FEP WORKING PAPERS
FEP WORKING PAPERS

Research Work in Progress

n. 356, Jan. 2010

Growth and Firm Dynamics with Horizontal and Vertical R&D

Pedro Rui Mazeda Gil [1]
Paulo Brito [2]
Óscar Afonso [1]

[1] CEF.UP, Faculdade de Economia, Universidade do Porto
[2] Instituto Superior de Economia e Gestão, UECE, Universidade Técnica de Lisboa

U.PORTO
FEP FACULDADE DE ECONOMIA UNIVERSIDADE DO PORTO

Growth and Firm Dynamics with Horizontal and Vertical R&D*

Pedro Mazeda Gil†, Paulo Brito‡, Oscar Afonso§

January 4, 2010

This paper develops a tournament model of horizontal and vertical R&D under a lab-equipment specification. A key feature is that the overall growth rate is endogenous, as the splitting of the growth rate between the intensive and the extensive margin is itself endogenous. This setup gives rise to strong inter-R&D composition effects, while making economic growth and firm dynamics closely related, both along the balanced-growth path and transition. The model hence offers a (qualitative) explanation for the negative or insignificant empirical correlation between aggregate R&D intensity and both firm size and economic growth, a well-known puzzle in the growth literature.

Keywords: endogenous growth, vertical and horizontal R&D, firm dynamics, transitional dynamics

JEL Classification: O41, D43, L16

---

*This paper is a substantially revised version of Gil, Brito, and Afonso (2008).

†Faculty of Economics, University of Porto, and CEF.UP. Corresponding author: please email to pgil@fep.up.pt or address to Rua Dr Roberto Frias, 4200-464, Porto, Portugal.

‡School of Economics and Management, Technical University of Lisbon, and UECE

§Faculty of Economics, University of Porto, and CEF.UP

1. Introduction

A well-known puzzle in the growth literature is the insignificant or negative correlation between aggregate R&D intensity and the per-capita GDP growth rate found by several empirical studies (e.g., Bassanini, Scarpetta, and Visco, 2000); a related puzzle occurs with respect to the relation between R&D intensity and firm size (e.g., Pagano and Schivardi, 2003). The literature often underlines the complexity of the link between these variables (e.g., Audretsch and Keilbacha, 2008), and the several conceptual and measurement problems that afflict empirical analysis in this field (e.g., Bartelsman, Haltiwanger, and Scarpetta, 2005) to explain the lack of robust results. This paper highlights the role of inter-R&D composition effects in accounting for the above facts, by developing a model of simultaneous horizontal and vertical R&D by entrants within a lab-equipment specification. In the existing models, featuring a knowledge-driven specification (e.g., Peretto, 1998; Howitt, 1999; Peretto and Smulders, 2002; Peretto and Connolly, 2007), re-compositions between vertical and horizontal R&D are not able to produce a negative correlation between total R&D intensity and economic growth, as the latter only changes in response to vertical innovation. Our results hence show the compatibility between innovation as the ultimate source of economic growth and the empirical evidence concerning the link between aggregate R&D intensity and both firm size and economic growth.

The joint consideration of horizontal and vertical R&D in a framework that allows for the study of their dynamic interaction fits the vision that growth proceeds along an extensive (introduction of new goods) and an intensive margin (increase of good quality and reduction of marginal costs). This is in accord with the description of the process of industrial growth in both economic history and industry life-cycle literature (e.g., Freeman and Soete, 1997; Klepper, 1996). Furthermore, this approach allows us to consider economic growth and firm dynamics, measured by size and/or the number of firms, under the same framework.

This paper follows the "creative destruction", or tournament, tradition in the simultaneous treatment of horizontal and vertical R&D: the entrant must choose between vertical and horizontal R&D, whereas the incumbent does not perform any R&D (e.g., Aghion and Howitt, 1998; Howitt, 1999; Segerstrom, 2000). Thus, we depart from the non-tournament approach, which postulates that vertical R&D is done by the incumbent and horizontal R&D is carried out by entrants (e.g., Young, 1998; Peretto, 1998, 1999; Dinopoulos and Thompson, 1998; Peretto and Smulders, 2002). Indeed, according to the stylised facts on entry (e.g., Geroski, 1995; Klepper, 1996), it (i) occurs mostly in already existing sectors, (ii) is driven mostly by new firms (instead of incumbents enduring diversification processes), and (iii) is closely related to R&D activities. In a model of R&D and firm dynamics under monopolistic competition, like ours, where R&D activity is the only mechanism by which entrants can make a competitive entry (by introducing either a new variety or a higher-quality version of an existing good), those facts imply that not only horizontal but also vertical R&D must be performed by entrants.[1]

[1]Vertical R&D by entrants is neither logically nor empirically incompatible with vertical R&D by

The entry mechanism is a key component of the model. A potential entrant has to choose between introducing an innovation in an existing industry, specialised in the production of a particular variety, or introducing a new variety. In contrast to Howitt (1999), among others, we assume that there are no intersectoral spillovers in vertical R&D (e.g., Segerstrom, 2007; Etro, 2008). The entrant then faces a double uncertainty at each point in time: he/she does not know if a successful innovation will occur and in which industry it will take place. In particular, with respect to horizontal entry, the potential entrant has to compare the costs of creating a new variety with the expected benefits from introducing an innovation in any of the existing industries. We assume that the arrival of the innovation in any industry follows the Schumpeterian creative-destruction process with Poisson distribution, while its occurrence in a particular industry follows a uniform distribution.

Also crucial is the way R&D technology is characterised in the model. We consider a lab-equipment specification, whereas the literature predominantly assumes that R&D is knowledge-driven.[2] In the latter case, the choice between vertical and horizontal R&D implies a division of labour between the two types of R&D. Since the total labour level is determined exogenously, there is a shortcoming: in the end, the rate of extensive growth is exogenous, i.e., the balanced-growth-path (BGP) flow of new goods occurs at the same rate as (or proportional to) population growth. The alternative assumption that R&D is of the lab-equipment type implies that the choice between vertical and horizontal innovation is related to the splitting of R&D expenditures, which are fully endogenous. Therefore, we endogenise the overall growth rate and thus the rate of extensive growth.

In particular, we develop a dynamic general equilibrium model where, from the point of view of households, wealth can be accumulated either by creating new firms or by accumulating capital (e.g., Brito and Dixon, 2009). A quality-ladders mechanism provides the non-physical capital accumulation (technological knowledge), whilst an expanding-variety mechanism offers the flow of new firms (new good lines). Average firm size is thus measured as technological-knowledge stock per firm, which, however, relates closely to production (sales) or financial assets per firm.

This creates a channel between vertical R&D (hence aggregate growth) and firm dynamics, both in the long run (BGP) and in the medium run (transition), due to the nature of horizontal entry technology. We consider simultaneously dynamic and static decreasing returns to horizontal R&D: the former capture the negative spillovers associated with the number of varieties (e.g., due to the existence of proportional barriers to entry) and the latter implies new varieties are brought to the market gradually, instead of through a lumpy adjustment. This is in line with the stylised facts on entry (e.g., Geroski, 1995),

incumbents, as attested by the extant papers on vertical R&D simultaneously by entrants and incumbent (e.g., Etro, 2004; Segerstrom, 2007). To simplify the structure of our model, we consider R&D by only one type of firm; in the light of the aforementioned empirical facts, we let that type be the entrants. A very recent paper with vertical R&D only by entrants is Francois and Lloyd-Ellis (2009).

[2] Using Rivera-Batiz and Romer (1991)'s terminology, the assumption that homogeneous final good is the R&D input means that one adopts the "lab-equipment" version of R&D, instead of the "knowledge-driven" specification, in which labour is ultimately the only input.

according to which entry occurs mostly at small scale (adjustment costs penalise large-scale initial entry). So far, the literature has treated the two features separately (the former in, e.g., Evans, Honkapohja, and Romer, 1998; Barro and Sala-i-Martin, 2004; the latter in, e.g., Howitt, 1999; Jones and Williams, 2000). The first feature is crucial for the BGP properties of the model. The negative spillovers in horizontal entry per se determine a constant number of varieties/firms along the BGP; however, vertical R&D sustains variety expansion and aggregate growth at a positive finite rate. The second feature is key with respect to transitional dynamics. We show static decreasing returns to horizontal R&D introduce dynamic second-order effects in entry: the entry cost increases with the number of firms entering the market at a given instant.This friction in horizontal entry generates transitional dynamics, which operates through adjustments in average firm size. We also show that, given the interrelation between the two types of R&D, the dynamics is explicitly obtained from a piecewise-smooth dynamical system.

In the end, our framework gives rise to strong endogenous inter-R&D composition effects and makes economic growth and firm dynamics closely related: vertical R&D is the ultimate growth engine, whilst horizontal R&D builds an explicit link between aggregate and firm-dynamics variables.[3]

One of the most interesting findings of the paper is the mixed result with respect to the BGP relation between R&D intensity and both economic growth and firm size, thereby lending theoretical support to the previously highlighted lack of clear-cut empirical relationship between those variables. Shifts in the BGP due to changes in preferences or in vertical-entry costs give rise to a positive correlation between R&D intensity and both growth and firm size, whilst changes in horizontal-entry costs yield a negative relationship. This is explained by an endogenous inter-R&D composition effect: an increase in horizontal-entry costs induces a shift of resources from horizontal to vertical R&D, but the effect on the growth rate of the increment in the vertical type is dominated by the decrease in the horizontal one. That is, there is a negative correlation between the growth rate and vertical R&D whose effect outweights the positive correlation between the former and horizontal R&D.

Furthermore, in transitional dynamics the relationship between R&D intensity and both economic growth and firm size is unequivocally negative. Consider an emergent-market economy displaying a shallow market of differentiated goods, i.e., with a number of varieties too low relative to the technological-knowledge stock, hence firm size that is too large. Such an economy starts with a smaller vertical-innovation rate and higher economic growth than a mature (deep-) market economy. A lower vertical-innovation rate implies a higher real interest rate and lower vertical R&D, which secure the larger resources allocated to horizontal R&D. Along the transition path, the growth rate decreases, while the resources allocated to horizontal R&D are gradually re-targeted to vertical R&D, thus

[3] This point has been made recently by Peretto and Connolly (2007), in a different analytical setup. Like Peretto (1998, 1999) and others, the authors build an endogenous growth model where incumbents do deterministic cost-reducing R&D, while entrants bring new products to the market, under a knowledge-driven specification. They consider fixed operating costs, which, unless the exogenous population growth rate is positive, determine a constant number of varieties along the BGP. Under this setting, only vertical R&D allows for growth "unconstrained by endowments".

increasing the vertical-innovation rate. Thus, the latter and both the growth rate and firm size are negatively related. Due to a composition effect identical to that described for the BGP, R&D intensity moves in tandem with the vertical-innovation rate.[4]

Our model is close to Evans, Honkapohja, and Romer (1998), who build an expanding-variety mechanism with a lab-equipment specification exhibiting dynamic decreasing returns to scale. Here, variety expansion is ultimately sustained by physical-capital accumulation. Like us, Arnold (1998) and Funke and Strulik (2000) obtain an expanding-variety mechanism that is driven by non-physical capital accumulation, in their case in the form of human capital. However, there is no inter-R&D composition effect in these models, as they only feature horizontal R&D. In contrast, several recent models consider both horizontal and vertical innovation, focusing on the study of economic growth and firm dynamics under a common framework (e.g., van de Klundert and Smulders, 1997; Peretto, 1998, 1999, 2007; Thompson, 2001; Peretto and Smulders, 2002). As already stated, in as much as these models feature a knowledge-driven specification, shifts between vertical and horizontal R&D are unable to produce a negative correlation between total R&D intensity and growth, as the latter only changes in response to the intensive margin. The noteworthy exception is Peretto (2007), where R&D has a lab-equipment specification. However, this is a non-tournament model with deterministic vertical R&D.

