DEPARTEMENT TOEGEPASTE ECONOMISCHE WETENSCHAPPEN

ONDERZOEKSRAPPORT NR 9647

DEPENDENCIES IN MULTI-LIFE STATUSSES

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D/1996/2376/47

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August 20, 1996

Abstract

The usual assumption of independence of the remaining life times involved in joint-life and last survivor statusses is ommitted. Given the marginal distributions of the remaining life times, lower and upper bounds are derived for the single premiums of multi-life insurances and annuities.

Keywords: joint-life statusses, last survivor statusses, independence, single premiums.

1 Introduction

Usually in the theory of multilife contingencies, the remaining life times of the lives involved are assumed to be mutually independent. Computational feasability rather than realism seem to be the major reason for making this assumption. Indeed, a husband and his wife are more or less exposed to the same risks. The "broken hart syndrome" causes an increase of the mortality rate after the mortality of one's spouse. Such effects may have a significant influence on present values related to multilife actuarial functions.

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In this paper, we will use some results from Dhaene & Goovaerts (1995) which were obtained for portfolios where the risks involved are not necessarily mutually independent. We will show that we can use some these general results for evaluating the effect of dependencies in case of multilife functions. We will restrict our discussion to situations involving two lives.

In the following section we will give some definitions and results, as obtained in Dhaene & Goovaerts (1995). In section 3 these results will be used for deriving ordering relations between multilife insurances and annuities on two lives. In the sections 4 and 5 we derive bounds for the expected payments for the different types of multilife insurances and annuities. In section 6 , we will discuss a particular case of dependency. Finally, in section 7 we will give some numerical illustrations of the results obtained in the previous sections.

2 Correlation order and positive quadrant dependency

Let $R(F, G)$ be the set of all bivariate distributed random variables (X, Y) with given marginal distribution functions F and G for X and Y respectively. We interpret X and Y as the remaining life times of persons (x) and (y) respectively. We will assume that all random variables involved are non-negative.

Definition 1 *Let (X_1, Y_1) and (X_2, Y_2) be two elements of R(F,G). (X_1, Y_1) is said to be less correlated than (X_2, Y_2), written as $(X_1, Y_1) \leq_c (X_2, Y_2)$ if*

$$cov(f(X_1), g(Y_1)) \leq cov(f(X_2), g(Y_2))$$

for all non-decreasing functions f and g for which the covariances exist.

The correlation order is a partial order between the joint distributions of the risks in R(F,G). It expresses the notion that some elements of R(F,G) are more positively correlated than others.

The following theorem gives an alternative definition for correlation order in terms of the joint probability distributions.

Theorem 1 *Let (X_1, Y_1) and (X_2, Y_2) be two elements of $R(F,G)$. Then the following statements are equivalent :*

(a) $(X_1, Y_1) \leq_c (X_2, Y_2)$

(b) $Prob(X_1 \leq x, Y_1 \leq y) \leq Prob(X_2 \leq x, Y_2 \leq y)$ *for all* $x, y \geq 0$

A proof of this theorem can be found in Dhaene & Goovaerts (1995).

Often certain insured risks tend to act similarly, they possess some "positive" dependency. In order to describe such situations we introduce the well-known notion of "positive quadrant dependency".

Definition 2 *Two random variables X and Y are said to be positively quadrant dependent, written as $PQD(X, Y)$, if*

$$Prob(X \leq x, Y \leq y) \geq Prob(X \leq x)Prob(Y \leq y)$$

for all $x, y \geq 0$.

Hence, the probability that X and Y both realize small values is larger than in the case of independent random variables. In terms of correlation order (definition 1) we can say that (X, Y) is actually more correlated than in the independent case.

3 Actuarial functions on two dependent lives

Let $v = \frac{1}{1+i}$ denote the discounting factor and $d = 1 - v$.

3.1 The joint-life status

In this subsection we consider insurances and annuities issued on the joint-life status (xy) which exists as long as (x) and (y) are both alive. We will consider a pure endowent (where an amount is paid after n years if both (x) and (y) are alive at that moment), a perpetuity (where an amount is paid in the beginning of each year, as long as both (x) and (y) are alive) and a whole life insurance (where an amount is paid at the end of the year of the first death).

