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STOCHASTIC APPROXIMATIONS OF PRESENT VALUE FUNCTIONS

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Abstract

The aim of this paper is to apply the method proposed by Denuit, Genest and Marceau (1999) for deriving stochastic upper and lower bounds on the present value of a sequence of cash flows, where the discounting is performed under a given stochastic return process. The convex approximation provided by Goovaerts, Dhaene and De Schepper (1999) and Goovaerts and Dhaene (1999) is then compared to these stochastic bounds. On the basis of several numerical examples, it will be seen that the convex approximation seems reasonable.

Key words and phrases: Dependence, Stochastic Dominance, Stochastic Annuities

1 Introduction

Let V_t be the present value at time 0 of an amount of α_t paid at time t. The stochastic discounted value at time 0 of payments of amount α_t made at times $t = 1, 2, \cdots, n$ is then given by

$$Z_n = V_1 + V_2 + \cdots + V_n. \tag{1.1}$$

Consider for instance an insurance company facing payments of amount α_t at times $t = 1, 2, \cdots, n$; the present value of these n deterministic payments is given by (1.1).

The V_i's involved in (1.1) are obviously correlated, so that the convenient independence assumption for the summands in Z_n is not realistic. As a consequence, an exact expression for the cumulative distribution function of Z_n requires the knowledge of the joint distribution of the random vector $(V_1, V_2, \cdots, V_n)$, which is in general not available. Goovaerts, Dhaene and De Schepper (1999) recently proposed to circumvent this problem by approximating Z_n by means of a random variable $\widetilde{Z}_n$ dominating the original Z_n in the convex sense. If we denote by $F_1, F_2, \cdots, F_n$ the respective distribution functions of $V_1, V_2, \ldots, V_n$ involved in (1.1), $\widetilde{Z}_n$ is given by

$$\widetilde{Z}_n = F_1^{-1}(U) + F_2^{-1}(U) + \cdots + F_n^{-1}(U),$$

where U is a unit uniform random variable and the F_i^{-1}'s are the quantile functions associated to the F_i's. We obviously have that $EZ_n = E\widetilde{Z}_n$ and it can be shown that the inequalities

$$E \max\{Z_n - d, 0\} \leq E \max\{\widetilde{Z}_n - d, 0\} \tag{1.2}$$

hold for any $d \geq 0$ (that is, Z_n is smaller than $\widetilde{Z}_n$ in the convex order).

Since $\widetilde{Z}_n$ precedes Z_n in the convex sense, the approximation $\widetilde{Z}_n$ is considered as less favorable by all the risk-averse decision-makers, and the method is thus conservative. Moreover, the cumulative distribution function of $\widetilde{Z}_n$ enjoys an explicit expression and is particularly easy to handle. On the basis of numerical illustrations performed in a situation where the exact cumulative distribution function of Z_n can be obtained, Goovaerts *et al.* (1999) showed that the cumulative distribution functions of Z_n and $\widetilde{Z}_n$ seem to be rather close.

The problem of estimating the distribution of Z_n has been studied, among others, by Beekman and Fuelling (1991), De Schepper and Goovaerts (1992), Dufresne (1990), Frees (1990), Parker (1994c,1997), De Schepper, Teunen, Goovaerts (1994) and Vanneste, Goovaerts and Labie (1994). This paper aims to carry on with Goovaerts *et al.*'s (1999) approach by providing lower and upper bounds on Z_n in the stochastic dominance sense, using the method proposed in Denuit, Genest and Marceau (1999). This approach also provides upper and lower bounds on the quantiles of Z_n. In risk management, these quantiles correspond to the Value at Risk at different probability levels. Such bounds cannot be obtained with the aid of the convex approximation $\widetilde{Z}_n$. Indeed, we see from (1.2) that the stop-loss premium of $\widetilde{Z}_n$ is an upper bound of the stop-loss premium of Z_n; more generally, $E\phi(\widetilde{Z}_n)$ is an upper bound for $E\phi(Z_n)$ for any convex function ϕ. However, there is in general no relation between $P[Z_n \leq z]$ and $P[\widetilde{Z}_n \leq z]$ (since indicator functions are not convex).