The remainder of the paper is organised as follows. In Section 2, we present the model, giving a detailed account of the production, price and R&D decisions, and derive the dynamic general equilibrium. In Sections 3 and 4, we analyse the interior BGP and local-dynamics properties, and discuss their consistency with the empirical literature. Section 5 presents some concluding remarks.

2. The model

We build a dynamic general equilibrium model of a closed economy where a single competitively-produced final good can be used in consumption, production of intermediate goods and R&D. The final good is produced by a (large) number of firms, each using labour and a continuum of intermediate goods indexed by $\omega \in [0, N]$. The economy is populated by L identical dynastic families, each endowed with one unit of labour that is inelastically supplied to final-good firms. Thus, the total labour level is L, which, by assumption, is constant over time. In turn, families make consumption decisions and invest in firms' equity.

Potential entrants can devote resources either to horizontal or to vertical R&D. Horizontal R&D increases the number of intermediate-goodindustries N, while vertical R&D increases the quality of the good of an existing industry, indexed by $j(\omega)$. Quality level $j(\omega)$ translates into productivity of the final producer from using the good produced by industry ω, $\lambda^{j(\omega)}$, where $\lambda > 1$ is a parameter measuring the size of each quality upgrade. By improving on the current best quality index j, a successful R&D firm will introduce the leading-edge quality $j(\omega) + 1$ and hence render inefficient the existing input supplied

[4]The negative relationship between economic growth and the vertical-innovation rate is also apparent in Aghion and Howitt (1998), but only for a specific set of parameter values.

by the producer of ω. Therefore, the successful innovator will become a monopolist in ω. However, this monopoly, and the monopolist earnings that come with it, are temporary, because a new successful innovator will eventually substitute the incumbent.

2.1. Production and price decisions

The final-good firm has a constant-returns-to-scale technology using labour and a continuum of intermediate goods with measure $N(t)$, changing over time t, which is well-known from Barro and Sala-i-Martin (2004, ch. 6 and ch. 7)

$$Y(t) = A \cdot L^{1-\alpha} \cdot \int_0^{N(t)} \left[\lambda^{j(\omega,t)} \cdot X(\omega,t)\right]^{\alpha} d\omega,\ 0 < \alpha < 1,\ \lambda > 1, \tag{1}$$

where $A > 0$ is the total factor productivity, L is the labour input and $1-\alpha$ is the labour share in production,and $\lambda^{j(\omega,t)} \cdot X(\omega,t)$ is the input of intermediate good ω measured in efficiency units at time t.[5]That is, we integrate the final-producer technology that is considered in variety-expansion (Barro and Sala-i-Martin, 2004, ch. 6) and quality-ladders (Barro and Sala-i-Martin, 2004, ch. 7) models.

Final producers are price-takers in all the markets they participate. They take wages, $w(t)$, and input prices $p(\omega,t)$ as given and sell their output at a price equal to unity. From the profit maximisation conditions, we determine the aggregate demand of intermediate good ω as

$$X(\omega,t) = L \cdot \left(\frac{A \cdot \lambda^{j(\omega,t)\alpha} \cdot \alpha}{p(\omega,t)}\right)^{\frac{1}{1-\alpha}},\ \omega \in [0, N(t)]. \tag{2}$$

The intermediate-good sector consists of a continuum $N(t)$ of industries. There is monopolistic competition if we consider the whole sector: the monopolist in industry ω fixes the price $p(\omega,t)$ but faces the isoelastic demand curve (2). We assume that the intermediate good is non-durable and entails a unit marginal cost of production, in terms of the final good, whose price is taken as given. Profit in industry ω is thus $\pi(\omega,t) = (p(\omega,t) - 1) \cdot X(\omega,t)$, and the profit maximising price is a constant markup over marginal cost[6]

$$p(\omega,t) \equiv p = \frac{1}{\alpha} > 1 \tag{3}$$

The quality of the intermediate good ω can be characterised by the quality index

$$q(\omega,t) \equiv \lambda^{j(\omega,t)\frac{\alpha}{1-\alpha}}. \tag{4}$$

Then, from (2) and (3), the aggregate quantity produced of ω is

[5]In equilibrium, only the top quality of each ω is produced and used; thus, $X(j,\omega,t) = X(\omega,t)$. Henceforth, we only use all arguments (j,ω,t) if they are useful for expositional convenience.

[6]We assume that $\frac{1}{\alpha} < \lambda$; i.e., if $\frac{1}{\alpha}$ is the price of the top quality, the price of the next lowest grade, $\frac{1}{\alpha\lambda}$, is less than the unit marginal cost. In this case, lower grades are unable to provide any effective competition, and the top-quality producer can charge the unconstrained monopoly price.

$$X(\omega,t) = L \cdot \left(A \cdot \alpha^2\right)^{\frac{1}{1-\alpha}} q(\omega,t), \tag{5}$$

and the profit accrued by the monopolist in ω is

$$\pi(\omega,t) = \pi_0 \cdot L \cdot q(\omega,t), \tag{6}$$

which are both linear functions of $q(\omega,t)$. The constant $\pi_0 \equiv \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}}$ can be seen as the "basic" profit flow.[7]

As is well known, if there are no intersectoral spillovers, the aggregate quality index

$$Q(t) = \int_0^{N(t)} q(\omega,t)d\omega \tag{7}$$

measures the technological-knowledge level of the economy. This implies that aggregate output,

$$Y(t) = \left(A^{\frac{1}{\alpha}} \alpha^2\right)^{\frac{\alpha}{1-\alpha}} \cdot L \cdot Q(t), \tag{8}$$

total resources devoted to intermediate-goods production,

$$X(t) = \int_0^{N(t)} X(\omega,t)d\omega = \left(A\alpha^2\right)^{\frac{1}{1-\alpha}} \cdot L \cdot Q(t), \tag{9}$$

and total profits,

$$\Pi(t) = \int_0^{N(t)} \pi(\omega,t)d\omega = \pi_0 \cdot L \cdot Q(t), \tag{10}$$

are all linear functions of $Q(t)$.

2.2. R&D

We assume there is both vertical and horizontal R&D (e.g., Howitt, 1999; Segerstrom, 2000), that R&D is only performed by (potential) entrants, and that successful R&D leads to the set up of a new firm in either an existing or in a new industry. There is perfect competition among entrants and free entry in R&D business. As already stated, these assumptions are consistent with a number of stylised facts on entry (e.g., Geroski, 1995; Klepper, 1996).

The mechanism of entry is a key aspect of our model. A potential entrant has to choose between introducing an innovation in an existing industry, which is specialised in the production of a particular existing variety, or introducing a new variety. Differently from Howitt (1999) and others, we assume that there are no intersectoral spillovers in vertical R&D (e.g., Segerstrom, 2007; Etro, 2008). The entrant then faces a double uncertainty at each point in time: he/she does not know if a successful innovation will occur and in which industry it will take place. First, the arrival of an innovation in any extant industry follows the Schumpeterian creative-destruction process with Poisson

[7] This is the profit flow, constant over time, that accrues when $j = 0$ (i.e., $q = 1$).

distribution. This process may lead to entry, but it will also erode the monopoly power stemming from the innovation, regardless of which type, vertical or horizontal. Then, the dilemma for the potential entrant lies in introducing an innovation that will increase quality in an uncertain but existing industry or creating a new variety. In order to compare the benefits from the two types of entry, we assume that the quality level for the new industry will be the mean quality of the existing industries, while vertical entry will bring about an increase in quality by $\lambda^{\alpha/(1-\alpha)}$ for an existing but uncertain industry.

2.2.1. Vertical R&D free-entry and dynamic arbitrage conditions

As is common in the literature, each new design is granted a patent and thus a successful innovator retains exclusive rights over the use of his/her good. By improving on the current top quality level $j(\omega, t)$, a successful R&D firm earns monopoly profits from selling the leading-edge input of $j(\omega, t) + 1$ quality to final-good firms. A successful innovation will instantaneously increase the quality index in ω from $q(\omega, t) = q(j)$ to $q^{+}(\omega, t) = q(j+1) = \lambda^{\alpha/(1-\alpha)} q(\omega, t)$. In equilibrium, lower qualities of ω are priced out of business.

Let $I_i(j)$ denote the Poisson arrival rate of vertical innovations (vertical-innovation rate) by potential entrant i in industry ω when the highest quality is j. Rate $I_i(j)$ is independently distributed across firms, across industries and over time, and depends on the flow of resources $R_{vi}(j)$ committed by entrants at time t. As in, e.g., Barro and Sala-i-Martin (2004, ch. 7), $I_i(j)$ features constant returns in R&D expenditures, $I_i(j) = R_{vi}(j) \cdot \Phi(j)$, where $\Phi(j)$ is the R&D productivity factor, which is assumed to be homogeneous across i in ω. We assume

$$\Phi(j) = \frac{1}{\zeta \cdot L \cdot q(j+1)}, \tag{11}$$

where $\zeta > 0$ is a constant (flow) fixed vertical-R&D cost. Equation (11) incorporates a congestion effect (e.g., Segerstrom, 2007; Etro, 2008), implying vertical-R&D dynamic decreasing returns to scale (i.e., decreasing returns to cumulated R&D).[8] To avoid the usual scale effect arising from labour level, (11) implies that an increase in market scale dilutes the effect of R&D outlays on innovation probability. Aggregating across i in ω, we get $R_v(j) = \sum_i R_{vi}(j)$ and $I(j) = \sum_i I_i(j)$, and thus

$$I(j) = R_v(j) \cdot \frac{1}{\zeta \cdot L \cdot q(j+1)}. \tag{12}$$

Observe that here $I(\omega, t) = I(j)$ is time-varying, and not constant as is usual in the literature.

As the terminal date of each monopoly arrives as a Poisson process with frequency $I(j)$ per (infinitesimal) increment of time, the present value of a monopolist's profits is a

[8] The way Φ depends on j implies that the increasing difficulty of creating new qualities exactly offsets the increased rewards from marketing higher qualities - see (11) and (6). This allows for constant vertical-innovation rate over t and across ω along the BGP, i.e., a symmetric equilibrium (on asymmetric equilibria in quality-ladders models and its growth consequences, see Cozzi, 2007).

random variable. Let $V(j)$ denote the expected value of an incumbent firm with current quality level $j(\omega,t)$,[9]

$$V(j) = \pi(j) \int_t^{\infty} e^{-\int_t^s [r(\nu)+I(j)]d\nu} ds, \tag{13}$$

where r is the equilibrium market real interest rate and $\pi(j)$, given by (6), is constant in-between innovations. Free-entry prevails in vertical R&D such that the condition $I(j) \cdot V(j+1) = R_v(j)$ holds, which implies that

$$V(j+1) = \frac{1}{\Phi(j)} = \zeta \cdot L \cdot q(j+1). \tag{14}$$

Next, we determine $V(j+1)$ analogously to (13), then consider (14) and time-differentiate the resulting expression. Thus, if we also consider (6), we get the arbitrage condition facing a vertical innovator

$$\zeta = \frac{\pi_0}{r(t) + I(t)}. \tag{15}$$

It has several implications:[10] firstly, the present value of "basic" profits, using the effective rate of interest $r + I$ as a discount factor, should be equal to the fixed cost of entry; secondly, the rates of entry are symmetric across industries $I(\omega,t) = I(t)$; and, finally, the effective discount rate is constant over time. Then, the vertical-innovation rate is perfectly negatively correlated to the rate of return $I(t) = \pi_0/\zeta - r(t)$.