Let X and Y be the remaining life times of (x) and (y) respectively. The bivariate remaining life time of the couple consisting of (x) and (y) is then given by (X, Y). The single premiums of these insurances and annuities are given by

(a) **Pure Endowment**

$$_nE_{xy} = v^n Prob(X > n, Y > n)$$

(b) **Perpetuity**

$$\ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k Prob(X > k, Y > k)$$

(c) **Whole Life Insurance**

$$A_{xy} = \sum_{k=0}^{\infty} v^{k+1} Prob(k < min(X, Y) \leq k + 1)$$

In the following theorem we will consider two bivariate remaining life times in $R(F, G)$ which are ordered by the correlation order. We will show that a correlation order between these remaining life times implies an ordering of the corresponding single premiums.

Theorem 2 *Let (X_1, Y_1) and (X_2, Y_2) be two bivariate remaining life times, both elements of $R(F, G)$. If $(X_1, Y_1) \leq_c (X_2, Y_2)$ then*

$$_nE_{xy}^{(1)} \leq {}_nE_{xy}^{(2)}$$

$$\ddot{a}_{xy}^{(1)} \leq \ddot{a}_{xy}^{(2)}$$

$$A_{xy}^{(1)} \geq A_{xy}^{(2)}$$

We have added the superscript (i) $(i = 1,\ 2)$ *to the single premium symbols to denote that the bivariate remainig life time of the couple involved is given by* (X_i, Y_i).

Proof.

To proof the stated relations, we use the equivalent definition of correlation order :

$$Prob(X_1 \leq x, Y_1 \leq y) \leq Prob(X_2 \leq x, Y_2 \leq y)$$

or equivalently

$$Prob(X_1 > x, Y_1 > y) \leq Prob(X_2 > x, Y_2 > y)$$

Using this inequality and the following relations

$$\ddot{a}_{xy} = \sum_{k=0}^{\infty} E_{xy}$$

$$A_{xy} = 1 - d\ddot{a}_{xy}$$

we find the stated results.

□

Theorem 2 can be interpreted as follows. Assume that the marginal distributions of the remaining life times of (x) and (y) are given. If the bivariate remaining life time of the couple increases in correlation order, then the single premiums of endowment insurances and annuities on the joint life status increase. For the whole life insurance, the opposite conclusion holds.

3.2 The last survivor status

Let us now consider insurances and annuities that are based on the last survivor status $(\overline{xy})$ which exists as long as at least one of (x) and (y) is alive.

(a) **Pure Endowment**

$$ {}_nE_{\overline{xy}} = v^n \left(1 - Prob(X \leq n, Y \leq n)\right) $$

(b) **Perpetuity**

$$ \ddot{a}_{\overline{xy}} = \sum_{k=0}^{\infty} v^k \left(1 - Prob(X \leq k, Y \leq k)\right) $$

(c) **Whole Life Insurance**

$$ A_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1} Prob(k < max(X,Y) \leq k+1) $$

For the last survivor status we find the following result.

Theorem 3 *Let (X_1, Y_1) and (X_2, Y_2) be two bivariate remaining life times, both elements of $R(F,G)$. If $(X_1, Y_1) \leq_c (X_2, Y_2)$ then*

$$ {}_nE^{(1)}_{\overline{xy}} \geq {}_nE^{(2)}_{\overline{xy}} $$
$$ \ddot{a}^{(1)}_{\overline{xy}} \geq \ddot{a}^{(2)}_{\overline{xy}} $$

$$ A^{(1)}_{\overline{xy}} \leq A^{(2)}_{\overline{xy}} $$

Proof. The proof is similar to the proof of Theorem 2. □

Theorem 3 can be also interpreted as follows. Assume that the marginal distributions of the remaining life times of (x) and (y) are given. If the bivariate remaining life time of the couple increases in correlation order, then the single premiums of endowment insurances and annuities on the last survivor status $(\overline{xy})$ decrease. For the whole life insurance, the opposite conclusion holds.

4 Independent lives versus PQD

In this section, we will again assume that the marginal distributions of the remaining life times X and Y of (x) and (y) respectively, are given. We will compare the case where the remaining life times are mutually independent with the case where they are PQD. The independent case will be denoted by the superscript "ind". The PQD-case will be denoted by the superscript "PQD".