Another purpose of this work is to provide several numerical illustrations which enhance the practical interest of our approach. In these illustrations, we will examine the position of the cumulative distribution function corresponding to the convex approximation $\widetilde{Z}_n$ in the admissible region delimitated by the stochastic bounds on Z_n. As a byproduct of our results, the error in the approximation of Z_n by $\widetilde{Z}_n$ can be evaluated (in other words, we get an upper bound for the Kolmogorov distance between Z_n and $\widetilde{Z}_n$).

2 Stochastic bounds on Z_n

In this section, we recall how to build two functions $F_{\min}$ and $F_{\max}$ such that the inequalities

$$F_{\min}(t) \leq P[Z_n \leq t] \leq F_{\max}(t) \text{ for all } t \geq 0, \tag{2.1}$$

hold, as well as

$$F_{\min}(t) \leq P[\widetilde{Z}_n \leq t] \leq F_{\max}(t) \text{ for all } t \geq 0. \tag{2.2}$$

To this end, we use the following result due to Denuit *et al.* (1999, Proposition 2). Let $F_1, F_2, \cdots, F_n$ be the respective cumulative distribution functions of $V_1, V_2, \ldots, V_n$. Then, the cumulative distribution function F_{Z_n} of $Z_n = V_1 + V_2 + \ldots + V_n$ is constrained by (2.1) with

$$F_{\min}(t) = \sup_{(v_1,v_2,\ldots,v_n)\in\Sigma(t)} \max\left\{\sum_{i=1}^{n} P[V_i < v_i] - (n-1), 0\right\},$$

and

$$F_{\max}(t) = \inf_{(v_1,v_2,\ldots,v_n)\in\Sigma(t)} \min\left\{\sum_{i=1}^{n} F_i(v_i), 1\right\},$$

where

$$\Sigma(t) = \{(v_1, v_2, \ldots, v_n) \in I\!R^n | v_1 + v_2 + \ldots + v_n = t\}, \ \ t \in I\!R.$$

Note that $F_{\max}$ is a *bona fide* cumulative distribution function, whereas $F_{\min}$ is the left-continuous version of some cumulative distribution function. The bounds in (2.1) and (2.2) are the best-possible bounds on Z_n and $\widetilde{Z}_n$ in the sense of stochastic dominance when we know the distribution functions

$F_1, F_2, \dots, F_n$, but no assumption is made on the dependence structure between the V_i's. Equivalently, these bounds hold for all sums (1.1) with given cumulative distribution functions for $V_1, V_2, \dots, V_n$.

Closed form expressions for the bounds (2.1) can in general not be obtained for distributions of the V_i's and one must resort to numerical evaluation. For more details, see Denuit *et al.* (1999).

Now, assume we have at our disposal some partial knowledge of the dependence existing between the V_i's, namely that there exists a multivariate cumulative distribution function G satisfying

$$G(v_1, v_2, \cdots, v_n) \leq P[V_1 \leq v_1, V_2 \leq v_2, \cdots, V_n \leq v_n] \text{ for all } v_1, v_2, \dots, v_n \in I\!R, \tag{2.3}$$

and a joint decumulative distribution function $\overline{H}$ such that

$$P[V_1 > v_1, V_2 > v_2, \cdots, V_n > v_n] \geq \overline{H}(v_1, v_2, \cdots, v_n) \text{ for all } v_1, v_2, \cdots, v_n \in I\!R. \tag{2.4}$$

From Denuit *et al.* (1999, Proposition 5), the inequalities

$$\sup_{(x_1,x_2,\cdots,x_n)\in\Sigma(t)} G(x_1, x_2, \cdots, x_n) \leq F_{Z_n}(t) \leq 1 - \sup_{(x_1,x_2,\cdots,x_n)\in\Sigma(t)} \overline{H}(x_1, x_2, \cdots, x_n), \tag{2.5}$$

hold for all $t \in I\!R$. The bounds in (2.5) are obviously more accurate than those in (2.1).

