1}{4}\gamma E|z|E_{;\beta;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

10. $E\left(h_{t;\gamma,\gamma}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\gamma)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

11. $E\left(h_{t;\gamma,\theta}\right)=0$

12. $E\left(h_{t;\gamma,\mu}\right)=\frac{-EIE_{-\frac{1}{2}}+\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\gamma}-\frac{1}{2}E|z|E_{;\mu}+\frac{1}{4}\gamma E|z|E_{;\gamma;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

13. $E\left(h_{t;\theta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\theta)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

14. $E\left(h_{t;\theta,\mu}\right)=\frac{-E_{-\frac{1}{2}}+\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\theta}+\frac{1}{4}\gamma E|z|E_{;\theta;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

15. $E\left(h_{t;\mu,\mu}\right)=\frac{(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\mu}+\frac{1}{4}\gamma E|z|E_{(;\mu)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$. ■

H. Symbols

The next symbols are used in the paper and more specifically in the expressions of the expected values of all the derivatives.

$$E\left(\ln^2\left(h_t\right)\right)=L^2 \quad E\left(\ln\left(h_t\right)\right)^3=L^3 \quad etc.$$

$$E\left(h_{t;\beta}\right)=E_{;\beta} \quad E\left(h_{t;\alpha}\right)=E_{;\alpha} \quad E\left(h_{t;\beta}\right)^2=E_{(;\beta)^2} \quad E\left(h_{t;\mu}\right)^3=E_{(;\mu)^3} \quad etc.$$

$$\begin{aligned} E\left(\ln\left(h_t\right)h_{t;\mu}\right) &= LE_{;\mu} \quad E\left(\ln\left(h_t\right)h_{t;\alpha}\right)=LE_{;\alpha} \quad E\left(h_{t;\beta}\ln\left(h_t\right)\right)=LE_{;\beta} \quad etc \\ E\left(h_{t;\beta}\left(\ln\left(h_t\right)\right)^2\right) &= L^2E_{;\beta} \quad E\left(\ln\left(h_t\right)h_{t;\beta}^2\right)=LE_{(;\beta)^2} \end{aligned}$$

$$\begin{aligned} E\left(\exp\left(\kappa\ln h_t\right)h_{t;\beta}\right) &= E_kE_{;\beta} \quad E\left(\exp\left(k\ln h_t\right)h_{t;\alpha}\right)=E_kE_{;\alpha} \\ E\left(\exp\left(k\ln h_t\right)h_{t;\mu}\right) &= E_kE_{;\mu} \quad etc. \\ E\left(\exp\left(\kappa\ln h_t\right)h_{t;\mu}^2\right) &= E_\kappa E_{(;\mu)^2} \quad E\left(\exp\left(\kappa\ln h_t\right)h_{t;\mu}^3\right)=E_\kappa E_{(;\mu)^3} \quad etc. \end{aligned}$$

$$E\left(h_{t;\beta}\left(h_{t;\mu}\right)^{2}\right)=E_{;\beta(;\mu)^{2}}$$

$$\begin{aligned} E\left(h_{t-1;\beta}h_{t-1;\mu}\right) &= E_{;\beta;\mu} \quad E\left(h_{t;\beta}h_{t;\alpha}\right)=E_{;\beta;\alpha} \quad E\left(h_{t;\beta}h_{t;\gamma}\right)=E_{;\beta;\gamma} \quad etc. \\ E\left(\exp\left[\kappa\ln\left(h_{t}\right)\right]h_{t;\beta}h_{t;\mu}\right) &= E_{\kappa}E_{;\beta;\mu} \quad etc \end{aligned}$$

$$E\left(h_{t;\beta,\mu}\right)=E_{;\beta,\mu} \quad E\left(h_{t;\mu,\mu}\right)=E_{;\mu,\mu} \quad etc.$$

$$\begin{aligned} E\left(h_{t;\mu}h_{t;\mu,\mu}\right) &= E_{;\mu;\mu,\mu} \\ E\left(\ln\left(h_{t}\right)h_{t;\beta,\beta}\right) &= LE_{;\beta,\beta} \end{aligned}$$

$$E\left(\exp\left(\kappa\ln h_{t}\right)h_{t;\mu,\mu}\right)=E_{\kappa}E_{;\mu,\mu}$$

$$E\left(h_{t;\beta}h_{t;\beta,\beta}\right)=E_{;\beta;\beta,\beta}$$

$$E\left(\exp\left(\kappa\ln h_{t}\right)\ln\left(h_{t}\right)h_{t;\mu}\right)=E_{\kappa}LE_{;\mu}$$

$$\begin{aligned} E\left(h_{t;\beta}h_{t;\mu,\mu}\right) &= E_{;\beta;\mu,\mu} \quad E\left(h_{t;\mu}h_{t;\mu,\beta}\right)=E_{;\mu;\mu,\beta} \\ E\left(\exp\left(\kappa\ln h_{t}\right)h_{t;\mu,\beta}\right) &= E_{\kappa}E_{;\mu,\beta} \end{aligned}$$