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Improved Error Bounds for Approximations to the Stop Loss Transform of Compound Distributions

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Abstract

In the present note we deduce a class of bounds for the difference between the stop loss transforms of two compound distributions with the same severity distribution. The class contains bounds of any degree of accuracy in the sense that the bounds can be chosen as close to the exact value as desired; the time required to compute the bounds increases with the accuracy.

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1. Introduction

In the present note we generalise a result from Dhaene & Sundt (1994) which gives bounds for the difference between the stop loss transforms of two compound distributions with the same severity distribution. We generalise these bounds to a class that contains bounds of any degree of accuracy in the sense that the bounds can be chosen as close to the exact value as desired; the time required to compute the bounds increases with the accuracy.

2. Notation and conventions

The following notation and conventions are to a great extent taken from Dhaene & Sundt (1994).

In the present paper we shall be concerned with probability distributions on the non-negative integers. Identifying such a distribution by its discrete density, we shall for convenience usually mean its discrete density when we talk about a distribution.

Let $\mathcal{P}$ denote the class of (discrete densities of) probability distributions on the non-negative integers with finite mean. For a distribution $f \in \mathcal{P}$ we denote by Γ_f the corresponding cumulative distribution, by Π_f the stop loss transform, and by μ_f the mean, that is,

$$\Gamma_f(x) = \Sigma_{y=0}^{x} f(y) \qquad (x=0,1,2,\ldots)$$
$$\Pi_f(x) = \Sigma_{y=x+1}^{\infty} (y-x) f(y) = \Sigma_{y=x}^{\infty} (1-\Gamma_f(y)) \qquad (x=0,1,2,\ldots)$$
$$\mu_f = \Pi_f(0) = \Sigma_{y=1}^{\infty} y f(y) = \Sigma_{y=0}^{\infty} (1-\Gamma_f(y)).$$

We shall denote a compound distribution with counting distribution $p \in \mathcal{P}$ and severity distribution $h \in \mathcal{P}$ by $p \vee h$, that is,

$$p \vee h = \Sigma_{n=0}^{\infty} p(n) h^{n*}.$$

For a distribution $f \in \mathcal{P}$ and a positive integer r, we define the approximation $\tilde{f}^{(r)}$ by

$$\tilde{f}^{(r)}(x) = \begin{cases} f(x) & (x=0\,,1,\ldots,r\text{-}1) \\ 1\text{-}\Gamma_f(r-1) & (x=r) \\ 0. & (x=r+1,r+2,\ldots) \end{cases}$$

This approximation can be interpreted as the distribution obtained by setting all observations greater than r equal to r.

By the notation x_+ we shall mean the maximum of x and zero.

We denote by I the indicator function defined by $I(A)=1$ if the condition A is true and $I(A)=0$ if it is false.

We shall interpret $\Sigma_{i=a}^{b} v_i = 0$ and $\Pi_{i=a}^{b} v_i = 1$ when $b<a$.

3. Main results

3A. We shall need the following lemma.

Lemma 1. *For $h \in \mathcal{P}$ and x, r, and m non-negative integers such that $r \leq m$, we have*

$$(m-r)\Pi_h(x) \le \Pi_{h^{m*}}(x) - \Pi_{h^{r*}}(x) \le (m-r)\mu_h.$$

Lemma 1 is proved as formula (38) in De Pril & Dhaene (1992) for the special case $r=1$; the proof is easily extended to the general case.

Lemma 2. *For $p,h\in\mathcal{P}$ and r a positive integer, we have*

$$\Pi_h(x)\Pi_p(r) \le \Pi_{p\vee h}(x) - \Pi_{\tilde{p}^{(r)}\vee h}(x) \le \mu_h\Pi_p(r). \qquad (x=0,1,2,\dots) \tag{1}$$

Proof. For $x=0,1,2,\dots$ we have

$$\Pi_{p\vee h}(x) - \Pi_{\tilde{p}^{(r)}\vee h}(x) = \Sigma_{n=r}^{\infty} p(n)\Pi_{h^{n*}}(x) - (1-\Gamma_p(r-1))\Pi_{h^{r*}}(x) =$$
$$\Sigma_{n=r}^{\infty} p(n)[\Pi_{h^{n*}}(x)-\Pi_{h^{r*}}(x)].$$

Application of Lemma 1 gives

$$\Sigma_{n=r}^{\infty} p(n)(n-r)\Pi_h(x) \le \Pi_{p\vee h}(x) - \Pi_{\tilde{p}^{(r)}\vee h}(x) \le \Sigma_{n=r}^{\infty} p(n)(n-r)\mu_h,$$

from which we obtain (1). Q.E.D.

The second inequality in (1) was proved under more general assumptions by Sundt (1991), who also showed that $0\le\Pi_{p\vee h}(x)-\Pi_{\tilde{p}^{(r)}\vee h}(x)$, which is weaker than the first inequality in (1).

If $\tilde{p}^{(r)}=p$, that is, $p(n)=0$ for all $n>r$, then the bounds in (1) become equal to zero.

Lemma 1 appears as a special case of Lemma 2 by letting p be the distribution concentrated in m.

