# PERFECT COMMUNICATION EQUILIBRIA IN REPEATED GAMES WITH IMPERFECT MONITORING

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Abstract. The aim of this paper is to study a notion of perfectness for communication equilibira in repeated games, to characterize the corresponding set of equilibrium payoffs and finally to derive a Folk theorem.

## 1. Introduction

The central result in the theory of repeated games is the Folk Theorem which states that when players are patient enough, every feasible and individually rational payoff vector can be sustained by an equilibrium. The equilibrium construction is well known: players agree on a contract that specifies the actions to play and in case of deviation from the contract, the deviating player is punished to his individually rational level. This reasoning relies on the assumption that actions are publicly observable. Since many economic situations are modelled by games with imperfect observation, a challenging problem is to characterize the set of equilibrium payoffs when players get imperfect and private signals.

Much is known about games with public signals and perfect public equilibria who have a natural recursive structure studied in Abreu, Pearce and Stachetti [APS90]. An adaptation of the Bellmann equation from dynamic programming to those games permits to characterize the set of payoffs associated to perfect public equilibria of the $\delta$-discounted game as the largest fixed point of some correspondance. The asymptotic behavior of this set as the discount factor $\delta$ goes to 1 is also well known and due to Fundenberg, Levine and Maskin [FLM94] and Fudenberg and Levine [FL94]. It has been recognized (see the special JET issue) that generally, restricting to public strategies might bound equilibrium payoffs away from efficiency, even in games with public signals. However, without public strategies, sequential equilbria lose their recursive properties. A way to circumvent this difficulty is to allow for some kind of communication between players to restore efficiency of equilibria: Ben-Porath and Kahneman [BPK96], Compte [Com98] and Kandori and Matsushima [KM98], consider models where players communicate by making public announcments.

The aim of the present paper is to study communication equilibria (Myerson [Mye82], Forges [For86]) of repeated games with general signals since this notion covers all kinds of possible communication technologies and thus provides an upper bound on the set of equilibrium payoffs; and further to study them by recursive methods *à la* [APS90], [FLM94] and [FL94]. It thus combines two approaches: the systematic study of communication equilibria of repeated games with imperfect monitoring, conducted in Renault and Tomala [RT04] for undiscounted games, and the use of Fundenberg and Levine's [FL94] characterization in games with *public* communication made by Kandori and Matsushima [KM98].

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We introduce an agent called *mediator* who is external to the game and has the ability to communicate with players between stages. The mediator has no interest in the outcome of the game an does not observe neither the actions played nor the signals observed by the players. After each stage, the mediator receives messages (inputs) from the players and sends them back some outputs. No restriction in placed on the set of inputs/outputs or on the way the mediator selects outputs, thus any specification of a communication technology can be seen as a particular instance of the mediator. A communication equilibrium is then a specification of the technology used by the mediator (a communication device) and a Nash equilibrium of the game defined by the repeated game with communication through the device. A direct application of the revelation principle says that any communication equilibrium outcome can be obtained by a canonical communication equilibrium where: (1) the device recommends privately each player which action to play at each stage, each player reports a signal to the mediator and (2) each player has an incentive to always play the recommended action and to report the observed signal, i.e. to play *faithfully.* A canonical communication equilibrium is thus a relatively simple object: it is simply a strategy of the mediator who chooses profiles of recommended actions and observes profiles of reported signals, such that the profile of faithful strategies is a Nash equilibrium of the game induced by the communication device.

We introduce the concept of *perfect communication equilibrium* which simply a canonical communication device such that after every history of the mediator, the communication device that starts with this history is a canonical communication equilibrium. For such a device, playing faithfully is a sequential equilibrium of the extended game. This concept is stronger than sequential communication equilibrium of Myerson [Mye86]. Indeed, playing faithfully is a *belief-free* equilibrium of the extended game as introduced in Ely *et al.* [Ely03].

The set of payoffs associated to perfect communication equilibria $C(\delta)$ behaves like a set of perfect public equilibrium payoffs, i.e. it has at recursive structure *à la* [APS90], the asymptotic properties are similar and most of the analysis made by [FLM94] and [FL94] carries out. We thus give a characterization of the limit of $C(\delta)$ as $\delta$ goes to one, we study the limit set and, as Kandori and Matsushima [KM98] did for public communication, we derive sufficient conditions for a Folk theorem to hold.

The model of repeated games and communication equilibria are described in section 2. In section 3, we introduce our perfect communication equilibria and discuss links with sequential equilibria and perfect public equilibria. The recursive structure is studied in section 4. The characterization of the limit set of perfect communication equilibrium payoffs is given in section 5. We give properties of the limit set in section 6. Our version of the Folk Theorem is in section 7.

## 2. Model and definitions

2.1. **The game.** We present a general model of repeated games with complete information and imperfect monitoring. The data of the game are: a finite set of players $\{1, \ldots, n\}$; for each player $i$ a finite set of actions $A^i$, a finite set of signals $Y^i$ and a reward function $r^i : \prod_j A^j \times \prod_j Y^j \to \mathbb{R}$; a signaling technology given by a transition probability $q : \prod_j A^j \to \prod_j Y^j$.

The game is played in discrete time. At each stage $t = 1, 2, \ldots$, each player $i = 1, \ldots, n$ chooses an action $a^i$ from his own set of action and if $a = (a^1, \ldots, a^n)$ is the action profile played, a profile of signals $y = (y^1, \ldots, y^n)$ is selected with probability $q(y|a)$. Each player $i$ observes then $y^i$ and the game goes to stage $t+1$. Player $i$ realized payoff at this stage is $r^i(a, y)$ and is not directly observed. We denote $A = \prod_j A^j$ the set of action profiles and $Y = \prod_j Y^j$ the set of signal profiles. Players discount payoff at a common rate $0 \le \delta < 1$ and if $(a_t, y_t)_{t\ge 1}$ is the sequence of realized action and signal profiles, the discounted payoff of player $i$ is

$$\sum\nolimits_{t\ge 1}(1-\delta)\delta^{t-1}r^i(a_t, y_t).$$

This model encompasses several well known models and in particular, games with deterministic and/or public signals. The signals are said to be deterministic when for each action profile $a$, there is a unique signal profile such that $q(y|a) = 1$. In such case, the signaling structure can be represented by mappings $(f^i)_{i=1,\dots,n}$ with $f^i : A \to Y^i$. A particular case considered in some papers (Ben-Porath and Kahnemann [BPK96], Renault and Tomala [RT98]) is when players are arranged in a network, i.e. are vertices of a graph and monitor the action of their neighbors. Another particular case thoroughly studied in the literature is games with public signals. Signals have a *public* part when each players signal set can be written as $Y^i = Z^i \times Z^p$ and when the signalling technology selects for each player a private signal $z^i$ in $Z^i$ and a public signal $z^p$ in $Z^p$. Each player $i$ then observes his private signal and all players observe the public signal. In our model, this is simply a restriction imposed on the transition probability $q$ which must select the same element of $z^p$ for each player.

Note that in the previous description, the stage-game is not completely defined. The expected payoff to player $i$ when the action profile $a = (a^1, \dots, a^n)$ is played is:

$$g^i(a) = \sum\nolimits_y q(y|a)r^i(a, y)$$

The normal form game $G = (A^1, \dots, A^n; g^1, \dots, g^n)$ will be thus refered to as the stage-game or one-shot game. Also in the sequel we shall deal with correlated distribution on actions. Given a probability distribution $p$ on $A$, the expected payoff to player $i$ is:

$$g^i(p) = \sum\nolimits_a p(a)g^i(a)$$

and the payoff vector yielded by $p$ is $g(p) = (g^1(p), \dots, g^n(p))$. The set of feasible payoffs (the feasible set) is the set of such vectors as $p$ varies:

$$V = \{g(p) \mid p \in \Delta(A)\}$$

This is also the convex hull of payoff vectors associated to action profiles, i.e. setting $g(a) = (g^1(a), \dots, g^n(a))$, $V$ is the convex hull of the set $\{g(a) \mid a \in A\}$.

Throughout the paper, we will use the following notations. If $(E^i)_{i=1,\dots,n}$ is a collection of sets indexed on players, $E$ will denote $\prod_{i\in N} E^i$, an element $(e^i)_{i\in N}$ in $E$ will be denoted by $e$, and $e^{-i}$ will stand for the current element of $E^{-i} = \prod_{j\ne i} E^j$. We write $e = (e^i, e^{-i})$ when the $i$-th component is stressed. If $E$ is a finite set, $|E|$ will denote its cardinality and $\Delta(E)$ the set of probability distributions over $E$. An element $e$ in $E$ will be identified with the Dirac mass on $e$. For $p = (p(e))_{e\in E}$ in $\Delta(E)$, $\operatorname{supp} p$ will denote the support of $p$. We also let $\operatorname{int}\Delta(E)$ be the interior of $\Delta(E)$, i.e. the set of $p$'s with full support.

2.2. **Communication equilibria.** We describe now the notion of communication equilibria due to Myerson [Mye82], [Mye86] and Forges [For86].

We add to the game an external player called *the mediator* who has no interest in the outcome of the game, who cannot observe actions or signals but who can communicate with players between stages. At the end of each stage, each player can send privately *a message* (called an input in Forges) to the mediator. The mediator reads all his messages and produces for each player *a recommendation* (called output in Forges) which he sends privately to this player. The mediator knows only past messages and recommendations and does not observe the play of the game. No restriction is posed on the sets of messages and recommendations, in particular they might be infinite, stage or history dependent. No restriction is made either on the way the mediator forms recommendations from messages. This allows the model to cover all possible methods of communication, for instance models where each player makes a public announcment (see e.g. [BPK96], [Com98], [KM98]) are particular cases since all are specifications of unmediated talk between players. A *communication device* is a specification of the communication method chosen by the mediator.

