Essays in Microeconomic Theory

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Essays in Microeconomic Theory

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1^{st} **Promotor**:
Prof. dr. P. Jean-Jacques Herings

2^{nd} **Promotor**:
Prof. dr. Bettina-Elisabeth Klaus

Co-Promotor:
Dr. Kristof Bosmans

Beoordelingscommissie:
Prof. dr. Hans (H.J.M.) Peters (Maastricht University) (voorzitter)
Prof. dr. Harrie (H.C.M.) de Swart (Tilburg University)
Dr. Ton (A.J.A.) Storcken (Maastricht University)

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My birthplace, Adana, is in a region known as Çukurova (the Lowlands) which makes me a Çukurovalı (a Lowlander). So in a sense I have been a Lowlander (Nederlander) myself since I was born. Ahmed Arif, my favorite poet, once wrote: "In the infamous prisons of my Anatolia, it is the Lowlanders that are imprisoned most". The reason was simply that Lowlanders never liked to bow down to authority. So I was raised in a culture where obedience to hierarchy was not considered as a virtue. That is why, as a typical "Çukurovalı", it would -normally- be impossible for me to finish this dissertation. However I was lucky enough to have met the people, whose names I would like to mention below. Thanks to them, I am a "free man" again after four years.

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Burak CAN
Maastricht, December 2011

Contents

Chapter 1

Introduction

Since Samuelson (1947), microeconomic theory has moved from a rather normative position to a positive framework. This can be perceived as good, since nobody argues anymore whether a theorem is nice or not. Instead, the debate takes place on the assumptions that make a theory work. Hence, this has been considered, by many, as a virtue of economics, i.e., involving mathematical rigor and solid proofs of theorems, whereas some considered this to be a secret sin peculiar to economics (McCloskey, 2002). This dissertation comprises three self-contained chapters which are hopefully profound examples of these sins.

The three chapters ahead share the same foundation of economic analysis but aim to answer three different questions. In each chapter, individual preferences establish the main framework of the analysis. We investigate the interaction between individual preferences in different contexts. In particular, we focus on three scenarios wherein; i) individuals are matched with each other to share a room according to their preferences, ii) individuals are assigned some preferences which represent them collectively, iii) ideological distances of individuals are measured via the differences in their preferences.

Chapters 2 and 3 are examples of two fields in modern Samuelsonian microeconomic theory: matching theory and social choice theory. Chapter 4, however, focuses on the main framework of the foundation of the theory itself, i.e., individual preferences. In Chapter 4, we introduce a distance (dissimilarity) function for these preferences, which can be applied to many problems in modern microeconomic theory.

In Chapter 2, we use individual preferences in the context of matching theory. We study the roommate markets introduced in Gale and Shapley (1962) which are one-to-one matching markets in which agents can either be matched as pairs to share a room or remain single. In particular we analyze variable population properties where agents are perceived as consumers and resources at the same time. Klaus (2011) introduced two new "population sensitivity" properties that capture the effect newcomers have on incumbent agents: competition sensitivity and resource sensitivity. On various roommate market domains (marriage markets, no odd rings roommate markets, solvable roommate markets), we characterize the core using either of the population sensitivity properties in addition to weak unanimity and consistency. On the domain of all roommate markets, we obtain two associated impossibility results.

In Chapter 3, individual preferences over some set of alternatives are considered as a collective decision making problem. In particular, collective decisions are modeled by preference correspondences (rules). We focus on a new condition: "update monotonicity" for preference rules. This condition, roughly speaking, requires that when individual preferences change in favor of the outcome of a preference rule, the outcome should be still assigned by the preference rule to the new preference profile of the individuals. Although many so-called impossibility theorems for the choice rules are based on -or are related to- monotonicity conditions, this appealing condition is satisfied by several non-trivial preference rules. In fact, in case of pairwise, Pareto optimal, neutral, and consistent rules; the Kemeny-Young rule is singled out by this condition. In case of convex valued, Pareto optimal, neutral and replication invariant rules; strong update monotonicity implies that the rule equals the union of preferences which extend all preference pairs unanimously agreed upon by k agents, where k is related to the number of alternatives and agents. In both cases, it therewith provides a characterization of these rules.

In Chapter 4, unlike the two previous chapters, we focus only on preferences. The use of distance functions to measure dissimilarity between individual preferences is a common practice in the social choice literature. The well-known Kemeny distance, for instance, counts the pairwise disagreements between two individual preferences. In this context, we propose a class of weighted distance functions which is based on some distribution of weights over the positions of the pairs of disagreement. Examples of members of this class are the Kemeny distance, the Lehmer distance, the inverse Lehmer distance, and the path-minimizing distance. We analyze the implications of changing weights on the structure of these distances. It turns out that the path-minimizing distance is the weighted generalization of the Kemeny distance, in the sense that it is the only one satisfying the triangular inequality condition for any non-degenerate weight vector. Furthermore, we show that this distance can easily be calculated when the weights over the positions of disagreements are monotonically increasing or decreasing.

Chapter 2

Consistency and Population Sensitivity Properties in Marriage and Roommate Markets

2.1 Introduction

We consider one-to-one matching markets in which agents can either be matched as pairs or remain single[1]. These markets are known as roommate markets and they include, as special cases, the well-known marriage markets (Gale and Shapley, 1962; Roth and Sotomayor, 1990). Furthermore, a roommate market is a simple example of *hedonic coalition* as well as *network formation*: in a "roommate coalition" situation, only coalitions of size one or two can be formed and in a "roommate network" situation, each agent is allowed or able to form only one link (for surveys and current research of coalition and network formation see Demange and Wooders, 2004; Jackson, 2008).

As discussed in Klaus (2011), in these markets the commodities to be traded are the agents themselves and agents are consumers and resources

[1]This chapter is based on a paper by Can and Klaus (2010).

at the same time. Two new "population sensitivity" properties, introduced in Klaus (2011), that capture the effect newcomers have on incumbent agents are *competition* and *resource sensitivity*: competition sensitivity requires that some incumbents will suffer if competition is caused because newcomers initiate new trades and resource sensitivity requires that some incumbents will benefit if the extra resources are consumed. The corresponding weak population sensitivity properties only consider situations when newcomers join one by one.

Both population sensitivity properties are closely related to population monotonicity, a solidarity property that requires that additional agents affect the incumbents in a similar way (either all incumbents are weakly better off or all incumbents are weakly worse off). Because of the polarization of interests that occurs in marriage markets, two specific versions of population monotonicity exist: own-side and other-side population monotonicity (Toda, 2006, introduced the first and Klaus, 2011, the second of these specifications).[2] Klaus (2011) shows that in marriage markets, essentially own-side population monotonicity implies weak competition sensitivity and other-side population monotonicity implies weak resource sensitivity. Furthermore, Klaus (2011) presents the first characterizations of the core for solvable roommate markets using weak unanimity[3], Maskin monotonicity[4], and either weak competition or weak resource sensitivity for marriage markets and solvable roommate markets and two associated impossibility results on the domain of all roommate markets. These characterizations can be seen as corresponding results for roommate markets to one of Toda's (2006, Theorem 3.1) core characterizations for marriage markets by weak unanimity, Maskin monotonicity, and own-side population monotonicity.

In a second characterization of the core for marriage markets, Toda (2006,

[2] *Own-side population monotonicity*: if additional men (women) enter the market, then all incumbent men (women) are weakly worse off.

Other-side population monotonicity: if additional men (women) enter the market, then all incumbent women (men) are weakly better off.

[3] *Weak unanimity*: if a complete unanimously best matching exists, then it is chosen.

[4] *Maskin monotonicity*: if a matching is chosen in one market, then it is also chosen in a market that results from a Maskin monotonic transformation (which essentially means that the matching improved in the ranking of all agents).

Theorem 3.2) uses consistency[5] instead of Maskin monotonicity. In this chapter, we show how Toda's "consistency results" can be extended to roommate markets. As main results, we obtain new characterizations of the core on the domains of marriage markets, no odd rings roommate markets, and solvable roommate markets: on any of these domains, a solution φ satisfies weak unanimity, consistency, and either of the population sensitivity properties if and only if it equals the core (Theorems 2.4.1 and 2.4.2). Two associated impossibility results on the domain of all roommate markets are also established (Lemmas 2.4.1 (d) and 2.4.2 (d)). Our results imply two corresponding "population monotonicity" results for marriage markets (Corollary 2.4.1): a solution φ satisfies individual rationality, weak unanimity, consistency, and either own-side or other-side population monotonicity if and only if it equals the core (the characterizations using own-side population monotonicity is the one obtained by Toda, 2006, Theorem 3.2). Apart from establishing new core characterizations for marriage and roommate markets (as well as some impossibilities), we obtain new insights into the working of one-sided-markets: to extend Toda's (2006, Theorem 3.2) core characterization from marriage to roommate markets, we not only had to use one of the new population sensitivity properties, we also had to develop a new proof strategy because the original two-sided market proof could not be adapted (Example 2.1 proves this). Another aspect of our results is the validation of population sensitivity properties as fundamental core properties. We discuss these issues in more detail in our conclusion (Section 2.5).

The chapter is organized as follows. In Section 2.2 we present the roommate model and basic properties of solutions. In Section 2.3, we introduce the variable population properties consistency, (weak) competition sensitivity, and (weak) resource sensitivity. Section 2.4 contains the main results. Section 2.5 concludes by discussing the importance of our results for the study of one-sided markets.

[5] *Consistency*: if a set of matched agents leaves, then the solution should still match the remaining agents as before.

2.2 Roommate Markets

The following Subsections 2.2.1 and 2.2.2 mainly follow Klaus (2011).

2.2.1 The Model

We consider Gale and Shapley's (1962, Example 3) roommate markets with variable sets of agents, e.g., because the allocation of dormitory rooms at a university occurs every year for different sets of students.

Let $\mathbb{N}$ be the set of potential agents.[6] For a non-empty finite subset $N \subsetneq \mathbb{N}$, $L(N)$ denotes the set of all linear orders over N.[7] For $i \in N$, we interpret $R_i \in L(N)$ as agent i's strict preferences over sharing a room with any of the agents in $N \setminus \{i\}$ and having a room for himself; e.g., $R_i : j,k,i,l$ means that i would first like to share a room with j, then with k, and then i would prefer to stay alone rather than sharing the room with l. If $j\,P_i\,i$, then agent i finds agent j *acceptable* and if $i\,P_i\,j$, then agent i finds agent j *unacceptable*. $\mathscr{R}^N = \prod_N L(N)$ denotes the set of all preference profiles of agents in N (over agents in N). A *roommate market* consists of a finite set of agents $N \subsetneq \mathbb{N}$ and their preferences $R \in \mathscr{R}^N$ and is denoted by (N,R). A *marriage market* (Gale and Shapley, 1962) is a roommate market (N,R) such that N is the union of two disjoint sets M and W and each agent in M (respectively W) prefers being single to being matched with any other agent in M (respectively W).

A *matching* μ for roommate market (N,R) is a function $\mu : N \to N$ of order two, i.e., for all $i \in N$, $\mu(\mu(i)) = i$. Thus, at any matching μ, the set of agents is partitioned into pairs of agents who share a room and singletons (agents who do not share a room). Agent $\mu(i)$ is agent i's *match* and if $\mu(i) = i$ then i is matched to himself or *single*. For notational convenience, we often denote a matching in terms of the induced partition, e.g., for $N = \{1,2,3,4,5\}$ and matching μ such that $\mu(1) = 2$, $\mu(3) = 3$ and $\mu(4) = 5$ we

[6] Most results remain valid for a finite set of potential agents. We will explain throughout the article, which results depend on the set of potential agents to be infinite.

[7] A linear order over N is a binary relation $\bar{R}$ that satisfies *antisymmetry* (for all $i,j \in N$, if $i\,\bar{R}\,j$ and $j\,\bar{R}\,i$, then $i = j$), *transitivity* (for all $i,j,k \in N$, if $i\,\bar{R}\,j$ and $j\,\bar{R}\,k$, then $i\,\bar{R}\,k$), and *comparability* (for all $i,j \in N$, $i\,\bar{R}\,j$ or $j\,\bar{R}\,i$). By $\bar{P}$ we denote the asymmetric part of $\bar{R}$. Hence, given $i,j \in N$, $i\,\bar{P}\,j$ means that i is strictly preferred to j; $i\,\bar{R}\,j$ means that $i\,\bar{P}\,j$ or $i = j$ and that i is weakly preferred to j.

write $\mu = \{(1,2),3,(4,5)\}$. For $S \subseteq N$, we denote by $\mu(S)$ the set of agents that are matched to agents in S, i.e., $\mu(S) = \{i \in N \mid \mu^{-1}(i) \in S\}$. We denote the set of all matchings for roommate market (N,R) by $\mathscr{M}(N,R)$ (even though this set does not depend on preferences R). If it is clear which roommate market (N,R) we refer to, matchings are assumed to be elements of $\mathscr{M}(N,R)$. Since agents only care about their own matches, we use the same notation for preferences over agents and matchings: for all agents $i \in N$ and matchings μ,μ', $\mu R_i \mu'$ if and only if $\mu(i) R_i \mu'(i)$.

Given a roommate market (N,R) and $N' \subseteq N$, we define the *reduced preferences* $R' \in \mathscr{R}^{N'}$ *of* R *to* N' as follows:

(i) for all $i \in N'$, $R'_i \in L(N')$ and

(ii) for all $j,k,l \in N'$, $j R'_l k$ if and only if $j R_l k$.

We also denote the reduced preferences of R to N' by $R_{N'}$.

Given a roommate market (N,R), a matching $\mu \in \mathscr{M}(N,R)$, and $N' \subseteq N$ such that $\mu(N') = N'$, the *reduced (roommate) market of* (N,R) *at* μ *to* N' equals $(N',R_{N'})$.

Given a roommate market (N,R), a matching $\mu \in \mathscr{M}(N,R)$, and $N' \subseteq N$ such that $\mu(N') = N'$, we define the *reduced matching* μ' *of* μ *to* N' as follows:

(i) $\mu' : N' \to N'$ and

(ii) for all $i \in N'$, $\mu'(i) = \mu(i)$.

We also denote the reduced matching of μ to N' by $\mu_{N'}$. Note that $\mu_{N'} \in \mathscr{M}(N',R_{N'})$.

In the sequel, we consider various domains of roommate problems: the domain of all roommate markets $\mathfrak{D}$, the domain of marriage markets $\mathfrak{D}_M$, and later the domains of solvable and of no odd rings roommate markets. To avoid notational complexity when introducing solutions and their properties, we use the domain of all roommate markets $\mathfrak{D}$ with the understanding that any other domain could be used as well.

A *solution* φ on $\mathfrak{D}$ is a correspondence that associates with each roommate market $(N,R) \in \mathfrak{D}$ a nonempty subset of matchings, i.e., for all $(N,R) \in \mathfrak{D}$, $\varphi(N,R) \subseteq \mathscr{M}(N,R)$ and $\varphi(N,R) \neq \emptyset$. A *subsolution* ψ of φ on $\mathfrak{D}$ is a correspondence that associates with each roommate market $(N,R) \in \mathfrak{D}$ a nonempty subset of matchings in $\varphi(N,R)$, i.e., for all roommate markets $(N,R) \in \mathfrak{D}$, $\psi(N,R) \subseteq \varphi(N,R)$ and $\psi(N,R) \neq \emptyset$. A *proper subsolution* ψ of φ on $\mathfrak{D}$ is a subsolution of φ on $\mathfrak{D}$ such that $\psi \neq \varphi$.

2.2.2 Basic Properties and the Core

We first introduce a voluntary participation condition based on the idea that no agent can be forced to share a room.

Individual Rationality: Let $(N,R) \in \mathfrak{D}$ and $\mu \in \mathscr{M}(N,R)$. Then, μ is *individually rational* if for all $i \in N$, $\mu(i)\,R_i\,i$. $IR(N,R)$ denotes the set of all these matchings. A solution φ on $\mathfrak{D}$ is *individually rational* if it only assigns individually rational matchings, i.e., for all $(N,R) \in \mathfrak{D}$, $\varphi(N,R) \subseteq IR(N,R)$.