Solving equation (12) for $R_v(\omega,t) = R_v(j)$ and aggregating across industries ω, we determine total resources devoted to vertical R&D, $R_v(t)$, $R_v(t) = \int_0^{N(t)} R_v(\omega,t)\, d\omega = \int_0^{N(t)} \zeta \cdot L \cdot q^+(\omega,t) \cdot I(\omega,t)\, d\omega$. As the innovation rate is industry independent, then

$$R_v(t) = \zeta \cdot L \cdot \lambda^{\frac{\alpha}{1-\alpha}} \cdot I(t) \cdot Q(t). \tag{16}$$

2.2.2. Horizontal R&D free-entry and dynamic arbitrage conditions

Variety expansion arises from R&D aimed at creating a new intermediate good. Again, innovation is performed by a potential entrant, which means that, because there is free entry, the new good is produced by new firms. Under perfect competition among R&D firms and constant returns to scale at the firm level, instantaneous entry is obtained as $\dot{N}_e(t) = \frac{1}{\eta} R_{ne}(t)$, where $\dot{N}_e(t)$ is the contribution to the instantaneous flow of the new good by R&D firm e at a cost of η units of the final good and $R_{ne}(t)$ is the flow of

---

[9] We assume that entrants are risk-neutral and, thus, only care about the expected value.

[10] Observe that, from (6) and (12), we have $\frac{\dot{\pi}(\omega,t)}{\pi(\omega,t)} = I(\omega,t) \cdot \left[\dot{j}(\omega,t) \cdot \left(\frac{\alpha}{1-\alpha}\right) \cdot \ln\lambda\right]$ and $\frac{\dot{R}_v(\omega,t)}{R_v(\omega,t)} - \frac{\dot{I}(\omega,t)}{I(\omega,t)} = I(\omega,t) \cdot \left[\dot{j}(\omega,t) \cdot \left(\frac{\alpha}{1-\alpha}\right) \cdot \ln\lambda\right]$. Then, if we time-differentiate (14) considering (13) and the equations above, we get $r(t) = \frac{\pi(j+1) \cdot I(j)}{R_v(j)} - I(j+1)$, which can then be re-written as (15).

resources devoted to horizontal R&D by innovator e at time t. The cost η is assumed to be symmetric. Then, $R_n = \sum_e R_{ne}$ and $\dot{N}(t) = \sum_e \dot{N}_e(t)$, implying

$$R_n(t) = \eta \cdot \dot{N}(t). \tag{17}$$

We assume that the cost of setting up a new variety (cost of horizontal entry) is increasing in both the number of existing varieties, N, and the number of new entrants, $\dot{N}$,

$$\eta(t) = \phi \cdot N(t)^{\sigma} \cdot \dot{N}(t)^{\gamma}, \tag{18}$$

where $\phi > 0$ is a fixed (flow) cost, and $\sigma > 0$ and $\gamma > 0$. The dependence of η on N captures the assumption there are dynamic decreasing returns to scale. That is, the scale of the economy induces a negative externality in the form of a barrier to entry because it becomes costlier to introduce new varieties in large growing economies. The dependence of η on $\dot{N}$ means that the entry technology displays static decreasing returns to scale at the aggregate level, which we assume to be entirely external to the firm, but are compatible with the previous assumption of constant returns to scale at the firm level. The existing endogenous-growth literature deals with the two features separately: some models only display dynamic negative externalities (e.g., Evans, Honkapohja, and Romer, 1998; Barro and Sala-i-Martin, 2004, ch. 6) while others only assume static decreasing returns to scale (e.g., Arnold, 1998; Howitt, 1999; Jones and Williams, 2000).[11]

The dependence of the entry cost on the number of entrants introduces dynamic second-order effects in entry, implying that new varieties are brought to the market gradually, instead of through a lumpy adjustment. This is in line with the stylised facts on entry (e.g., Geroski, 1995): entry occurs mostly at small scale since adjustment costs penalise large-scale entry.

We assume the innovator enters with the average quality level of the existing varieties,

$$\bar{q}(t) = \int_0^{N(t)} \frac{q(\omega, t)}{N(t)} d\omega = \frac{Q(t)}{N(t)}, \tag{19}$$

which is tantamount to having a uniform distribution with probability $1/N(t)$. As his/her monopoly power will be also terminated by the arrival of a successful vertical innovator in the future, the benefits from entry are given by

$$V(\bar{q}) = \bar{\pi}(t) \int_t^{\infty} e^{-\int_t^s [r(\nu)+I(\bar{q})]d\nu} ds, \tag{20}$$

where $\bar{\pi} = \pi_0 L \bar{q}$. The free-entry condition is now $\dot{N} \cdot V(\bar{q}) = R_n$, which simplifies to

$$V(\bar{q}) = \eta(t). \tag{21}$$

[11] We also departure from Howitt (1999) (see also Segerstrom, 2000) in that he hypothesises decreasing returns to scale to R&D at the firm level. Such an entry technology implies that (keeping our notation) $V = \eta = \frac{dR_n}{d\dot{N}} > \frac{R_n}{\dot{N}}$. In contrast, given our assumption of constant returns to scale, we have $V = \eta = \frac{R_n}{\dot{N}} = \frac{dR_n}{d\dot{N}}$ (see (21), below). Note that the price of entry, V, equals the marginal cost of entry, η, in both cases considered above; nevertheless, the assumption of constant returns eschews positive profits from entering, since $V = \frac{R_n}{\dot{N}}$.

Substituting (20) into (21) and time-differentiating the resulting expression, yields the arbitrage equation facing a horizontal innovator

$$r\,(t) + I(t) = \frac{\bar{\pi}(t)}{\eta\,(t)} \tag{22}$$

2.2.3. Consistency arbitrage condition

Before deciding which type of R&D to perform, the potential entrant should evaluate the best type of entry. At the margin, she/he should be indifferent between the two types, implying there is a consistency condition between them. If we equate the effective rate of return $r + I$ for both types of entry by considering (15) and (22), we get the arbitrage condition

$$\overline{q}(t) = \frac{Q\,(t)}{N\,(t)} = \frac{\eta\,(t)}{\zeta \cdot L}. \tag{23}$$

This condition is one of the key ingredients of the model. It equates the cost of the horizontal, η, to the average cost of vertical R&D, $\bar{q}\zeta L$. In response to a given exogenous shock, a lower (higher) N enhances (reduces) relative average quality, $1/N$, received by a newly-created variety, in order to compensate for a higher (lower) relative entry cost. The adjustment of the relative average quality $1/N$ guarantees that (23) has a finite and determined solution with respect to N without requiring (static) decreasing returns to horizontal R&D. This reflects our assumption of the absence of intersectoral spillovers in vertical innovation. In contrast, models that feature intersectoral spillovers in vertical innovation display a relative average quality which is independent of the number of varieties and, thus, the assumption of static decreasing returns to horizontal R&D (given the constant returns to vertical R&D) is necessary to ensure an interior equilibrium with non-zero vertical and horizontal innovative activity (e.g., Howitt, 1999).

Equation (23) can be equivalently recast as

$$\dot{N}(t) = x(Q(t), N(t)) \cdot N(t), \tag{24}$$

where

$$x(Q, N) = \left(\frac{\zeta \cdot L}{\phi}\right)^{\frac{1}{\gamma}} \cdot Q^{\frac{1}{\gamma}} \cdot N^{-\left(\frac{\sigma+\gamma+1}{\gamma}\right)}, \tag{25}$$

which clarifies the mechanism of entry adopted in our model by explicitly incorporating a channel between vertical innovation and firm dynamics. It shows that the horizontal-entry rate, $\frac{\dot{N}}{N}$, depends positively on Q and negatively on N: the first effect is an implication of the complementarity between the horizontal-entry rate and the technological-knowledge stock, and the second results from the barriers-to-entry effect incorporated in equation (18).

In a small time interval, the rate of growth of average quality is equal to the expected arrival rate of a successful innovation multiplied by the quality shift it introduces:

$\dot{\overline{q}}/\overline{q} = I(t)(q^+ - q)/q$, where both the innovation rate and the quality shift are industry-independent. Time-differentiating (19), and using (24) yields

$$\dot{Q}(t) = (I(t) \cdot \Xi + x(N(t), Q(t))) \cdot Q(t). \tag{26}$$

where the quality shift is denoted by $\Xi \equiv (q^+ - q)/q = \left(\lambda^{\frac{\alpha}{1-\alpha}} - 1\right)$. The innovation rate is endogenous and will be determined as an economy-wide function below. Equation (26) introduces a second dynamic interaction between the two types of entry, in this case between the number of varieties and the rate of growth of the quality index of the economy.

2.3. Households

The economy has L identical dynastic families who consume and collect income (dividends) from investments in financial assets (equity) and from labour. Each family is endowed with one unit of labour that is inelastically supplied. Thus, total labour supply, L, is exogenous and constant. We assume consumers have perfect foresight concerning the technological change over time and every household chooses the path of consumption $\{C(t), t \geq 0\}$ to maximise discounted lifetime utility

$$U = \int_0^\infty \left(\frac{C(t)^{1-\theta} - 1}{1-\theta}\right) e^{-\rho t} dt, \tag{27}$$

where $\rho > 0$ is the subjective discount rate and $\theta > 0$ is the inverse of the intertemporal elasticity of substitution, subject to the flow budget constraint

$$\dot{a}(t) = r(t) \cdot a(t) + w(t) - C(t), \tag{28}$$

where a denotes household's real financial assets holdings. The initial level of wealth $a(0)$ is given and the non-Ponzi games condition $lim_{t\to\infty} e^{-\int_0^t r(s)ds} a(t) \geq 0$ is also imposed. The optimal path of consumption Euler equation and the transversality condition are standard:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta} \cdot (r(t) - \rho) \tag{29}$$

$$\lim_{t\to\infty} e^{-\rho t} \cdot C(t)^{-\theta} \cdot a(t) = 0 \tag{30}$$

2.4. Macroeconomic aggregation and equilibrium innovation rate

The aggregate financial wealth held by all households is $L \cdot a(t) = \int_0^{N(t)} V(\omega, t) d\omega$, which, from the arbitrage condition between vertical and horizontal entry, yields $L \cdot a(t) = \eta(t) \cdot N(t)$. Taking time derivatives and comparing with (28), we get an expression for the aggregate flow budget constraint which is equivalent to the product market equilibrium condition (see Appendix A)

$$Y(t) = L \cdot C(t) + X(t) + R_v(t) + R_n(t). \tag{31}$$

If we substitute the expressions for the aggregate outputs (8) and (9) and for total R&D expenditures (16) and (17), we have

$$L \cdot H_Y \cdot Q(t) = L \cdot H_X \cdot Q(t) + L \cdot C(t) + \eta(t) \cdot \dot{N}(t) + \zeta \cdot L \cdot \lambda^{\frac{\alpha}{1-\alpha}} \cdot I(t) \cdot Q(t) \quad (32)$$

where $H_Y \equiv \left(A^{\frac{1}{\alpha}} \cdot \alpha^2\right)^{\frac{\alpha}{1-\alpha}}$ and $H_X \equiv \left(A \cdot \alpha^2\right)^{\frac{1}{1-\alpha}}$. Solving for I, and using (23) and (24), we get the endogenous vertical-innovation rate at equilibrium

$$I(Q,N,C) = \frac{1}{\zeta \cdot \lambda^{\frac{\alpha}{1-\alpha}}} \cdot \left(H_Y - H_X - \frac{C}{Q} - \zeta \cdot x(Q,N)\right) \quad (33)$$

as a function which is decreasing in consumption, increasing in the number of varieties and related in an ambiguous way with the aggregate quality level.[12] However, if the quality level is high, the innovation rate will tend to be negatively related with Q. As function $I(Q,N,C)$ may be negative, the relevant innovation rate at the macroeconomic level is

$$I^+(Q,N,C) = \max\{I(Q,N,C), 0\}. \quad (34)$$

Again, we emphasise the complementarity between horizontal and vertical innovation, here made clear by the fact that, if N is too low, vertical R&D shuts down.[13] From (15), we get the rate of return of capital as $r(Q,N,C) = r_0 - I^+(Q,N,C)$, where $r_0 \equiv \pi_0/\zeta$.