**Definition 2.1.** *A communication device is a tuple* $\mathcal{C} = (M_t^i, R_t^i, c_t)_{i=1,\dots,n;t\geq 1}$ *where for each player* $i$ *and stage* $t$*:*

- $M_t^i$ *is the set of messages for player* $i$ *at stage* $t$ *and* $R_t^i$ *is the set of recommendations for player* $i$ *at stage* $t$*. All these sets are measurable sets, the* $\sigma$*-fields are left unspecified for notational simplicity.*
- $c_t$ *is the rule used by the mediator at stage* $t$ *to select new recommendations:*
  - $c_1$ *is a probability distribution on* $\prod_i R_1^i$ *endowed with the product* $\sigma$*-field.*
  - *For each* $t > 2$, $c_t$ *is a measurable mapping from* $\prod_{s<t}(\prod_i R_s^i \times \prod_i M_s^i)$ *to the set of probabilities on* $\prod_i R_t^i$*, all product sets being endowed with the product* $\sigma$*-field.*

Given a communication device $\mathcal{C}$, the repeated game extended by communication by $\mathcal{C}$ is played as follows. At the beginning of stage 1, the mediator selects $r_1 = (r_1^1, \dots, r_1^n)$ according to $c_1$ and sends privately the recommendation $r_1^i$ to player $i$. Each player $i$ then chooses an action $a_1^i$. A profile of signals $y_1$ is drawn according to $q(\cdot|a_1)$ and player $i$ privately observes $y_1^i$. Each player $i$ then sends privately a message $m_1^i$ to the mediator and this concludes the first stage. Stage $t$ is described similarly except that the recommendation profile $r_t = (r_t^1, \dots, r_t^n)$ is selected according to $c_t(\cdot)$ which depends on past recommendations and messages. The game is infinitely repeated and payoffs are discounted at the common discount factor $\delta$.

A behavioral strategy for player $i$ in this game is a sequence $\sigma^i = (\alpha_t^i, \mu_t^i)_{t\geq 1}$ where for each $t$, the first component $\alpha_t^i$ gives the mixed action played by player $i$ at stage $t$ depending on his whole past history: the recommendations he got, the actions he played, the signals he observed and the messages he sent to the mediator; it is thus a mapping $\alpha_t^i : \prod_{s<t}(R_s^i \times A^i \times Y^i \times M_s^i) \times R_t^i \longrightarrow \Delta(A^i)$. The second component $\mu_t^i$ gives the probability distribution on messages sent by player $i$ at stage $t$, depending also on the past, it is a mapping $\mu_t^i : \prod_{s<t}(R_s^i \times A^i \times Y^i \times M_s^i) \times R_t^i \times A^i \times Y^i \longrightarrow \Delta(M_t^i)$. As above, product sets are endowed with the product $\sigma$-field and all mappings are measurable.

A communication device and a profile of behavior strategy induce a probability distribution on the set of infinite sequences of action and signal profiles. Since payoffs are uniformly bounded, the discounted expected payoff is well defined and the definition of the repeated game extended by $\mathcal{C}$ is complete.

**Definition 2.2.** *A communication equilibrium of the repeated game is a pair* $(\mathcal{C}, \sigma)$ *where* $\mathcal{C}$ *is a communication device and* $\sigma$ *is a Nash equilibrium of the game extended by* $\mathcal{C}$*.*

This concept generalizes both Nash equilibria and correlated equilibria of the repeated game: If all set of messages and recommendations are singletons, the mediator is inactive and the correpsonding communication equilibrium is simply a

Nash equilibrium. A correlated equilibrium is simply a communication equilibrium where all sets of recommendations and messages are singletons except for the $R_1^i$'s.

At first sight, the concept may look too general to be tractable. Fortunately, a revelation principle applies which considerably simplifies the analysis.

**Definition 2.3.** (1) *A communication device is called canonical if for each player $i$ and stage $t$, $R_t^i = A^i$ and $M_t^i = Y^i$, i.e. the mediator suggests player $i$ an action to play at stage $t$, and player $i$ reports a signal to the mediator.*

(2) *Consider the strategy of player $i$ that plays at each stage after each history the action recommended by the mediator at that stage and as messages sends at stage $t$ the value of the signal he observed at that stage. This strategy is called the* faithful strategy.

(3) *A communication equilibrium is called canonical if the communication device is canonical and the profile of faithful strategies is a Nash equilibrium of the game extended by the device.*

Obviously, a canonical communication equilibrium is a communication equilibrium. Consider a communication equilibrium $(C, \sigma)$ and assume now that the mediator not only operates $C$ but also performs the computations and randomizations defined by $\sigma^i$ and just informs player $i$ of the results of these randomizations. This means that player $i$, instead of randomizing himself, relies on the mediator to do this job. This defines a canonical communication equilibrium that induces the same payoff for each player as $(C, \sigma)$. These arguments are standard and we summarize this discussion by the following lemma.

**Lemma 2.4** (The revelation principle)**.** *For every communication equilibrium, there is a canonical communication equilibrium that induces the same payoff for each player.*

## 3. The solution concept

3.1. **Definition.** We concentrate now on canonical communication equilibria. Observe that since the sets of recommendations and messages are now fixed as well as the equilibrium strategy, a canonical communication equilibrium is only determined by a "strategy" of the mediator $c = (c_t)_{t \geq 1}$ with $c_t : (A \times Y)^{t-1} \to \Delta(A)$ for $t \geq 2$ and $c_1 \in \Delta(A)$ and we indentify the communication equilibrium with it. We would like the equilibrium to have perfectness properties, namely $c$ would be called perfect if the faithful strategy profile were a sequential equilibrium of the extended repeated game. To get a recursive structure, we give a more stringent definition of perfectness.

Consider the repeated game extended by some canonical communication device. A history $h^i$ of length $t$ for player $i$ (a $i$-history) consists of all the actions recommended to player $i$, all the actions played by player $i$, all the signals observed by him and all signals reported by him up to stage $t$ and we let $H^i = \cup_t (A^i \times A^i \times Y^i \times Y^i)^t$ be the set of all $i$-histories. A pure strategy of player $i$ will prescribe (1) which action to play after $h^i$ and the next recommendation $a_{t+1}^i$ of the mediator and (2) which signal to report after $(h^i, a_{t+1}^i)$, having played the action $b_{t+1}^i$ and observed the signal $y_{t+1}^i$.

Let us call *action rule* a mapping $\alpha^i : A^i \to A^i$ and *reporting rule* a mapping from $A^i \times Y^i$ to $Y^i$. Finally call *decision rule* a pair $(\alpha^i, \rho^i)$ and let $D^i$ be the set of decision rules for player $i$. A strategy for player $i$ in the extended game is then a mapping which associates to every $i$-history a decision rule, i.e. it is a mapping $\sigma^i : H^i \to D^i$. The *faithful decision rule* $\varphi^i$ is the one that plays the recommended action, i.e. the action rule is the indentity on $A^i$ and reports the correct signal i.e. the reporting rule is the projection on $Y^i$. The *faithful strategy* of player $i$ is then the one that selects the faithful decision rule after each $i$-history. The canonical communication device $c$ is a canonical communication equilibrium iff the profile of faithful strategies is a Nash equilibrium of the game extended by $c$.

A history of length $t$ for the mediator (a $m$-history) consists of all action profiles recommended by the mediator and all signal profiles reported by the players up to stage $t$, thus an element of $(A \times Y)^t$ and we let $H^m = \cup_t (A \times Y)^t$ be the set of all $m$-histories. For every $m$-history $h$ of the mediator, we let $c(\cdot|h)$ be the canonical communication device induced by $c$ after history $h$, i.e. if $h'$ is a $m$-history, $c(h'|h) = c(hh')$ where $hh'$ is the concatenated history.

**Definition 3.1.** *The canonical communication device $c$ is a* perfect communication equilibrium *if for every history $h \in H^m$ of the mediator, $c(\cdot|h)$ is a canonical communication equilibrium.*

Notice that, although perfectly well defined, the concept has more punch when interpreted from the point of view of the mediator. Indeed, since player $i$ is not aware of the history of the mediator, a strategy for player $i$ after $h$ is not defined and player $i$ is not able to compute his continuation payoff. However, if all player know that the communication device is perfect, they all are confident that at any point in the game, playing the faithful strategy would be a best-reply if they had more information, i.e. if they did observe the history of the mediator. Consequently, if $c$ is a perfect communication equilibrium, then the faithful strategy is a sequential equilibrium of the extended game: whatever player $i$'s beliefs are, his best response is to play the faithful strategy. This is deeply related to *belief-free* equilibria introduced in [Ely03]. Indeed, for a perfect communication equilibrium, the faithful strategy profile is a belief-free equilibrium of the extended game.

Perfect communication equilibria clearly exist: if $c(h)$ is a correlated equilibrium of the stage game for every $m$-history $h$, then $c$ is a perfect communication equilibrium. We let $C(\delta)$ be the set of payoff vectors associated to perfect communication equilibria of the game with discount factor $\delta$. All such payoffs are feasible thus, $C(\delta) \subset V$.

3.2. **Perfect communication equilibria and perfect public equilibria.** We give here an intuition as to why the methods of [FLM94] and [FL94] apply to our concept. Indeed, perfect communication equilibria may be seen as particular perfect public equilibria of an auxiliary game which we describe now.

There are $n+1$ players, player $i = 1, \ldots, n$ is the same as in the original game and player $n+1$ is the mediator. The stage-game is in extensive form and unfolds as follows.

- At the first round the mediator selects an action profile $a$ and $a^i$ is privately announced to player $i$.
- At the second round, players $i = 1, \ldots, n$ simultaneously select actions. If $b$ is the action profile selected, a profile of signals $z$ is drawn with probability $q(z|b)$ and $z^i$ is privately announced to player $i$.
- At the third round each player reports privately a signal $y^i$ to the mediator.
- Finally, the profile of recommended actions $a$ and of reported signals $y$ are publicly announced.
- Player $i = 1, \ldots, n$ receives the payoff $r^i(b, z)$ (without observing it directly) and player $n+1$ gets a payoff of zero.

Players discount payoffs at the common rate $\delta$ and the description of the repeated game is complete. To describe it in the usual way, let us reduce the extensive stage-game to its normal form. The action set for player $n+1$ in the new stage-game is the set of action profiles $A$ in the original game while the action set for player $i=1,\dots,n$ in the new stage-game is the set of decision rules $D^i$. Given an action profile in the new game $(a,d^1,\dots,d^n)$, the payoff to player $n+1$ is zero and the payoff to player $i=1,\dots,n$ is:

$$G^i(a,d^1,\dots,d^n) := \sum q(z|\alpha^1(a^1),\dots,\alpha^n(a^n))r^i(\alpha^1(a^1),\dots,\alpha^n(a^n),z)$$

Signals in the new game are public and if the action profile $(a,d^1,\dots,d^n)$, the signal $s=(a,y)$ is drawn with probability:

$$Q(a,y|a,d^1,\dots,d^n) := \sum\nolimits_{z\in Y,\ \forall i,\rho^i(a^i,z^i)=y^i} q(z|\alpha^1(a^1),\dots,\alpha^n(a^n))$$

and is publicly announced.