Remark 2.2.1 (*Individual Rationality and (Classical) Marriage Markets*).
An individually rational matching for a marriage market $(N,R) \in \mathfrak{D}_M$ respects the partition of agents into two types and never matches two men or two women. Hence, we embed marriage markets into our roommate market framework by an assumption on preferences (same gender agents are unacceptable) and individual rationality to ensure that no two agents of the same gender are matched. We refer to a marriage market for which matching agents of the same gender is not feasible as a classical marriage market. $\triangle$

Next, we introduce the well-known condition of Pareto optimality and the weaker conditions of unanimity and weak unanimity.

Pareto Optimality: Let $(N,R) \in \mathfrak{D}$ and $\mu \in \mathscr{M}(N,R)$. Then, μ is *Pareto optimal* if there is no other matching $\mu' \in \mathscr{M}(N,R)$ such that for all $i \in N$, $\mu'\,R_i\,\mu$ and for some $j \in N$, $\mu'\,P_j\,\mu$. $PO(N,R)$ denotes the set of all these matchings. A solution φ on $\mathfrak{D}$ is *Pareto optimal* if it only assigns Pareto optimal matchings, i.e., for all $(N,R) \in \mathfrak{D}$, $\varphi(N,R) \subseteq PO(N,R)$.

(Weak) Unanimity: Let $(N,R) \in \mathfrak{D}$ and $\mu \in \mathscr{M}(N,R)$ be such that for all $i,j \in N$, $\mu(i)\,R_i\,j$. Then, μ is the *unanimously best matching* for (N,R). If μ is complete,[8] then, μ is the *unanimously best complete matching* for (N,R). A solution φ on $\mathfrak{D}$ is *unanimous* if it assigns the unanimously best matching whenever it exists, i.e., for all roommate markets $(N,R) \in \mathfrak{D}$ with a unanimously best matching μ, $\varphi(N,R) = \{\mu\}$. A solution φ on $\mathfrak{D}$ is *weakly unanimous* if it assigns the unanimously best complete matching whenever it

[8] A matching is *complete* if it partitions the set of agents into pairs, i.e., it contains no singletons.

exists, i.e., for all roommate markets $(N,R) \in \mathfrak{D}$ with a unanimously best complete matching μ, $\varphi(N,R) = \{\mu\}$.

Pareto optimality implies unanimity and unanimity implies weak unanimity.

The next property requires that two agents who are *mutually best agents* are always matched with each other.

Mutually Best: Let $(N,R) \in \mathfrak{D}$ and $i,j \in N$ [possibly $i = j$] such that for all $k \in N$, $i\,R_j\,k$ and $j\,R_i\,k$. Then, i and j are *mutually best agents for* (N,R). A matching is a *mutually best matching* if all mutually best agents are matched. $MB(N,R)$ denotes the set of all these matchings[9]. A solution φ on $\mathfrak{D}$ is *mutually best* if it only assigns matchings at which all mutually best agents are matched, i.e., for all roommate markets $(N,R) \in \mathfrak{D}$, $\varphi(N,R) \subseteq MB(N,R)$.

Our notion of mutually best is slightly stronger than that used in Toda (2006) (because he considers mutually best man-woman pairs, he does not allow for a single mutually best agent $i = j$). Furthermore, mutually best implies (weak) unanimity, and Pareto optimality and mutually best are logically unrelated.

The above properties can be used to define solutions, the most prominent one being the *Pareto solution PO* that assign to each roommate market the set of Pareto optimal matchings.

Next, we define stability for roommate markets. A matching μ for roommate market $(N,R) \in \mathfrak{D}$ is blocked by a pair $\{i,j\} \subseteq N$ [possibly $i = j$] if $j\,P_i\,\mu(i)$ and $i\,P_j\,\mu(j)$. If $\{i,j\}$ blocks μ, then $\{i,j\}$ is called a *blocking pair for* μ. A matching μ for roommate market $(N,R) \in \mathfrak{D}$ is individually rational if there is no blocking pair $\{i,j\}$ with $i = j$ for μ.

Stability, Solvability, and the Domain of Solvable Roommate Markets: Let $(N,R) \in \mathfrak{D}$ and $\mu \in \mathcal{M}(N,R)$. Then, μ is *stable* if there is no blocking pair for μ. $S(N,R)$ denotes the set of all these matchings. A roommate market is *solvable* if stable matchings exist, i.e., (N,R) is solvable if and only

[9]If there are no mutually best agents in a roommate market (N,R), then $MB(N,R) = \mathcal{M}(N,R)$.

if the set of stable matchings $S(N,R) \neq \emptyset$. The domain of solvable roommate markets is denoted by $\mathfrak{D}_S$. Furthermore, on the domain of solvable roommate markets $\mathfrak{D}_S$, a solution φ is *stable* if it only assigns stable matchings, i.e., for all $(N,R) \in \mathfrak{D}_S$, $\varphi(N,R) \subseteq S(N,R)$.

Gale and Shapley (1962) showed that all marriage markets are solvable, i.e., $\mathfrak{D}_M \subseteq \mathfrak{D}_S$, and they gave an example of an unsolvable roommate market (Gale and Shapley, 1962, Example 3).

For many of our results we need the solvability of roommate markets and their reduced markets (Remark 2.3.1 in Section 2.3.1 explains the reason for this assumption); e.g., the domain of marriage markets is such a domain of roommate markets because it is closed with respect to the reduction operator, i.e., starting from a marriage market $(N,R) \in \mathfrak{D}_M$, any reduced market $(N',R_{N'})$ of (N,R) is a marriage market.

Chung (2000) introduced a sufficient condition for solvability that also applies to the larger domain of weak preferences. We formulate his well-known *no odd rings* condition for our strict preference setup and refer to it as the *no odd rings* condition. This roughly means that there are no odd number of agents who prefer one another in a cyclical manner.[10]

Odd Rings and the Domain of No Odd Rings Roommate Markets: Let $(N,R) \in \mathfrak{D}$. Then, a *ring* for roommate market (N,R) is an ordered subset of agents $\{i_1, i_2, \ldots, i_k\} \subseteq N$, $k \geq 3$, such that for all $t \in \{1,2,..,k\}$, $i_{t+1} \, P_{i_t} \, i_{t-1} \, P_{i_t}$ i_t (subscript modulo k). If k is odd, then $\{i_1, i_2, \ldots, i_k\}$ is an *odd ring* for roommate market (N,R). A roommate market $(N,R) \in \mathfrak{D}$ is a *no odd rings roommate market* if there exists no odd ring for roommate market (N,R). The domain of all such roommate markets is called *the domain of no odd rings roommate markets* and denoted by $\mathfrak{D}_{NOR}$. Note that $\mathfrak{D}_M \subsetneq \mathfrak{D}_{NOR} \subsetneq \mathfrak{D}_S$.

Another well-known concept for matching problems is the core.

[10] Among the no odd rings domains listed by Chung (2000) are the Beckerian domain, single-peaked domains, single-dipped domains, and preference domains that are based on agents' representability in a metric space with the assumption that any agents prefers a match that is closer to a match that is further away.

Core: A matching is in the *(strict or strong) core* if no coalition of agents can improve their welfare by rematching among themselves. For roommate market $(N,R) \in \mathfrak{D}$, $core(N,R) = \{\mu \in \mathscr{M}(N,R) \mid$ there exists no $S \subseteq N$ and no $\mu' \in \mathscr{M}(N,R)$ such that $\mu'(S) = S$, and for all $i \in S$, $\mu'(i) R_i \, \mu(i)$, and for some $j \in S, \mu'(j) P_j \, \mu(j)\}$.

Remark 2.2.2 (*Stability and the Core*).
Similarly as in other matching models (e.g., marriage markets and college admissions markets), the core equals the set of stable matchings, i.e., for all $(N,R) \in \mathfrak{D}$, $core(N,R) = S(N,R)$. Hence, the core is a solution on the domain of solvable roommate markets $\mathfrak{D}_S$ and all its subdomains, but not on the domain of all roommate markets $\mathfrak{D}$. △

It is well-known that the core satisfies all properties introduced in this subsection.

Proposition 2.2.1. *On the domain of solvable roommate markets (and on any of its subdomains), the core satisfies individual rationality, Pareto optimality, (weak) unanimity, mutually best, and stability.*

2.3 Variable Population Properties

In this section we introduce and analyze properties that concern population changes.

2.3.1 Consistency

Consistency is one of the key properties in many frameworks with variable sets of agents. Thomson (2009) provides an extensive survey of consistency for various economic models, including marriage markets. For roommate markets, consistency essentially requires that when a set of matched agents leaves, then the solution should still match the remaining agents as before.

Consistency: A solution φ on $\mathfrak{D}$ is *consistent* if the following holds. For each $(N,R) \in \mathfrak{D}$, each $N' \subseteq N$, and each $\mu \in \varphi(N,R)$, if $(N', R_{N'}) \in \mathfrak{D}$ is a reduced market of (N,R) at μ to N' (i.e., $\mu(N') = N'$), then $\mu_{N'} \in \varphi(N', R_{N'})$.

For solutions defined on $\mathfrak{D}' \subseteq \mathfrak{D}$, consistency only applies to reduced markets $(N',R_{N'}) \in \mathfrak{D}'$. Of the four domains $(\mathfrak{D}_M,\mathfrak{D}_{NOR},\mathfrak{D}_S,\mathfrak{D})$ that we consider, only three are closed with respect to the reduction operator, i.e., for $\mathfrak{D}' \in \{\mathfrak{D}_M,\mathfrak{D}_{NOR},\mathfrak{D}\}$, if $(N,R) \in \mathfrak{D}'$, $\mu \in \mathcal{M}(N,R)$, and $(N',R_{N'})$ is a reduced market of (N,R) at μ, then $(N',R_{N'}) \in \mathfrak{D}'$. For the domain of solvable roommate markets $\mathfrak{D}_S$, non-solvable reduced markets exist and therefore consistency "loses some of its bite" (because it makes no predictions whenever market reduction leads to unsolvable reduced markets).

Remark 2.3.1 (*Solvability when Studying the Core and Domain Restrictions*).
Since stable matchings need not exist for the general domain of all roommate markets, we have to restrict attention to subdomains of solvable roommate markets when studying the core. Considering the whole domain of solvable roommate markets when studying consistency is difficult because a solvable roommate market might well have unsolvable reduced markets. Requiring that a solution only selects matchings that guarantee the solvability of all restricted markets, would already steer results forcefully towards the core. However, two domains of roommate markets we consider, $\mathfrak{D}_M$ and $\mathfrak{D}_{NOR}$, satisfy "closedness and solvability under the reduction operation", i.e., for any roommate market in $\mathfrak{D}' \in \{\mathfrak{D}_M,\mathfrak{D}_{NOR}\}$, all possible reduced markets are (i) elements of the domain $\mathfrak{D}'$ and (ii) solvable. △

Proposition 2.3.1. *On the domain of solvable roommate markets (and on any of its subdomains), the core satisfies consistency.*

Proof. Let $\mathfrak{D}'$ be a (sub)domain of solvable roommate markets. Let $(N,R) \in \mathfrak{D}'$, $\mu \in core(N,R)$ and assume that $(N',R_{N'}) \in \mathfrak{D}'$ is a reduced market of (N,R) at μ to N'. Thus, $core(N',R'_N) \neq \emptyset$.

Assume that the *core* is not consistent and $\mu_{N'} \notin core(N',R_{N'})$. Hence, there exists a blocking pair $\{i,j\} \subseteq N'$ for $\mu_{N'}$, i.e., $j\,P_i\,\mu_{N'}(i)$ and $i\,P_j\,\mu_{N'}(j)$. However, since $\mu_{N'}(i) = \mu(i)$ and $\mu_{N'}(j) = \mu(j)$, $\{i,j\} \subseteq N$ is a also a blocking pair for μ; contradicting $\mu \in core(N,R)$. □

Lemma 2.3.1.

(a) *On the domain of marriage markets (see also Toda, 2006, Lemma 3.6),*

(b) *On the domain of no odd rings roommate markets,*

(c) *On the domain of solvable roommate markets,*

no proper subsolution of the core satisfies consistency.

(d) *On the domain of all roommate markets, no solution is a subsolution of the core for solvable problems and satisfies consistency (Sanver, 2010, Proposition 4.3).*

We prove Lemma 2.3.1 (a), (b), and (c) in Appendix 2.6.1.

Lemma 2.3.2.
(a) *On the domain of marriage markets,*
(b) *On the domain of no odd rings roommate markets,*
(c) *On the domain of solvable roommate markets,*
(d) *On the domain of all roommate markets,*

mutually best and consistency imply individual rationality.

Proof. Let φ be a solution on any of the domains $\mathfrak{D}'$ of Lemma 2.3.2 that satisfies mutually best and consistency. Assume, by contradiction, that there exists a roommate market $(N,R) \in \mathfrak{D}'$, a matching $\mu \in \varphi(N,R)$, and an agent $i \in N$ such that $i\, P_i\, \mu(i)$. Hence, $\mu(i) \neq i$.

Let $N' = \{i, \mu(i)\}$ and consider the reduced market $(N', R_{N'}) \in \mathfrak{D}'$ of μ to N'. By consistency, $\mu_{N'} \in \varphi(N', R')$. However, at $(N', R_{N'})$ agent i is mutually best with himself and by mutually best, $\mu_{N'} \notin \varphi(N', R')$; a contradiction. $\square$

Note that the proof of Lemma 2.3.2 does not contain any steps that are sensitive with respect to domain restrictions (except that all two-agent restricted markets used in the proof should be included in the subdomain that is considered).

2.3.2 Population Sensitivity Properties

The following two population sensitivity properties were introduced and analyzed by Klaus (2011).

Consider the change of a roommate market (N,R) when a finite set of agents or *newcomers* $\hat{N} \subsetneq \mathbb{N} \setminus N$ shows up. Then, the new set of agents is $N' = N \cup \hat{N}$ and (N', R'), $R' \in \mathscr{R}^{N'}$, is an *extension of* (N,R) if $R'_N = R$.

Adding a set of newcomers $\hat{N}$ might be a positive or a negative change for any of the incumbents in N because it might mean

a negative change with more competition or
a positive change with more resources.

First, with competition sensitivity we formulate a property that captures the possible negative effect newcomers might have on some agents. Essentially, competition sensitivity requires that if two incumbents are newly matched after a set of newcomers arrived, then one of them suffers from the increased competition by the newcomers and is worse off (for the detailed derivation of competition sensitivity and its relation to own-side population monotonicity for marriage markets we refer to Klaus, 2011).

(Weak) Competition Sensitivity (Klaus, 2011): A solution φ on $\mathfrak{D}$ is *competition sensitive* if the following holds. Let $(N,R) \in \mathfrak{D}$ be a roommate market and assume that $(N',R') \in \mathfrak{D}$, $N' = N \cup \hat{N}$, is an extension of (N,R). Then, for all $\mu \in \varphi(N,R)$ there exists $\mu' \in \varphi(N',R')$ such that for all $i,j \in N$ [possibly $i = j$] that are *newly matched at* μ', at least one is worse off, i.e., if $i,j \in N$, $\mu(i) \neq j$, and $\mu'(i) = j$, then $\mu(i) P'_i \mu'(i)$ or $\mu(j) P'_j \mu'(j)$.[11] A solution φ on $\mathfrak{D}$ is *weakly competition sensitive* if we require competition sensitivity only when adding one newcomer at a time, i.e., $\hat{N} = \{n\}$. Note that the competition sensitivity defined in Klaus (2011, Definition 9) equals the weak competition sensitivity here.

Klaus (2011, Lemma 3') shows that on the domains of marriage markets, solvable roommate markets, and all roommate markets, weak unanimity and weak competition sensitivity imply mutually best. We list these results below and add a corresponding result for the subdomain of no odd rings roommate markets.

Lemma 2.3.3.

(a) *On the domain of marriage markets (Klaus, 2011, Lemma 3' (a)),*
(b) *On the domain of no odd rings roommate markets,*
(c) *On the domain of solvable roommate markets (Klaus, 2011, Lemma 3' (b)),*
(d) *On the domain of all roommate markets (Klaus, 2011, Lemma 3' (c)),*

[11]Equivalently, if agents in $\hat{N}$ are leaving: for all $i,j \in N$ [possibly $i = j$] that *are not matched at* μ *anymore*, at least one is better off, i.e., if $i,j \in N$, $\mu'(i) = j$, and $\mu(i) \neq j$, then $\mu(i) P_i \mu'(i)$ or $\mu(j) P_j \mu'(j)$.

weak unanimity and weak competition sensitivity imply mutually best.