2.5. The dynamic general equilibrium

The dynamic general equilibrium is defined by the allocation $\{X(\omega,t), \omega \in [0,N(t)], t \geq 0\}$, by the prices $\{p(\omega,t), \omega \in [0,N(t)], t \geq 0\}$ and by the aggregate paths $\{C(t), N(t), Q(t), I(t), r(t), t \geq 0\}$, such that: (i) consumers, final-good firms and intermediate-good firms solve their problems; (ii) vertical, horizontal and consistency free-entry conditions are met; and (iii) markets clear. The equilibrium paths can be obtained from the piecewise-smooth system

$$\dot{C} = \begin{cases} \frac{1}{\theta} \cdot (r_0 - \rho) \cdot C & \text{if } I(Q,N,C) \leq 0 \\ \frac{1}{\theta} \cdot (r_0 - I(Q,N,C) - \rho) \cdot C & \text{if } I(Q,N,C) > 0 \end{cases} \quad (35a)$$

$$\dot{Q} = \begin{cases} x(Q,N) \cdot Q & \text{if } I(Q,N,C) \leq 0 \\ (I(Q,N,C) \cdot \Xi + x(Q,N)) \cdot Q & \text{if } I(Q,N,C) > 0 \end{cases} \quad (35b)$$

$$\dot{N} = x(Q,N) \cdot N \quad (35c)$$

given $Q(0)$ and $N(0)$, and the transversality condition (30), which may be re-written as

$$\lim_{t \to \infty} e^{-\rho t} C(t)^{-\theta} \zeta \cdot L \cdot Q(t) = 0. \quad (36)$$

[12] The partial derivative with Q has the sign of $C(t)/Q(t) - \zeta/\gamma$, meaning that it may change over t.

[13] This contrasts with, e.g., Peretto (2007) and Peretto and Connolly (2007). In these models, vertical R&D falls to zero when the number of varieties becomes too *high*, a result that basically reflects the assumption that horizontal R&D competes away scarce resources from vertical R&D.

3. Equilibrium dynamics

3.1. The balanced-growth path

Let $g_y \equiv \dot{y}/y$ represent the growth rate of variable $y(t)$. As the functions in system (35a)-(35c) are homogeneous, a BGP may exist if the further necessary conditions are verified (see Appendix B): (i) the growth rates of consumption and of the quality index are equal, $g_C = g_Q = g$; (ii) the vertical-innovation rate is trendless $g_I = 0$; and (iii) the growth rates of the quality index and the number of varieties are monotonously related, $g_Q = (\sigma + \gamma + 1) \cdot g_N$. Observe that $x = g_N$ is always positive if $N > 0$.

Proposition 1. Assume that $\mu \equiv (r_0 - \rho)/\theta > 0$ and that $\theta \geq 1$. Then there is a BGP only if the long-run level of the vertical-innovation rate is positive,

$$I^* = \frac{\mu(\sigma + \gamma)}{\Xi(\sigma + \gamma + 1) + \frac{1}{\theta}(\sigma + \gamma)} > 0, \tag{37}$$

and where the endogenous growth rates are also positive

$$g^* = \frac{\mu\Xi(\sigma + \gamma + 1)}{\Xi(\sigma + \gamma + 1) + \frac{1}{\theta}(\sigma + \gamma)} > 0, \tag{38}$$

$$g_N^* = \frac{\mu\Xi}{\Xi(\sigma + \gamma + 1) + \frac{1}{\theta}(\sigma + \gamma)} > 0. \tag{39}$$

We should emphasise the result that a BGP will only exist if $I > 0$ and there is no BGP in which $I = 0$. It is convenient to recast system (35a)-(35c), by using variable x as in (25), and variable $z \equiv C/Q$ into an equivalent system in detrended variables

$$\dot{x} = \begin{cases} -\left(\frac{\sigma}{\gamma} + 1\right) \cdot x^2 & \text{if } I(x,z) \leq 0 \\ \left[I(x,z) \cdot \Xi \cdot \frac{1}{\gamma} - \left(\frac{\sigma}{\gamma} + 1\right) \cdot x\right] \cdot x & \text{if } I(x,z) > 0 \end{cases} \tag{40a}$$

$$\dot{z} = \begin{cases} (\mu - x)\, z & \text{if } I(x,z) \leq 0 \\ \left[\mu - \left(\frac{1}{\theta} + \Xi\right) \cdot I(x,z) - x\right] \cdot z & \text{if } I(x,z) > 0 \end{cases} \tag{40b}$$

Proposition 2. Let $H_Y - H_X > \zeta\lambda^{\frac{\alpha}{1-\alpha}}(r_0 - \rho)/(1 + \theta\Xi)$. Then there is a unique BGP, such that $I(x^*, z^*) > 0$, $C^*(t) = z^* \cdot Q^*(t)$, $Q^*(t) = \phi/(\zeta L) \cdot (x^*)^\gamma \cdot N(0)^{\sigma+\gamma+1} \cdot e^{g^* \cdot t}$, $N^*(t) = N(0) \cdot e^{g_N^* \cdot t}$, where

$$x^* = g_N^*, z^* = H_Y - H_X - \zeta \cdot L \cdot \left(1 + \frac{\sigma + \gamma}{\Xi} \cdot \lambda^{\frac{\alpha}{1-\alpha}}\right) \frac{\mu\Xi}{\Xi(\sigma + \gamma + 1) + \frac{1}{\theta}(\sigma + \gamma)}. \tag{41}$$

Observe that $\underset{\sigma \to \infty}{lim}\, g^* = \underset{\gamma \to \infty}{lim}\, g^* = g^*_{no-entry}$ and that $g^* > 0$ requires $r^* - \rho > 0$. This condition also guarantees $g_N^* > 0$. Thus, under a sufficiently productive technology, our model predicts a BGP with constant positive g and g_N, where the former exceeds the latter by an amount corresponding to the growth of intermediate-good quality, driven by vertical innovation; to verify this, just replace (24) in (26) and solve to get $\dot{\bar{q}}/\bar{q} = \dot{Q}/Q - \dot{N}/N = I \cdot \Xi$, which is positive if $I > 0$. This implies that the consumption growth rate equals the growth rate of the number of varieties plus the growth rate of intermediate-good quality, which is consistent with the idea that industrial growth proceeds both along an intensive and an extensive margin. A similar result can be found in, e.g., Arnold (1998), Peretto (1998) and Howitt (1999).

But differently from the knowledge-driven literature, g_N^* is not a function of the (exogenous) population growth rate. The negative externality effect in (18) per se determines a constant N along the BGP (provided population growth is zero; see Barro and Sala-i-Martin, 2004, ch. 6);[14]however, the variety expansion is sustained by technological-knowledge accumulation (independently of population growth), as the expected growth of intermediate-good quality due to vertical R&D makes it attractive, in terms of intertemporal profits, for potential entrants to always put up an (horizontal) entry cost, in spite of its increase with N. Hence, vertical R&D is the ultimate growth engine. In this sense, it is not necessarily the larger economy, measured by population size, that produces the greater number of varieties, but that with the larger technological-knowledge stock, which thus emerges as the relevant endogenous economic size measure.

Arnold (1998) and Funke and Strulik (2000) also obtain a positive g_N^* that is solely driven by knowledge accumulation. However, this occurs in the form of human-capital production, with the latter counterbalancing the increasing entry cost due to rising real wages caused by the positive impact of a growing N in labour marginal productivity. Evans, Honkapohja, and Romer (1998) build an expanding-variety mechanism with a lab-equipment specification exhibiting dynamic decreasing returns to scale, which is analogous to the one assumed here. In their model, the expansion of varieties is ultimately sustained by physical-capital accumulation. The monotonic positive relationship between g^* and g_N^* is qualitatively similar to the one observed in the models cited above.

We interpret the technological-knowledge stock per firm, $\bar{q}$, as a measure of average firm size (or $1/N$ as firm size relative to market size). Because $\dot{\bar{q}}/\bar{q} = I \cdot \Xi$, it expands at the growth rate of intermediate-good quality along the BGP. Alternative measures of firm size such as production (or sales) per firm, $X/N = H_X Q/N$ (see (9)), or financial assets per firm, $a/N = \eta = \zeta LQ/N$ (see (23)) produce the same result. Peretto (1998) takes employment per firm as an explicit measure of firm size, while physical capital per firm in efficiency units and human capital per firm can be interpreted as measures of firm size in Aghion and Howitt (1998, ch. 12), and Arnold (1998) and Funke and Strulik (2000), respectively.

These results are broadly supported by historical empirical evidence. The increase in sales per firm over time is referred to, e.g., by Jovanovic (1993) for the US, while Ehrlich (1985) finds a "relative stability of establishment sizes", measured as employment per

[14]In fact, the dependence of η on N is necessary to eschew the explosive growth that would occur if η were constant over t, or depended solely on $\dot{N}$, thus implying that a BGP would not exist. This is not the case in Barro and Sala-i-Martin (2004)'s basic model of pure expanding variety. It can be shown that the specification $\eta \equiv \eta(Q)$, $\eta' > 0$, $\eta'' < 0$, produces a similar result in our model.

establishment, in a long time-series data base for the US, Japan and eight European countries. The increase in the number of firms and establishments over the long run is reported, e.g., by Maddison (1994). The last two empirical regularities are confirmed by Laincz and Peretto (2006), who analyse more recent data for the US.[15]

On the relationship between economic size and number of firms, Sherer and Ross (1990) state the robustness of the cross-sectional evidence that shows that large countries (measured by population size) tend to have a larger number of firms (and lower concentration rates); however, the relationship is far less clear when the comparison stands between intermediate and large-size countries. An explanation for this result may be that population size is not the best measure of economic size. Our model suggests that one should focus instead on endogenous measures of economic size, such as the technological-knowledge stock.

3.2. Aggregate transitional dynamics

Proposition 3. Under the previous assumptions, the BGP is determinate and is saddle-point stable. There is a piecewise-smooth continuous stable manifold which is positively sloped, implying there is a positive correlation between x and z along the transition to the BGP (see Figure 1). In addition, there is a magnitude $\tilde{x} \in \left(\mu, \frac{\Xi(\sigma+\gamma+1)+\sigma+\gamma}{\Xi(\sigma+\gamma+1)+\frac{1}{\theta}(\sigma+\gamma)}\mu\right)$, such that:

(a) If $x(0) > \tilde{x}$, then there is a point $z(0)$ so that the pair (x, z) diminishes until the economy crosses the point $(\tilde{x}, \tilde{z})$ at t_0. In $t \in [0, t_0]$, we have $I(t) = 0$; from t_0 onwards, $I(t)$ increases and the pair (x, z) will still diminish until it reaches the BGP levels (x^*, z^*).

(b) If $x^* < x(0) < \tilde{x}$, we have $0 < I(0) < I^*$ and $I(t)$ increases, while the pair (x, z) diminishes until it reaches the BGP levels.

(c) If $0 < x(0) < x^*$, then $I(0) > I^* > 0$ and $I(t)$ decreases, while the pair (x, z) increases until it reaches the BGP levels.

[Figure 1 goes about here]

[15] See Gil (2009) for a detailed discussion of the empirical literature relating firm dynamics with long-run economic growth.

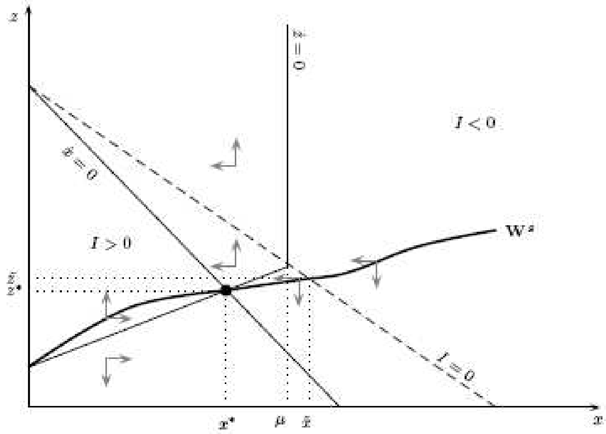

Figure 1: Phase diagram in the detrended variables (x, z) for the piecewise-smooth system (40a)-(40b). Curves $\dot{x} = 0$ and $\dot{z} = 0$ are the isoclines and curve $\mathbf{W}^s$ is the stable manifold, which is the only equilibrium trajectory.