3B. For $p,q,h\in\mathcal{P}$, r a positive integer, and x a non-negative integer, we introduce

$$B_r(x;p,q,h) = \Pi_{\tilde{p}(r)\vee h}(x) - \Pi_{\tilde{q}(r)\vee h}(x) + \mu_h\Pi_p(r) - \Pi_h(x)\Pi_q(r),$$

which can also be written as

$$B_r(x;p,q,h) = \Sigma_{n=0}^{r-1}\,(p(n)-q(n))\Pi_{h^{n*}}(x) - (\Gamma_p(r-1)-\Gamma_q(r-1))\Pi_{h^{r*}}(x) + \mu_h\Pi_p(r) - \Pi_h(x)\Pi_q(r). \tag{2}$$

Theorem 1. *For $p,q,h\in\mathcal{P}$, and r a positive integer, we have*

$$-B_r(x;q,p,h) \le \Pi_{p\vee h}(x) - \Pi_{q\vee h}(x) \le B_r(x;p,q,h). \qquad (x=0,1,2,\ldots) \tag{3}$$

Proof. Application of Lemma 2 gives for $x=0,1,2,\ldots$

$$\Pi_{p\vee h}(x) - \Pi_{q\vee h}(x) \le \Pi_{\tilde{p}(r)\vee h}(x) + \mu_h\Pi_p(r) - \Pi_{\tilde{q}(r)\vee h}(x) - \Pi_h(x)\Pi_q(r) = B_r(x;p,q,h),$$

which proves the second inequality in (3). The first inequality in (3) follows by symmetry.

This completes the proof of Theorem 1. Q.E.D.

We shall look at some special cases of Theorem 1:

1. As for $p,h \in \mathcal{P}$ and r a positive integer

$$B_r(x;p,\tilde{p}^{(r)},h) = \mu_h \Pi_p(r) \qquad B_r(x;\tilde{p}^{(r)},p,h) = -\Pi_h(x)\Pi_p(r), \qquad (x=0,1,2,\ldots)$$

we see that Lemma 2 (and thus also Lemma 1) is a special case of Theorem 1.

2. We have

$$B_1(x;p,q,h) = -(p(0)-q(0))\Pi_h(x) + \mu_h \Pi_p(1) - \Pi_h(x)\Pi_q(1) =$$
$$(\mu_h - \Pi_h(x))\Pi_p(1) + \Pi_h(x)(\mu_p - \mu_q).$$

Thus, when $r=1$, Theorem 1 gives the same bounds as Theorem 5.3 in Dhaene & Sundt (1994).

3. If $p(x)=q(x)=0$ for all $x>r$, then $\tilde{p}^{(r)}=p$ and $\tilde{q}^{(r)}=q$, and we obtain

$$B_r(x;p,q,h) = -B_r(x;q,p,h) = \Pi_{p \vee h}(x) - \Pi_{q \vee h}(x),$$

that is, in this case Theorem 1 becomes trivial.

3C. Let $D_r(x;p,q,h)$ denote the difference between the upper and lower bound in Theorem 1, that is,

$$D_r(x;p,q,h) = B_r(x;p,q,h) + B_r(x;q,p,h). \tag{4}$$

Then

$$D_r(x;p,q,h) = (\mu_h - \Pi_h(x))(\Pi_p(r) + \Pi_q(r)). \tag{5}$$

We see that $D_r(x;p,q,h)$ decreases to zero when r increases to infinity, that is, we can make the difference between the upper and lower bound in Theorem 1 as small as desired by making r sufficiently large.

On the other hand

$$D_r(x;p,p,h) = 2(\mu_h - \Pi_h(x))\Pi_p(r),$$

which is greater than zero except for the cases when $x=0$ or $\Pi_p(r)=0$.

We see that $D_r(x;p,q,h)$ increases from zero to $\mu_h(\Pi_p(r)+\Pi_q(r))$ when x increases from zero to infinity. Thus our bounds are most accurate for low values of x. Furthermore, if for some $\epsilon>0$ we choose r such that

$$\Pi_p(r)+\Pi_q(r) < \frac{\epsilon}{\mu_h},$$

then $D_r(x;p,q,h)<\epsilon$ for all x.

3D. For $p,q,h \in \mathcal{P}$, r a positive integer, and x a non-negative integer, let

$$b_r(x;p,q,h) = B_r(x;p,q,h) - B_{r+1}(x;p,q,h).$$

From (2) and trivial calculus we obtain

$$b_r(x;p,q,h) = (\Gamma_p(r)-\Gamma_q(r))[\Pi_{h^{(r+1)*}}(x)-\Pi_{h^{r*}}(x)] + \mu_h(1-\Gamma_p(r))-\Pi_h(x)(1-\Gamma_q(r)). \tag{6}$$

By rewriting (6) as

$$b_r(x;p,q,h) = (1-\Gamma_p(r))[\mu_h + \Pi_{h^{r*}}(x) - \Pi_{h^{(r+1)*}}(x)] +$$
$$(1-\Gamma_q(r))[\Pi_{h^{(r+1)*}}(x) - \Pi_{h^{r*}}(x) - \Pi_h(x)]$$

and application of Lemma 1, we see that $b_r(x;p,q,h)$ is non-negative. Thus $B_r(x;p,q,h)$ is non-increasing in r. This implies that in (3), the upper bound is non-increasing and the lower bound is non-decreasing in r, and as $D_r(x;p,q,h)$ goes to zero when r goes to infinity, both bounds converge to the estimation error $\Pi_{p\vee h}(x) - \Pi_{q\vee h}(x)$.

3E. Formula (6) can be applied for recursive evaluation of $B_r(x;p,q,h)$. Furthermore, when we have have found $B_r(x;p,q,h)$, we easily obtain $B_r(x;q,p,h)$ from (4) and (5).

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