Let $\varphi^i$ be the strategy of player $i$ that plays the faithful decision rule after each stage and every history. It follows from the definitions that a communication device $c$ is a perfect communication equilibrium if and only if in the auxiliary game, the strategy profile $(c,\varphi^1,\dots,\varphi^n)$ is a perfect public equilibrium.

## 4. The recursive structure

We develop in this section an analysis of perfect communication equilibria similar to the one made by [APS90] for public equilibria. The starting point is the well-known one-shot deviation principle. A one-shot deviation of player $i$ is a strategy that always plays the faithful decision rule except after one single $m$-history. Note that with this definition, a one-shot deviation of player $i$ is not a strategy for player $i$ who does not know the history of the mediator. It is however a well-defined strategy in the auxiliary game with public signals. A canonical communication device is *immune to one-shot deviations* if no player has a profitable one-shot deviation.

**Lemma 4.1** (The one-shot deviation principle). *A canonical communication device $c$ is a perfect communication equilibrium if and only if for every $m$-history $h$, $c(\cdot|h)$ is immune to one-shot deviations.*

*Proof.* Following the representation of perfect commnication equilibria by perfect public equilibria from section 3.2, this is a direct consequence of the one-shot deviation principle for perfect public equilibria. □

Using this result, one can write a kind of Bellmann equation for perfect communication equilibria. We introduce now a family of extensive games who serve as Bellmann operator.

**Definition 4.2.** *Let $p\in\Delta(A)$ be a correlated distribution of actions, $f: A\times Y\to\mathbb{R}^n$, and $0\le\delta<1$ a discount factor. We denote by $f^i$ the $i$-th coordinate of $f$. We define $\Gamma(p,f,\delta)$ as the following extensive form game.*

- *At the first round, the mediator selects $a\in A$ according to $p$ and informs privately each player $i$ of $a^i$.*
- *At the second round each player $i$ chooses an action $b^i$, these choices are made simultaneously. If $b$ is the profile of action chosen, a profile of signals $z$ is drawn according to $q(\cdot|b)$ and player $i$ observes $z^i$ solely.*
- *At the third round, each player $i$ reports privately a signal $y^i$ to the mediator, let $y$ be the profile of reported signals.*
- *The payoff for player $i$ is $(1-\delta)r^i(b,z)+\delta f^i(a,y)$.*

The strategy for player $i$ that always plays at the second round the action recommended at the first round, and always reports at the third round the signal observed at the second round shall be refered to as the *faithful strategy.* Remark that a pure strategy for player $i$ in this game is a decision rule $d^i = (\alpha^i, \rho^i)$ where $\alpha^i : A^i \to A^i$ is an action rule and $\rho^i : A^i \times Y^i \to Y^i$ is a reporting rule, thus the faithful strategy in $\Gamma(p, f, \delta)$ is the faithful decision rule $\varphi^i$.

We introduce the following notations. Given a probability $p \in \Delta(A)$ and a decision rule $d^i = (\alpha^i, \rho^i)$ for player $i$, we let $g^i(p, d^i)$ be the payoff to player when the mediator uses $p$ to recommend actions and player $i$ plays according to $d^i$ while other players play faithfully:

$$g^i(p, d^i) = \sum\nolimits_a p(a) g^i(\alpha^i(a^i), a^{-i})$$

We denote by $Q(p, d^i)$ the probability induced by $p$ and $d^i$ on the set of profiles of recommended actions and reported signals:

$$Q(p, d^i)(a, y) = p(a) \sum\nolimits_{z^i : \rho^i(a^i, z^i) = y^i} q(\rho^i(a^i, z^i), y^{-i} | \alpha^i(a^i), a^{-i})$$

We also denote by $Q(a, d^i)$ the probability induced by the recommended action $a$ and the decision rule $d^i$ on reported signals:

$$Q(a, d^i)(y) = \sum\nolimits_{z^i : \rho^i(a^i, z^i) = y^i} q(\rho^i(a^i, z^i), y^{-i} | \alpha^i(a^i), a^{-i})$$

Obviously, $Q(p, d^i)(a, y) = p(a) Q(a, d^i)(y)$. When $d^i$ is the faithful decision rule $\varphi^i$, we simply denote $Q(p, \varphi^i)$ by $Q(p)$ and $Q(a, \varphi^i)$ by $Q(a)$. All these definition extends to mixed decision rules. Given $\mu^i = (\mu^i(d^i))_{d^i} \in \Delta(D^i)$, we set:

$$\begin{aligned} g^i(p, \mu^i) &= \sum\nolimits_{d^i \in D^i} g^i(p, d^i) \mu^i(d^i) \\ Q(p, \mu^i)(a, y) &= \sum\nolimits_{d^i \in D^i} Q(p, d^i)(a, y) \mu^i(d^i) \\ Q(a, \mu^i)(y) &= \sum\nolimits_{d^i \in D^i} Q(a, d^i)(y) \mu^i(d^i) \end{aligned}$$

Finally, in the game $\Gamma(p, f, \delta)$, the payoff for player $i$ when he plays $d^i$ and other players play faithfully is:

$$v^i_\delta(p, f, d^i) = (1 - \delta) g^i(p) + \delta \sum\nolimits_{a,y} Q(p, d^i)(a, y) f^i(a, y)$$

For simplicity we shall denote $\sum_{a,y} Q(p, d^i)(a, y) f^i(a, y)$ by $Q(p, d^i) \cdot f^i$.

Consider a perfect canonical communication equilibrium $c$. Let $c_1 \in \Delta(A)$ be the probability distribution on recommended actions at the first stage under $c$ and for each profile of actions $a$ and of signals $z$, let $f^i_c(a, z)$ be the payoff for player $i$ yielded by the canonical communication equilibrium $c(\cdot | a, z)$, i.e. this is the continuation payoff for player $i$ after the first stage given that the mediator recommended $a$ and was reported $z$. If $c$ is a perfect communication equilibrium, one clearly has:

*Claim* 4.3. *The faithful strategy profile is a Nash equilibrium of* $\Gamma(c_1, f_c, \delta)$.

Let $v = (v^1, \dots, v^n)$ be the payoff vector yielded by $c$, the above claim states that the following equalities/inequalities are satisfied:

For every player $i$ and decision rule $d^i = (\alpha^i, \rho^i)$,

$$\begin{aligned} v_i &= (1 - \delta) \sum c_1(a^i, a^{-i}) q(y | a^i, a^{-i}) r^i(a, y) + \delta \sum c_1(a^i, a^{-i}) q(y | a^i, a^{-i}) f^i_c(a, y) \\ &\geq (1 - \delta) \sum c_1(a^i, a^{-i}) q(y | \alpha^i(a^i), a^{-i}) r^i(\alpha^i(a^i), a^{-i}, y) \\ &\quad + \delta \sum c_1(a^i, a^{-i}) q(y | \alpha^i(a^i), a^{-i}) f^i_c(a, \rho^i(a^i, y^i), y^{-i}) \end{aligned}$$

That is, with the notations introduced above:

$$\begin{aligned} v_i &= (1-\delta)g^i(c_1) + \delta Q(c_1)\cdot f_c^i \\ &\geq (1-\delta)g^i(c_1,d^i) + \delta Q(c_1,d^i)\cdot f_c^i \end{aligned}$$

The recursive analysis consists in characterizing $C(\delta)$ by using those equations and as a fixed point of a correspondence built on them.

**Definition 4.4.** *Let $p \in \Delta(A)$ be a correlated distribution of actions, $f : A \times Y \to \mathbb{R}^n$, and $0 \leq \delta < 1$ a discount factor.*

- *The pair $(p, f)$ is* $\delta$-balanced *if the faithful strategy if a Nash equilibrium of $\Gamma(p, f, \delta)$, i.e. for each player $i$ and decision rule $d^i$:*

$$(1-\delta)g^i(p) + \delta Q(p)\cdot f^i \geq (1-\delta)g^i(p,d^i) + \delta Q(p,d^i)\cdot f^i$$

- *For each pair $(p, f)$ $\delta$-balanced, we let $v^i_\delta(p, f)$ the payoff for player $i$ yielded by the faithful strategy in $\Gamma(p, f, \delta)$. We denote by $v_\delta(p, f)$ the corresponding payoff vector. Note that:*

$$v_\delta(p,f) = (1-\delta)g(p) + \delta Q(p)\cdot f$$

  *where $Q(p)\cdot f = (Q(p)\cdot f^i)_i$.*
- *Given a set of payoff vectors $W \subset \mathbb{R}^n$, we write $f \in W$ to mean that the range of $f$ is a subset of $W$, i.e. $f(a, y) \in W$, $\forall (a, y)$. We define a correspondence $F_\delta$ which associated to $W$ another subset of $\mathbb{R}^n$:*

$$F_\delta : W \mapsto \{v_\delta(p,f) \mid f \in W, (p,f)\ \delta - \text{balanced}\}$$

- *A payoff vector $v$ is* decomposable *with respect to $p$, $W$ and $\delta$ if there exists a mapping $f \in W$ such that $(p, f)$ is $\delta$-balanced and $v = v_\delta(p, f)$. A set $W$ is* self-decomposable *with respect to $\delta$ if $W \subset F_\delta(W)$.*

The recursive characterization of the set of equilibrium payoffs is the following:

**Theorem 4.5.** *$C(\delta)$ is the largest (for inclusion) bounded set which is self-decomposable with respect to $\delta$, i.e. $C(\delta) \subset F_\delta(C(\delta))$ and for every bounded $W$, $W \subset F_\delta(W)$ implies $W \subset C(\delta)$.*

*Proof.* We first show that $C(\delta)$ is self-decomposable with respect to $\delta$. Let $v \in C(\delta)$ be the payoff vector associated to the perfect communcation equilibrium $c$. Call $c_1$ the distribution of recommended actions at stage 1 under $c$ and $f^i_c(a, y)$ the payoff for player $i$ yielded by $c(\cdot|a, y)$. From claim 4.3, $(c_1, f_c)$ is $\delta$-balanced and $v = v_\delta(c_1, f_c)$ and since $c$ is perfect, for each pair $(a, y)$, $f_c(a, y) \in C(\delta)$.