We focus on the empirically relevant case of $\theta > 1$ and consider deviations of x above its steady-state value, i.e., $x(0) > x^*$. This would be the case of an economy with a N too low relative to Q and, thus, firm size too large: e.g., an emergent-market economy, displaying a shallow market of differentiated goods, in contrast to a mature-market economy, which is expected to boast a deep market of differentiated goods.[16]

Proposition 4. Consider an economy initially endowed with $N(0)$, such that$x^* < x(0) < \tilde{x}$, and $\theta > 1$. The transitional dynamics is characterised as follows. The economy experiences a decreasing z and $x\,(= g_N)$; this implies that more resources become available to R_v, boosting I and reducing r; consequently, g falls due to the downward movement of g_N but less so due to the effect of accelerated vertical innovation, reflecting the increase in I. In stationarised terms, both N and Q grow along the transition path,[17] but the former grows more than the latter, implying a falling firm size; also, C grows less than Q, whereas E decreases.

[16]We use the following set of baseline parameter values to illustrate the transitional dynamics: $\gamma = 1.2$, $\sigma = 1.2$, $\phi = 1$, $\zeta = 0.9$, $\lambda = 2.5$, $\rho = 0.02$, $\theta = 1.5$, $\alpha = 0.4$, $A = 1$, $L = 1$. Given that along the BGP, $g_Q - g_N = (\sigma + \gamma)g_N$, the choice of values for σ and γ is such that $(\sigma + \gamma) = 2.4$, which is the ratio between the growth rate of the average firm size and the growth rate of the number of firms we have found in the empirical data (the data, which is available from the authors upon request, concerns 23 European countries in the period 1995-2005 and was taken from the Eurostat on-line database). The values for λ, θ, ρ and α were set in line with previous work on growth and guided either by empirical findings or by theoretical specification, while the normalisation of A and L to unity at every t implies that all aggregate magnitudes can be interpreted as per capita magnitudes. The values of the remaining parameters were chosen in order to calibrate the BGP aggregate growth rate around 2.5 percent/year.

[17]In order to compute stationarised Q, Q_{stat}, let $Q(t) = Q_0 e^{g_Q(t)t}$ (with $Q_0 > 0$) and $Q_{stat}(t) = Q(t)e^{-g^* t}$; hence, $Q_{stat}(t) = Q_0 e^{(g_Q(t) - g^*)t}$. Stationarised N, C and E are computed from Q_{stat}, after numerical integration.

According to Proposition 4, our model exhibits the standard conditional convergence property (falling g and r towards the interior BGP), although through a specific mechanism. This consists of dynamic second-order effects in horizontal entry (see (18)), implying that new varieties are brought to the market gradually, as explained earlier. Since the aggregate production function exhibits constant returns in the accumulated factor, Q, this friction in horizontal entry is the only source of transitional dynamics. Its mechanism operates through imbalances between Q and N, i.e., adjustments in average firm size. An economy with very few varieties relative to the technological-knowledge stock (i.e., with very high firm size) starts with a smaller I, a higher g_N, a higher g and a higher r than a mature market economy. A lower I implies a higher r (see (15)) and a lower R_v (see (12)), which secure the larger resources allocated to R_n. Note also that the higher g is solely justified by the higher g_N. Along the transition path, g and g_N decrease, whereas part of the resources allocated to R_n is gradually re-targeted to R_v. The consequent increase in I implies a falling r.

Thus, g and I are unequivocally negatively related in transitional dynamics. Due to the described inter-R&D composition effect, R&D intensity, $\frac{R_v+R_n}{Y-X}$,moves in tandem with I. Dinopoulos and Thompson (1998) report a similar re-balancing effect between vertical and horizontal R&D; however, their vertical-innovation rate falls in parallel with aggregate growth and the real interest rate along the transition path (as in Peretto, 1998). On the contrary, the medium-run negative relationship between aggregate growth and the vertical-innovation rate is also apparent in Aghion and Howitt (1998), but only for a specific set of parameter values.[18]

The latter results offer a theoretical explanation for the lack of positive correlation between innovation intensity (measured as R&D intensity) and economic growth found recently (e.g., OECD, 2006), particularly in countries situated below the technological frontier. In the context of our model, these would be countries approaching the BGP from above, and exhibiting a low speed of transition.[19]

Along the transition path. Q/N and g are positively related (see Figure 2). In Aghion and Howitt (1998), firm size, measured as the physical-capital stock per firm in efficiency units, is commanded by the physical-capital stock along the transition path. As long as the model exhibits the convergence property, it produces a positive relationship between firm size and aggregate growth. The models by Peretto (1998), Arnold (1998) and Funke and Strulik (2000) also generate a positive relationship along the transition path between aggregate growth and firm size, measured as employment per firm in the former and the human-capital stock per firm in the other two. Our results then imply that R&D intensity and firm size display a negative relationship along the transition path, although

[18]The mechanism in Aghion and Howitt (1998) is different from ours. They develop a quality-ladders model with physical capital as an input to R&D. Innovation, and hence aggregate growth, is stimulated by a rise in capital intensity towards its BGP level, whilst diminishing marginal returns to physical capital imply per se a fall in the aggregate-growth rate. For some parameter values, economic growth and the innovation rate move in opposite directions along the transition path.

[19]Available empirical evidence suggests slow transitions are the case in general. With a standard set of baseline parameters (see fn. 16), our model predicts a speed of convergence to the BGP of 2.2 percent/year, implying a half-life of roughly thirty-two years. This is within the range of estimates given by Barro and Sala-i-Martin (2004, ch. 11) for the US, Japan and several European countries.

their correlation may be either positive or negative with respect to shifts in the BGP, as explained below (see Section 4).

Empirical evidence of a positive correlation between aggregate growth and firm size (employment per firm) in the medium-run is provided by Jovanovic (1993) and Laincz and Peretto (2006), whilst Campbell (1998) reports evidence on the positive correlation between aggregate growth and the rate of entry (corresponding, in our model, to g_N). With respect to the latter relationship, it is noteworthy that the value of less than one predicted by our model is clearly matched by the empirical findings in Campbell (1998).

[Figure 2 goes about here]

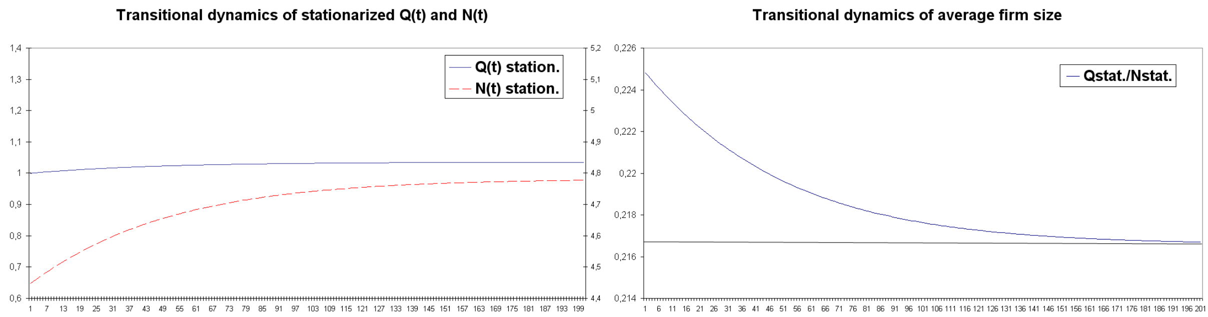

Figure 2: Transitional dynamics of N, Q and average firm size in stationarised terms, when $x(0) > x^*$.

4. Comparative-statics analysis

From (23) and Proposition 2, we have

$$N^*(t) = \left(\frac{\zeta \cdot L}{\phi}\right)^{\frac{1}{\sigma+\gamma+1}} \cdot (x^*)^{\frac{-\gamma}{\sigma+\gamma+1}} \cdot (Q^*(t))^{\frac{1}{\sigma+\gamma+1}} \tag{42}$$

$$E^*(t) = x^* \cdot N^*(t) \tag{43}$$

Differentiate (38), (39), (42) and (43) with respect to the relevant structural parameters. The following proposition summarises the main results.

Proposition 5.

(a) The aggregate growth rate, g^*, and the growth rate of the number of varieties, g_N^*, are: (i) decreasing in the fixed cost of vertical R&D, ζ, and in the elasticities of the horizontal entry cost function, σ and γ; (ii) increasing in the size of quality upgrade, λ; (iii) independent of the fixed cost of horizontal entry, ϕ.

(b) For a given Q^*, the number of firms in the differentiated-good sector, N^*, (alternatively, firm size as measured by Q^*/N^*) is decreasing (increasing) in σ, ϕ and λ, and is increasing (decreasing) in γ and ζ.[20] Instantaneous entry, E^*, for a given Q^*, is decreasing in σ, ϕ and ζ, and is increasing in γ and λ.

The results in part (a) pertain to the growth effects and part (b) to the level effects of changes in the structural parameters. The lack of relationship between g^* and ϕ is noteworthy. Intuitively, it results from the dominant effect exerted by the vertical-innovation mechanism (the intensive margin) over the horizontal-entry dynamics (the extensive margin). Given the postulated horizontal entry technology, an equilibrium BGP with positive net entry occurs ultimately because entrants expect incumbency value to grow propelled by quality-enhancing R&D. However, ϕ influences N^* and E^*, reducing both - and thus increasing firm size - for a given value of Q^*, due to its impact on (23). Arnold (1998) and Funke and Strulik (2000) obtain a similar result with respect to both the aggregate growth rate and firm size, while Peretto (1998) predicts a similar relationship with firm size but the author establishes a positive relationship with the aggregate growth rate.

Changes in σ and in γ have a negative impact on g^*, but their effects contrast when it comes to the impact on N^* and E^*. Parameter σ exerts a first-order effect on N^* and E^*, qualitatively similar to the impact due to changes in ϕ, whereas γ exerts a second-order effect. These results suggest that industry policies aimed at reducing the variable costs faced by entrants may have opposing outcomes with respect to their impact on the market structure and industry dynamics, depending on whether they target σ or γ.

For a given Q^*, the correlation between N^* and g_N^* tends to be negative. This conforms with some of the empirical evidence on the rate of entry and the number of firms reported, e.g., by Sherer and Ross (1990). Since the same relationship applies to g^*, our model also predicts that economies with higher long-run growth rates tend to have a market structure characterised by larger firms, in line with Peretto (1998) and Aghion and Howitt (1998).[21] A positive empirical relation between firm size (employment per firm) and the aggregate growth rate is found by Pagano and Schivardi (2003).

Our model offers mixed results with respect to the correlation between the vertical-innovation rate, I^* (see (37)), and both g^* and firm size. Shifts in the BGP due to changes in the size of quality upgrade, λ, or in the elasticities of the entry cost function, σ and γ, yield a *negative* relationship between the two variables, whilst changes in the preferences parameters or in ζ give rise to a *positive* correlation. The same results apply to the relationship between g^* (or firm size) and R&D intensity, $\frac{R_v^*+R_n^*}{Y^*-X^*}$.The impact of changes in σ and γ (respectively, λ) is explained by an endogenous inter-R&D composition

[20] In some particular cases, the alternative measures of firm size $\frac{X}{N} = H_X \frac{Q}{N}$ and $\frac{a}{N} = \zeta L \frac{Q}{N}$ yield different comparative static results from those in the text. For example, $\frac{a^*}{N^*}$ is increasing in ζ, whilst $\frac{Q^*}{N^*}$ and $\frac{X^*}{N^*}$ are both decreasing in that parameter.