The proof of Lemma 2.3.3 (b) is very similar to the proof of Lemma 2.3.3 (a) for marriage markets (Klaus, 2011, Lemma 3' (a)) because starting from a no odd rings market [marriage market] one can add the newcomers in the proof such that the resulting markets are again no odd rings markets [marriage markets].

On the domain of marriage markets, Toda (2006, Lemma 3.1) proves that weak unanimity and own-side population monotonicity imply mutually best. The proof of Lemma 2.3.3 (or, more precisely, Klaus, 2011, Lemma 3') follows similar arguments as Toda's (2006, Lemma 3.1) proof for the corresponding marriage market result.

Klaus (2011, Example 3) illustrates why Lemma 2.3.3 might not hold if the set of potential agents is finite (Example 2.6.1 in Appendix 2.6.2 also illustrates this).

Second, with resource sensitivity we formulate a property that captures the possible positive effect newcomers might have on some agents. Essentially, resource sensitivity requires that if two incumbents are unmatched after a set of newcomers arrived, then one of them benefits from the increase of resources by the newcomers and is better off (for the detailed derivation of resource sensitivity and its relation to other-side population monotonicity we refer to Klaus, 2011).

(Weak) Resource Sensitivity (Klaus, 2011): A solution φ on $\mathfrak{D}$ is *resource sensitive* if the following holds. Let $(N,R) \in \mathfrak{D}$ be a roommate market and assume that $(N',R') \in \mathfrak{D}$, $N' = N \cup \hat{N}$, is an extension of (N,R). Then, for all $\mu' \in \varphi(N',R')$ there exists $\mu \in \varphi(N,R)$ such that for all $i,j \in N$ [possibly $i = j$] that *are not matched at* μ' *anymore*, at least one is better off, i.e., if $i,j \in N$, $\mu(i) = j$, and $\mu'(i) \neq j$, then $\mu'(i)\,P'_i\,\mu(i)$ or $\mu'(j)\,P'_j\,\mu(j)$.[12] A solution φ on $\mathfrak{D}$ is *weakly resource sensitive* if we require resource sensitivity only when adding one newcomer at a time, i.e., $\hat{N} = \{n\}$. Note that the resource sensitivity defined in Klaus (2011, Definition 11) equals the weak resource sensitivity here.

[12] Equivalently, if agents in $\hat{N}$ are leaving: for all $i,j \in N$ [possibly $i = j$] that are *newly matched at* μ at least one is worse off, i.e., if $i,j \in N$, $\mu'(i) \neq j$, and $\mu(i) = j$, then $\mu'(i)\,P_i\,\mu(i)$ or $\mu'(j)\,P_j\,\mu(j)$.

Klaus (2011, Lemma 4' (a) and (c)) shows that on the domains of marriage markets and all roommate markets, weak unanimity and weak resource sensitivity imply mutually best. We list these results below and add a corresponding result for the subdomain of no odd rings roommate markets. Furthermore, Klaus (2011, Lemma 4 (b)) shows that on the domain of solvable roommate markets, weak unanimity and resource sensitivity imply mutually best. Here, we establish the new result that on the domain of solvable roommate markets, weak unanimity, weak resource sensitivity, and consistency, imply mutually best.

Lemma 2.3.4.

(a) *On the domain of marriage markets (Klaus, 2011, Lemma 4' (a)),*

(b) *On the domain of no odd rings roommate markets,*

(c) *On the domain of solvable roommate markets, consistency,*

(d) *On the domain of all roommate markets (Klaus, 2011, Lemma 4' (c)),*

weak unanimity and weak resource sensitivity imply mutually best.

Proof. The proof of (b) is very similar to the proof of Lemma 2.3.4 (a) for marriage markets (Klaus, 2011, Lemma 4' (a)) because starting from a no odd rings market [marriage market] one can add the newcomers in the proof such that the resulting markets are again no odd rings markets [marriage markets].

Proof of (c): Let φ be a solution on the domain of solvable roommate markets that satisfies consistency, weak unanimity, and weak resource sensitivity, but not mutually best. Thus, there exists a solvable roommate market (N,R) and a matching $\mu \in \varphi(N,R)$ such that agents i and j [possibly $i = j$] are mutually best and $\mu(i) \neq j$. Let $\tilde{N} = \{i,j,\mu(i),\mu(j)\}$ and consider the reduced market $(\tilde{N},R_{\tilde{N}})$. By consistency, $\mu_{\tilde{N}} \in \varphi(\tilde{N},R_{\tilde{N}})$, i and j are mutually best agents, and $\mu_{\tilde{N}}(i) \neq j$.

Let $\bar{N} = \{i,j\}$ and consider the reduced preferences $\bar{R} = R_{\bar{N}}$. If $i \neq j$, then there exists a unanimously best complete matching $\bar{\nu}$ for solvable roommate market $(\bar{N},\bar{R})$: $\bar{\nu}$ matches agent i with agent j. Hence, by weak unanimity, $\varphi(\bar{N},\bar{R}) = \{\bar{\nu}\}$ and $\bar{\nu}(i) = j$. If $i = j$, then $\varphi(\bar{N},\bar{R}) = \{\bar{\nu}\}$ and $\bar{\nu}(i) = j$ because $\bar{\nu}$ is the only possible matching. In the sequel we will not use the single-valuedness of $\varphi(\bar{N},\bar{R})$ but that for all $\mu' \in \varphi(\bar{N},\bar{R})$, $\mu'(i) = j$.

If $\mu(i) \neq i$, then consider the extension (N^1, R^1) of $(\bar{N}, \bar{R})$ that is obtained by adding newcomer $\mu(i)$ such that $N^1 = \bar{N} \cup \{\mu(i)\}$ and $R^1 = R_{N^1}$.[13] By weak resource sensitivity, for all $\mu^1 \in \varphi(N^1, R^1)$, there exists $\mu' \in \varphi(\bar{N}, \bar{R})$ such that if agents i and j (possibly $i = j$) are not matched at μ^1 anymore, then at least one is better off. Then, since for all $\mu' \in \varphi(\bar{N}, \bar{R})$ agents i and j are already mutually best matched, for all $\mu^1 \in \varphi(N^1, R^1)$, $\mu^1(i) = j$.

If $\mu(j) \neq j$, then we add newcomer $\mu(j)$ in a similar fashion. So we end up with the reduced market $(\tilde{N}, R_{\tilde{N}})$. By weak resource-sensitivity, for all $\mu^2 \in \varphi(\tilde{N}, R_{\tilde{N}})$, $\mu^2(i) = j$, contradicting $\mu_{\tilde{N}}(i) \neq j$. □

Lemma 2.3.4 (c) cannot be established without the addition of consistency: Klaus (2011, Example 4) provides a solution on the domain of solvable roommate markets that satisfies weak unanimity and weak resource sensitivity, but neither mutually best nor consistency.

The following result slightly generalizes Klaus (2011, Proposition 2) (the proof is insensitive with respect to the specific domain of solvable roommate markets used).

Proposition 2.3.2. *On the domain of solvable roommate markets (and on any of its subdomains), any stable solution satisfies competition and resource sensitivity. In particular, the core satisfies competition and resource sensitivity.*

2.3.3 Previous Results for Marriage Markets

We are aware of two papers that analyze consistency for the *domain of classical marriage markets* for which matching agents of the same gender is not feasible. First, Sasaki and Toda (1992) use the property together with Pareto optimality, anonymity,[14] and converse consistency[15] to characterize the core. Sanver (2010, Proposition 4.2) shows that on the domain of all roommate markets, no solution satisfies Pareto optimality, anonymity, and converse

[13]Note that (N^1, R^1) has a unique core allocation that matches agent i with agent j, and agent $\mu(i)$ is single.

[14]*Anonymity*: matchings assigned by the solution do not depend on agents' names.

[15]*Converse consistency*: matchings assigned by the solution are (conversely) related to the matchings the solution assigns to certain restricted roommate markets (with at most four agents).

consistency. Second, Toda (2006) shows that the core is characterized by weak unanimity, own-side population monotonicity,[16] and consistency.

Theorem 2.3.1 (Toda, 2006, Theorem 3.2)**.** *On the domain of classical marriage markets, a solution satisfies weak unanimity, own-side population monotonicity, and consistency if and only if it equals the core.*

Here we focus on Toda's characterization and analyze if and how the result extends from (classical) marriage markets to roommate markets. Before doing so, we obtain a new result by replacing own-side population monotonicity with other-side population monotonicity[17] in Theorem 2.3.1.

Theorem 2.3.2. *On the domain of classical marriage markets, a solution satisfies weak unanimity, other-side population monotonicity, and consistency if and only if it equals the core.*

We prove Theorem 2.3.2 in Appendix 2.7.

The proofs of Theorems 2.3.1 and 2.3.2 both rely on the following lemma.

Lemma 2.3.5 (Toda, 2006, Lemma 3.4)**.** *On the domain of classical marriage markets, if a solution satisfies mutually best and consistency, then it is a subsolution of the core.*

We show, in Section 2.4, that Lemma 2.3.5 cannot be extended to the domain of solvable roommate markets; Example 2.4.1 in Section 2.4 shows that there exists a solution satisfying mutually best and consistency, but which assigns unstable matchings to some solvable roommate markets.

2.4 Main Results

In this section we first explore some logical relations between the properties and the core. This analysis yields "subsolution of the core" results on the

[16]*Own-side population monotonicity*: if additional men (women) enter the market, then all incumbent men (women) are weakly worse off (for a formal definition see Appendix 2.7). In Klaus (2011) we argue that the proper extension of Toda's (2006) own-side population monotonicity to roommate markets is competition sensitivity.

[17]*Other-side population monotonicity*: if additional men (women) enter the market, then all incumbent women (men) are weakly better off (for a formal definition see Appendix 2.7). In Klaus (2011) we argue that the proper extension of other-side population monotonicity to roommate markets is resource sensitivity.

domains of marriage markets, no odd rings roommate markets, and solvable roommate markets (parts (a), (b), and (c) in Lemmas 2.4.1 and 2.4.2) and we establish two impossibility results on the domain of all roommate markets (parts (d) in Lemmas 2.4.1 and 2.4.2). Second, we establish various characterizations of the core (Theorems 2.4.1 and 2.4.2). Third, we derive two marriage market results using population monotonicity (Corollary 2.4.1).

2.4.1 "Subsolution of the Core" and Impossibility Results

Lemma 2.4.1.

(a) *On the domain of marriage markets,*

(b) *On the domain of no odd rings roommate markets,*

(c) *On the domain of solvable roommate markets,*

if a solution satisfies weak unanimity, competition sensitivity, and consistency, then it is a subsolution of the core.

(d) *On the domain of all roommate markets,*

no solution satisfies weak unanimity, competition sensitivity, and consistency.

Proof. Let φ be a solution on any of the domains of Lemma 2.4.1 that satisfies weak unanimity, competition sensitivity, and consistency. By Lemma 2.3.3, φ is mutually best and by Lemma 2.3.2, φ is individually rational.

To prove (a), (b), and (c) let (N,R) be a solvable roommate market [marriage / no odd rings roommate market] such that $\varphi(N,R) \nsubseteq core(N,R)$. To prove (d), let (N,R) be an unsolvable roommate market. In all cases there exists a matching $\mu \in \varphi(N,R)$ with a blocking pair $\{i,j\}$ [possibly $i = j$] for μ. By individual rationality, $i \neq j$.

Without loss of generality assume that $N \setminus \{i,j\} = \{1,2,\ldots,l\}$. Let $\hat{N} = \{k_1,k_2,\ldots,k_l\} \subsetneq \mathbb{N} \setminus N$ be a set of newcomers and assume that (N',R'), $N' = N \cup \hat{N}$, is an extension of (N,R) such that for all agents $m \in N \setminus \{i,j\}$, m and k_m are mutually best pairs and agent m is the only one that finds k_m acceptable and k_m finds only m acceptable [if (N,R) is a marriage / no odd rings roommate market, then the newcomers and preferences can be chosen such that (N',R') is also a marriage / no odd rings roommate market]. By mutually best, for all $\mu' \in \varphi(N',R')$ and for all $m \in N \setminus \{i,j\}$, $\mu'(m) = k_m$. By competition sensitivity, for $\mu \in \varphi(N,R)$ there exists $\hat{\mu}' \in \varphi(N',R')$ such that agents i and j are not matched, i.e., $\hat{\mu}'(i) \neq j$ (if not, then agents i and j are newly matched at $\hat{\mu}'$, but both are better off). Hence, $\hat{\mu}' \in \varphi(N',R')$ is the matching that mutually best matches all agents in $N' \setminus \{i,j\}$ and agents i and j are single.

Thus, $(\{i,j\},R_{\{i,j\}})$ is a reduced market of (N',R') at $\hat{\mu}'$ to $\{i,j\}$. Note that i and j are mutually best agents at $(\{i,j\},R_{\{i,j\}})$ and both single at $\hat{\mu}'_{\{i,j\}}$. By consistency, $\hat{\mu}'_{\{i,j\}} \in \varphi(\{i,j\},R_{\{i,j\}})$, which contradicts mutually best. □

In Appendix 2.6.2 we establish a stronger version of Lemma 2.4.1 (a) and (b) – Lemma 2.4.1' – using weak competition sensitivity. Whether we can strengthen Lemma 2.4.1 (c) by using weak competition sensitivity instead of competition sensitivity is an *open problem*.

With Example 2.6.1 in Appendix 2.6.2 we illustrate why Lemmas 2.4.1 and 2.4.1' might not hold if the set of potential agents is finite.

Lemma 2.4.2.

(a) *On the domain of marriage markets,*

(b) *On the domain of no odd rings roommate markets,*

(c) *On the domain of solvable roommate markets,*

if a solution satisfies weak unanimity, resource sensitivity, and consistency, then it is a subsolution of the core.

(d) *On the domain of all roommate markets,*

no solution satisfies weak unanimity, resource sensitivity, and consistency.

Proof. Let φ be a solution on any of the domains of Lemma 2.4.2 that satisfies weak unanimity, resource sensitivity, and consistency. By Lemma 2.3.4, φ is mutually best and by Lemma 2.3.2, φ is individually rational.

To prove (a), (b), and (c) let (N,R) be a solvable roommate market [marriage / no odd rings roommate market] such that $\varphi(N,R) \nsubseteq core(N,R)$. To prove (d), let (N,R) be an unsolvable roommate market. In both cases there exists a matching $\mu \in \varphi(N,R)$ with a blocking pair $\{i,j\}$ [possibly $i = j$] for μ. By individual rationality, $i \neq j$.

Without loss of generality assume that $N \setminus \{i,j\} = \{1,2,\ldots,l\}$ and consider the roommate market $(\{i,j\},R_{\{i,j\}})$. There exists a unanimously best complete matching $\bar{\mu}$ for (marriage, no odd rings, solvable) roommate market

$(\{i,j\},R_{\{i,j\}})$: $\bar{\mu}$ matches agent i with agent j. Hence, by weak unanimity, $\varphi(\{i,j\},R_{\{i,j\}})=\{\bar{\mu}\}$ and $\bar{\mu}(i)=j$. Consider the extension (N,R) of $(\{i,j\},R_{\{i,j\}})$ that is obtained by adding newcomers $\hat{N}=\{1,\dots,l\}$. Because $\mu(i)\neq j$ and $\bar{\mu}(i)=j$, by resource sensitivity, $\mu(i)P_i\,\bar{\mu}(i)=j$ or $\mu(j)P_j\,\bar{\mu}(j)=i$. This contradicts that $\{i,j\}$ is a blocking pair for μ. □

In Appendix 2.6.2 we establish a stronger version of Lemma 2.4.2 (a) and (b) – Lemma 2.4.2' – using weak resource sensitivity.

The following solution demonstrates that corresponding results to Lemmas 2.3.5 and 2.4.2' do not exist for solvable roommate markets.