Let now $W$ be a bounded set of payoffs such that $W \subset F_\delta(W)$. For each $w \in W$, there is a $\delta$-balanced pair $(p_w, f_w)$ with $f_w \in W$ such that $w = v_\delta(p_w, f_w)$. Let us fix $u \in W$ and construct inductively a perfect communication equilibrium $c$ with payoff $u$. By the above property,

$$u = v_\delta(p_u,f_u) = (1-\delta)g(p_u) + \delta \sum\nolimits_{a,y} p(a)q(y|a)f_u(a,y)$$

We set thus $c_1 = p_u$ which defines the device on histories of length zero and $f_u$ defines the continuation payoff after each history of length one, i.e. for each $(a, y)$ the continuation payoff is $u_2 = f_u(a, y)$. For this vector, we have

$$u_2 = v_\delta(p_{u_2},f_{u_2}) = (1-\delta)g(p_{u_2}) + \delta \sum\nolimits_{a,y} p(a)q(y|a)f_{u_2}(a,y)$$

thus we set $c_2(a, y) = u_2$. This defines $c$ on histories of length one as well as a continuation payoff after each history of length two, i.e. after $(a_1, y_1, a_2, y_2)$, the continuation payoff is $f_{u_2}(a_2, y_2)$ with $u_2 = f_u(a_1, y_1)$. We continue inductively in this way. Suppose that the device is defined on all histories of length $T-1$ and the continuation payoffs are defined after all histories of length $T$. Consider a history $h$ of length $T$ and let $u_T$ be the continuation payoff after $h$. We have,

$$u_T = v_\delta(p_{u_T}, f_{u_T}) = (1-\delta)g(p_{u_T}) + \delta \sum\nolimits_{a,y} p(a)q(y|a)f_{u_T}(a,y)$$

Thus the device after $h$ is defined as $p_{u_T}$ and the continuation payoff after each history $(h, a, y)$ of length $T+1$, is $f_{u_T}(a, y)$. □

## 5. The asmptotic analysis

This aim of this section is to give a characterization of $\lim_{\delta\to 1} C(\delta)$. We apply the method developed in [FLM94] and [FL94] and adapt it to our solution concept.

**Definition 5.1.** *Let $\lambda$ be a vector in $\mathbb{R}^n$ and $(p, f)$ be a $\delta$-balanced pair. This pair is $\lambda$-directed if:*

$$\lambda \cdot v_\delta(p, f) \geq \lambda \cdot f(a, y), \quad \forall (a, y) \in A \times Y$$

*where $u \cdot v$ is the inner product in $\mathbb{R}^n$.*

Remark that this implies $\lambda \cdot g(p) \geq \lambda \cdot v_\delta(p, f) \geq \lambda \cdot f(a, y)$, $\forall (a, y) \in A \times Y$, i.e. the current payoff vector is separated from continuation payoffs by the hyperplane $\{v \mid \lambda \cdot v = \lambda \cdot v_\delta(p, f)\}$. We shall make extensive use of the following properties of balanced pairs.

**Proposition 5.2.** *Let $(p, f)$ be a $\delta$-balanced pair, $\alpha > 0$ and $\beta \in \mathbb{R}^n$. We let $\alpha f + \beta$ be the mapping that assoiates to $(a, y)$, $\alpha f(a, y) + \beta$. Then,*

1. *$(p, f + \beta)$ is also $\delta$-balanced.*
2. *For each $\delta'$, $(p, \frac{\delta(1-\delta')}{(1-\delta)\delta'} f)$ is $\delta'$-balanced.*
3. *For each $\delta' > \delta$ and each $(a, y)$ set:*

$$f'(a, y) = \frac{\delta' - \delta}{\delta'(1-\delta)} v_\delta(p, f) + \frac{\delta(1-\delta')}{\delta'(1-\delta)} f(a, y)$$

   *Then:*
   (a) *$(p, f')$ is $\delta'$-balanced.*
   (b) *If $(p, f)$ is $\lambda$-directed then so is $(p, f')$.*
   (c) *$v_\delta(p, f) = v_{\delta'}(p, f')$.*
4. *There exists a positive constant $M = M(p, f, \delta)$ such that for each $\delta'$ and $w \in \mathbb{R}^n$ there is a mapping $f'$ for which $(p, f')$ is $\delta'$-balanced and $\forall (a, y)$, $\|f'(a, y) - w\| \leq M \frac{1-\delta'}{\delta'}$.*
5. *If $(p, f')$ is $\delta$-balanced and $0 \leq \alpha \leq 1$, then $(p, \alpha f + (1-\alpha) f')$ is also $\delta$-balanced.*

*Proof.* (1). The payoff for player $i$ in $\Gamma(p, f + \beta, \delta)$ is deduced from his payoff in $\Gamma(p, f, \delta)$ by adding the constant $\beta^i$.

(2). The payoff for player $i$ in $\Gamma(p, \frac{\delta(1-\delta')}{(1-\delta)\delta'} f), \delta'$ is deduced from his payoff in $\Gamma(p, f, \delta)$ by multiplying it by the constant $\frac{1-\delta'}{1-\delta}$.

(3). (a). Follows directly from (1) and (2).

(3). (b). Simple algebra shows, $v_{\delta'}(p, f') - f'(a, y) = \frac{\delta(1-\delta')}{(1-\delta)\delta'}(v_\delta(p, f) - f(a, y))$ hence the result.

(3). (c). Straightforward computation.

(4). Let $\bar{f} = \sum_{a,y} p(a)q(y|a)f(a, y)$, for $\delta'$ and $w \in \mathbb{R}^n$ fixed, set:

$$f'(a, y) = w + \frac{\delta(1-\delta')}{(1-\delta)\delta'}(f(a, y) - \bar{f})$$

From (1) and (2), $f'$ is $\delta'$-balanced. A simple computation shows $v_{\delta'}(p, f') = w$ and setting $M = \frac{\delta}{1-\delta} \max_{a,y} \left\| f(a, y) - \bar{f} \right\|$, for each $(a, y)$:

$$\|f'(a, y) - w\| \leq M \frac{1 - \delta'}{\delta'}$$

(5). It suffices to write the incentive constraints for $f$ $(ICf)$, for $f'$ $(ICf')$ and to form the combination $\alpha(ICf) + (1 - \alpha)(ICf')$.

□

**Definition 5.3.** *The maximal score in direction $\lambda$ is:*

$$k_\delta(\lambda) = \max \{\lambda \cdot v_\delta(p, f) \mid (p, f) \ \delta - \text{balanced and } \lambda - \text{directed}\}$$

**Lemma 5.4.** *The maximal score $k_\delta(\lambda)$ does not depend on $\delta$ and is denoted $k(\lambda)$.*

*Proof.* This follows directly from point (3) of proposition 5.2. □

For each vector $\lambda$, we let $H_\lambda$ be the half-space $\{v \mid \lambda \cdot v \leq k(\lambda)\}$ and denote:

$$C^* = \bigcap\nolimits_{\lambda \in \mathbb{R}^n} H_\lambda$$

**Theorem 5.5.** *(1) For each discount factor $\delta$, $C(\delta) \subset C^*$.*
*(2)* $\lim_{\delta \to 1} C(\delta) = C^*$.

A heuristic proof consists in recalling the analogy between perfect communcation equilibria made in section 3.2. The set $C(\delta)$ is the set of payoffs associated to some public equilibria in a game with public signals. Theorem 5.5 can thus be seen an application of theorem 3.1 p. 111 of [FL94].

This proof is not sufficient since: (1) $C(\delta)$ is not the set of all public equilibria in the auxiliary game but a subset of it, so one has to prove that their properties are similar; (2) [FL94] assume that (the analog of) the set $C^*$ has non-empty interior while (i) their method of proof perfectly applies when considering the relative interior of $C^*$ and (ii) since in the auxiliary game, player $n + 1$ has payoff zero, the set of feasible payoffs in this game will always have empty interior. We thus give in appendix a formal proof of theorem 5.5, verifying that most of [FL94] arguments apply.

## 6. Properties of the limit set

In this section we discuss properties of the set $C^*$. We start by recalling usual material on the one-shot game $G = (A^1, \ldots, A^n; g^1, \ldots, g^n)$. Recall that the set of feasible payoffs is:

$$V = \{g(p) \mid p \in \Delta(A)\}$$

A *correlated equilibrium* of the game $G$ is a distribution $p \in \Delta(A)$ such that for each player $i$ each action $a^i$ s.t. $p(a^i) > 0$ and each action $b^i$:

$$\sum\nolimits_{a^{-i}} p(a^i, a^{-i}) g^i(a^i, a^{-i}) \geq \sum\nolimits_{a^{-i}} p(a^i, a^{-i}) g^i(b^i, a^{-i})$$

Let $C(G)$ be the set of correlated equilibria of $G$ and $CP(G) = \{g(p) \mid p \in C(G)\}$ be the set payoffs associated to correlated equilibria of $G$. It is well known that $C(G)$ is closed and convex and contains all Nash equilibria of $G$. Note that $p \in C(G)$ if and only if the faithful strategy profile is a Nash equilibrium of the game $\Gamma(p, f, \delta)$ where $f \equiv 0$ is constantly equal to zero.

The individual rationality level (the minmax level) of player $i$ is the harshest punishment that players $-i$ can inflict to player $i$. In general it is defined as $\min_{p^{-i}} \max_{a^i} g^i(a^i, p^{-i})$ where the min ranges over the set of mixed action profiles for players $-i$, i.e. $p^{-i} \in \prod_{j \neq i} \Delta(A^j)$. It is well known that the punisment is even harsher when players $-i$ are correlated, i.e. may choose $p^{-i} \in \Delta(\prod_{j\neq i} A^j)$. This correlation is possible with the help of the mediator. We thus define the (correlated) minmax level of player $i$ as:

$$m^i = \min_{p^{-i}\in\Delta(\prod_{j\neq i} A^j)} \max_{a^i} g^i(a^i, p^{-i})$$

Remark that the minmax $m^i$ can be formulated as follows:

$$m^i = \min_p \max_{d^i} g^i(p, d^i)$$

Observe also that $m^i$ is the value of a two-player zero-sum game where: the maximizing player chooses $a^i$, the minimizing player chooses $a^{-i}$, the payoff is $g^i(a^i, a^{-i})$. It is also the payoff obtained by player $i$ at any correlated equilibrium of this zero-sum game (Forges, [For88]). Note that for each action $a^i$ s.t. $p(a^i) > 0$, the conditional distribution on $A^{-i}$, $p(\cdot|a^i)$ achieves the min in the definition of $m^i$ and $a^i$ is a best reply to $p(\cdot|a^i)$.