[21] A positive correlation between firm size and growth is the general case in our model. However, that correlation is negative when a shift in the BGP is due to a variation in σ. Likewise, in Aghion and Howitt (1998), a change in the exogenous labour growth rate (implying a change in the rate of creation of differentiated goods, an effect qualitatively similar to a change in σ in our model) also implies a negative relationship between these two variables.

effect: an increase in these two parameters (decrease) induces a shift of resources from horizontal R&D to vertical R&D, but the effect on g^* of the increment in the vertical type is dominated by the decrease in the horizontal one. In the end, the observed negative correlation between g^* and R&D intensity reflects the negative correlation between g^* and R_v^*, whose effect outweights the positive relation between g^* and R_n^*. A similar reasoning would apply to the relation between firm size and R&D intensity. Also in Peretto (1998), Dinopoulos and Thompson (1998) and others, a change in horizontal entry costs induces a change of the vertical-innovation rate in the same direction. However, this implies a positive correlation with aggregate growth, because the latter changes due solely to the intensive margin (the extensive margin is linked to the exogenous population growth rate).

According to, e.g., Bassanini, Scarpetta, and Visco (2000), empirical studies in general find a strong positive relationship between R&D intensity and growth at the sectoral and firm level, but a clear link is usually difficult to establish at the aggregate (cross-country) level. On the other hand, Pagano and Schivardi (2003) note that empirical firm-level studies on the relationship between firm size and innovation have failed to reach a clear conclusion. Nevertheless, those authors conduct a cross-section study at the aggregate level and find that average firm size matters for growth through its effects on R&D intensity, thus implicitly establishing a positive correlation between firm size and innovation intensity. The literature often places the emphasis on the complexity of the link between these variables (Audretsch and Keilbacha, 2008), and the several conceptual and measurement problems that still afflict empirical analysis in this field (Bartelsman, Haltiwanger, and Scarpetta, 2005) to explain the lack of robust results. By producing mixed results in this respect, connected to the existence of an endogenous inter-R&D composition effect, our model lends theoretical support to the lack of clear-cut empirical findings. In other words, it shows that the shift of resources between vertical and horizontal R&D may hide the underlying positive relationship between R&D intensity and growth.

5. Conclusion

In this paper, we develop a non-scale tournament endogenous-growth model, where the expanding variety and the quality-ladders mechanism are merged under a lab-equipment specification. This assumption implies that the choice between vertical and horizontal innovation is related to the splitting of R&D expenditures, which are fully endogenous. Therefore, we endogenise the overall growth rate and thus the rate of extensive growth. Our framework gives rise to strong endogenous inter-R&D composition effects and makes economic growth and firm dynamics closely related: vertical R&D is the ultimate growth engine, whilst horizontal R&D builds an explicit link between aggregate and firm-dynamics variables.

The model predicts, under a sufficiently productive technology, a stationary BGP with constant positive growth rates, and where the consumption growth rate equals the growth rate of the number of varieties plus the growth rate of intermediate-good quality, in line with the general view that industrial growth proceeds both along an intensive and an extensive margin. The growth of the number of varieties is sustained by (endogenous) technological-knowledge accumulation, as the expected growth of intermediate-good quality makes it attractive for potential entrants to always put up an entry cost, in spite of its upward trend. In this setting, it is not necessarily the larger economies, measured by population size, that produce the greater number of distinct goods. Instead, the relevant distinction herein is based on an endogenous measure of economic size, such as the technological-knowledge stock.

We obtain specific results with respect to the impact of changes in the entry-cost parameters both in the aggregate growth rate and in the market structure along the BGP. We highlight (i) the lack of relationship between economic growth and the fixed horizontal entry cost, but the positive relation between the latter and firm size, (ii) the contrasting effect of changes in the two elasticity parameters of the entry cost function in firm size, (iii) the positive correlation between firm size and economic growth, and (iv) the mixed result with respect to the BGP relation between R&D intensity and both the aggregate growth rate and firm size.

We also conclude that the model exhibits the standard convergence property, though based on a rather distinct mechanism. The model produces results that differ from (or expand the results of) the early models of quality ladders with expanding variety. In particular, we obtain as a general result that medium-term economic growth and firm size are positively correlated, whereas R&D intensity and both medium-term economic growth and firm size move in opposite directions. The former result adds to the theoretical predictions already found in the literature, of positive correlation between economic growth and firm size measured either as employment per firm, human-capital stock per firm or physical-capital stock per firm in efficiency units, and which have had wide empirical support. Importantly, the latter result - together with (iv), above - offers a theoretical explanation for the insignificant or negative empirical correlation between aggregate R&D intensity and both economic growth and firm size, a well-known puzzle in the growth literature.

References

Aghion, P., and P. Howitt (1998): *Endogenous Growth Theory.* Cambridge, Massachusetts: MIT Press.

Arnold, L. G. (1998): "Growth, Welfare, and Trade in an Integrated Model of Human-Capital Accumulation and Research," *Journal of Macroeconomics*, 20 (1), 81–105.

Audretsch, D. B., and M. Keilbacha (2008): "Resolving the Knowledge Paradox: Knowledge-Spillover Entrepreneurship and Economic Growth," *Research Policy*, 37, 1697–1705.

Barro, R., and X. Sala-i-Martin (2004): *Economic Growth.* Cambridge, Massachusetts: MIT Press, second edn.

Bartelsman, E., J. Haltiwanger, and S. Scarpetta (2005): "Measuring and Analyzing Cross-Country Differences in Firm Dynamics," Work in progress.

Bassanini, A., S. Scarpetta, and I. Visco (2000): "Knowledge, Technology and Economic Growth: Recent Evidence from OECD Countries," *OECD Economics Department Working Papers*, 259, 1–38.

Brito, P., and H. Dixon (2009): "Entry and the Accumulation of Capital: a Two State-Variable Extension to the Ramsey Model," *International Journal of Economic Theory*, 5, 333–357.

Caballé, J., and M. S. Santos (1993): "On Endogenous Growth with Physical and Human Capital," *Journal of Political Economy*, 101(6), 1042–1067.

Campbell, J. R. (1998): "Entry, Exit, Embodied Technology, and Business Cycles," *Review of Economic Dynamics*, 1, 371–408.

Cozzi, G. (2007): "Self-Fulfilling Prophicies in the Quality Ladders Economy," *Journal of Development Economics*, 84, 445–464.

Dinopoulos, E., and P. Thompson (1998): "Schumpeterian Growth Without Scale Effects," *Journal of Economic Growth*, 3 (December), 313–335.

Ehrlich, E. (1985): "The Size Structure of Manufacturing Establishments and Enterprises: An International Comparison," *Journal of Comparative Economics*, 9, 267–295.

Etro, F. (2004): "Innovation by Leaders," *Economic Journal*, 114, 281–303.

——— (2008): "Growth Leaders," *Journal of Macroeconomics*, 30, 1148–1172.

Evans, G. W., S. M. Honkapohja, and P. Romer (1998): "Growth Cycles," *American Economic Review*, 88, 495–515.

Francois, P., and H. Lloyd-Ellis (2009): "Schumpeterian Cycles with Pro-cyclical R&D," *Review of Economic Dynamics*, forthcoming.

Freeman, C., and L. Soete (1997): *The Economics of Industrial Innovation*. Cambridge, MA: MIT Press.

Funke, M., and H. Strulik (2000): "On Endogenous Growth with Physical Capital, Human Capital and Product Variety," *European Economic Review*, 44, 491–515.

Geroski, P. (1995): "What Do We Know About Entry?," *International Journal of Industrial Organization*, 13, 421–440.

Gil, P. M. (2009): "Stylised Facts and Other Empirical Evidence on Firm Dynamics, Business Cycle and Growth," *Research in Economics*, forthcoming.

Gil, P. M., P. Brito, and O. Afonso (2008): "A Model of Quality Ladders with Horizontal Entry," *FEP Working Papers*, 296, 1–52.

Howitt, P. (1999): "Steady Endogenous Growth with Population and R&D Inputs Growing," *Journal of Political Economy*, 107(4), 715–730.

Jones, C. I., and J. C. Williams (2000): "Too Much of a Good Thing? The Economics of Investment in R&D," *Journal of Economic Growth*, 5, 65–85.

Jovanovic, B. (1993): "The Diversification of Production," *Brookings Papers on Economic Activity: Microeconomics*, 1, 197–247.

Klepper, S. (1996): "Entry, Exit, Growth, and Innovation over the Product Life Cycle," *American Economic Review*, 86 (3), 562–583.

Laincz, C. A., and P. F. Peretto (2006): "Scale Effects in Endogenous Growth Theory: An Error of Aggregation Not Specification," *Journal of Economic Growth*, 11, 263–288.

Maddison, A. (1994): "Explaining the Economic Performance of Nations," in *Convergence and Productivity*. Oxford University Press.

OECD (2006): *OECD in Figures: 2006-2007 Edition*. Paris: Organisation for Economic Co-operation and Development.

Pagano, P., and F. Schivardi (2003): "Firm Size Distribution and Growth," *Scandinavian Journal of Economics*, 105 (2), 255–274.

Peretto, P. (1998): "Technological Change and Population Growth," *Journal of Economic Growth*, 3 (December), 283–311.

——— (1999): "Cost Reduction, Entry, and the Interdependence of Market Structure and Economic Growth," *Journal of Monetary Economics*, 43, 173–195.

——— (2007): "Corporate Taxes, Growth and Welfare in a Schumpeterian Economy," *Journal of Economic Theory*, 137, 353–382.

Peretto, P., and M. Connolly (2007): "The Manhattan Metaphor," *Journal of Economic Growth*, 12, 329–350.

Peretto, P., and S. Smulders (2002): "Technological Distance, Growth and Scale Effects," *Economic Journal*, 112 (July), 603–624.

Rivera-Batiz, L., and P. Romer (1991): "Economic Integration and Endogenous Growth," *Quarterly Journal of Economics*, 106 (2), 531–555.

Segerstrom, P. (2000): "The Long-Run Growth Effects of R&D Subsidies," *Journal of Economic Growth*, 5, 277–305.

——— (2007): "Intel Economics," *International Economic Review*, 48 (1), 247–280.

Sherer, F., and D. Ross (1990): *Industrial Market Structure and Economic Performance*. Boston: Houghton Mifflin.

Thompson, P. (2001): "The Microeconomics of an R&D-based Model of Endogenous Growth," *Journal of Economic Growth*, 6, 263–283.

van de Klundert, T., and S. Smulders (1997): "Growth, Competition and Welfare," *Scandinavian Journal of Economics*, 99(1), 99–118.

Young, A. (1998): "Growth without Scale Effects," *Journal of Political Economy*, 106 (1), 41–63.

A. Derivation of equation (31)

The aggregate financial wealth held by all households is $L \cdot a(t) = \int_0^{N(t)} V(\omega, t) d\omega$. If we consider the arbitrage condition between vertical and horizontal entry, we get

$$L \cdot a(t) = \eta(t) \cdot N(t) \tag{44}$$

Hence, we can characterise the change in the value of equity as

$$L \cdot \dot{a}(t) = \eta(t) \cdot \dot{N}(t) + \dot{\eta}(t) \cdot N(t) \tag{45}$$

Solve (22) in order to $\dot{\eta}$ and, together with (44) and (28), substitute in (45) to get

$$r(t) \cdot L \cdot a(t) + L \cdot w(t) - L \cdot C(t) = \eta(t) \cdot \dot{N}(t) - \bar{\pi} \cdot N(t) +$$

$$+ (r(t) + I(t)) \cdot \eta(t) \cdot N(t) + \frac{\dot{\bar{\pi}}(t)}{\bar{\pi}(t)} \cdot \eta(t) \cdot N(t) = 0 \Leftrightarrow$$

$$\Leftrightarrow (L \cdot w(t) + \bar{\pi}(t) \cdot N(t)) - L \cdot C(t) - \eta(t) \cdot \dot{N}(t) - \left(I(t) \cdot \eta(t) \cdot N(t) + \frac{\dot{\bar{\pi}}(t)}{\bar{\pi}(t)} \cdot \eta(t) \cdot N(t) \right) = 0 \Leftrightarrow$$

$$\Leftrightarrow Y(t) - X(t) - L \cdot C(t) - \eta(t) \cdot \dot{N}(t) - \left(I(t) \cdot \eta(t) \cdot N(t) + \frac{\dot{\bar{\pi}}(t)}{\bar{\pi}(t)} \cdot \eta(t) \cdot N(t) \right) = 0 \tag{46}$$

where we used $Y = w \cdot L + p \cdot X \Leftrightarrow Y - X = w \cdot L + \bar{\pi} \cdot N$.[22] Next, recall that $R_n = \eta \cdot \dot{N}$ and $R_v = I \cdot \eta \cdot N + \frac{\dot{\bar{\pi}}}{\bar{\pi}} \cdot \eta \cdot N$,[23] such that (46) reads

$$Y(t) = X(t) + L \cdot C(t) + R_n(t) + R_v(t)$$

which is (31).