In the literature, several notions of positive dependence have been introduced in order to express the fact that large values of one of the components of a random vector tend to be associated with large values of the others. In our context, one intuitively feels that in most situations the V_i's mainly "move together" (i.e. a large value of V_i is usually followed by a large value of V_{i+1}). For the numerical illustrations in this paper, we will assume that (2.3) and (2.4) are satisfied with

$$G(v_1, v_2, \cdots, v_n) = \prod_{i=1}^{n} F_i(v_i)$$

and

$$\overline{H}(v_1, v_2, \cdots, v_n) = \prod_{i=1}^{n} (1 - F_i(v_i)).$$

In such a case, the V_i's are said to be Positively Orthant Dependent (POD, in short). POD comes thus down to assume that the probability that all the V_i's assume "small" values (i.e. $V_i \leq v_i$, $i = 1, 2, \dots, n$) is larger than the corresponding probability under the assumption that the V_i's are mutually independent. The interpretation for $\overline{H}$ is similar by substituting "large" for "small". For more details, see, e.g., Szekli (1995, pp. 144-145).

3 Applications

3.1 Stochastic annuities

Let δ_s be the force of interest at time s and let Y_t denote the force of interest accumulation function at time t, i.e.

$$Y_t = \int_{s=0}^{t} \delta_s ds.$$

The random present value at time 0 of a payment of 1 monetary unit at time t is given by $\exp(-Y_t)$, $t \geq 0$.

As noticed by Parker (1994b), there are mainly two possible approaches to model the interest randomness, namely the modeling of Y_t and the modeling of δ_s. In the first approach, we could let Y_t be the sum of a deterministic drift of slope δ and a perturbation modeled by a Wiener process, i.e.

$$Y_t = \delta t + \sigma W_t, \;\; t \in I\!R^+, \tag{3.1}$$

where σ is a non-negative constant and $\{W_t, \;\; t \in I\!R^+\}$ is a standardized Brownian motion. In such a case, V_t is log-normally distributed with parameters $-\delta t$ and $\sigma^2 t$. This corresponds to the approach adopted by Goovaerts *et al.* (1999) who considered a discounted cash flow Z_n of the form

$$Z_n = \sum_{i=1}^{n} \exp(-\delta i - X_i),$$

where the X_i's are assumed to be normally distributed with mean 0 and variance $i\sigma^2$, and δ is the expected force of interest. The convex upper bound $\widetilde{Z}_n$ on Z_n obtained by Goovaerts *et al.* (1999) is

$$\widetilde{Z}_n = \sum_{i=1}^{n} \exp\left\{-\delta i - \sigma\sqrt{i}\Phi^{-1}(U)\right\}, \tag{3.2}$$

where Φ is the cumulative distribution function of a standard normal distribution and U is a random variable uniformly distributed on the unit interval $[0,1]$. The survival function of $\widetilde{Z}_n$ then follows from

$$P[\widetilde{Z}_n > x] = 1 - F_{\widetilde{Z}_n}(x) = \Phi(\nu_x),$$

with ν_x the root of the equation

$$\sum_{i=1}^{n} \alpha_i \exp(-\delta i - \sqrt{i}\sigma\nu_x) = x.$$

Let us now investigate the accuracy of the bounds (2.1) and (2.5) on the distribution function of Z_n in the model (3.1). In Figure 1, one sees the