Example 2.4.1. *We define solution $\hat{\varphi}$ on $\mathfrak{D}'\subseteq\mathfrak{D}_S$ using the following roommate market and matchings. Let $(\hat{N},\hat{R})$ be such that $\hat{N}=\{1,2,3,4\}$ and preferences $\hat{R}$ are given in Table 2.1.*

$\hat{R}_1$	$2,3,4,1$	$\hat{\mu}=(3,4,1,2)$
$\hat{R}_2$	$3,4,1,2$	$\hat{\mu}'=(2,1,4,3)$
$\hat{R}_3$	$4,1,2,3$	$\hat{\mu}''=(4,3,2,1)$
$\hat{R}_4$	$1,2,3,4$	$core(\hat{N},\hat{R})=\{\hat{\mu}\}$

Table 2.1: The roommate market $(\hat{N},\hat{R})$.

The unique stable matching $\hat{\mu}$ for $(\hat{N},\hat{R})$ matches agents 1 and 3 and agents 2 and 4. Removing any of the agents creates a "roommate cycle" for the remaining agents and the restricted roommate market is not solvable. Thus, the solvable roommate market $(\hat{N},\hat{R})$ cannot be reached from another solvable roommate market by adding one newcomer.

R_1	$2,3,4,1,\dots$
R_2	$3,4,1,2,\dots$
R_3	$4,1,2,3,\dots$
R_4	$1,2,3,4,\dots$

Table 2.2: A separable submarket of (N,R).

If (N,R) is a roommate market such that $\hat{N}\subseteq N$ and preferences R are given in Table 2.2, then we say that (N,R) is a roommate market with the separable submarket $(\hat{N},\hat{R})$ (note that agents in $\hat{N}$ find only agents in $\hat{N}$ acceptable and any individually rational matching will match agents in $\hat{N}$ among each *others).*

We now define $\hat{\varphi}$ as follows. Let (N,R) be a solvable roommate market. Whenever, $(\hat{N},\hat{R})$ is a separable submarket of (N,R), $\hat{\varphi}$ first assigns all stable matchings. Furthermore, for each stable matching μ (which matches all agents in $\hat{N}$ according to the restricted matching $\mu_{\hat{N}} = \hat{\mu}$), $\hat{\varphi}$ also assigns the two matchings μ' and μ'' that correspond to $\hat{\mu}'$ and $\hat{\mu}''$, i.e., μ' [μ''] matches all agents in $\hat{N}$ according to $\hat{\mu}'$ [$\hat{\mu}''$] and all agents in $N \setminus \hat{N}$ according to μ. For all other solvable roommate markets, $\hat{\varphi}$ assigns the set of stable matchings. Thus, core $\subsetneq \hat{\varphi}$. ◇

Proposition 2.4.1. *On the domain of solvable roommate markets (and on any of its subdomains), solution $\hat{\varphi}$ (defined in Example 2.4.1) satisfies individual rationality, Pareto optimality, (weak) unanimity, mutually best, consistency, and weak resource sensitivity.*

We prove Proposition 2.4.1 in Appendix 2.6.3.

2.4.2 Core Characterizations

Next, we strengthen the marriage market characterizations of the core presented in Theorem 2.3.1 (Toda, 2006, Theorem 3.2) and Theorem 2.3.2 in two ways. First, for marriage markets we replace the respective population monotonicity property with its corresponding population sensitivity property and second, we extend this characterization to the domains of no odd rings and of solvable roommate markets.

Theorem 2.4.1 (*Three Core Characterizations: Competition Sensitivity*).

(a) *On the domain of marriage markets,*

(b) *On the domain of no odd rings roommate markets,*

a solution satisfies weak unanimity, weak competition sensitivity, and consistency if and only if it equals the core.

(c) *On the domain of solvable roommate markets,*

a solution satisfies weak unanimity, competition sensitivity, and consistency if and only if it equals the core.

Proof. Let φ be a solution on any of the domains of Theorem 2.4.1. By Propositions 2.2.1, 2.3.1, and 2.3.2, the core satisfies weak unanimity, (weak) competition sensitivity, and consistency. Let φ be weakly unanimous, competition sensitive, and consistent. Then, by Lemma 2.4.1, φ is a subsolution of the core and by Lemma 2.3.1, φ equals the core. We establish (a) and (b) with weak competition sensitivity instead of competition sensitivity by using Lemma 2.4.1' instead of Lemma 2.4.1. □

Lemma 2.4.1 (d) establishes a corresponding impossibility result to Theorem 2.4.1 on the domain of all roommate markets. Whether we can strengthen Theorem 2.4.1 (c) by using weak competition sensitivity instead of weak competition sensitivity is an *open problem*.

With Example 2.6.1 in Appendix 2.6.2 we illustrate why Theorem 2.4.1 might not hold if the set of potential agents is finite.

Theorem 2.4.2 (*Three Core Characterizations: Resource Sensitivity*).

(a) *On the domain of marriage markets,*

(b) *On the domain of no odd rings roommate markets,*

a solution satisfies weak unanimity, weak resource sensitivity, and consistency if and only if it equals the core.

(c) *On the domain of solvable roommate markets,*

a solution satisfies weak unanimity, resource sensitivity, and consistency if and only if it equals the core.

Proof. Let φ be a solution on any of the domains of Theorem 2.4.2. By Propositions 2.2.1, 2.3.1, and 2.3.2, the core satisfies weak unanimity, (weak) resource sensitivity, and consistency. Let φ be weakly unanimous, resource sensitive, and consistent. Then, by Lemma 2.4.2, φ is a subsolution of the core and by Lemma 2.3.1, φ equals the core. We establish (a) and (b) with weak resource sensitivity instead of resource sensitivity by using Lemma 2.4.2' instead of Lemma 2.4.2. □

Lemma 2.4.2 (d) establishes a corresponding impossibility result to Theorem 2.4.2 on the domain of all roommate markets. Solution $\hat{\varphi}$ (defined in Example 2.4.1) demonstrates that Theorem 2.4.2 (c) for solvable roommate markets cannot be strengthened by using weak resource sensitivity instead of resource sensitivity.

Theorems 2.3.1 and 2.3.2 show that on the domain of classical marriage markets, the core is the unique solution satisfying weak unanimity, consistency, and own-side or other-side population monotonicity. Both results follow from our "population sensitivity characterizations of the core" for marriage markets (Theorems 2.4.1 (a) and 2.4.2 (a)).

Corollary 2.4.1 (*Two Core Characterizations for Marriage Markets*).
On the domain of marriage markets, a solution satisfies weak unanimity, consistency, and

(1) *own-side population monotonicity,*

(2) *other-side population monotonicity,*

if and only if it equals the core.

We prove Corollary 2.4.1 in Appendix 2.7. An example constructed along the lines of Example 2.6.1 in Appendix 2.6.2 illustrates why Corollary 2.4.1 (1) might not hold if the set of potential agents is finite.

We next show the independence of properties in Theorems 2.4.1 and 2.4.2 (these examples can also be used to show the independence of properties in Corollary 2.4.1).

The solution φ^s on the domains in Theorems 2.4.1 and 2.4.2 that always assigns the matching at which all agents are single satisfies (weak) competition and (weak) resource sensitivity, consistency, but not weak unanimity.

On the domains in Theorems 2.4.1 and 2.4.2 any proper subsolution of the core satisfies (weak) unanimity, (weak) competition and (weak) resource sensitivity (Proposition 2.3.2), but not consistency (Lemma 2.3.1).

The Pareto solution PO on the domains in Theorems 2.4.1 and 2.4.2 satisfies (weak) unanimity and consistency, but neither weak competition nor weak resource sensitivity (see Klaus, 2011, Example 2).

Remark 2.4.1 (*Pareto Optimality*).
Since Pareto optimality implies weak unanimity, we can use this stronger efficiency property in all of our results (the same solutions that establish the independence of properties in Theorems 2.4.1 and 2.4.2 can be used again). $\triangle$

2.5 Conclusion

In this chapter, we make some positive contribution to the study of one-sided markets: a marriage market characterization (Toda, 2006, Theorem 3.2) is extended to various (solvable) roommate market domains (Theorem 2.4.1 with (weak) competition sensitivity) and some new complementary characterizations are established (Theorem 2.4.2 with (weak) resource sensitivity). Furthermore, corresponding impossibility results (Lemmas 2.4.1 (d) and 2.4.2 (d)) and corresponding marriage market results (Corollary 2.4.1) are obtained. At first sight, this chapter thus parallels Klaus (2011) with the difference that it extends Toda's "consistency core characterization" (Toda, 2006, Theorem 3.2) instead of his "Maskin monotonicity core characterization" (Toda, 2006, Theorem 3.1). Of course, we find it important to validate the new population sensitivity properties introduced by Klaus (2011) by establishing the results mentioned above, but there are some interesting and important differences between this chapter and Klaus (2011) that go beyond the use of consistency instead of Maskin monotonicity. In the following, we discuss the need and the use of a new proof strategy to obtain marriage market results for roommate markets, but another new feature in our article is the addition of the no odd rings domain (in particular, single-peaked, single-dipped, and distance-based preference domains constitute relevant no odd rings roommate market domains).

All results in Klaus (2011) could essentially be established by following Toda's (2006, Theorem 3.1) proof steps (although adjusting those steps to work with the new population sensitivity properties did require a lot of work and technical skills). However, this is not true anymore when extending Toda's (2006, Theorem 3.2) second result: it is impossible to extend Toda's crucial Lemma 3.4, i.e., one cannot show that a rule that satisfies individual rationality, mutually best, and consistency must be a subset of the core – see counter-Example 2.4.1. Hence, we have developed a new proof strategy for the original result (Toda, 2006, Theorem 3.2) that also works for the extension to roommate markets.[18] An example of a result (or results) where

[18]Two-sided markets show a lot of regularity that is missing from roommate markets: one main feature often used in "marriage market proofs" is the polarization that occurs within the core; in particular, the existence of side-optimal stable matchings and the possibility to

a similar asymmetry of proof techniques can be observed between two-sided and one-sided markets are the random paths to stability result(s) by Roth and Vate (1990) and by Diamantoudi, Miyagawa, and Xue (2004). We believe that the study of "one-sided (roommate) market techniques" also provides valuable insights for more general models such as coalition formation or network formation.

2.6 Appendix A: Proofs of Lemmas 2.3.1, 2.4.1', 2.4.2', and Proposition 2.4.1

2.6.1 Proof of Lemma 2.3.1

Before proving Lemma 2.3.1, we state and prove a so-called Bracing Lemma (which is a typical consistency result for many economic models, see Thomson, 2009). This result is a key element in the proof of Lemma 2.3.1.

Lemma 2.6.1 (*Bracing Lemma*).

(a) *Let (N,R) be a marriage market. For each $\mu \in core(N,R)$, there exists a marriage market (N',R') (see also Toda, 2006, Lemma 5.8),*

(b) *Let (N,R) be a no odd rings roommate market. For each $\mu \in core(N,R)$, there exists a no odd rings roommate market (N',R'),*

(c) *Let (N,R) be a solvable roommate market. For each $\mu \in core(N,R)$, there exists a solvable roommate market (N',R'),*

such that $N \subseteq N'$, $R'_N = R$, $core(N',R') = \{\mu'\}$, and $\mu'_N = \mu$.

Proof. For the proof of (a), let $(N,R) \in \mathfrak{D}_M$, for the proof of (b), let $(N,R) \in \mathfrak{D}_{NOR}$, and for the proof of (c), let $(N,R) \in \mathfrak{D}_S$. If $|core(N,R)| = 1$, then there is nothing to prove. Let $core(N,R) = \{\mu,\mu_1,\ldots,\mu_k\}$ for some $k \geq 1$. Since the core is Pareto optimal, there exists $i^* \in N$ such that $\mu(i^*) P_{i^*} \mu_1(i^*)$.

First, consider the extension (N^*,R^*) of (N,R) that is obtained by adding a newcomer $n^* \in \mathbb{N}\backslash N$ such that $N^* = N \cup \{n^*\}$ and $R^* \in R^{N^*}$ is such that:

use side-monotonic (or side-greedy) arguments is key in many proofs for two-sided matching markets. This polarization essentially creates the marriage market core regularity used by Toda (2006) when adding or removing agents. For roommate markets (i.e., in this chapter), the absence of polarization and regularity (not so surprisingly) causes "trouble" in that the core basically can suddenly collapse or expand (in contrast, in marriage markets, the core lattice is only truncated or expanded on one of its "sides").

(i) $R^*_N = R$,

(ii) for all $i \in N \setminus \{i^*\}$ and all $j \in N$ (possibly $i = j$), $j P^*_i n^*$, i.e., for every agent in N – except agent i^* – agent n^* is the least preferred agent,

(iii) $\mu(i^*) P^*_{i^*} n^* P^*_{i^*} \mu_1(i^*)$, i.e., agent i^* ranks the newcomer n^* between agents $\mu(i^*)$ and $\mu_1(i^*)$, and

(iv) for all $j \in N \setminus \{i^*\}$, $i^* P^*_{n^*} n^* P^*_{n^*} j$, i.e., the newcomer finds only agent i^* acceptable.

For the proof of (a), (N^*, R^*) is a marriage market by choosing agent n^*'s gender to be opposite of agent i^*'s gender.

For the proof of (b), we show that (N^*, R^*) is also a no odd rings roommate market. Suppose not, then there exists an odd ring $K = \{i_1, i_2, \ldots, i_k\} \subseteq N^*$ for (N^*, R^*) with $k \geq 3$. If $n^* \notin K$ then $K \subseteq N$, which contradicts that (N, R) is a no odd rings roommate market. Hence, $n^* = i_t$ for some $t \in \{1, 2, \ldots, k\}$. Then, by the definition of an odd ring, $i_{t+1} P_{n^*} i_{t-1}$ and $n^* P_{i_{t-1}} i_{t-2}$. By (ii) in the construction of preference profile R^*, $i_{t-1} = i^*$ (for all other agents in N, n^* is the least preferred agent). Thus, $i_{t+1} P_{n^*} i^*$. However, by (iv) in the construction of preference profile R^*, agent n^* does not strictly prefer any agent in $N \setminus \{i^*\}$ to agent i^*; a contradiction.

For the proof of (c), we show that (N^*, R^*) is also a solvable roommate market. Note that by construction $\mu \cup \{n^*\} \in core(N^*, R^*)$. Thus (N^*, R^*) is solvable.

Second, we prove that (N^*, R^*) has fewer stable matchings than (N, R). By construction, $\mu \cup \{n^*\} \in core(N^*, R^*)$. Note that by the so-called Lone Wolf Theorem (e.g., Klaus and Klijn, 2010, Theorem 1), any agent who is single in one stable matching is single in all other stable matchings. Thus, only matchings of the form $\mu^* \cup \{n^*\}$, $\mu^* \in \mathcal{M}(N, R)$, can be stable for roommate market (N^*, R^*). Furthermore, since the core is consistent (Proposition 2.3.1), if for any $\tilde{\mu} \in \mathcal{M}(N, R)$, $\tilde{\mu} \cup \{n^*\} \in core(N^*, R^*)$, then $\tilde{\mu} \in core(N, R)$. Hence, $|core(N^*, R^*)| \leq |core(N, R)|$. Finally, since (i^*, n^*) blocks $\mu_1 \cup \{n^*\}$, $\mu_1 \cup \{n^*\} \notin core(N^*, R^*)$. We conclude that $|core(N^*, R^*)| < |core(N, R)|$ and $\mu \cup \{n^*\} \in core(N^*, R^*)$.

Repeating this process of adding a newcomer to reduce the number of stable matchings at most k times results in:

(a) a marriage market (N',R'),
(b) a no odd rings roommate market (N',R'),
(c) a solvable roommate market (N',R'),

such that $N \subseteq N'$, $R'_N = R$, $core(N',R') = \{\mu'\}$, and $\mu'_N = \mu$. □

Lemma 2.3.1.

(a) *On the domain of marriage markets (see also Toda, 2006, Lemma 3.6),*
(b) *On the domain of no odd rings roommate markets,*
(c) *On the domain of solvable roommate markets,*

no proper subsolution of the core satisfies consistency.