**Definition 6.1.** *A distribution $p \in \Delta(A)$ is a* minmax distribution *for player $i$ if it is a correlated equilibrium of the zero-sum game $(A^i, A^{-i}, g^i, -g^i)$. Let $M^i$ be the (convex) set of minmax distributions for player $i$.*

We let $IR = \left\{v \in \mathbb{R}^n \mid \forall i,\ v^i \geq m^i\right\}$ be the set of individually rational payoffs and $V^* = V \cap IR$. We get easily:

**Lemma 6.2.** (1) $C^* \subset V^*$
(2) $CP(G) \subset C^*$

*Proof.* (1). First $C^* \subset V$. Take $v \in C^*$, for each direction $\lambda$, there exists a pair $(p, f)$ $\delta$-balanced and $\lambda$-directed, s.t. $\lambda \cdot v \leq \lambda \cdot v_\delta(p, f) \leq \lambda \cdot g(p)$. Thus $\forall\lambda$, $\lambda \cdot v \leq \max_{x\in V} \lambda \cdot x$ which implies $C^* \subset V$ since $V$ is convex.

We prove now $C^* \subset IR$. For each player $i$, let $e^i$ be the unit vector whose $i$-component is 1 and other components are zero. Take $v \in C^*$, for $\lambda = -e^i$, $\lambda \cdot v \leq k(\lambda)$ i.e. there exists a pair $(p, f)$ $\delta$-balanced s.t. $v^i \geq v^i_\delta(p, f)$ and $f^i(a, y) \geq v^i_\delta(p, f)$ for each $(a, y)$. This implies that for every decision rule $d^i$, $Q(p, d^i) \cdot f^i \geq v^i_\delta(p, f)$. Since the pair is balanced, for each $d^i$:

$$\begin{aligned} v^i &\geq v^i_\delta(p, f) \\ &\geq (1-\delta)g^i(p, d^i) + \delta Q(p, d^i) \cdot f^i \\ &\geq (1-\delta)g^i(p, d^i) + \delta v^i_\delta(p, f) \end{aligned}$$

thus $v^i_\delta(p, f) \geq g^i(p, d^i)$. So there exists $p$ s.t. for each $d^i$, $v^i \geq g^i(p, d^i)$ i.e. $v^i \geq \min_p \max_{d^i} g^i(p, d^i) = m^i$.

(2). Let $p$ be a correlated equilibrium of $G$. It is plain that for each $\delta$ and for each constant mapping $f$, the pair $(p, f)$ is $\delta$-balanced. In particular, letting for each $(a, y)$, $f(a, y) = g(p)$, we get $v_\delta(p, f) = g(p)$ and for each $\lambda$, $\lambda \cdot v_\delta(p, f) = \lambda \cdot f(a, y)$. Thus $g(p) \in C^*$. □

An easy case to study is when signals are *trivial* that is, they reveal nothing about the action profile played. This can be modelled by letting the signal sets be singletons or by assuming that $q(y|a)$ does not depend on $a$. In such a case, there is no link between stages so one expects that at each stage, the players play a static (correlated) equilibrium. This intuitive reasoning is formalized in the next lemma:

**Lemma 6.3.** *If signals are trivial, $C^* = CP(G)$.*

*Proof.* We assume that signal sets are singletons for simplicity. Thanks to the previous lemma, we only need to prove $C^* \subset CP(G)$. For each $v \in C^*$ and each direction $\lambda$, there exists $(p, f)$ $\delta$-balanced, $\lambda$-directed s.t. $\lambda \cdot v \leq \lambda \cdot v_\delta \leq \lambda \cdot g(p)$. Since signals are trivial, $f(a, y)$ does not depend on actions *played* and on signals. The payoff for player $i$ when he plays $d^i$ is:

$$(1-\delta)g^i(p, d^i) + \delta Q(p) \cdot f^i$$

therefore playing faithfully is a best reply for player $i$ in $\Gamma(p, f, \delta)$ if and only if it maximizes $g^i(p, d^i)$. Thus, $p \in C(G)$. Then, for each $v \in C^*$ and each direction $\lambda$, there exists $p \in C(G)$ s.t. $\lambda \cdot v \leq \lambda \cdot g(p)$ which implies $C^* \subset CP(G)$. □

6.1. **Enforceability.** We examine when a distribution $p$ may be part of a balanced pair $(p, f)$. Namely, given $p$, does there exists a mapping $f$ such that playing faithfully is a Nash equilibrium of $\Gamma(p, f, \delta)$. Assume that player $i$ has a decision rule $d^i$ that increases his stage-payoff: $g^i(p, d^i) > g^i(p)$ and does not affect the reported signals: $Q(p, d^i) = Q(p)$, i.e. $\forall a \in \operatorname{supp} p$, $Q(a, d^i) = Q(a)$. In such case, no $f$ can make $(p, f)$ a balanced pair. Indeed, if no player can increase his payoff without affecting signals, then there exists $f$ such that $(p, f)$ is balanced.

**Definition 6.4.** (1) *A distribution $p \in \Delta(A)$ is* enforceable *if there is $\delta$ and a mapping $f$ s.t. $(p, f)$ is $\delta$-balanced.*
(2) *A distribution $p \in \Delta(A)$ is* immune to undetectable deviations *if for each player $i$ and (mixed) decision rule $\mu^i \in \Delta(D^i)$:*

$$Q(p, \mu^i) = Q(p) \Longrightarrow g^i(p, \mu^i) \leq g^i(p)$$

**Proposition 6.5.** *The correlated ditribution $p$ is enforceable if and only if $p$ is immune to undetectable deviations.*

*Proof.* As mentioned above, if $p$ is not immune to undetectable deviations, there is a player $i$, a mixed decision rule $\mu^i$ s.t. $Q(p, \mu^i) = Q(p)$ and $g^i(p, \mu^i) > g^i(p)$. Then $\mu^i$ is a profitable deviation in $\Gamma(p, f, \delta)$ for any $\delta$ and $f$.

Assume now that $p$ is immune to undetectable deviations. Let $\delta$ be a discount factor and $f : A \times Y \to \mathbb{R}^n$. The pair $(p, f)$ is $\delta$-balanced if for each player $i$ and decision rule $d^i$:

$$(1-\delta)g^i(p) + \delta Q(p) \cdot f^i \geq (1-\delta)g^i(p, d^i) + \delta Q(p, d^i) \cdot f^i$$

That is, for each player $i$ we must solve the system:

$$\left\{ (Q(p) - Q(p, d^i)) \cdot f^i \geq \frac{1-\delta}{\delta}(g^i(p, d^i) - g^i(p)) \quad (\forall d^i) \right. \tag{1}$$

We apply the alternative theorem (see e.g. Ky Fan [Fan56] or Rockafellar [Roc70]): A linear system of inequalities $Mx \geq c$ has a solution iff ($\beta \geq 0$ and $\beta M = 0$ imply $\beta \cdot c \leq 0$) where $\geq$ means component-wise weak inequality. Letting $M$ be the matrix with row vectors $(Q(p) - Q(p, d^i))$, $d^i \in D^i$ and $c$ be the column vector with $d^i$-component $\frac{1-\delta}{\delta}(g^i(p, d^i) - g^i(p))$, the system (1) has a solution $f^i$ iff $p$ is immune to undetectable deviations from player $i$. □

Let $\mathcal{P}$ be the set of enforceable distributions and $g(\mathcal{P})$ the corresponding set of payoffs.

**Lemma 6.6.** $C^* \subset \operatorname{co} g(\mathcal{P})$

*Proof.* Take $v \in C^*$, for each direction $\lambda$, there exists a pair $(p, f)$ $\delta$-balanced and $\lambda$-directed, s.t. $\lambda \cdot v \leq \lambda \cdot v_\delta(p, f) \leq \lambda \cdot g(p)$. Obviously, $p \in \mathcal{P}$. Thus $\forall \lambda$, $\lambda \cdot v \leq \sup_{x \in g(\mathcal{P})} \lambda \cdot x$ and therefore $C^* \subset \operatorname{co} g(\mathcal{P})$. □

We examine now a stronger notion of enforceability which will be useful for the Folk Theorem.

**Definition 6.7.** (1) *Given a direction $\lambda$, the distribution $p$ is enforceable with respect to $\lambda$-hyperplanes if there exists a mapping $f$ and a discount factor $\delta$ such that $(p, f)$ is $\delta$-balanced and there exists a real number $k$ s.t. $\forall(a, y)$, $\lambda \cdot f(a, y) = k$.*

(2) *A vector $\lambda = (\lambda^1, \ldots, \lambda^n)$ is* singular *if there is a unique $i$ s.t. $\lambda^i \neq 0$. Otherwise, $\lambda$ is called* regular. *Given a pair of players $i \neq j$, a $ij$-vector is a regular $\lambda$ s.t. $\lambda^k = 0$ for $k \neq i$, $k \neq j$.*

(3) *Given a pair of players $i \neq j$, $p$ is enforceable with respect to $ij$-hyperplanes if $p$ is enforceable with respect to $\lambda$-hyperplanes for each $ij$-vector $\lambda$.*

These definitions bear the following properties.

**Lemma 6.8.** (1) *The distribution $p$ is enforceable with respect to $\lambda$-hyperplanes if and only if there exists a mapping $f$ and a discount factor $\delta$ such that $(p, f)$ is $\delta$-balanced and $\forall(a, y)$, $\lambda \cdot f(a, y) = 0$.*

(2) *If $p$ is enforceable with respect to $\lambda$-hyperplanes, then $\lambda \cdot g(p) \leq k(\lambda)$.*

(3) *If for all pair of players $i \neq j$, $p$ is enforceable with respect to $ij$-hyperplanes, then $p$ is enforceable with respect to $\lambda$-hyperplanes for all regular $\lambda$.*

*Proof.* (1). If $(p, f)$ is $\delta$-balanced, so is $(p, f + \beta)$ for each $\beta \in \mathbb{R}^n$. Thus if $\forall(a, y)$, $\lambda \cdot f(a, y) = k$, for each real $k'$ one can choose $\beta$ such that $\forall(a, y)$, $\lambda \cdot (f(a, y) + \beta) = k'$.