[22] Having in mind (1), (3), (8), (9) and (10), and that, in equilibrium, w and p are equated to the marginal product of labour and the marginal product of intermediate goods, respectively, it is easily shown that $w \cdot L = (1-\alpha) \cdot Y$, $X = \alpha^2 \cdot Y$, $p \cdot X = \alpha \cdot Y$ and total profits $\Pi = X \cdot (p-1) = \alpha \cdot Y - \alpha^2 \cdot Y$. Also, have in mind that given (10) and (19), total profits can be represented as $\Pi = \bar{\pi} \cdot N$.

[23] To see this, recall that $\frac{\dot{\bar{\pi}}}{\bar{\pi}} = I \cdot \left[\dot{j} \cdot \left(\frac{\alpha}{1-\alpha} \right) \cdot \ln \lambda \right] = I \cdot \left(\lambda^{\frac{\alpha}{1-\alpha}} - 1 \right)$ for $\dot{j} = 1$ and small λ, and use it together with the consistency condition (23), solved in order to ηN, in (16), to get $R_v = I \cdot \zeta \cdot L \cdot \lambda^{\frac{\alpha}{1-\alpha}} \cdot Q = I \cdot \eta \cdot N + \frac{\dot{\bar{\pi}}}{\bar{\pi}} \cdot \eta \cdot N$.

B. Necessary conditions for the existence of a BGP

First, we prove (i) and (ii). Substituting $R_v(t) = I(t) \cdot \zeta \cdot L \cdot \lambda^{\frac{\alpha}{1-\alpha}} \cdot Q(t)$, (8), (9) and (17) in (31), and simplifying the resultant considering (23) and the fact that, along the BGP, $\dot{N} = g_N N$, we have

$$H_Y Q(t) = H_X Q(t) + C(t) + \zeta g_N + I(t)\zeta\lambda^{\frac{\alpha}{1-\alpha}} Q(t) \tag{47}$$

where $H_Y \equiv \left(A^{\frac{1}{\alpha}}\alpha^2\right)^{\frac{\alpha}{1-\alpha}}$ and $H_X \equiv \left(A\alpha^2\right)^{\frac{1}{1-\alpha}}$. Time-differentiating (47) and solving in order to $\dot{I}$, yields

$$\dot{I}(t) = \frac{1}{\zeta\lambda^{\frac{1}{1-\alpha}} Q(t)} \left\{ \dot{Q}(t) \left[H_Y - H_X - \zeta g_N - I(t)\zeta\lambda^{\frac{\alpha}{1-\alpha}} \right] - \dot{C}(t) \right\} \Leftrightarrow$$

$$\Leftrightarrow g_I = \frac{1}{\zeta\lambda^{\frac{\alpha}{1-\alpha}} I(t)} \left\{ g_Q \left[H_Y - H_X - \zeta g_N - I(t)\zeta\lambda^{\frac{\alpha}{1-\alpha}} \right] - g_C \frac{C(t)}{Q(t)} \right\} \tag{48}$$

Along the BGP, g_I, g_Q and g_C must be constant. Consider three scenarios:

(a) Suppose $g_I, g_Q, g_C \neq 0$ and $g_I \neq g_Q \neq g_C$. From (48), this implies that g_I changes as I and $\frac{C}{Q}$ change over time, which is a contradiction.

(b) Suppose $g_I, g_Q, g_C \neq 0$ and $g_Q = g_C = g$, then we have, from (48),

$$g_I = \frac{1}{\zeta\lambda^{\frac{\alpha}{1-\alpha}} I(t)} \left\{ g \left[H_Y - H_X - \zeta g_N - I(t)\zeta\lambda^{\frac{\alpha}{1-\alpha}} - \frac{C(t)}{Q(t)} \right] \right\} \Leftrightarrow g_I = 0$$

which is a contradiction.

(c) Suppose $g_I = 0$ and $g_Q \neq g_C$, then we have, from (48),

$$g_C = \frac{Q}{C} g_Q \left[H_Y - H_X - \zeta g_N - I(t)\zeta\lambda^{\frac{\alpha}{1-\alpha}} \right]$$

This implies that g_C changes as $\frac{Q}{C}$ changes over time, which is also a contradiction.

Then, it must be true that, along the BGP, $g_Q = g_C = g$ and $g_I = 0$. □

The proof of (iii) is as follows. From (24), using (23), we have

$$\frac{\dot{N}(t)}{N(t)} = g_N(t) = \left(\frac{\zeta L}{\phi}\right)^{\frac{1}{\gamma}} Q^{\frac{1}{\gamma}} N^{-\left(\frac{\sigma+\gamma+1}{\gamma}\right)} \equiv x(Q, N) \tag{49}$$

Since g_N must be constant along the BGP, then, by time-differentiating the equation above and equating to zero, we find

$$x \left[\frac{1}{\gamma} \frac{\dot{Q}(t)}{Q(t)} - \left(\frac{\sigma+\gamma+1}{\gamma}\right) \frac{\dot{N}(t)}{N(t)} \right] = 0 \Leftrightarrow \frac{g_Q}{g_N} = (\sigma+\gamma+1)$$

provided $x \neq 0$. □

C. Proof of Proposition 1

Proof. Using the Caballé and Santos (1993) decomposition, we write $C(t) = c(t)e^{g_C t}$, $Q(t) = \tilde{q}(t)e^{g_Q t}$ and $N(t) = n(t)e^{g_N t}$. We can build a detrended system in $(c, \tilde{q}, n)$ only if $g_C = g_Q = g$ and $g_Q = g_N(\sigma+\gamma+1)$. If those conditions hold, then $x(\tilde{q}, n) = x(Q, N)$ and $I^+(C, Q, N) = I^+(c, \tilde{q}, n)$, and an equivalent system for the detrended variables is obtained: $\dot{n} = (x(\tilde{q}, n) - g_N) \cdot n$, $\dot{c} = (\mu - I^{\pm}(c, \tilde{q}, n) - \theta \cdot g) \cdot (c/\theta)$, $\dot{\tilde{q}} = (I^{\pm}(c, \tilde{q}, n) \cdot \Xi + x(\tilde{q}, n) - g_Q) \cdot \tilde{q}$. Now, consider the steady state of this system $(c^*, \tilde{q}^*, n^*)$. Firstly, observe that if we have $H_y - H_x - c^*/\tilde{q}^* - \zeta n^*/\tilde{q}^* < 0$, then $I = I^- = 0$. In this case, there is no BGP because we should have $g_Q = g_N = 0$, which contradicts the fact that $g_N = x(\tilde{q}, n) > 0$ if $\tilde{q} > 0$ and $n > 0$. Secondly, consider $H_y - H_x - c^*/\tilde{q}^* - \zeta n^*/\tilde{q}^* > 0$. In this case, we get a steady state for $I = I^+ = I^*$, as in (37), which is positive if $\mu > 0$. Then, there is a unique BGP, $C^*(t) = c^* e^{g^* t}$, $Q^*(t) = \tilde{q}^* e^{g^* t}$, $N^*(t) = n^* e^{g_N^* t}$, in which the long run growth rates g^* and g_N^*, as in (38) and (39), are positive if $\mu > 0$. At last, the transversality condition $\lim_{t\to\infty} \zeta L(c^*)^{-\theta} \tilde{q}(t) e^{-(\rho+(\theta-1)g^*)t} = 0$ holds if $\theta \geq 1$. □

D. Proof of Proposition 2

Proof. Consider the change in variables, $z = C/Q$ and $x = \phi/(L\zeta)Q^{1/\gamma}N^{-(\sigma+\gamma+1)}$, and the equivalent system (40a)-(40b) in detrended variables and Proposition 1. If the BGP is unique in the initial system (35a)-(35c), it is also unique in this new system. We can also prove that, in this new system, an admissible steady state $(x^*, z^*) \in \mathbb{R}^2_{++}$, can also exist if $I^* = I^+(x^*, z^*) > 0$. However, those steady-state level variables should be positive. Sufficient conditions for this are that $\mu > 0$ and that the isocline $\dot{z} = 0$ crosses the axis z, for $x = 0$, at non-negative values, that is $H_y - H_x > \zeta\lambda^{\alpha/(1-\alpha)}(r_0 - \rho)/(1 + \theta\Xi)$. At last, as is usual in endogenous growth models, if we consider the three original variables, C, Q, and N, the dynamic system in under-determinate. As a natural choice for a determinate variable is the number of varieties, we can determine explicit values along the BGP by choosing $N(0)$ as given. Therefore, the economy will be placed along a BGP if $Q(0) = (\phi/\zeta)(x^*)^\gamma N(0)^{\sigma+\gamma+1}$. □

E. Proof of Proposition 3

Proof. Consider the system (40a)-(40b) and its admissible steady state (x^*, z^*), where $I^* = I^+(x^*, z^*) > 0$. The Jacobian evaluated at that steady state is

$$J(x^*, z^*) = \begin{pmatrix} a_{11}z^* & a_{12}x^* \\ a_{21}z^* & a_{22}x^* \end{pmatrix} = \frac{1}{\lambda^{\frac{\alpha}{(1-\alpha)}}} \begin{pmatrix} (1+\theta\Xi)z^*/(\theta\zeta) & (1-\theta)x^*/\theta \\ -\Xi z^*/(\gamma\zeta) & -[\Xi(\sigma+\gamma+1)+\sigma+\gamma]x^*/\gamma \end{pmatrix}$$

As $\det(J(x^*, z^*)) = -[\Xi(\sigma+\gamma+1)+\sigma+\gamma]/(\gamma\zeta\lambda^{\frac{\alpha}{(1-\alpha)}}) < 0$, then $J(x^*, z^*)$ has one negative, $\lambda_s < 0$, and one positive eigenvalue. The slope of the eigenspace associated to the negative eigenvalue is $-(a_{22}x^* - \lambda_s)/(a_{21}z^*)$. As $a_{21} < 0$ then the slope has the same sign as $a_{22}x^* - \lambda_s$. After some algebra, we get

$$a_{22}x^* - \lambda_s = -\frac{1}{2\lambda^{\frac{\alpha}{(1-\alpha)}}}\left[\left(\Xi(\sigma+\gamma+1)+\frac{1}{\theta}(\sigma+\gamma)\right)\frac{x^*}{\gamma}+\frac{(1+\theta\Xi)z^*}{\theta\zeta}\right]+\Delta(J^*)^{1/2}$$

where the discriminant of the Jacobian is

$$\begin{aligned}\Delta(J^*) &= \left\{\frac{1}{2\lambda^{\frac{\alpha}{(1-\alpha)}}}\left[\left(\Xi(\sigma+\gamma+1)+\frac{1}{\theta}(\sigma+\gamma)\right)\frac{x^*}{\gamma}+\frac{(1+\theta\Xi)z^*}{\theta\zeta}\right]\right\}^2+\\ &+\frac{4z^*x^*}{\zeta\gamma}\Xi\left(\sigma+\gamma+1-\frac{1}{\theta}\right).\end{aligned}$$

Then $a_{22}x^* - \lambda_s > 0$ if $\theta \geq 1$ and the local stable manifold is positively sloped in the neighbourhood of (x^*, z^*).