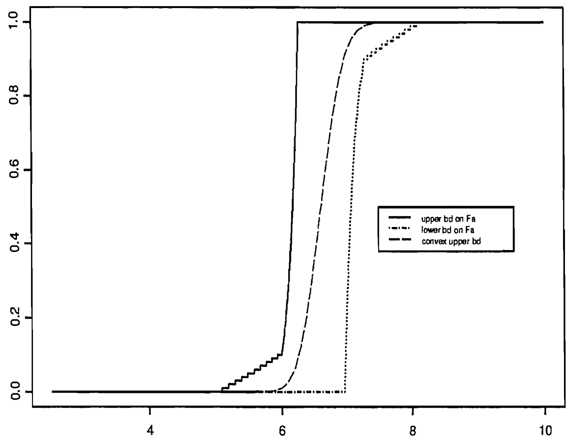

Figure 1: *Graph of the bounds (2.1) and cumulative distribution function of* $\tilde{Z}_{10}$ *for (3.1) with* $\delta = 0.08$ *and* $\sigma = 0.02$.

functions $F_{\min}$ and $F_{\max}$ involved in (2.1). with in between the approximation $F_{\tilde{Z}_n}$ of the unknown F_{Z_n} for $n = 10$, $\delta = 0.08$ and $\sigma = 0.02$. Figure 3 is the analog for $n = 20$. Comparing the cumulative distribution function of the convex approximation (3.2) with the stochastic bounds (2.1), we see from Figures 1 and 3 that (3.2) lies in the very middle of the admissible region bordered by $F_{\min}$ and $F_{\max}$. This indicates that (3.2) could be reasonable. In Figures 2 and 4, we further assume that the V_i's are POD and we computed the improved bounds furnished in (2.1). Only the lower bound got improved. As it is observed in Example 3 of Denuit *et al.* (1999), both upper and lower bounds on the distribution of a sum of random variables got improved when the supports of the random variables are of the form $[a_i, b_i]$ with $-\infty < a_i < b_i < +\infty$. If b_i is equal to $+\infty$ as in Example 1 of Denuit *et al.* (1999), only the lower bound will be improved with the assumption of POD. In our examples, the random variables are lognormally distributed with supports corresponding to $[0, +\infty)$. If, as in Goovaerts and Dhaene (1999), δ_t is defined by a CIR model, then Y_t will be strictly positive, $V_t = \exp(-Y_t)$ will take values between 0 and 1, and therefore upper and lower bounds on the distribution of Z_t will have been improved.

A second approach to model interest randomness is to model δ_s. For instance, the force of interest can be defined by the differential equation

$$d\delta_t = -\alpha(\delta_t - \delta)dt + \sigma dW_t, \tag{3.3}$$

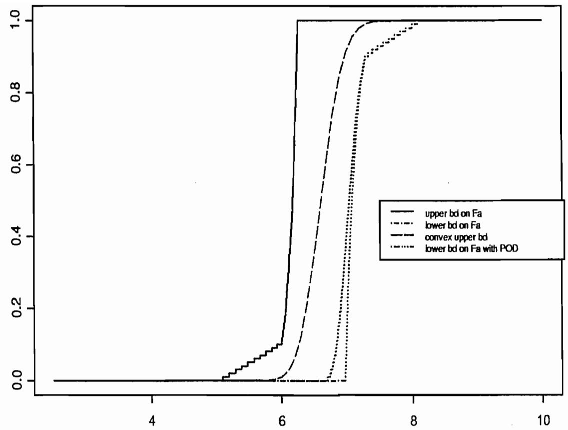

Figure 2: *Graph of the bounds (2.5) and cumulative distribution function of* $\tilde{Z}_{10}$ *for (3.1) with* $\delta = 0.08$ *and* $\sigma = 0.02$.

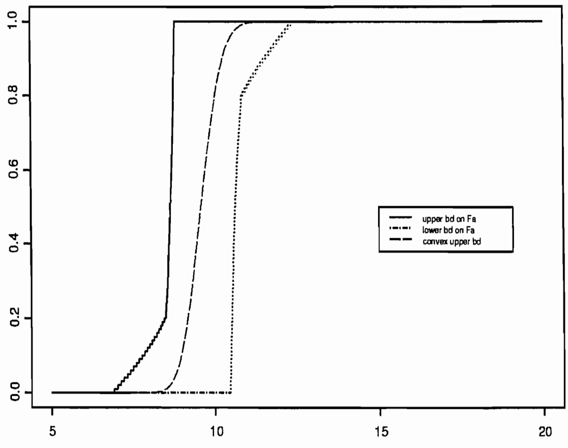

Figure 3: *Graphs of the bounds (2.1) and cumulative distribution function of* $\tilde{Z}_{20}$ *for (3.1) with* $\delta = 0.08$ *and* $\sigma = 0.02$.