(2). In the above proof, choose $\beta$ s.t. $k' = \lambda \cdot g(p)$ and set $f'(a, y) = f(a, y) + \beta$. Then $\lambda \cdot ((1 - \delta)g(p) + \delta Q(p) \cdot f')) = \lambda \cdot g(p)$. Thus, $(p, f)$ is $\lambda$-directed and $\lambda \cdot g(p) \leq k(\lambda)$.

(3). Fix the discount factor. Assume that for each $ij$-vector $\lambda$, $p$ is enforceable with respect to $\lambda$-hyperplanes. Let $\lambda$ be a regular vector. Assume first that the number of players s.t. $\lambda_i \neq 0$ is even, i.e. up to a relabelling of players, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K, \lambda_{K+1}, 0, \ldots, 0)$. For the $2K$ first players, for each pair $(k, k+1)$ choose $(f^k, f^{k+1})$ that solves the system:

(2)
$$\begin{cases} (Q(p) - Q(p, d^k)) \cdot f^k & \geq & \frac{1-\delta}{\delta}(g^k(p, d^k) - g^k(p)) & (\forall d^k) \\ (Q(p) - Q(p, d^{k+1})) \cdot f^{k+1} & \geq & \frac{1-\delta}{\delta}(g^{k+1}(p, d^{k+1}) - g^{k+1}(p)) & (\forall d^{k+1}) \\ \lambda^k f^k(a, y) + \lambda^{k+1} f^{k+1}(a, y) & = & 0 & (\forall(a, y)) \end{cases}$$

This exists from point (1). For other players $i$, choose $f^i$ that solves:

$$\begin{cases} (Q(p) - Q(p, d^i)) \cdot f^i \geq \frac{1 - \delta}{\delta}(g^i(p, d^i) - g^i(p)) \quad (\forall d^i) \end{cases} \tag{3}$$

This exists since $p$ is enforceable. This defines a mapping $f$ which has all the required properties by construction.

Assume now that the number of players s.t. $\lambda_i \neq 0$ is odd, i.e. $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K, \lambda_{K+1}, \lambda_{K+2}, 0, \ldots, 0)$. For players $i$ s.t. $\lambda_i = 0$, solve the system (3). For players $k$ s.t. $\lambda_k \neq 0$ and $k > 3$, solve the system (2) for the pair $(k, k+1)$. For players 1 and 2 choose $(f^1_*, f^2_*)$ that solves:

$$\begin{cases} (Q(p) - Q(p, d^1)) \cdot f^1 & \geq & \frac{1-\delta}{\delta}(g^1(p, d^1) - g^1(p)) & (\forall d^1) \\ (Q(p) - Q(p, d^2)) \cdot f^2 & \geq & \frac{1-\delta}{\delta}(g^2(p, d^2) - g^2(p)) & (\forall d^2) \\ \lambda^1 f^1(a, y) + \frac{\lambda^2}{2} f^2(a, y) & = & 0 & (\forall(a, y)) \end{cases}$$

and for players 2 and 3 choose $(f^2_{**}, f^3_{**})$ that solves:

$$\begin{cases} (Q(p) - Q(p, d^2)) \cdot f^2 & \geq & \frac{1-\delta}{\delta}(g^2(p, d^1) - g^2(p)) & (\forall d^2) \\ (Q(p) - Q(p, d^3)) \cdot f^3 & \geq & \frac{1-\delta}{\delta}(g^3(p, d^3) - g^3(p)) & (\forall d^3) \\ \frac{\lambda^2}{2} f^2(a, y) + \lambda^3 f^3(a, y) & = & 0 & (\forall(a, y)) \end{cases}$$

Finally set $f^1 = f^1_*$, $f^2 = \frac{1}{2}f^2_* + \frac{1}{2}f^2_{**}$ and $f^3 = f^3_*$. As before, the so-defined mapping $f$ has all required properties: recall that if $f^2_*$ and $f^2_{**}$ satisfy the incentive constraints for player 2, then so does their average $f^2$. □

## 7. The Folk Theorem

We give here sufficient conditions to obtain a Folk Theorem: $C^* = V^*$. We indeed give conditions under which for each $v \in V^*$ and direction $\lambda$, there exists $p$ enforceable with respect to $\lambda$-hyperplanes such that $\lambda \cdot v \leq \lambda \cdot g(p)$.

We say that a mixed decision rule $\mu^i$ for player $i$ is an *undetectable deviation* if $\forall a$, $Q(a, \mu^i) = Q(a)$, that is for every profile of recommended actions, the mediator cannot tell from the reported signals whether player $i$ is playing faithfully or deviating to $\mu^i$. If some player has an undetectable deviation, he might gain from it by increasing his stage payoff without affcting future payoffs. We thus give a condition that copes with undetectable deviations.

Fix a pair of players $(i, j)$ and assume that an agent (call him *deviator*, to differentiate him from the mediator) takes control of player $i$ or of player $j$ and makes one of them deviate but not both. That is, the deviator can choose any decision rule in $D^i \cup D^j$. A mixed strategy for the deviator is a probability distribution on $D^i \cup D^j$ and can be represented by a tuple $(t, \mu^i, \mu^j)$ where: $t \in [0, 1]$ is the probability that the deviator takes control of player $i$, $\mu^i$ is the distribution induced on $D^i$ given that the deviator takes control of player $i$ and $\mu^j$ is the distribution induced on $D^j$ given that the deviator takes control of player $j$. We say that the deviator is *faithful* if, whichever player he takes control of, he makes this player play his faithful strategy, i.e. $t\mu^i(\varphi^i) + (1-t)\mu^j(\varphi^j) = 1$. A tuple $(t, \mu^i, \mu^j)$ which is not faithful is called a deviation. A deviation is detectable by the mediator if there exists an action profile $a$, such that the distribution of reported signals under the deviation is different from the distribution induced by the faithful strategies, i.e. there exists $a$ s.t. $Q(a) \neq tQ(a, \mu^i) + (1-t)Q(a, \mu^j)$. Otherwise, the deviation in undetectable. The first condition says that the deviator has no undetectable deviations.

**Condition C1.** *For each pair of players $i, j$, for each mixed decision rules $\mu^i \in \Delta(D^i)$, $\mu^j \in \Delta(D^j)$ and $t \in [0, 1]$,*

$$\left[t\mu^i(\varphi^i) + (1-t)\mu^j(\varphi^j) < 1\right] \Longrightarrow \left[\exists a \in A, Q(a) \neq tQ(a, \mu^i) + (1-t)Q(a, \mu^j)\right].$$

Since for each $p \in \Delta(A)$, $Q(p, \mu^i))(a, y) = p(a)Q(a, \mu^i)(y)$, $Q(a) = tQ(a, \mu^i) + (1-t)Q(a, \mu^j)$, $(\forall a)$ holds if and only if there exists $p \in \operatorname{int}\Delta(A)$ such that $Q(p) = tQ(p, \mu^i) + (1-t)Q(p, \mu^j)$ and in turn, this holds for all $p \in \operatorname{int}\Delta(A)$. That is:

*Claim* 7.1. *Condition (C1) holds if and only if for each pair of players $(i, j)$, there exists $p \in \operatorname{int}\Delta(A)$ such that:*

$$(\mathrm{C1}(i,j,p)) \qquad Q(p) \notin \operatorname{co}\left(\left\{Q(p, d^i), d^i \neq \varphi^i\right\} \cup \left\{Q(p, d^j), d^j \neq \varphi^j\right\}\right)$$

*where* co *is the convex hull operator. In such case,* $(\mathrm{C1}(i,j,p))$ *holds for all* $p \in \operatorname{int}\Delta(A)$.

Condition (C1) implies clearly that no single player has an undetectable deviation:

*Claim* 7.2. *Under (C1), for each player $i$ and each $\mu^i \in \Delta(D^i)$:*

$$\left[\mu^i(\varphi^i) < 1\right] \Longrightarrow \left[\exists a \in A, Q(a) \neq Q(a, \mu^i)\right].$$

Therefore, under (C1) for each $p \in \operatorname{int}\Delta(A)$ no player $i$ has a profitable undetectable deviation from $p$. From 6.5, $p$ is enforceable.

Suppose now that a deviation has been detected by the mediator. If he can not ascribe it to a particular player, he has to punish several players simultaneously, which may cause a loss of efficiency. We will then ask the mediator to be able to differentiate between player $i$ and player $j$'s deviations.

**Condition C2.** *For each pair of players $i, j$, for each mixed decision rules $\mu^i \in \Delta(D^i)$, $\mu^j \in \Delta(D^j)$,*

$$\left[\forall a \in A, Q(a, \mu^i) = Q(a, \mu^j)\right] \Longrightarrow \left[\forall a \in A, Q(a, \mu^i) = Q(a, \mu^j) = Q(a)\right].$$

This says that if two mixed decision rules induce the same distribution of reported signals, then this distribution is the one induced by the faithful strategy. Or, if one player deviates in a detectable way ($\exists a$, s.t. $Q(a, \mu^i) \neq Q(a)$), then he creates a distribution which differ from one created by another player ($\exists a$, s.t. $Q(a, \mu^i) \neq Q(a, \mu^j)$). Again, since for each $p \in \Delta(A)$, $Q(p, \mu^i)(a, y) = p(a)Q(a, \mu^i)(y)$, condition (C2) holds if and only if for each pair of players $(i, j)$, there exists $p \in \operatorname{int}\Delta(A)$ s.t.

$$\left[Q(p, \mu^i) = Q(p, \mu^j)\right] \Longrightarrow \left[Q(p, \mu^i) = Q(p, \mu^j) = Q(p)\right]$$

and in such case, this holds for every $p \in \operatorname{int}\Delta(A)$. Thus,

*Claim* 7.3. *Condition (C2) holds if and only if for each pair of players $(i, j)$, there exists $p \in \operatorname{int}\Delta(A)$ such that:*

$$(\mathrm{C2}(i,j,p)) \qquad \operatorname{co}\left\{Q(p, d^i), d^i \in D^i\right\} \cap \operatorname{co}\left\{Q(p, d^j), d^j \in D^j\right\} = \{Q(p)\}$$

*and in such case,* $(\mathrm{C2}(i,j,p))$ *holds for all $p \in \operatorname{int}\Delta(A)$.*

To get the Folk Theorem, we need two additional conditions.