Proof. Let φ be a solution on any of the domains $\mathfrak{D}'$ of Lemma 2.3.1 that is a consistent subsolution of the core. Let $(N,R) \in \mathfrak{D}'$ and $\mu \in core(N,R)$. Then, by the Bracing Lemma (Lemma 2.6.1), there exists a roommate market $(N^*,R^*) \in \mathfrak{D}'$ with $core(N^*,R^*) = \{\mu^*\}$ such that (N,R) is a reduced market of (N^*,R^*) at μ^* and $\mu^*_N = \mu$. Since φ is a subsolution of the core, $\varphi(N^*,R^*) = \{\mu^*\}$. As φ is consistent, $\mu \in \varphi(N,R)$. So, $core(N,R) \subseteq \varphi(N,R)$. Since φ is a subsolution of the core, $\varphi(N,R) \subseteq core(N,R)$. Hence, $\varphi(N,R) = core(N,R)$. □

2.6.2 Lemmas 2.4.1' and 2.4.2' and their Proofs

Lemma 2.4.1'.

(a) *On the domain of marriage markets,*
(b) *On the domain of no odd rings roommate markets,*

if a solution satisfies weak unanimity, weak competition sensitivity, and consistency, then it is a subsolution of the core.

Lemma 2.4.2'.

(a) *On the domain of marriage markets,*
(b) *On the domain of no odd rings roommate markets,*

if a solution satisfies weak unanimity, weak resource sensitivity, and consistency then it is a subsolution of the core.

Because the first parts of the proofs of the above lemmas are identical, we prove both lemmas together and indicate the steps when either weak competition sensitivity or weak resource sensitivity are used. Throughout the proof we will list partial information on agents' preferences. For instance, $j R_i \, k R_i \, l$ will be represented as $R_i | \; j \; k \; l$ in a partial preference table.

Proof. Let $\mathfrak{D}' \in \{\mathfrak{D}_M, \mathfrak{D}_{NOR}\}$. Assume that φ satisfies weak unanimity, weak competition sensitivity [weak resource sensitivity], and consistency, but that it is not a subsolution of the core. Thus, there exists $(N,R) \in \mathfrak{D}'$ and a matching $\mu \in \varphi(N,R)$ such that $\mu \notin core(N,R)$. By Lemma 2.3.3 [Lemma 2.3.4] (a) and (b), φ satisfies mutually best. By Lemma 2.3.2 (a) and (b), φ satisfies individual rationality. Hence, there exists a blocking pair $\{i,j\}$, $i \neq j$, for μ such that $j P_i \, \mu(i) R_i \, i$ and $i P_j \, \mu(j) R_j \, j$. Let $N' = \{i,j,\mu(i),\mu(j)\}$ and consider the reduced market $(N',R_{N'}) \in \mathfrak{D}'$ of (N,R) at μ. By consistency,

$$\mu_{N'} \in \varphi(N',R_{N'}). \tag{2.1}$$

We consider three cases depending on the cardinality of N'.

Case 1 (|N'|=2): Consider the reduced market $(N',R_{N'})$ with the set of agents $N' = \{i,j\}$ (note that $(N',R_{N'}) \in \mathfrak{D}_M$). Agents i and j are mutually best agents for $(N',R_{N'})$. However, at the reduced matching $\mu_{N'}$ they are not matched. Hence, (2.1) contradicts mutually best.

Case 2 (|N'|=3): Consider the reduced market $(N',R_{N'})$ with the set of agents $N' = \{i,j,\mu(i)\}$. It is without loss of generality that we assume that agent j is single. By individual rationality and $\{i,j\}$ being a blocking pair for μ, agents' preferences are as follows:

R_i	$j \; \mu(i) \; i$
$R_{\mu(i)}$	$i \; \mu(i)$
R_j	$i \; j$

Weak Competition Sensitivity Step (Lemma 2.4.1'). Assume that $(\hat{N}',\hat{R}')$, $\hat{N}' = N' \cup \{n\}$, is an extension of $(N',R_{N'})$ such that agents $\mu(i)$ and n are mutually best agents for $(\hat{N}',\hat{R}')$ and agent $\mu(i)$ is the only one that finds n acceptable and n finds only $\mu(i)$ acceptable [the newcomer and preferences can be chosen such that $(\hat{N}',R_{\hat{N}'})$ is also a marriage / no odd rings roommate market]. By mutually best, for all $\mu' \in \varphi(\hat{N}',\hat{R}')$, $\mu'(n) = \mu(i)$. By weak competition sensitivity, for $\mu_{N'} \in \varphi(N',R_{N'})$ there exists $\hat{\mu}' \in \varphi(\hat{N}',\hat{R}')$ such that agents i and j are not matched, i.e., $\hat{\mu}'(i) \neq j$ (if not, then agents i and j are newly matched at $\hat{\mu}'$, but both are better off). Hence, $\hat{\mu}' \in \varphi(\hat{N}',\hat{R}')$ is the matching that mutually best matches agents $\mu(i)$ and n and agents i and j are single.

Thus, $(\{i,j\},R_{\{i,j\}})$ is a reduced market of $(\hat{N}',\hat{R}')$ at $\hat{\mu}'$ to $\{i,j\}$. Note that i and j are mutually best agents at $(\{i,j\},R_{\{i,j\}})$ and both single at $\hat{\mu}'_{\{i,j\}}$. By consistency, $\hat{\mu}'_{\{i,j\}} \in \varphi(\{i,j\},R_{\{i,j\}})$, which contradicts mutually best.

Weak Resource Sensitivity Step (Lemma 2.4.2'). Consider the roommate market $(\{i,j\},R_{\{i,j\}})$. There exists a unanimously best complete matching $\bar{\mu}$ for (marriage, no odd rings) roommate market $(\{i,j\},R_{\{i,j\}})$: $\bar{\mu}$ matches agent i with agent j. Hence, by weak unanimity, $\varphi(\{i,j\},R_{\{i,j\}}) = \{\bar{\mu}\}$ and $\bar{\mu}(i) = j$.

Consider the extension $(N',R_{N'})$ of $(\{i,j\},R_{\{i,j\}})$ that is obtained by adding newcomer $\mu(i)$. Because $\mu(i) \neq j$ and $\bar{\mu}(i) = j$, by weak resource sensitivity, $\mu(i)\,P_i\,\bar{\mu}(i) = j$ or $\mu(j)\,P_j\,\bar{\mu}(j) = i$. This contradicts that $\{i,j\}$ is a blocking pair for μ.

Case 3 (|N'|=4): Consider the reduced market $(N',R_{N'})$ with the set of agents $N' = \{i,j,\mu(i),\mu(j)\}$. By individual rationality and $\{i,j\}$ being a blocking pair for μ, agents' preferences are as follows:

$$\begin{array}{c|c} R_i & j\ \mu(i)\ i \\ R_j & i\ \mu(j)\ j \\ R_{\mu(i)} & i\ \mu(i) \\ R_{\mu(j)} & j\ \mu(j) \end{array}$$

If agents i and j are mutually best agents for $(N',R_{N'})$, then (2.1) contradicts mutually best. If $(N',R_{N'})$ is a marriage market, then agents j and $\mu(i)$ and agents i and $\mu(j)$ have the same gender. But then, agents i and j are mutually best agents for $(N',R_{N'})$ that are not matched at $\mu_{N'}$; a contradiction. Hence, $\mathfrak{D}' = \mathfrak{D}_{NOR}$ and agents i and j not being mutually best agents for $(N',R_{N'})$ implies $\mu(j)\,P_i\,j\,P_i\,\mu(i)\,P_i\,i$ or $\mu(i)\,P_j\,i\,P_j\,\mu(j)\,P_j\,j$. Without loss of generality we assume that $\mu(i)\,P_j\,i\,P_j\,\mu(j)\,P_j\,j$. Thus, agents' preferences can be further restricted to:

R_i	$j\ \mu(i)\ i$
R_j	$\mu(i)\ i\ \mu(j)\ j$
$R_{\mu(i)}$	$i\ \mu(i)$
$R_{\mu(j)}$	$j\ \mu(j)$

Weak Competition Sensitivity Step (Lemma 2.4.1’). Assume that $(\hat{N}',\hat{R}')$, $\hat{N}' = N' \cup \{n\}$, is an extension of $(N',R_{N'})$ such that agents $\mu(j)$ and n are mutually best agents for $(\hat{N}',\hat{R}')$ and agent $\mu(j)$ is the only one that finds n acceptable and n finds only $\mu(j)$ acceptable [the newcomer and preferences can be chosen such that $(N',R_{N'})$ is a no odd rings roommate market]. By mutually best, for all $\mu' \in \varphi(\hat{N}',\hat{R}')$, $\mu'(n) = \mu(j)$. By weak competition sensitivity, for $\mu_{N'} \in \varphi(N',R_{N'})$ there exists $\hat{\mu}' \in \varphi(\hat{N}',\hat{R}')$ such that agents i and j are not matched, i.e., $\hat{\mu}'(i) \neq j$ (if not, then agents i and j are newly matched at $\hat{\mu}'$, but both are better off).

(*) If $j\,P_{\mu(i)}\,i\,P_{\mu(i)}\,\mu(i)$ and $\hat{\mu}'(j) = \mu(i)$, then agents j and $\mu(i)$ are newly matched and both better off; contradicting the choice of $\hat{\mu}'$ to satisfy weak competition sensitivity.

Consider the reduced market $(\{i,j,\mu(i)\},R_{\{i,j,\mu(i)\}}) \in \mathfrak{D}_{NOR}$ of $(\hat{N}',\hat{R}')$ at $\hat{\mu}'$.

R_i	$j\ \mu(i)\ i$
R_j	$\mu(i)\ i\ j$
$R_{\mu(i)}$	$i\ \mu(i)$

By weak competition sensitivity and consistency, we have that $\bar{\mu}' \equiv \hat{\mu}'_{\{i,j,\mu(i)\}} \in \varphi(\{i,j,\mu(i)\},R_{\{i,j,\mu(i)\}})$ such that $\bar{\mu}'(i) \neq j$.

We continue with the joint proof of Lemmas 2.4.1’ and 2.4.2’ after establishing a corresponding proof step for weak resource sensitivity.

Weak Resource Sensitivity Step (Lemma 2.4.2’). Consider the reduced market $(\{i,j,\mu(i)\},R_{\{i,j,\mu(i)\}}) \in \mathfrak{D}_{NOR}$ obtained from agent $\mu(j)$ leaving $(N',R_{N'})$.

R_i	$j\ \mu(i)\ i$
R_j	$\mu(i)\ i\ j$
$R_{\mu(i)}$	$i\ \mu(i)$

By weak resource sensitivity, for the matching $\mu_{N'} \in \varphi(N',R_{N'})$ there exists $\bar{\mu}' \in \varphi(\{i,j,\mu(i)\},R_{\{i,j,\mu(i)\}})$ such that agents i and j are not matched, i.e.,

$\bar{\mu}'(i) \neq j$ (if not, then agents i and j are not matched at $\mu_{N'}$ anymore, but both are worse off).

(*) If $j\, P_{\mu(i)}\, i\, P_{\mu(i)}\, \mu(i)$ and $\hat{\mu}'(j) = \mu(i)$, then agents j and $\mu(i)$ are not matched at $\mu_{N'}$ anymore and both are worse off; contradicting the choice of $\hat{\mu}'$ to satisfy weak resource sensitivity.

We now finish the proof of Lemmas 2.4.1' and 2.4.2' with a joint step. In the previous steps we have established the existence of

$$\bar{\mu}' \in \varphi(\{i,j,\mu(i)\}, R_{\{i,j,\mu(i)\}}) \text{ such that } \bar{\mu}'(i) \neq j.$$

If $\bar{\mu}'(i) = \mu(i)$, then $\{i,j\}$ is a blocking pair for $\bar{\mu}'$. If $\bar{\mu}'(i) = i$ and $\bar{\mu}'(j) = j$, then $\{i,j\}$ is a blocking pair for $\bar{\mu}'$. The remaining case to discuss is $\bar{\mu}'(i) = i$ and $\bar{\mu}'(j) = \mu(i)$. If $i\, P_{\mu(i)}\, j\, P_{\mu(i)}\, \mu(i)$, then $\{i,j,\mu(i)\}$ constitutes an odd ring; contradicting $(\{i,j,\mu(i)\}, R_{\{i,j,\mu(i)\}}) \in \mathfrak{D}_{NOR}$. By (*), $j\, P_{\mu(i)}\, i\, P_{\mu(i)}\, \mu(i)$ is not possible. Hence, $i\, P_{\mu(i)}\, \mu(i)\, P_{\mu(i)}\, j$ and $\bar{\mu}'(j) = \mu(i)$. However, $\bar{\mu}'(j) = \mu(i)$ violates individual rationality, a contradiction.

To summarize, we either obtain a contradiction, or blocking pair $\{i,j\}$ for $\bar{\mu}' \in \varphi(\{i,j,\mu(i)\}, R_{\{i,j,\mu(i)\}})$. Since $|\{i,j,\bar{\mu}'(i),\bar{\mu}'(j)\}| \leq 3$, Cases 1 and 2 now imply a contradiction. □

The following three-agent example demonstrates why Lemmas 2.4.1 and 2.4.1' might not hold if the set of potential agents is finite. Example 2.6.1 also illustrates why Lemma 2.3.3 and Theorem 2.4.1 might not hold if the set of potential agents is finite. Note that the simple idea of Example 2.6.1 (namely to add a non-core matching to all roommate markets containing the finite set of potential agents) can be extended to any finite set of potential agents (if the set of potential agents is even, then one should only add the additional matching for roommate markets without a unanimously best complete matching).

Example 2.6.1. *Assume that the set of potential agents is $\{1,2,3\}$ and denote by μ^{12} the matching where agents 1 and 2 are matched. Then, for all roommate markets $(N,R) \in \mathfrak{D}' \subseteq \mathfrak{D}_S$,*

$$\tilde{\varphi}(N,R) = \begin{cases} core(N,R) \cup \{\mu^{12}\} & \textit{if } |N| = 3 \textit{ and } \mu^{12} \in IR(N,R), \\ core(N,R) & \textit{otherwise.} \end{cases}$$

It is easy to check that $\tilde{\varphi}$ satisfies weak unanimity, (weak) competition sensitivity, and consistency, but it is not a subsolution of the core. $\diamond$

2.6.3 Proof of Proposition 2.4.1

Proposition 2.4.1. *On the domain of solvable roommate markets (and on any of its subdomains), solution $\hat{\varphi}$ (defined in Example 2.4.1) satisfies individual rationality, Pareto optimality, (weak) unanimity, mutually best, consistency, and weak resource sensitivity.*

Proof. We prove Proposition 2.4.1 for $\mathfrak{D}' = \mathfrak{D}_S$. It is easy to see that solution $\hat{\varphi}$ (defined in Example 2.4.1) satisfies individual rationality, Pareto optimality, (weak) unanimity, and mutually best.

We partition the domain of solvable roommate markets $\mathfrak{D}_S$ into the subdomain $\mathfrak{D}_{SS}$ of solvable roommate markets with the separable submarket $(\hat{N},\hat{R})$ and its complement solvable domain $\mathfrak{D}_{CS} = \mathfrak{D}_S \setminus \mathfrak{D}_{SS}$ (without separable submarket $(\hat{N},\hat{R})$).

Note that for roommate markets with a separable submarket ($\mathfrak{D}_{SS}$), the set of agents $\hat{N}$ constitutes an even ring of size 4. Hence removing an agent from this submarket dissolves the solvability of the market due to an odd ring at the top of preferences of 3 agents in $\hat{N}$. Therefore, it is not possible to obtain an extension $(N',R') \in \mathfrak{D}_{SS}$ of $(N,R) \in \mathfrak{D}_{CS}$ by adding a newcomer $n \in \mathbb{N}\backslash N$. Furthermore, since each agent in $\hat{N}$ finds only agents in $\hat{N}$ acceptable, for all extensions $(N',R') \in \mathfrak{D}_S$ of $(N,R) \in \mathfrak{D}_{SS}$ that are obtained by adding a newcomer $n \in \mathbb{N}\backslash N$, and for all $\mu' \in \hat{\varphi}(N',R')$, $\mu'(\hat{N}) = \hat{N}$.

Weak Resource Sensitivity. In order to show that $\hat{\varphi}$ is weakly resource sensitive, let $(N,R) \in \mathfrak{D}_S$ and consider the extension $(N',R') \in \mathfrak{D}_S$ of (N,R) obtained by adding a newcomer $n \in \mathbb{N}\backslash N$.