Theorem 4 *If the marginal distributions of the remaining life times of (x) and (y) are given, then*

$$\begin{array}{rr} {}_nE_{xy}^{ind} \leq {}_nE_{xy}^{PQD} & {}_nE_{\overline{xy}}^{ind} \geq {}_nE_{\overline{xy}}^{PQD} \\ \ddot{a}_{xy}^{ind} \leq \ddot{a}_{xy}^{PQD} & \ddot{a}_{\overline{xy}}^{ind} \geq \ddot{a}_{\overline{xy}}^{PQD} \\ A_{xy}^{ind} \geq A_{xy}^{PQD} & A_{\overline{xy}}^{ind} \leq A_{\overline{xy}}^{PQD} \end{array}$$

Proof. The proof follows immediately from Definition 2 and the Theorems 2 and 3. □

These inequalities have been derived in Norberg (1989).

5 Lower or upper bounds for the single premiums

In this section we will look at an extremal element in $R(F, G)$, namely the one which is larger in correlation order than any other element in $R(F, G)$.

Lemma 1 *For any element (X, Y) in $R(F, G)$, we have that*

$$Prob(X \leq x, Y \leq y) \leq min\,(F(x), G(y))$$

with this upper bound being the bivariate distribution function of an an element contained in $R(F, G)$.

This result can be found in Fréchet (1951).

From Lemma 1 and the Theorems 2 and 3 we immediately find the following result.

Theorem 5 *For any bivariate remaining life time (X, Y) in $R(F, G)$ we have*

$$
\begin{array}{cc}
{}_nE_{xy} \leq {}_nE^*_{xy} & {}_nE_{\overline{xy}} \geq {}_nE^*_{\overline{xy}} \\
\ddot{a}_{xy} \leq \ddot{a}^*_{xy} & \ddot{a}_{\overline{xy}} \geq \ddot{a}^*_{\overline{xy}} \\
A_{xy} \geq A^*_{xy} & A_{\overline{xy}} \leq A^*_{\overline{xy}}
\end{array}
$$

where the single premiums with a superscript "" are computed with the bivariate distribution of the remaining life time of the couple given by* $\min(F(x), G(y))$.

Now assume that the given remaining life times of (x) and (y) are PQD. In this case the bounds obtained in the Theorems 4 and 5 complement each other in the sense that we have an upper and a lower bound for each type of insurance or annuity on two lives. One of the bounds corresponds to the independence case. The other bound corresponds to the case where the bivariate distribution is the minimum of the two marginal distributions involved.

6 A particular type of dependency

Let X and Y be the remaining life times of (x) and (y) respectively. Assume that the bivariate remaining life time (X, Y) is an element of $R(F, G)$. The following inequalities can easily be derived.

$$
\begin{array}{cc}
{}_nE_{xy} \leq {}_nE_x & {}_nE_{\overline{xy}} \geq {}_nE_y \\
\ddot{a}_{xy} \leq \ddot{a}_x & \ddot{a}_{\overline{xy}} \geq \ddot{a}_y \\
A_{xy} \geq A_x & A_{\overline{xy}} \leq A^*_y
\end{array}
$$

Now we will prove that these bounds for the multilife single premiums correspond to the bounds denoted with a superscript "*" in Theorem 5 provided that Y stochastically dominates X. The well-known definition of stochastic dominance is repeated below.

Definition 3 *Let (X, Y) be an element of $R(F, G)$. We say that Y stochastically dominates X, written as $X \leq_{st} Y$ if*

$$F(t) \geq G(t)$$

for all $t \geq 0$.

Note that $\leq_{st}$ is an order between distributions. This implies that all elements of $R(F, G)$ are stochastically ordered or not.