**Condition C3.** *For each player $i$, there exists $p$ enforceable that maximises $i$'s payoff, i.e. $p \in \mathcal{P}$ and $g^i(p) = \max_{a \in A} g^i(a)$.*

**Condition C4.** *For each player $i$, there exists $p$ enforceable that is a minmax distribution for player $i$, i.e. $\mathcal{P} \cap M^i \neq \emptyset$.*

Conditions (C1) and (C2) are generalizations of conditons (A2) and (A3) respectively of Kandori and Matsushima [KM98]. Our conditions are weaker: Kandori and Matsushima assume that (the analog of) conditions $(C1(i,j,a))$ and $(C2(i,j,a))$[1] hold for every pure action profile $a$ whose payoff vector is extremal in $V$ and for every pair of players. That is, at every pure (extremal) action profile, one can detect all deviations and ascribe it to a single player. Conditions (C1) and (C2) ask that for every pair of players $(i, j)$ and pair of decision rules $(\mu^i, \mu^j)$, there exists a profile $a$, depending on $(i, j, \mu^i, \mu^j)$, such that at $a$, the mediator can detect the deviation and differentiate $\mu^i$ from $\mu^j$. Note that under (C1) (resp. (C2)), $(C1(i,j,p))$ (resp. $(C2(i,j,p))$ hold for every pair $(i, j)$ and completely mixed $p$. The conditions (A2) and (A3) of Kandori and Matsushima clearly imply (C1) and (C2). Intuitively, we need weaker conditions since the mediator, knowing the profile of recommended actions and of reported signals has a better information on the deviation than any player has. Condition (C3) is implied by (A2) of Kandori and Matsushima and

[1] we indentify $a$ and the Dirac measure on $a$

(C4) is the essentially the same as (A1) of [KM98]. We give in the next section examples of signals that verify our conditions but not those of [KM98].

On another hand, [KM98] conditions are formulated on the signaling technology $q$ of the stage game whereas our conditions are formulated on distributions of reported signals which are more difficult to handle. However, since each player has finitely many decision rules, our conditions are reasonably tractable.

Let $\operatorname{rel} V$ be the relative interior of $V$, since $g$ is linear on $\Delta(A)$, $\operatorname{rel} V = g(\operatorname{int} \Delta(A))$.

**Theorem 7.4.** *Under conditions (C1), (C2), (C3) and (C4), if* $\operatorname{rel} V \cap IR \neq \emptyset$, $C^* = V^*$.

*Proof.* Since $C^* \subset V^*$, we prove that for each $v \in \operatorname{rel} V \cap IR$ and each direction $\lambda$ there exists $p \in \Delta(A)$ which is enforceable with respect to $\lambda$-hyperplanes such that $\lambda \cdot v \leq \lambda g(p)$. From lemma 6.8, this implies $\lambda \cdot v \leq k(\lambda)$ and thus $v \in C^*$. Since $C^*$ is closed and the closure of $\operatorname{rel} V \cap IR$ is $V^*$, this gives the result. Fix thus $v \in \operatorname{rel} V \cap IR$. We treat first singular vectors $\lambda$ then regular vectors.

*Singular vectors.*
Case 1. $\lambda = e^i$, the unit vector whose $i$-component is 1 and other components are zero. Let $p$ be an enforceable distribution that maximizes $i$'s payoff, this exists from (C3). Then $v^i \leq g^i(p)$. Since $p$ is enforceable, for each player $j \neq i$, there exists $f^j$ that solves the system:

$$\left\{(Q(p) - Q(p, d^j)) \cdot f^j \geq \frac{1-\delta}{\delta}(g^j(p, d^j) - g^j(p)) \quad (\forall d^j)\right.$$

Under $p$, each $a^i$ s.t. $p(a^i) > 0$ is a best response to $p(\cdot|a^i)$, thus one can choose $f^i$ to be constant, so let $f^i(a, y) = g^i(p)$ for each $(a, y)$ and $(p, f)$ is $\delta$-balanced. Clearly $\lambda \cdot f^i(a, y)$ is constant, thus $p$ is enforceable with respect to $\lambda$-hyperplanes.

Case 2. $\lambda = -e^i$. Let $p$ be an enforceable distribution that is also a minmax distribution for player $i$, this exists from (C4). Then $v^i \geq g^i(p) = m^i$. Again, since $p$ is enforceable, for each player $j \neq i$, there exists $f^j$ that solves the system:

$$\left\{(Q(p) - Q(p, d^j)) \cdot f^j \geq \frac{1-\delta}{\delta}(g^j(p, d^j) - g^j(p)) \quad (\forall d^j)\right.$$

Since $p$ is in $M^i$, each $a^i$ s.t. $p(a^i) > 0$ is a best response to $p(\cdot|a^i)$, thus one can choose $f^i$ to be constant, so let $f^i(a, y) = m^i$ for each $(a, y)$ and $(p, f)$ is $\delta$-balanced. As in the previous case $\lambda \cdot f^i(a, y)$ is constant, thus $p$ is enforceable with respect to $\lambda$-hyperplanes.

*Regular vectors.*
Let $\lambda$ be a regular vector. Since $v$ belongs to the relative interior of $V$, there exists $p \in \operatorname{int} \Delta(A)$ such that $v = g(p)$. We prove that (A1) and (A2) imply that $p$ is enforceable with respect to $ij$-hyperplanes for all pair $(i, j)$. Fix thus a pair of players $(i, j)$ and a vector $\lambda$ s.t. $\lambda^i \lambda^j \neq 0$ and $\lambda^k = 0$, for each $k \neq i$, $k \neq j$. Since $p$ is enforceable, for each such players $k$, there exists $f^k$ that solves the system:

$$\left\{(Q(p) - Q(p, d^k)) \cdot f^k \geq \frac{1-\delta}{\delta}(g^k(p, d^k) - g^k(p)) \quad (\forall d^k)\right.$$

Case 1. $\lambda^i \lambda^j > 0$. From condition (C2), (C2$(i, j, p)$) holds and by the separation theorem, there exists a mapping $\ell : A \times Y \to \mathbb{R}^n$ s.t.

$$Q(p, d^j) \cdot \ell < Q(p) \cdot \ell < Q(p, d^i) \cdot \ell$$

for all decision rules $d^i \neq \varphi^i$ ,$d^j \neq \varphi^j$. For $t > 0$, set $f^i = t\ell$ and $f^j = -\frac{\lambda^i}{\lambda^j} f^i$. Obviously, $\lambda^i f^i + \lambda^j f^j = 0$. The system of incentive constraints for player $i$ now writes:

$$\left\{ t(Q(p) - Q(p, d^i)) \cdot \ell \geq \frac{1-\delta}{\delta}(g^i(p, d^i) - g^i(p)) \ \ (\forall d^i) \right.$$

which is verified for large enough $t$ since the left-hand side is positive for $d^i \neq \varphi^i$. The system of incentive constraints for player $j$ writes:

$$\left\{ -t\frac{\lambda^i}{\lambda^j}(Q(p) - Q(p, d^j)) \cdot \ell \geq \frac{1-\delta}{\delta}(g^j(p, d^j) - g^j(p)) \ \ (\forall d^j) \right.$$

which is also verified for large enough $t$ since $(Q(p) - Q(p, d^j)) \cdot \ell$ is negative for $d^j \neq \varphi^j$.
Case 2. $\lambda^i \lambda^j < 0$. From condition (C1), (C1$(i, j, p)$) holds and by the separation theorem, there exists a mapping $\ell : A \times Y \to \mathbb{R}^n$ s.t.

$$Q(p, d^h) \cdot \ell < Q(p) \cdot \ell$$

for $h = i, j$ and all decision rules $d^h \neq \varphi^h$. As before, for $t > 0$, set $f^i = t\ell$ and $f^j = -\frac{\lambda^i}{\lambda^j} f^i$. Again, $\lambda^i f^i + \lambda^j f^j = 0$ and the system of incentive constraints for player $i$ writes:

$$\left\{ t(Q(p) - Q(p, d^i)) \cdot \ell \geq \frac{1-\delta}{\delta}(g^i(p, d^i) - g^i(p)) \ \ (\forall d^i) \right.$$

which holds large enough $t$. The system of incentive constraints for player $j$ writes:

$$\left\{ -t\frac{\lambda^i}{\lambda^j}(Q(p) - Q(p, d^j)) \cdot \ell \geq \frac{1-\delta}{\delta}(g^j(p, d^j) - g^j(p)) \ \ (\forall d^j) \right.$$

which is also verified for large enough $t$. The proof of the theorem is thus complete. □

## 8. Example: a partnership game

Consider a firm employing 4 workers. Each worker has two actions: work or shirk. The output of the firm depends on the action profile and the associated profit increases with the number of agents who work. The manager offers a day-to-day salary to each worker, which might depend on the output. The wage contract thus defines a (one-shot) game between the workers. This game is played each day and the players discount payoffs at a common rate. We assume here that the manager is a non-strategic agent, that is, the wage contract is known by all agents and is fixed once and for all throughout the game. For example, the workers are the owners/shareholders of the firm and the manager is paid a fixed amount independent of outcomes. This assumption can also be the result of legal dispositions regarding wages. We may thus regard the manager as a trustworthy mediator for this game.

The goal of the manager is to get the employees to work each day. The problem he faces is how to verify this? A simple way to do it is to crowd all employees in the same office, so that each worker's action is monitored by all his peers, and to ask each person to report his observation at the end of the day. Under unilateral deviations, i.e. assuming that at most one player shirks and produces false reports, a strict majority of players will report to the manager the action profile actually played. A punishing strategy can thus be triggered in case of deviation.

Suppose now that the firm's building has two offices, each containing at most two persons. Say, players 1 and 2 are office-mates and so are players 3 and 4. If the manager/mediator still asks each player to report his observations at the end of the day, he might not be able to tell which player is deviating. Suppose player 2 claims that player 1 has deviated. Either this statement is true or it is a deviation of player 2 to induce a punishment against player 1.