Case 1. Let $(N,R) \in \mathfrak{D}_{CS}$. It follows that $(N',R') \in \mathfrak{D}_{CS}$. By construction of $\hat{\varphi}$, $\hat{\varphi}(N,R) = core(N,R)$ and $\hat{\varphi}(N',R') = core(N',R')$. By Proposition 2.3.2, the core is weakly resource sensitive. Hence, for all $\mu' \in core(N',R') = \hat{\varphi}(N',R')$ there exists $\mu \in core(N,R) = \hat{\varphi}(N,R)$ such that for all $i,j \in N$ [possibly $i = j$] that are not matched at μ' anymore, at least one is better off.

Case 2. Let $(N,R) \in \mathfrak{D}_{SS}$ and assume $(N',R') \in \mathfrak{D}_{CS}$. By construction of $\hat{\varphi}$, $\hat{\varphi}(N,R) \supsetneq core(N,R)$ and $\hat{\varphi}(N',R') = core(N',R')$. By Proposition 2.3.2, the core is weakly resource sensitive. Hence, for all $\mu' \in core(N',R') = \hat{\varphi}(N',R')$ there exists $\mu \in core(N,R) \subsetneq \hat{\varphi}(N,R)$ such that for all $i,j \in N$ [possibly $i = j$] that are not matched at μ' anymore, at least one is better off.

Case 3. Let $(N,R) \in \mathfrak{D}_{SS}$ and assume $(N',R') \in \mathfrak{D}_{SS}$. Since each agent in $\hat{N}$ finds only agents in $\hat{N}$ acceptable, for all $\mu \in \hat{\varphi}(N,R)$, $\mu(\hat{N}) = \hat{N}$, and for all $\mu' \in \hat{\varphi}(N',R')$, $\mu'(\hat{N}) = \hat{N}$. Therefore, we treat the set of agents $\hat{N}$ separately from the set of agents $N \setminus \hat{N}$.

For agents in $\hat{N}$: Note that in both roommate markets (N,R) and (N',R'), $\hat{\varphi}$ matches agents in $\hat{N}$ according to $\hat{\mu}, \hat{\mu}'$, or $\hat{\mu}''$. Therefore, for all $\mu' \in \hat{\varphi}(N',R')$ there exists $\mu \in \hat{\varphi}(N,R)$ such that $\mu'_{\hat{N}} = \mu_{\hat{N}} \in \{\hat{\mu}, \hat{\mu}', \hat{\mu}''\}$. In particular, $\mu \in \hat{\varphi}(N,R)$ is such that for all $i,j \in \hat{N}$ [possibly $i = j$] that are not matched at μ' anymore, at least one is better off.

For agents in $N \setminus \hat{N}$: Note that in both roommate markets (N,R) and (N',R'), $\hat{\varphi}$ matches agents in $N \setminus \hat{N}$ (respectively $N' \setminus \hat{N}$) according to $\hat{\varphi}(N \setminus \hat{N}, R_{N \setminus \hat{N}}) = core(N \setminus \hat{N}, R_{N \setminus \hat{N}})$ (respectively $\hat{\varphi}(N' \setminus \hat{N}, R_{N' \setminus \hat{N}}) = core(N' \setminus \hat{N}, R_{N' \setminus \hat{N}})$). By Proposition 2.3.2, the core is weakly resource sensitive. Hence, for all $\mu' \in \hat{\varphi}(N',R')$, $\mu'_{N' \setminus \hat{N}} \in core(N' \setminus \hat{N}, R_{N' \setminus \hat{N}})$ and there exists a matching $\mu_{N \setminus \hat{N}} \in core(N \setminus \hat{N}, R_{N \setminus \hat{N}})$ such that for all $i,j \in N \setminus \hat{N}$ [possibly $i = j$] that are not matched at $\mu'_{N' \setminus \hat{N}}$ anymore, at least one is better off. In particular, we can choose $\mu \in \hat{\varphi}(N,R)$ such that $\mu'_{\hat{N}} = \mu_{\hat{N}}$.

Cases 1 - 3 imply that for all $(N,R),(N',R') \in \mathfrak{D}_S$ such that (N',R') is an extension of (N,R) obtained by adding a newcomer $n \in \mathbb{N} \setminus N$ and for all $\mu' \in \hat{\varphi}(N',R')$ there exists $\mu \in \hat{\varphi}(N,R)$ such that for all $i,j \in N$ [possibly $i = j$] that are not matched at μ' anymore, at least one is better off.

Consistency. In order to show that $\hat{\varphi}$ is consistent, let $(N,R) \in \mathfrak{D}_S$, $\mu \in \hat{\varphi}(N,R)$, and assume that $(N',R_{N'}) \in \mathfrak{D}_S$ is a reduced market of (N,R) at μ to N'.[19]

Case 1. Let $(N,R) \in \mathfrak{D}_{CS}$. By construction of $\hat{\varphi}$, $\hat{\varphi}(N,R) = core(N,R)$ and $core(N',R') \subseteq \hat{\varphi}(N',R')$. By Proposition 2.3.1, the core is consistent. Hence, for all $\mu \in core(N,R) = \hat{\varphi}(N,R)$, $\mu_{N'} \in core(N',R') \subseteq \hat{\varphi}(N,R)$.

[19]Note that solvability is closed under reduction for $\hat{\varphi}$, i.e., for all $(N,R) \in \mathfrak{D}_S$ and for all matchings $\mu \in \hat{\varphi}(N,R)$, the reduced market (N',R') of (N,R) at μ is also solvable.

Case 2. Let $(N,R) \in \mathfrak{D}_{SS}$. By construction of $\hat{\varphi}$, $\mu = (\mu^*, \tilde{\mu})$ for some $\mu^* = \mu_{N \setminus \hat{N}} \in core(N \setminus \hat{N}, R_{N \setminus \hat{N}})$ and for some $\tilde{\mu} = \mu_{\hat{N}} \in \{\hat{\mu}, \hat{\mu}', \hat{\mu}''\}$.

Case 2.1. Let $(N',R') \in \mathfrak{D}_{SS}$. Thus, $\hat{N} \subseteq N'$ and $\mu_{N'} = (\mu^*_{N' \setminus \hat{N}}, \tilde{\mu})$. By Proposition 2.3.1, the core is consistent. Hence, $\mu^*_{N' \setminus \hat{N}} \in core(N' \setminus \hat{N}, R_{N' \setminus \hat{N}})$. Thus, by construction of $\hat{\varphi}$, $\mu_{N'} = (\mu^*_{N' \setminus \hat{N}}, \tilde{\mu}) \in \hat{\varphi}(N',R')$.

Case 2.2. Let $(N',R') \in \mathfrak{D}_{CS}$. By construction of $\hat{\varphi}$, $\hat{\varphi}(N',R') = core(N',R')$. By Proposition 2.3.1, the core is consistent. Hence, since $\mu^* \in core(N \setminus \hat{N}, R_{N \setminus \hat{N}})$, $\mu^*_{N' \setminus \hat{N}} \in core(N' \setminus \hat{N}, R_{N' \setminus \hat{N}})$. Since $(N',R') \in \mathfrak{D}_{CS}$, either $\hat{N} \cap N' = \emptyset$ or $|\hat{N} \cap N'| = 2$. If $\hat{N} \cap N' = \emptyset$, then $N' \setminus \hat{N} = N'$, which implies $\mu_{N'} = \mu^*_{N' \setminus \hat{N}} \in core(N',R') = \hat{\varphi}(N',R')$. Assume that $|\hat{N} \cap N'| = 2$ and $\hat{N} \cap N' = \{i,j\}$. Since the reduced market (N',R') is obtained from $(N,R) \in \mathfrak{D}_{SS}$, $\tilde{\mu}(i) = j$. Furthermore, agents i and j are mutually best agents for (N',R'). Hence, for all $\mu' \in \hat{\varphi}(N',R') = core(N',R')$, $\mu'(i) = j$. Thus, by construction of $\hat{\varphi}$, $\mu_{N'} = (\mu^*_{N' \setminus \hat{N}}, \tilde{\mu}_{\{i,j\}}) \in core(N',R') = \hat{\varphi}(N',R')$. □

2.7 Appendix B: Consistency and Population Monotonicity for (Classical) Marriage Markets

A *classical marriage market* (Gale and Shapley, 1962) (N,R) is such that N is the union of two disjoint sets M and W and each agent in M (respectively W) has restricted preferences over being matched to agents in the set W (respectively M) and being single (instead of having preferences over N). Furthermore, matching agents of the same gender is not feasible (instead of matching agents of the same gender being individually irrational). Definitions and results labeled "on the *domain of (classical) marriage markets*" in this appendix apply to the classical marriage market domain as well as to our "marriage-roommate market" domain. We first introduce two population monotonicity properties.

Own-side population monotonicity (simply called population monotonicity by Toda, 2006) states that if additional men (women) enter the market, then all incumbent men (women) are weakly worse off. We formalize a somewhat weaker version of own-side population monotonicity by restricting population changes to one newcomer at a time (this "weak own-side population monotonicity" implies the original own-side population monotonicity by adding men (women) one by one).

Own-Side Population Monotonicity for Marriage Markets: A solution φ on the domain of (classical) marriage markets is *own-side population monotonic* if the following holds. Let (N,R) be a marriage market and assume that (N',R'), $N' = N \cup \{n\}$, is an extension of (N,R) and the newcomer n is a man [woman]. Then, for all $\mu \in \varphi(N,R)$ there exists $\mu' \in \varphi(N',R')$ such that for all men $m \in N$, $\mu(m) R_m \, \mu'(m)$ [for all women $w \in N$, $\mu(w) R_w \, \mu'(w)$].

Other-side population monotonicity states that if additional men (women) enter the market, then all incumbent women (men) are weakly better off. We formalize a somewhat weaker version of other-side population monotonicity by restricting population changes to one newcomer at a time (this "weak other-side population monotonicity" implies the original other-side population monotonicity by adding men (women) one by one).

Other-Side Population Monotonicity for Marriage Markets: A solution φ on the domain of (classical) marriage markets is *other-side population monotonic* if the following holds. Let (N,R) be a marriage market and assume that (N',R'), $N' = N \cup \{n\}$, is an extension of (N,R) and the newcomer n is a man [woman]. Then, for all $\mu' \in \varphi(N',R')$ there exists $\mu \in \varphi(N,R)$ such that for all women $w \in N$, $\mu'(w) R_w \, \mu(w)$ [for all men $m \in N$, $\mu'(m) R_m \, \mu(m)$].

Proposition 2.7.1. *On the domain of (classical) marriage markets, the core satisfies own-side and other-side population monotonicity.*

Proof. By Toda (2006, Proposition 2.1) the core satisfies own-side population monotonicity for the domain of classical marriage markets. However, we show both properties in a symmetric way using results by Crawford (1991). Note that the classical marriage market results used in this proof also hold for "marriage-roommate" markets (see Remark 2.2.1).

Let $(N,R),(N',R')$ be such that (N',R') is an extension of (N,R) with $N' = N \cup \{n\}$, without loss of generality newcomer n being a man. We denote the men-optimal stable matchings and the women-optimal stable matchings (Gale and Shapley, 1962) by μ_M, μ_W for roommate market (N,R) and μ'_M, μ'_W for roommate market (N',R').

By Roth and Sotomayor (1990, Corollary 2.14), for all $\mu \in core(N,R)$ and all men $m \in N$, $\mu R_m \, \mu_W$. By Crawford (1991, Theorem 1), for all men $m \in$

N, $\mu_W \, R_m \, \mu'_W$. Hence for all $\mu \in core(N,R)$, there exists $\mu' \in core(N',R')$, namely $\mu' \equiv \mu'_W$, such that for all men $m \in N$, $\mu R_m \, \mu'$. This implies own-side population monotonicity.

By Roth and Sotomayor (1990, Corollary 2.14), for all $\mu' \in core(N',R')$ and all women $w \in N$, $\mu' \, R_w \, \mu'_M$. By Crawford (1991, Theorem 2), for all women $w \in N$, $\mu'_M \, R_w \, \mu_M$. Hence for all $\mu' \in core(N',R')$, there exists $\mu \in core(N,R)$, namely $\mu \equiv \mu_M$, such that for all women $w \in N$, $\mu' \, R_w \, \mu$. This implies other-side population monotonicity. □

We next establish some relations between properties for marriage markets.

Lemma 2.7.1. *On the domain of (classical) marriage markets, weak unanimity, other-side population monotonicity, and consistency imply individual rationality.*

Proof. Assume that φ satisfies weak unanimity and consistency, but not individual rationality. Then, there exists a marriage market (N,R), a matching $\mu \in \varphi(N,R)$, and without loss of generality a man $m \in N$, such that $m \, P_m \, \mu(m)$ (alternatively we could assume that there exists a woman $w \in N$ such that $w \, P_w \, \mu(w)$).

Let $N' = \{m, \mu(m)\}$. Then, marriage market $(N', R_{N'})$ is a reduced market of (N,R) at μ and by consistency, $\mu_{N'} \in \varphi(N', R_{N'})$ and $\mu_{N'}(m) = \mu(m)$.

Let $\bar{N} = \{m\}$. Marriage market $(\bar{N}, R_{\bar{N}})$ is a one agent market with only one possible matching. Hence, $\varphi(\bar{N}, R_{\bar{N}}) = \{\bar{\mu}\}$ with $\bar{\mu}(m) = m$. Furthermore, (N',R') is an extension of $(\bar{N}, R_{\bar{N}})$ and the newcomer $\mu(m)$ is a woman. Thus, by other-side population monotonicity, for all $\mu' \in \varphi(N',R')$, $\mu'(m) \, R_m \, \bar{\mu}(m) = m$. This contradicts $\mu_{N'} \in \varphi(N', R_{N'})$ and $m \, P_m \, \mu_{N'}(m)$. □

Klaus (2011) shows that on the domain of marriage markets, weak resource sensitivity is essentially a weaker property than other-side population monotonicity (individual rationality is added to ensure that no two agents of the same gender are matched, see Remark 2.2.1).

Lemma 2.7.2 (Klaus, 2011, Lemma 2)**.** *On the domain of (classical) marriage markets, individual rationality and other-side population monotonicity imply weak resource sensitivity.*

Strictly speaking Klaus (2011) does not give the proof for classical marriage markets, but the proof of Lemma 2.7.2 is essentially the same as Klaus (2011, Lemma 2).

The following two lemmas by Toda (2006) are used in the proof of Theorem 2.3.2.

Lemma 2.3.5 (Toda, 2006, Lemma 3.4)**.** *On the domain of classical marriage markets, if a solution φ satisfies individual rationality, mutually best, and consistency, then it is a subsolution of the core.*

Lemma 2.7.3 (Toda, 2006, Lemma 3.6)**.** *On the domain of classical marriage markets, no proper subsolution of the core satisfies consistency.*

Next, we restate and prove Theorem 2.3.2.

Theorem 2.3.2. *On the domain of classical marriage markets, a solution satisfies weak unanimity, other-side population monotonicity, and consistency if and only if it equals the core.*

Proof. The core satisfies weak unanimity and consistency (Toda, 2006). By Proposition 2.7.1, the core is other-side population monotonic.

Let φ satisfy weak unanimity, other-side population monotonicity, and consistency. Then, by Lemma 2.7.1, φ is individually rational. Thus, by Lemma 2.7.2, φ satisfies weak resource sensitivity. Hence, by Lemma 2.3.4 (a) (the proof essentially remains the same on the domain of classical marriage markets), φ satisfies mutually best. Thus, by Lemma 2.3.5, φ is a subsolution of the core. Then, Lemma 2.7.3 implies that φ equals the core. □

We conclude this appendix by restating and proving Corollary 2.4.1. For completeness, we first state the following lemma.

Lemma 2.7.4 (Klaus, 2011, Lemma 1)**.** *On the domain of marriage markets, individual rationality and own-side population monotonicity imply weak competition sensitivity.*

Corollary 2.4.1 (*Two Characterizations of the Core for Marriage Markets*).
On the domain of marriage markets, a solution satisfies weak unanimity, consistency, and

(1) *own-side population monotonicity;*

(2) *other-side population monotonicity;*

if and only if it equals the core.