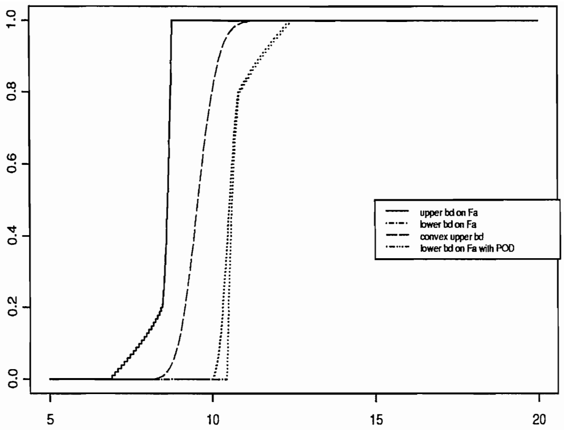

Figure 4: *Graphs of the bounds (2.5) and cumulative distribution function of* $\tilde{Z}_{20}$ *for (3.1) with* $\delta = 0.08$ *and* $\sigma = 0.02$.

with non-negative constants α and σ, and with initial value $\delta_0 = \delta \geq 0$; $\{\delta_t,\ t \geq 0\}$ is thus an Ornstein-Uhlenbeck process. The force of interest accumulation function $\{Y_t,\ t \geq 0\}$ is therefore a Gaussian process with mean function

$$t \mapsto \mu_t = \delta t + (\delta_0 - \delta)\frac{1 - \exp(-\alpha t)}{\alpha},$$

and autocovariance $(s,t) \mapsto \mathrm{Cov}[Y_s, Y_t] \equiv \omega(s,t)$, where

$$\begin{aligned} \omega(s,t) &= \frac{\sigma^2}{\alpha^2}\min(s,t) + \frac{\sigma^2}{2\alpha^3}\{-2 + 2\exp(-\alpha s) + 2\exp(-\alpha t) \\ &\qquad - \exp(-\alpha(t-s)) - \exp(-\alpha(t+s))\}\,; \end{aligned}$$

see e.g. Parker (1994a, Section 6). Then,

$$Z_n = \sum_{i=1}^{n} \exp(-Y_i),$$

where Y_i is a Normal random variable with mean μ_i and variance $\omega(i,i)$. In such a case, the convex upper bound $\tilde{Z}_n$ follows from Goovaerts *et al.* (1999):

$$\widetilde{Z}_n = \sum_{i=1}^{n} \exp\left\{-\mu_i - \sqrt{\omega(i,i)}\Phi^{-1}(U)\right\},$$

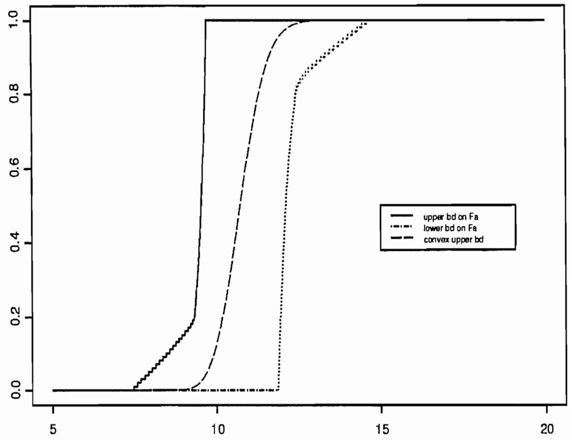

Figure 5: *Graphs of the bounds (2.1) and cumulative distribution function of* $\tilde{Z}_{10}$ *for (3.3) with* $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ *and* $\sigma = 0.01$.

where U is a random variable uniformly distributed on the unit interval $[0, 1]$. In Figure 5, you can see the bounds on the cumulative distribution function of Z_{10} in the model (3.3) with $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ and $\sigma = 0.01$, together with the cumulative distribution function of $\tilde{Z}_{10}$. Figure 7 is the analog for $n = 20$. The comments inspired from Figures 1 and 3 still apply. In Figures 6 and 8, we assumed that the V_i's were POD. Again, the improvement with POD is moderate.