The manager thus hires an inspector who chooses each day which office to inspect or may shirk. The manager then has to handle a five player game where: each worker has two actions, *work*, *shirk*, and the inspector has three actions: *inspect room A, inspect room B* or *shirk.* The signaling structure is as follows: each worker monitors the action of his office-mate and knows whether the inspector is in the office or not. The inspector monitors the actions of the players in the office he is inspecting, if he inspects, and observes nothing if he shirks.

This signaling structure does not verify the conditions of [KM98]. Given any pure action profile, there is an office which is not inspected and thus deviations by players in this office cannot be differentiated. On the other hand our conditions are fulfilled. Take a pair of workers and a deviation for each. If they are in different offices, say player 1 or player 3 deviates, then their deviations created different reported signals: a deviation of player 1 wil be reported by player 2 while a deviation of player 3 will be reported by player 4. If the two players are in the same office, say player 1 or 2 deviates, then their deviations are detected and differentiated at an action profile where the inspector inspects their office. Last case, assume that the pair of players under consideration contains the inspector. Deviations by the inspector can be detected and indentifed: if the inspector shirks all players will report not having seen him, if the inspector inspects the wrong office, the mediator will know it from the worker's reports. Therefore conditions C1 and C2 hold. Since every unilateral deviation can be detected at every action profile, every correlated distribution on actions is enforcable and therefore C4 and C5 also hold.

## 9. Appendix

We give now a proof of theorem 5.5 along the lines of [FLM94] and [FL94].

*Proof of theorem 5.5, point (1).* We need to prove that $C(\delta) \subset \bigcap_\lambda H_\lambda$. By contradiction, assume that for some $\lambda$ there exists $v \in C(\delta)$ such that $v \cdot \lambda > k(\lambda)$. Let then $v^* \in C(\delta)$ that maximizes $v \cdot \lambda$ on $C(\delta)$, one has $v^* \cdot \lambda > k(\lambda)$ and $v^* \cdot \lambda \geq v \cdot \lambda$ for each $v \in C(\delta)$. Since $C(\delta) \subset F_\delta(C(\delta))$, there exists $(p, f)$ $\delta$-balanced, $\lambda$-directed such that $f \in C(\delta)$, $v^* = v_\delta(p, f)$ which contradicts the definition of $k(\lambda)$. □

*Proof of point (2).* Let $E$ be the affine subspace of $\mathbb{R}^n$ spanned by the convex $C^*$. In the proof that follows, $E$ is endowed with the norm induced by the euclidean norm on $\mathbb{R}^n$ and all topological notions such as neighborhood, interior, boundary are taken with respect to the topology of $E$. For example, the interior of $C^*$ with respect to this topology (the relative interior of $C^*$) is non-empty.

**Definition 9.1.** *A set $W \subset E$ is locally self-decomposable if for each $v \in W$, there exists an open neighborhood $U$ of $v$ and a discount factor $\delta_U$ such that $U \subset F_{\delta_U}(W)$.*

**Proposition 9.2.** *If a convex and compact set $W \subset E$ is locally self-decomposable then there exists $\delta'$ such that for each $\delta \geq \delta'$, $W \subset C(\delta)$.*

*Proof.* Take a locally self-decomposable convex compact set $W$, for each for each $v \in W$, there exists an open neighborhood $U$ of $v$ and a discount factor $\delta_U$ such that $U \subset F_{\delta_U}(W)$. The $U$'s form an open cover of $W$. Take a finite subcover and let $\delta'$ be the maximum of $\delta_U$ over this subcover. For each $\delta \geq \delta'$ and $v \in W$ there is $U$ in the subcover such that $v \in U$ and thus $v \in F_{\delta_U}(W)$, i.e. there exists $(p, f_U)$ $\delta_U$-balanced with $f_U \in W$ and $v = v_{\delta_U}$. From proposition 5.2, setting for each $(a, y)$,

$$f(a,y) = \frac{\delta - \delta_U}{\delta(1-\delta_U)} v_{\delta_U}(p,f) + \frac{\delta_U(1-\delta)}{\delta(1-\delta_U)} f_U(a,y)$$

the pair $(p, f)$ is $\delta$-balanced, $v = v_\delta(p, f)$ and since $v \in W$, $f_U \in W$ and $W$ convex, $f \in W$. Thus, $v \in F_\delta(W)$ i.e. $W \subset F_\delta(W)$ and from theorem 4.5, $W \subset C(\delta)$.

□

To complete the proof of theorem 5.5, we approximate the convex $C^*$ by smooth sets. A subset $W \subset E$ is *smooth* if it is convex compact, has non-empty interior and its boundary is a $C^2$ submanifold of $E$. The last step of the proof is to show that for each smooth $W \subset \text{int } C^*$, there exists $\delta'$ such that for each $\delta \geq \delta'$, $W \subset C(\delta)$. This follows directly from the next proposition.

**Proposition 9.3.** *Let $W$ be a smooth set $W \subset \text{int } C^*$, then $W$ is locally self-decomposable.*

*Proof.* We need to prove that for each for each $v \in W$, there exists an open neighborhood $U$ of $v$ and a discount factor $\delta_U$ such that $U \subset F_{\delta_U}(W)$.

Case 1. Take $v$ in the interior of $W$. Let $\varepsilon > 0$ such that $U = B(v, 3\varepsilon)$, the ball centered at $v$ with radius $3\varepsilon$ is a subset of $W$ and fix $p$ a correlated equilibrium of the stage game.

*Claim.* $\exists \delta < 1$, $\forall u \in B(v, \epsilon)$, $\exists w \in W$ such that $u = (1-\delta)g(p) + \delta w$.

*Proof of the claim.* Given $u \in B(v, \epsilon)$ and $\delta < 1$, set $w = \frac{1}{\delta}u - \frac{1-\delta}{\delta}g(p)$. Then, $\|w - u\| \leq \frac{1-\delta}{\delta}\|u - g(p)\|$ and $\|w - v\| \leq \|u - v\| + \frac{1-\delta}{\delta}\|u - g(p)\|$. Now, $\|u - g(p)\|$ is bounded by some $M > 0$ as $u$ varies in $W$, thus $\|w - v\| \leq \varepsilon + \frac{1-\delta}{\delta}M$ which is less than $2\varepsilon$ for $\delta$ close enough to one. □

Define now the mapping $f(a, y) = w$ for each $(a, y)$. Since $p$ is a correlated equilibrium, the pair $(p, f)$ is $\delta$-balanced from proposition 5.2 and $u = v_\delta(p, f)$. Thus $U \subset F_\delta(W)$. This ends the first case.

Case 2. Consider now $v$ on the boundary of $W$. Since $W$ is smooth, there is a unique hyperplane tangent on $W$ at $v$ and let $\lambda$ be normal to $W$ at $v$. Let $(p, f)$ be a pair that achieves the maximum in the definition of $k(\lambda)$. Then $(p, f)$ is $\delta_0$-balanced for some $\delta_0$ and the tangent hyperplane to $W$ at $v$ strictly separates $g(p)$ from $W$. The line connecting $g(p)$ and $v$ thus crosses the boundary of $W$ and therefore there exists $\delta' < 1$ such that for each $\delta > \delta'$, the point $w_\delta$ defined by $v = (1-\delta)g(p) + \delta w_\delta$ is in the interior of $W$. Let us now define the mapping $f_\delta$ for each $(a, y)$ by:

$$f_\delta(a,y) = w_\delta + \frac{(1-\delta)\delta_0}{\delta(1-\delta_0)}(f(a,y) - \bar{f})$$

From proposition 5.2, $(p, f_\delta)$ is $\delta$-balanced, $v = v_\delta(p, f_\delta)$ and for each $(a, y)$, $\|f_\delta(a, y) - w_\delta\| \leq M\frac{1-\delta}{\delta}$ with $M = M(p, f, \delta_0)$. We have then the following geometric result.

**Lemma 9.4.** *$\exists \delta''$, s.t. $\forall \delta > \delta''$, the ball centered at $w_\delta$ with radius $M\frac{1-\delta}{\delta}$ is in the interior of $W$.*

*Proof.* Choose a new coordinates system. Assume $v = 0$, let $H$ be the hyperplane $\{x \mid \lambda \cdot x = \lambda \cdot v\}$ and $L$ be the line connecting $g(p)$ and $v$. Write each $x \in \mathbb{R}^n$ as $x = (x^H, x^L)$ with $x^H \in H$ and $x^L \in L$. There is $X$ a neighborhood of the origin such that $X \cap W$ is the hypograph of a concave $C^2$ function $\varphi : X \cap H \to \mathbb{R}$ s.t.

$\varphi(0) = 0$ and the gradient of $\varphi$ at zero is zero. By Taylor's theorem, there is a neighborhood of the origin $X' \subset X$ and a positive constant $K$ s.t. $\forall x \in X' \cap H$, $\|x^L\| \geq K \|x^H\|^2$ implies $x \in \text{int } W$. There exist two positive constants $\alpha, \beta$ s.t. for each $x \in B(w_\delta, M\frac{1-\delta}{\delta})$, $\|x^L\| \geq \alpha\frac{1-\delta}{\delta}$ and $\|x^H\| \leq \beta\frac{1-\delta}{\delta}$. Thus $\|x^L\| \geq K \|x^H\|^2$ holds if $\alpha\frac{1-\delta}{\delta} \geq K\beta^2(\frac{1-\delta}{\delta})^2$, which is true for $\delta$ close enough to one. □

We have proved so far that for the payoff vector $v$, there exists a discount factor $\delta$ and a $\delta$-balanced pair $(p, f)$ s.t. $v = v_\delta(p, f_\delta)$ and $f \in B(w_\delta, M\frac{1-\delta}{\delta}) \subset \text{int}\, W$. Then for $v'$ in a neighborhood of $v$, define for each $(a, y)$, $f'(a, y) = f_\delta(a, y) + \frac{1}{\delta}(v' - v)$. Clearly, $(p, f')$ is $\delta$-balanced, $v_\delta(p, f') = v'$ and $f'(a, y)$ lies in the interior of $W$ for $\|v' - v\|$ is small enough. The proof is thus complete. □

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