If $F(t) \geq G(t)$ for all $t \geq 0$ we have that

$$min\,(F(t), G(t)) = G(t)$$

for all $t \geq 0$. After some straightforward derivations we find that in this case

$$\begin{aligned} {}_nE^*_{xy} &= {}_nE_x & {}_nE^*_{\overline{xy}} &= {}_nE_y \\ \ddot{a}^*_{xy} &= \ddot{a}_x & \ddot{a}^*_{\overline{xy}} &= \ddot{a}_y \\ A^*_{xy} &= A_x & A^*_{\overline{xy}} &= A^*_y \end{aligned}$$

We can conclude that if $X \leq_{st} Y$ for all (X, Y) in $R(F, G)$ then the bounds derived in Theorem 5 all reduce to single premiums of insurances and annuities on a single life.

7 Numerical illustration

In this section we will illustrate the bounds derived in the previous sections by some numerical examples. The technical interest rate equals 0.475. Further, (x) and (y) are a male and a female respectively. The marginal distribution functions of the remaining life times of (x) and (y) follow from the Belgian mortality tables MR and FR respectively. Finally, we assume that the remaining life times of (x) and (y) are positive quadrant dependent.

In Table 1 bounds are given for perpetuities on (xy) and $(\overline{xy})$ with $x = y$ for different values of x. The bounds follow from Theorems 4 and 5.

	$\ddot{a}_{xx}$		$\ddot{a}_{\overline{xx}}$	
x	LB	UB	LB	UB
20	19.73491	20.16667	20.65737	21.08913
25	19.25552	19.75987	20.33743	20.84178
30	18.66676	19.25966	19.9384	20.53131
35	17.94998	18.64924	19.44297	20.14223
40	17.08711	17.9114	18.83157	19.65585
45	16.06302	17.03007	18.08316	19.05021
50	14.86913	15.9929	17.17676	18.30054
55	13.50804	14.79454	16.09438	17.38088
60	11.9987	13.44083	14.82536	16.26748
65	10.38052	11.95296	13.37225	14.94469

Table 1. Bounds for perpetuities on (xx) and $(\overline{xx})$.

The differences between the upper and lower bounds are relatively small. This means that the knowledge of the marginal distributions of the (not necessarily independent) remaining life times involved, gives already a lot of information concerning the multilife annuity values. We also remark that the absolute difference between the upper and the lower bound increases with the age.

In Table 2 we compare the single premiums for endowment insurances on

$(25:20)$ and $(\overline{25:20})$ respectively, for varying durations of the endowment.

	${}_n E_{25\,20}$		${}_n E_{\overline{25}\,\overline{20}}$	
n	LB	UB	LB	UB
5	0.7877	0.78926	0.79135	0.79291
10	0.61963	0.62223	0.62609	0.6287
15	0.48632	0.48965	0.49513	0.49847
20	0.38028	0.38418	0.39128	0.39518
25	0.29557	0.29998	0.30883	0.31324
30	0.22746	0.23243	0.24321	0.24819
35	0.17219	0.17784	0.19081	0.19645
40	0.12689	0.13333	0.14872	0.15515
45	0.08945	0.09672	0.11458	0.12186

Table 2. Bounds for endowment insurances on $(25:20)$ and $(\overline{25:20})$.

In both cases, the difference between the upper and the lower bound seems to be an increasing function of the duration.

Finally, in Table 3 we compare perpetuities on $(x:20)$ and $(\overline{x:20})$ with x varying from 20 to 55.

		$\ddot{a}_{xy}$		$\ddot{a}_{\overline{xy}}$	
x	y	LB	UB	LB	UB
20	20	19.73491	20.16667	20.65737	21.08913
25	20	18.97906	19.25966	20.65737	21.00743
30	20	18.97906	19.25966	20.65737	20.93798
35	20	18.42589	18.64924	20.65737	20.88073
40	20	17.7345	17.9114	20.65737	20.83428
45	20	16.89073	17.03007	20.65737	20.79672
50	20	15.88407	15.9929	20.65737	20.76621
55	20	14.71068	14.79454	20.65737	20.74124

Table 3. Bounds for perpetuities on $(x:20)$ and $(\overline{x:20})$.

For the last survivor annuity the lower bound equals a_y and hence is constant. From Table 3 we see that increasing the difference in age between (x) and (y) decreases the absolute difference between the bounds.

Acknowledgement

Jan Dhaene and Marleen Vanneste would like to thank the financial support of grant OT/93/5 of Onderzoeksfonds K.U.Leuven.

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