3.2 Life insurance

Consider a temporary life annuity issued to an individual aged x with curtate-future-lifetime K and denote $P[k < K \le k+1] = {}_{k|}q_x$ and $P[K > n] = {}_n p_x$. We assume that K is independent of the random discount factors $V_1, V_2, V_3, \ldots$. The net single premium relating to this contract is given by

$$a_{x;\overline{n|}} = E[a^{\circ}_{x;\overline{n|}}],$$

with

$$a^{\circ}_{x;\overline{n|}} = \begin{cases} 0 \text{ if } K = 0, \\ Z_K \text{ if } K = 1, \ldots, n-1, \\ Z_n \text{ if } K \ge n, \end{cases}$$

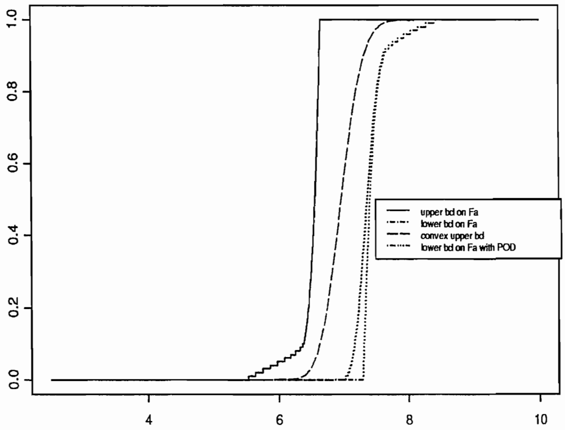

Figure 6: *Graphs of the bounds (2.5) and cumulative distribution function of* $\tilde{Z}_{10}$ *for (3.3) with* $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ *and* $\sigma = 0.01$.

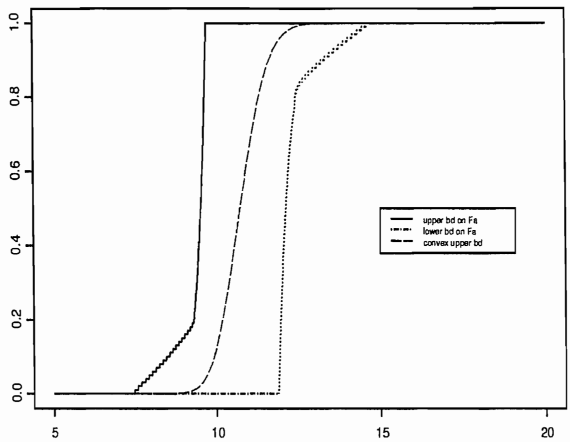

Figure 7: *Graphs of the bounds (2.1) and cumulative distribution function of* $\tilde{Z}_{20}$ *for (3.3) with* $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ *and* $\sigma = 0.01$.

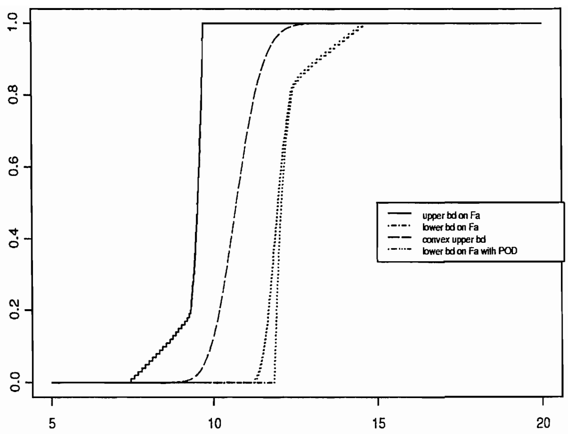

Figure 8: *Graphs of the bounds (2.5) and cumulative distribution function of* $\tilde{Z}_{20}$ *for (3.3) with* $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ *and* $\sigma = 0.01$.