Proof. Let φ be a solution on the domain of marriage markets. Let φ be weakly unanimous, consistent, and (1) own-side population monotonic or (2) other-side population monotonic. By Lemma 2.7.1, φ is individually rational. Thus, (1) by Lemma 2.7.4, φ is weakly competition sensitive and (2) by Lemma 2.7.2, φ is weakly resource sensitive. Hence, (1) by Theorem 2.4.1 (a), φ equals the core and (2) by Theorem 2.4.2 (a), φ equals the core. □

Chapter 3

Update Monotone Preference Rules

3.1 Introduction

Consider a collective decision making problem where, by means of a preference correspondence[1], individual strict preferences, i.e. complete, transitive, and antisymmetric binary relations over a set of alternatives, are aggregated into a set of strict preferences[2]. For instance, to appoint a new dean, a committee of department representatives collectively ranks three candidates: A(lice), B(ill) and C(aroline). Let the outcome be such that A is ranked the best, B is second best and C is ranked the lowest. Suppose now that one of the representatives, preferring C above B and both these above A, is substituted by another member of that department with preference A above C above B. Clearly, the latter preference agrees on more ordered pairs of the outcome than the former does and it only differs with the outcome on pairs where the former differs with the outcome. So to speak, the latter preference is an "update" of the former towards the outcome. The question raised here is whether in the new hypothetical situation the outcome would still be A above B above C.

[1]Set valuedness of an aggregation rule may reflect the possible outcomes of neutral tie-breaking.

[2]This chapter is based on a paper by Can and Storcken (2011b).

Take a (preference) profile, i.e. a combination of individual preferences, and a possible outcome of a preference correspondence. By *updating*, we refer to transformations of the profile such that the individual preferences change only in parts where they differ from the given outcome. Therewith, these transformations yield profiles which are more similar to the outcome and which are, in a sense, updates towards it. A preference correspondence is said to be *update monotone* if the outcome is still chosen at such a transformation. *Strong update monotonicity*, moreover, requires that the outcomes at the updated profile form a subset of the previous set of outcomes.

For collective choice functions and choice correspondences (for simplicity referred to as choice rules) various monotonicity conditions have been studied intensively. Well-known impossibility theorems, such as in Muller and Satterthwaite (1977), express that only trivial choice rules such as dictatorial rules or constant rules are monotone. Therefore, imposing this type of condition to choice rules can seem too restrictive. Furthermore, when defining a monotonicity condition for a choice rule, the description of an increase in the support of a winning alternative is not free of contamination. For instance, extending the lower contour set of a winning alternative, as in Maskin monotonicity, assumes that these choice rules should not depend on how these lower contour sets are ordered individually. In the case of preference rules and update monotonicity, however, demanding that individual preferences can only change as far as they differ from the outcome is at least intuitively more sound. Furthermore monotonicity conditions have not yet been analyzed for preference rules.

In comparison to these monotonicity conditions for choice rules, we draw the conclusion that the ones presented here for preference rules seem less restrictive. Indeed, imposing update monotonicity on preference rules along with other conditions, which are satisfied by many well-known preference rules, yields characterizations of "Kemeny-Young" and "super majority" preference rules. The main reason for this is that compared to choice rules, the framework of preference rules allows for more detail in expressing monotonicity.

In the choice function framework it is also well-known that the aforementioned impossibility theorems based on monotonicity conditions correspond to some impossibility theorems based on strategy-proofness conditions. In the preference rule framework, Bossert and Storcken (1992) showed that there exists no coalitional strategy-proof welfare function, i.e., single valued preference correspondence, which is in addition nonimposed and weak extrema independent. However, for welfare functions update monotonicity is compatible with these two latter conditions as is discussed in Section 3.5.

Update monotonicity is discussed for two disjoint cases; the first in which we assume that preference rules are convex valued and the second where this is not required. In the first case we determine the class of Pareto optimal, neutral, replication invariant, convex valued and strongly update monotone preference rules. It consists of super majority rules. At these rules, depending on the number of alternatives and the number of individuals, a number k close to the number of agents n is fixed. Now these correspondences assign all preferences which extend all ordered pairs unanimously agreed upon by any set of at least k individuals. This result characterizes super majority rules as the only preference rules that satisfy the five conditions mentioned above. Pareto optimality, neutrality and replication invariance are conditions met by many well-known preference correspondences. Whenever set valued outcomes stem from tie breaking indifferences, the convexity requirement on these outcome sets means that indifferences are broken in every possible way. Many well-known preference correspondences are convex valued. The characterization of super majority rules therefore implies that many well-known rules, such as the Borda (1784) and the Copeland (1951) rule, are not strongly update monotone. Moreover, in deducing this characterization of super majority rules we have the following result based on Pareto optimality, anonymity, neutrality, convex valuedness and update monotonicity only. Consider profiles where the set of agents is partitioned into two groups such that all members in each group report the same preference and the two reported preferences differ on precisely one consecutively ordered pair of alternatives only. At these profiles, which in fact resemble a decision situation on two alternatives, the outcome is both of these preferences if no group has more than $\frac{m-1}{m}n$ agents, where m is the number of alternatives and n the number of agents. So, in case of 3 alternatives a $\frac{2}{3}$ majority is not decisive in such a situation. This result shows that many well-known rules do not satisfy update monotonicity. In the setting of convex valued preference rules, the strong version of update monotonicity is necessary for the characterization of super majority rules. The logical independence of these characterizing conditions is discussed by examples; and furthermore, the necessity of the stronger version of update monotonicity is shown.

Next, consider the situation of possibly non-convex valued outcomes. The Kemeny-Young preference correspondence (simply the Kemeny rule), assigning those strict preferences to a given profile which minimize the sum of Kemeny distance to all of the individual preferences in the profile, is update monotone. It is, however, not convex valued. For instance in case of three alternatives at a Condorcet profile, unlike many preference correspondences having all six strict preferences in the outcome, it only assigns the three preferences reported in the profile. The Kemeny rule is the only rule which is Pareto optimal, consistent, pairwise, neutral and update monotone. The former four conditions are well-known and satisfied by various preference rules. Comparing the two characterizations, we see a substitution of convexity by pairwiseness and a trade off between a strengthening of replication invariance to consistency and a weakening of update monotonicity.

Young and Levenglick (1978) characterized the Kemeny rule by neutrality, consistency and the Condorcet condition. A major step in their proof is that given a profile, outcomes assigned by rules satisfying these three conditions are determined by a pairwise difference matrix which records the numerical differences of pairwise comparisons in its cell. That is, if two profiles yield the same pairwise difference matrix, then the outcomes at these two profiles are equal. Herewith such rules are pairwise. A similar result can be found in Lemma 3.4.5. From that point on, because of the differences in the characterizing conditions, the line of arguments and, hence, the proofs differ completely. In addition to the logical independence of the five conditions we employ, we also show their logical equivalence to those used by Young and Levenglick

The chapter proceeds as follows: in Section 3.2, we provide some notation for our model and briefly discuss certain properties including update monotonicity, and provide examples of some rules to which we refer in the later sections. Section 3.3 is devoted to the class of convex valued rules. An intuition is provided on why many convex valued rules are not update monotone. Thereafter, we provide a characterization of the family of super majority correspondences as indicated above. Section 3.4 discusses preference correspondences where convex valuedness is dropped and here we provide a new characterization of the Kemeny rule based on update monotonicity. In Section 3.5, we discuss the findings, such as the logical independence of the characterizing conditions and their logical equivalence to the conditions of Young and Levenglick. Furthermore, some weakening and strengthening of update monotonicity are formulated and discussed briefly. Finally, we show that in the social welfare setting of Bossert and Storcken (1992), update monotonicity is less demanding than strategy-proofness.

3.2 The Model

3.2.1 Basic Notation

Let A be a finite set of alternatives with cardinality $\#A = m > 1$. Preferences are taken to be linear orders over the set of alternatives A. Let $\mathbb{L}$ denote the set of all preferences over A. Let $\mathcal{N}$ be a countable infinite set of potential agents. For non-empty and finite subsets N of $\mathcal{N}$ with cardinality n, $\mathbb{L}^N$ denotes the set of all preference profiles p, i.e., an n-dimensional vector of preferences where its i^{th} component $p(i)$ refers to individual i's preference. The restriction of profile p to a subset of agents, say S, is denoted by $p|_S$. For situations where two disjoint sets of agents $N, N' \in \mathcal{N}$, with preference profiles $p \in \mathbb{L}^N$, and $q \in \mathbb{L}^{N'}$ over the same set of alternatives are united, we interpret (p,q) as the union[3] of these two profiles, i.e., $(p,q) \in \mathbb{L}^{N \cup N'}$ with $(p,q)(i) = p(i)$ if $i \in N$ and $(p,q)(i) = q(i)$ if $i \in N'$. Furthermore, for a *coalition* S, a nonempty subset of N, and a linear order R, let R^S denote the profile p where $p(i) = R$ for all i in S. So, R^N denotes the the unanimous profile in which all agents in N have preference R. A *preference correspondence* or simply a *rule* φ, is a function such that for every finite and non-empty set $N \subset \mathcal{N}$, φ assigns a nonempty subset $\varphi(p)$ of $\mathbb{L}$, to each preference profile p in $\mathbb{L}^N$.

[3] We abuse notation by dropping parentheses whenever it is clear that we refer to the union of these two set of agents.

Let R be a linear order on A. Consider three different alternatives $a,b,c \in A$. Note that because of anti-symmetry of R, $(a,b) \in R$ means that a is strictly preferred to b, which hereafter will be denoted by "$.a.b. = R$". The case where a and b are *consecutive* at R, i.e., there is no alternative c that is ordered in between a and b, is denoted by $.ab. = R$. Notations like $.a.b.c. = R$ and $.ab.c. = R$ have the obvious interpretation. Furthermore $a... = R$ means that alternative a is ordered best at R and likewise $...b = R$ means that alternative b is ordered worst at R. Let P be a partial order, i.e. an anti-symmetric, reflexive, and transitive binary relation on A. Then, $\mathbb{L}_P = \{R \in \mathbb{L} : P \subseteq R\}$ denotes the set of linear extensions of P. To save parenthesis, we write $\mathbb{L}_{ab}$ instead of $\mathbb{L}_{\{(a,b)\}}$. Let $a_1a_2...a_{l-1}a_la_{l+1}...a_m = R$ and $a_1a_2...a_{l-1}a_{l+1}a_l...a_m = \overline{R}$. Then we say R (respectively $\overline{R}$) is an *elementary change* of $\overline{R}$ (R) in pair a_la_{l+1} ($a_{l+1}a_l$) or R and $\overline{R}$ form an elementary change (in a_l and a_{l+1}). Furthermore this elementary change is in position l, i.e., the l^{th} and $(l+1)^{th}$ alternatives are swapped.

Given a nonempty subset of alternatives $B \subseteq A$, let $R|_B$ denote the *restriction* of an order R to B, i.e., $R|_B = \{(x,y) \in R : x,y \in B\}$. In a similar way let $p|_B$ be the *restriction of a profile* p to B, i.e., $(p|_B)(i) = p(i)|_B$ for every agent i in N.

For preferences R^1, R^2, and R^3, we say R^3 is *between*[4] R^1 and R^2 if and only if $R^1 \cap R^2 \subseteq R^3$. The Kemeny distance (Kemeny, 1959; Kemeny and Snell, 1962) for two linear orders R^1 and R^2 is defined by $\delta(R^1,R^2) = \frac{1}{2}\#((R^1-R^2)\cup(R^2-R^1)) = \#(R^1-R^2)$, i.e., half of the cardinality of the symmetric difference of R^1 and R^2. Note that δ is a distance function and hence satisfies the triangular inequality. Furthermore, for any three linear orders; R^1, R^2 and R^3, we have:

$$\delta(R^1,R^3)+\delta(R^3,R^2) = \delta(R^1,R^2) \text{ if and only if } R^3 \text{ is between } R^1 \text{ and } R^2.$$

Finally let $\mathbb{S}$ be a nonempty subset of $\mathbb{L}$. $\mathbb{S}$ is *convex* if for all R^1 and R^2 in $\mathbb{S}$, and for all R^3 in $\mathbb{L}$, R^3 is also in $\mathbb{S}$ if R^3 is between R^1 and R^2. In Bogart (1973) there is a discussion of betweenness and the concept of convexity. In Storcken (2008), the following characterization can be found:

[4]For general relations also the condition $R^3 \subseteq R^1 \cup R^2$ is needed. In case of linear orders, this, however, is equivalent to $R^1 \cap R^2 \subseteq R^3$.

Proposition 3.2.1. *Let $\mathbb{S}$ be a subset of $\mathbb{L}$. Then the following two are equivalent:*

(i) *$\mathbb{S}$ is convex,*

(ii) *There is a partial order P such that $\mathbb{L}_P = \mathbb{S}$.*

3.2.2 Properties of Rules

Next we discuss some properties for preference correspondences that we will use in the following sections.

Single valuedness: A rule is single valued if the outcome at every profile is a singleton, i.e. is a set of precisely one linear order. Clearly, this condition means that the rule is a function. In the literature these functions are also known as welfare functions.

Pareto Optimality: The Pareto condition requires that all preferences assigned to a profile are Pareto optimal. Formally: $\varphi(p) \subseteq \mathbb{L}_{\cap\{p(i):i\in N\}}$ for all profiles p in $\mathbb{L}^N$.

Anonymity: Anonymity requires that all individuals are treated equally; hence, renaming them should not change the outcome, i.e. $\varphi(p) = \varphi(p \circ \pi)$ for all profiles p in $\mathbb{L}^N$ and all permutations π of N, where $p \circ \pi$ is the profile such that $p \circ \pi(i) = p(\pi(i))$ for all $i \in N$.

Neutrality: A rule is said to be neutral whenever it treats alternatives in a neutral way: $\tau(\varphi(p)) = \varphi(\tau p)$ for all profiles p in $\mathbb{L}^N$ and all permutation τ of A, where τ extends to any relation R on A by $\tau(R) = \{(\tau(a), \tau(b)) : (a,b) \in R\}$, $\tau(\mathbb{S}) = \{\tau(R) : R \in \mathbb{S}\}$ denotes the complete image of a set $\mathbb{S}$ under τ and τp is defined for an individual i by $(\tau p)(i) = \tau(p(i))$.

Replication Invariance: Replication invariance means that the outcome of a rule is the same before and after the replication of a profile: $\varphi(p) = \varphi(p^1, p^2, ..., p^k)$ for all profiles p, $p^1, p^2, ...$ and p^k in respectively $\mathbb{L}^N, \mathbb{L}^{N_1}, \mathbb{L}^{N_2}...$ and $\mathbb{L}^{N_k}$ such that N, $N_1, N_2...N_k$ are all pairwise disjoint and there are bijections σ_t from N to N_t for all $t \in \{1,2,...k\}$ such that $p(i) = p^t(\sigma_t(i))$ for all i in N and $(p^1, p^2, ..., p^k)$ is the profile say q on $N_1 \cup N_2 \cup ...N_k$, with $q(i) = p^t(i)$ for all $i \in N_t$.

Consistency: Consistency requires that the outcome assigned to a profile, which is composed of two disjoint sets of individuals that are merged, equals the intersection of the outcomes assigned to the profiles of these sets of individuals separately, whenever this intersection is non-empty. Formally: $\varphi(p)\cap\varphi(q)=\varphi(p,q)$ for all profiles p and q in respectively $\mathbb{L}^N$ and $\mathbb{L}^M$ where N and M are disjoint and $\varphi(p)\cap\varphi(q)\neq\emptyset$.

Pairwiseness: A rule is pairwise if the outcomes at two profiles are equal whenever every pairwise comparison in a profile is numerically equal to that in the other profile. Formally: $\varphi(p)=\varphi(q)$ for all profiles p and q in $\mathbb{L}^N$ such that $M(p)=M(q)$, where $M(p)$ is a m by m matrix such that for alternatives a and b, cell (a,b) is defined by the number of agents preferring a to b:

$$\begin{aligned} M(p)_{(a,b)} &= \#\{i\in N:(a,b)\in p(i)\} \text{ if } a\neq b \\ &= 0 \text{ if } a=b. \end{aligned}$$

Convex valuedness: A rule satisfies convex valuedness if it assigns to each profile a convex set of linear orders: $\varphi(p)$ is convex for all profiles p in $\mathbb{L}^N$.