The phase diagram in Figure 1 allows for a geometric characterisation of the dynamics of transition. Observe that curve $I^+(x,z) = 0$, that we will call switching curve, divides the state space into two zones: in the northeast area, where, as $I^+(x,z) < 0$, we set $I = 0$ and the dynamics is given by the first branch, and in the southwest area, where we have $I^+(x,z) > 0$ and the dynamics is given by the second branch. The isocline $\dot{x} = 0$ is negatively sloped and lies entirely in the second branch. The isocline $\dot{z} = 0$ passes through the two branches: it has a positive intercept in the second branch if $H_y - H_x > \zeta\lambda^{\alpha/(1-\alpha)}(r_0-\rho)/(1+\theta\Xi)$, has a positive slope and cuts the switching curve at point $x = \mu$, where it is continuous but piecewise-smooth, and is vertical in the first branch. As x is a predetermined variable, this means that if $x(0) < \mu$ then the transition path lies entirely in the second branch, and approaches the BGP values (x^*, z^*) as shown in the figure because the slope of the stable manifold is flatter than the isocline $\dot{z} = 0$. This can be easily proved if it is observed that

$$\left.\frac{dz}{dx}\right|_{\dot{z}=0} - \left.\frac{dz}{dx}\right|_{W^s} = -\frac{a_{12}x^*}{a_{11}z^*}+\frac{a_{22}x^*-\lambda_s}{a_{21}z^*} = \frac{\lambda_u + a_{11}z^*}{a_{11}a_{21}(z^*)^2}\lambda_s > 0.$$

If $x(0) > \tilde{x}$, the transition path is a concatenation of a transition path lying in the first branch with the transition path in the second branch. This can be proved if we observe that: first, the paths in the first branch to the right of the isocline $\dot{z} = 0$ cross the switching curve in a point $\tilde{x} > \mu$ but smaller than $\frac{\Xi(\sigma+\gamma+1)+\sigma+\gamma}{\Xi(\sigma+\gamma+1)+\frac{1}{\theta}(\sigma+\gamma)}\mu$, which is the projection of the steady state z^* into the switching curve, because of the slope of the stable manifold; second, the paths starting in the first branch to the right of the isocline $\dot{z} = 0$ approach the switching curve and cross it. We can prove that there is no other form of collision, by computing the projections of the vector fields in both sides of the switching curve

$$\frac{\partial I^+}{\partial z}\dot{z}^- + \frac{\partial I^+}{\partial x}\dot{x}^- = \frac{\partial I^+}{\partial z}\dot{z}^+ + \frac{\partial I^+}{\partial x}\dot{x}^+ = \frac{1}{\zeta\lambda^{\frac{\alpha}{(1-\alpha)}}}\left[\left(\frac{\sigma+\gamma}{\gamma}x^2+\zeta(x-\mu)z\right)\right] > 0$$

because the collision should take place for $x > \mu$. This means that the path coming from the first branch approaches the switching curve with the same direction as the one defined by the vector field for $I^+ < 0$, that is, both x and z decrease, which means that it will cross the switching boundary and continue with the same direction inside the second branch. We cannot determine the crossing point exactly, but we know that it should be in the intersection of the stable manifold in the second branch with the switching boundary. This means that the stable manifold is piecewise-smooth and lies in the two branches as depicted in Figure 1. □

Recent FEP Working Papers

Nº 355	Aurora A.C. Teixeira and José Miguel Silva, "*Emergent and declining themes in the Economics and Management of Innovation scientific area over the past three decades*", January 2010
Nº 354	José Miguel Silva and Aurora A.C. Teixeira, "*Identifying the intellectual scientific basis of the Economics and Management of Innovation Management area*", January 2010
Nº 353	Paulo Guimarães, Octávio Figueiredo and Douglas Woodward, "*Accounting for Neighboring Effects in Measures of Spatial Concentration*", December 2009
Nº 352	Vasco Leite, Sofia B.S.D. Castro and João Correia-da-Silva, "*A third sector in the core-periphery model: non-tradable goods*", December 2009
Nº 351	João Correia-da-Silva and Joana Pinho, "*Costly horizontal differentiation*", December 2009
Nº 350	João Correia-da-Silva and Joana Resende, "*Free daily newspapers: too many incentives to print?*", December 2009
Nº 349	Ricardo Correia and Carlos Brito, "*Análise Conjunta da Dinâmica Territorial e Industrial: O Caso da IKEA – Swedwood*", December 2009
Nº 348	Gonçalo Faria, João Correia-da-Silva and Cláudia Ribeiro, "*Dynamic Consumption and Portfolio Choice with Ambiguity about Stochastic Volatility*", December 2009
Nº 347	André Caiado, Ana Paula Africano and Aurora A.C. Teixeira, "*Firms' perceptions on the usefulness of State trade missions: an exploratory micro level empirical analysis*", December 2009
Nº 346	Luís Pinheiro and Aurora A.C. Teixeira, "*Bridging University-Firm relationships and Open Innovation literature: a critical synthesis*", November 2009
Nº 345	Cláudia Carvalho, Carlos Brito and José Sarsfield Cabral, "*Assessing the Quality of Public Services: A Conceptual Model*", November 2009
Nº 344	Margarida Catarino and Aurora A.C. Teixeira, "*International R&D cooperation: the perceptions of SMEs and Intermediaries*", November 2009
Nº 343	Nuno Torres, Óscar Afonso and Isabel Soares, "*Geographic oil concentration and economic growth – a panel data analysis*", November 2009
Nº 342	Catarina Roseira and Carlos Brito, "*Value Co-Creation with Suppliers*", November 2009
Nº 341	José Fernando Gonçalves and Paulo S. A. Sousa, "*A Genetic Algorithm for Lot Size and Scheduling under Capacity Constraints and Allowing Backorders*", November 2009
Nº 340	Nuno Gonçalves and Ana Paula Africano, "*The Immigration and Trade Link in the European Union Integration Process*", November 2009
Nº 339	Filomena Garcia and Joana Resende, "*Conformity based behavior and the dynamics of price competition: a new rational for fashion shifts*", October 2009
Nº 338	Nuno Torres, Óscar Afonso and Isabel Soares, "*Natural resources, economic growth and institutions – a panel approach*", October 2009
Nº 337	Ana Pinto Borges, João Correia-da-Silva and Didier Laussel, "*Regulating a monopolist with unknown bureaucratic tendencies*", October 2009
Nº 336	Pedro Rui Mazeda Gil, "*Animal Spirits and the Composition of Innovation in a Lab-Equipment R&D Model*", September 2009
Nº 335	Cristina Santos and Aurora A.C. Teixeira, "*The evolution of the literature on entrepreneurship. Uncovering some under researched themes*", September 2009
Nº 334	Maria das Dores B. Moura Oliveira and Aurora A.C. Teixeira, "*Policy approaches regarding technology transfer: Portugal and Switzerland compared*", September 2009
Nº 333	Ana Sofia Ferreira, Leonídio Fonseca and Lilian Santos, "*Serão os 'estudantes empreendedores' os empreendedores do futuro? O contributo das empresas juniores para o empreendedorismo*", August 2009
Nº 332	Raquel Almeida, Marina Silva and Tiago Soares, "*Coesão Territorial - As relações de fronteira entre Portugal e Espanha*", August 2009
Nº 331	Custódia Bastos, Suzi Ladeira and Sofia Silva, "*Empreendedorismo nas Artes ou Artes do Empreendedorismo? Um estudo empírico do 'Cluster' da Rua Miguel Bombarda*", August 2009
Nº 330	Filipe A. Ribeiro, Ana N. Veloso and Artur V. Vieira, "*Empreendedorismo Social: Uma*

	análise via associativismo juvenil", August 2009
Nº 329	Argentino Pessoa, "*Outsourcing And Public Sector Efficiency: How Effective Is Outsourcing In Dealing With Impure Public Goods?"*, July 2009
Nº 328	Joana Almodovar, Aurora A.C. Teixeira, "*Conceptualizing clusters through the lens of networks: a critical synthesis"*, July 2009
Nº 327	Pedro Mazeda Gil, Fernanda Figueiredo and Óscar Afonso, "*Equilibrium Price Distribution with Directed Technical Change"*, July 2009
Nº 326	Armando Silva, Ana Paula Africano and Óscar Afonso, "*Which Portuguese firms are more innovative? The importance of multinationals and exporters"*, June 2009
Nº 325	Sofia B. S. D. Castro, João Correia-da-Silva and Pascal Mossay, "*The core-periphery model with three regions"*, June 2009
Nº 324	Marta Sofia R. Monteiro, Dalila B. M. M. Fontes and Fernando A. C. C. Fontes, "*Restructuring Facility Networks under Economy of Scales"*, June 2009
Nº 323	Óscar Afonso and Maria Thompson, "*Costly Investment, Complementarities and the Skill Premium"*, April 2009
Nº 322	Aurora A.C. Teixeira and Rosa Portela Forte, "*Unbounding entrepreneurial intents of university students: a multidisciplinary perspective"*, April 2009
Nº 321	Paula Sarmento and António Brandão, "*Next Generation Access Networks: The Effects of Vertical Spillovers on Access and Innovation"*, April 2009
Nº 320	Marco Meireles and Paula Sarmento, "*Incomplete Regulation, Asymmetric Information and Collusion-Proofness"*, April 2009
Nº 319	Aurora A.C. Teixeira and José Sequeira, "*Determinants of the international influence of a R&D organisation: a bibliometric approach"*, March 2009
Nº 318	José Sequeira and Aurora A.C. Teixeira, "*Assessing the influence of R&D institutions by mapping international scientific networks: the case of INESC Porto"*, March 2009
Nº 317	João Loureiro, Manuel M. F. Martins and Ana Paula Ribeiro, "*Cape Verde: The Case for Euroization"*, March 2009
Nº 316	Ester Gomes da Silva and Aurora A.C. Teixeira, "*Does structure influence growth? A panel data econometric assessment of 'relatively less developed' countries, 1979-2003"*, March 2009
Nº 315	Mário A. P. M. Silva, "*A Model of Growth with Intertemporal Knowledge Externalities, Augmented with Contemporaneous Knowledge Externalities"*, March 2009
Nº 314	Mariana Lopes and Aurora A.C. Teixeira, "*Open Innovation in firms located in an intermediate technology developed country"*, March 2009
Nº 313	Ester Gomes da Silva, "*Capital services estimates in Portuguese industries, 1977-2003"*, February 2009
Nº 312	Jorge M. S. Valente, Maria R. A. Moreira, Alok Singh and Rui A. F. S. Alves, "*Genetic algorithms for single machine scheduling with quadratic earliness and tardiness costs"*, February 2009
Nº 311	Abel Costa Fernandes, "*Explaining Government Spending: a Cointegration Approach"*, February 2009
Nº 310	João Correia-da-Silva, "*Uncertain delivery in markets for lemons"*, January 2009
Nº 309	Ana Paula Ribeiro, "*Interactions between Labor Market Reforms and Monetary Policy under Slowly Changing Habits"*, January 2009
Nº 308	Argentino Pessoa and Mário Rui Silva, "*Environment Based Innovation: Policy Questions"*, January 2009

Editor: Sandra Silva (sandras@fep.up.pt)
Download available at:
http://www.fep.up.pt/investigacao/workingpapers/
also in http://ideas.repec.org/PaperSeries.html

www.fep.up.pt

Faculdade de Economia da Universidade do Porto
Rua Dr. Roberto Frias, 4200-464 Porto | Tel. 225 571 100
Tel. 225571100 | www.fep.up.pt