where Z is defined as in (1.1). By conditioning on K, the net single premium relating to such a contract is

$$a_{x;\overline{n}|} = \sum_{k=1}^{n-1} E[Z_k]_{k|}q_x + E[Z_n]_n p_x.$$

The cumulative distribution function of $a^\circ_{x;\overline{n}|}$ is also obtained by conditioning on K:

$$P[a^\circ_{x;\overline{n}|} \leq y] = q_x + \sum_{k=1}^{n-1} P[Z_k \leq y]_{k|}q_x + P[Z_n \leq y]_n p_x.$$

No explicit expression exists for $P[a^\circ_{x;\overline{n}|} \leq y]$, but we use the approach developed above allows us to find stochastic dominance bounds on $a^\circ_{x;\overline{n}|}$. In Figure 9, we depicted the graph of the bounds on $P[a^\circ_{x;\overline{n}|} \leq y]$ for an individual aged 45 in the model (3.1) with $\delta = 0.08$ and $\sigma = 0.02$. Figure 10 is the analog in model (3.3) with $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ and $\sigma = 0.01$. For these numerical illustrations, we used the standard mortality table (Makeham model) given in Bowers *et al.* (1996). The bounds in Figures 9 and 10 give a good idea of the danger inherent to the stochastic interest rate combined with the stochastic mortality. Let us mention that the convex approximation of Goovaerts *et al.* (1999) also applies in this situation.

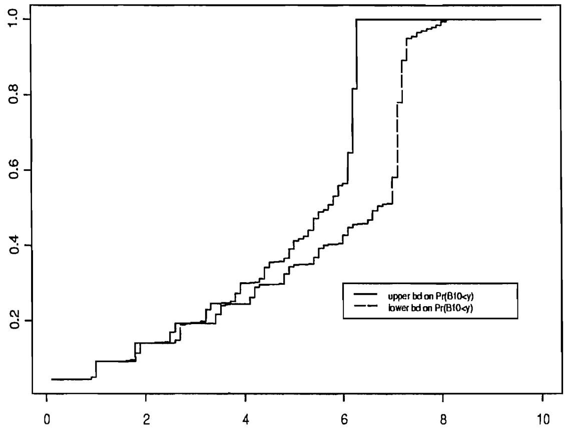

Figure 9: *Bounds on* $P[a^{\circ}_{x;\overline{10}|} \leq y]$ *for* $x = 45$ *and (3.1) with* $\delta = 0.08$ *and* $\sigma = 0.02$.

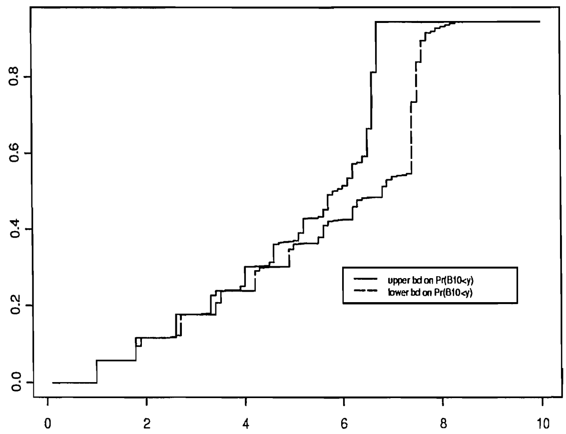

Figure 10: *Bounds on* $P[a^{\circ}_{x;\overline{10}|} \leq y]$ *for* $x = 45$ *and (3.3) with* $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ *and* $\sigma = 0.01$.

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