Remark 3.2.1. *Note that replication invariance implies anonymity, and consistency implies replication invariance.* △

Below we provide some examples of convex valued rules:

Example 3.2.1. *(**Score rules**) Let $\overrightarrow{s}=(s_1,...s_m)$ be the score vector over the set of alternatives such that $s_1\geq...\geq s_m$ and $s_1>s_m$. Let $rank(a,p(i))=\#\{b\in A:.b.a.=p(i)$ or $a=b\}$, i.e. the rank of alternative a in i^{th} preference $p(i)$ equals the the number of alternatives preferred or indifferent to a. For each alternative $a\in A$, let $score\left(\overrightarrow{s},a,p\right)=\sum_{i\in N}s_{rank(a,p(i))}$ be the sum of scores of alternative a in each individual preference $p(i)$ in the profile p. Consider the partial order $P_{\overrightarrow{s}}(p)=(a,b):score\left(\overrightarrow{s},a,p\right)>score\left(\overrightarrow{s},b,p\right)$ or $a=b\}$. The score rule $\varphi_{\overrightarrow{s}}$ induced by the score vector $\overrightarrow{s}$ is defined for profile p by*

$$\varphi_{\overrightarrow{s}}(p)=\mathbb{L}_{P_{\overrightarrow{s}}(p)},$$

where $\mathbb{L}_{P_{\overrightarrow{s}}(p)}=\left\{R\in\mathbb{L}:P_{\overrightarrow{s}}(p)\subseteq R\right\}$. ⋄

Example 3.2.2. *(**Copeland rule**) Let $s_C(a,p)$ denote the Copeland score of alternative a in preference profile p which is the number of alternatives a beats in pairwise comparisons, i.e., $\#\{b\in A\setminus\{a\}:\#\{i\in N:.a.b.=p(i)\}>n/2\}$. Consider the partial order $P_c(p)=\{(a,b):s_C(a,p)>s_C(b,p)$ or $a=b\}$. The Copeland rule φ_C is defined for profile p by*

$$\varphi_C(p)=\mathbb{L}_{P_c(p)},$$

where $\mathbb{L}_{P_c(p)} = \{R \in \mathbb{L} : P_c(p) \subseteq R\}$. ◇

Note that by Proposition 3.2.1, score rules and the Copeland rule, as defined in the examples above, satisfy convex valuedness.

Example 3.2.3. ***(Pareto rule)*** *For each profile* p*, the Pareto rule* φ_{Par} *is defined* $\varphi_{Par}(p) = \mathbb{L}_{\cap\{p(i):i \in N\}}$. ◇

Example 3.2.4. ***(k-majority rule)*** *Let* A *consist of* m *alternatives and* N *be a set of* n *agents. Let* k *be a positive integer such that* $\frac{m-1}{m}n < k \leq n$*. For any profile* p *define* k*-majority-relation*$(p) = \{(x,y) \in A \times A$: *there are at least* k *agents* i *such that* $.x.y. = p(i)\}$ *and define the* k*-majority rule for a profile* p *by* $\varphi_k(p) = \mathbb{L}_{k-majority-relation(p)}$. *Since* k *is strictly larger than* $\frac{m-1}{m}n$*, the* k*-majority-relation*(p) *is acyclic. Indeed a cycle say* (a_1,a_2), $(a_2,a_3),\ldots,(a_{l-1},a_l),(a_l,a_1)$ *in the* k*-majority-relation*(p) *would imply the existence of subsets* $S^1,S^2,\ldots,S^{l-1},S^l$ *of* N *such that* $\#S^j \geq k$ *and* $S^j = \{i \in N : .a_j.a_{j+1}. = p(i)\}$ *for* $j \in \{1,\ldots,l\}$ *and where further* a_{l+1} *is set equal to* a_1*. As preferences* $p(i)$ *are transitive and therefore in particular acyclic it follows that* $\cap\{S^j : j \in \{1,\ldots,l\}\} = \emptyset$*. But as* $\#S^j \geq k > \frac{m-1}{m}n \geq \frac{l-1}{l}n$*, because* $l \leq m$*, this result would contradict Proposition 3.6.1 (see the Appendix). Therefore* $\mathbb{L}_{k-majority-relation(p)}$ *is non-empty and* φ_k *is well-defined. Rule* φ_k *is convex valued by definition and it is obvious that it is neutral, Pareto optimal and anonymous. Note that in case* $k = n$ *rule* φ_k *equals the Pareto rule,* φ_{Par}*. Further note that* $\mathbb{L}_{k-majority-relation(p)} = \{R \in \mathbb{L} : \cap\{p(i) : i \in S\} \subseteq R$ *for all* $\#S \geq k\}$. ◇

3.2.3 Update Monotonicity

Consider a profile p and a linear order R. We call profile q an *update of* p *towards* R if $p(i) \cap R \subseteq q(i)$ for all agents $i \in N$. So, for all agents i this means that R and $q(i)$ have at least in common what $p(i)$ and R have. Or to put it differently, for all agents i, preference $q(i)$ only differs from R on pairs where $p(i)$ differs from R. Loosely speaking, this boils down to $q(i)$ and R having more pairs in common than $p(i)$ and R have. Note that in that case $q(i)$ is between $p(i)$ and R for all agents i. Based on this update, the following notions of monotonicity are defined.

(Update) Monotonicity: A rule φ is update monotone if for all p in $\mathbb{L}^N$, for all $R \in \varphi(p)$, and for all updates q of p towards R,

$$R \in \varphi(q).$$

Furthermore, φ is strongly update monotone if it is update monotone and $\varphi(q) \subseteq \varphi(p)$ for all such outcomes R in $\varphi(p)$ and profiles q.

From this point, on we use the word monotonicity instead of update monotonicity whenever it is clear that we mean the latter. Note that betweenness implies that the Kemeny distance between the collective preference R and each individual preference is not increased from profile p to profile q. Indeed monotonicity can be defined in a stronger way by only demanding that these distances do not increase and therewith dropping the betweenness condition. In Section 3.5 this is discussed in more detail.

Because the set of profiles is connected by elementary changes we have the following immediate result.

Proposition 3.2.2. *A rule φ is monotone if $\mathbb{L}_{ab} \cap \varphi(p) \subseteq \varphi(q)$ for all profiles $p, q \in \mathbb{L}^N$, such that $\mathbb{L}_{ab} \cap \varphi(p) \neq \emptyset$ and there are agents $j \in N$ and alternatives $a, b \in N$ with $p|_{N-\{j\}} = q|_{N-\{j\}}$, $.ab. = q(j)$, $.ba. = p(j)$ and $q(j)$ is an elementary change of $p(j)$ in pair ab. Furthermore, φ is strongly monotone if and only if it is monotone and $\varphi(q) \subseteq \varphi(p)$ for such p, q and alternatives a and b.*

Remark 3.2.2. *As the empty set is contained in any set, it follows immediately that φ is monotone if and only if $\mathbb{L}_{ab} \cap \varphi(p) \subseteq \varphi(q)$ for all profiles $p, q \in \mathbb{L}^N$, such that there are agents $j \in N$ and alternatives $a, b \in N$ with $p|_{N-\{j\}} = q|_{N-\{j\}}$, $.ab. = q(j)$, $.ba. = p(j)$ and $q(j)$ is an elementary change of $p(j)$ in pair ab. The requirement $\mathbb{L}_{ab} \cap \varphi(p)$ being non-empty is important, though more for strong monotonicity. Actually, monotonicity implies both $\mathbb{L}_{ab} \cap \varphi(p) \subseteq \varphi(q)$ and $\mathbb{L}_{ba} \cap \varphi(q) \subseteq \varphi(p)$. If $\mathbb{L}_{ab} \cap \varphi(p)$ is non-empty, then $\varphi(q) \subseteq \varphi(p)$. If in addition $\mathbb{L}_{ba} \cap \varphi(q) = \emptyset$, a situation which holds for the Kemeny rule, we have $\mathbb{L}_{ab} \cap \varphi(p) \subseteq \varphi(q)$, $\varphi(q) \subseteq \varphi(p)$ and $\varphi(q) \subseteq \mathbb{L}_{ab}$. Hence, then $\mathbb{L}_{ab} \cap \varphi(p) = \varphi(q)$. Note also that a rule φ is strongly monotone if for all alternative a and b and all profiles like p and q we have $\mathbb{L}_{ab} \cap \varphi(p) = \varphi(q)$ whenever $\mathbb{L}_{ab} \cap \varphi(p)$ is non-empty.* $\triangle$

3.3 Convex Valued Rules

As pointed out in the introduction and in the examples of the previous section, many preferences rules, such as score rules, the Kramer rule and the Copeland rule, are convex valued. In this section, we discuss the consequences of strong monotonicity under the assumption of convex valued outcomes. To structure this discussion further, we only consider rules satisfying the following basic conditions: Pareto optimality, neutrality and replication invariance. The latter condition is a strengthening of anonymity and links different sets of agents. These five conditions together characterize the class of super majority rules. To each profile, these rules assign all linear orders that extend the ordered pairs for which there are at least k agents who unanimously agree upon them. The number k is chosen in such a way that the pairs for the possibly different sets of k agents cannot form a cyclical decision. Therefore k depends on the number of alternatives and the number of agents.

To illustrate how convex valuedness and monotonicity under the other three conditions bring about this characterization, consider the outcome of an arbitrary rule φ satisfying these conditions at a standard Condorcet profile p for three agents 1,2, and 3 and three alternatives a, b, and c defined by:

$$\begin{aligned} p(1) &= abc \\ p(2) &= bca \\ p(3) &= cab. \end{aligned}$$

Here preferences are denoted by their representation. So, abc denotes the linear order at which a is the most preferred, b the second most and c is considered the worst. Assume $abc \in \varphi(p)$. Then by neutrality $bca, cab \in \varphi(p)$. Now $abc, bca, cab \in \varphi(p)$ and convex valuedness imply $\varphi(p) = \mathbb{L}$. Assume $acb \in \varphi(p)$. Then by neutrality $bac, cba \in \varphi(p)$. Now $acb, bac, cba \in \varphi(p)$ and convex valuedness again imply $\varphi(p) = \mathbb{L}$. This means that such rules will assign the set of all possible preferences to the Condorcet profile p. In particular $bac \in \varphi(p)$. Next, consider the preference profile q defined by

$$\begin{aligned} q(1) &= abc = p(1) \\ q(2) &= bac \\ q(3) &= abc = p(1). \end{aligned}$$

Note that q is an update of p towards both bac and abc. Because of monotonicity, φ therefore assigns abc and bac to profile q, which by Pareto optimality then implies that $\varphi(q) = \{abc, bac\}$. Note that bac is among the outcome at profile p where only one-third of the population preferences is in line with this ranking and two-third is opposing it. Many commonly used rules, however, take only the two third majority point of view abc at this profile q. This means that these rules do not satisfy all five conditions. In fact they defect on monotonicity.

The question now is "how large a coalition, say S, should be" in order to ensure that φ only assigns the opinion of this coalition S and therewith ignores that of the minority $N - S$. The example above clarifies that in case of three alternatives S should consist of strictly more than two third of the agents present. Indeed, this makes sense for k-majority rules, because in the three alternatives three agents case letting $k = 2$ would yield a cycle since two agent coalition $\{1,2\}$ is unanimous on bc, coalition $\{2,3\}$ on ca and coalition $\{1,3\}$ on ab. But this cycle cannot be extended to a linear order.

Unless stated otherwise, for the remainder of this section, we switch to the stronger version of monotonicity. This is because we were not able to determine the class of Pareto optimal, neutral, replication invariant, convex valued and monotone rules. It is clear that this class is larger than that of super majority rules. For instance, biased super majority rules, introduced in the discussion section, are not strongly monotone but they are monotone.

The class of Pareto optimal, neutral, convex valued, replication invariant and strongly monotone rules is described by some threshold function g which assigns to every number of agents n a minimal number of agents $g(n)$ needed to form a decisive coalition. Formally, the *super majority rule* φ_g is defined for every profile p in $\mathbb{L}^N$ by:

$$\varphi_g = \mathbb{L}_{g(\#N)-majority-relation(p)}.$$

Here g is a so-called *threshold function* from $\mathbb{N}-\{0\}$ to $\mathbb{N}-\{0\}$ defined by the following two conditions:

$$\begin{array}{lll} a) & \frac{m-1}{m} < \frac{g(n)}{n} \leq 1 & \text{for all } n \text{ in } \mathbb{N}, \\ b) & \frac{g(n)-1}{n} < \frac{g(kn)}{kn} \leq \frac{g(n)}{n} & \text{for all } k \text{ and } n \text{ in } \mathbb{N}. \end{array}$$

Clearly φ_g is well-defined as by condition a), $g(\#N) \leq \#N$ and further $\frac{m-1}{m} \cdot \#N < g(\#N)$ implies that $\mathbb{L}_{g(\#N)-majority-relation(p)}$ is non-empty. See also Example 3.2.4 and Proposition 3.6.1. By condition b), minimality of the threshold is carried to different numbers of agents. So, φ_g assigns to a profile p in $\mathbb{L}^N$, the linear extensions of all pairs of alternatives on which at least $g(\#N)$ agents in N unanimously agree. The fraction $\frac{g(n)}{n}$ can be seen as a minimal fraction for a coalition to be decisive. That is $\frac{g(k)-1}{k} < \frac{g(l)}{l}$ for all numbers k and l in $\mathbb{N}$. Indeed by condition b) on g it follows that $\frac{g(k)-1}{k} < \frac{g(kl)}{kl} \leq \frac{g(k)}{k}$ and $\frac{g(l)-1}{l} < \frac{g(kl)}{kl} \leq \frac{g(l)}{l}$ for all such numbers k and l. Hence, $\frac{g(k)-1}{k} < \frac{g(kl)}{kl} \leq \frac{g(l)}{l}$ and therewith $\frac{g(k)-1}{k} < \frac{g(l)}{l}$ for all numbers k and l. It is therefore tempting to define φ_g on the basis of $\beta = \inf\{\frac{g(n)}{n} : n \in \mathbb{N}\}$, and assigning to a profile p in $\mathbb{L}^N$ the convex set $\mathbb{L}_{k-majority-relation(p)}$ for k such that $\frac{k}{\#N} \geq \beta$. But also assigning $\mathbb{L}_{l-majority-relation(p)}$ for l such that $\frac{l}{\#N} > \beta$ would yield a rule satisfying the five conditions mentioned above. In case β is rational these two definitions yield (slightly) different rules. Therefore to capture them both in one formulation we chose for the description based on a function like g.

Clearly, by definition, φ_g is Pareto optimal and neutral. As $\mathbb{L}_{\cap\{p(i):i\in S\}}$ is convex by Lemma 3.2.1 and the fact that intersection of convex sets is convex, it follows that φ_g is convex valued. Next, we argue that φ_g is replication invariant.

Lemma 3.3.1. *φ_g is replication invariant.*

Proof. Consider profile p in $\mathbb{L}^N$ and its l^{th} replication $q = (p^1, p^2, ..., p^l)$ in $\mathbb{L}^{N_1\cup N_2\cup\ldots\cup N_l}$ for some integer $l \geq 2$. It is sufficient to prove that $\varphi_g(p) = \varphi_g(q)$. That is $g(\#N)$-majority-relation(p) = $g(l \cdot \#N)$-majority-relation(q). Since $\frac{g(l\cdot\#N)}{l\cdot\#N} \leq \frac{g(\#N)}{\#N}$, it follows that $g(\#N)$-majority-relation(p) $\subseteq g(l \cdot \#N)$-majority-relation(q). Indeed if $S = \{i \in N : .a.b. = p(i)\}$ for different alternatives a and b such that $\#S \geq g(\#N)$ and therewith $(a,b) \in g(\#N)$-majority-relation(p), then we may identify S_1 up to S_l in respectively N_1 up to N_l such that $\#S = \#S_j$ and $.a.b. = q(i)$ for all $j \in \{1,...,l\}$ and all $i \in S_j$. Therefore

$\#\{i \in N_1 \cup N_2 \cup \ldots \cup N_l : .a.b. = q(i)\} \geq l \cdot \#S = l \cdot g(\#N)$ and $(a,b) \in g(l \cdot \#N)$-majority-relation(q). Further, because of $\frac{g(\#N)-1}{\#N} < \frac{g(l \cdot \#N)}{l \cdot \#N}$, $g(l \cdot \#N)$-majority-relation$(q) \subseteq g(\#N)$-majority-relation(p). To see this let $\widetilde{S} = \{i \in N_1 \cup N_2 \cup \ldots \cup N_l : .a.b. = q(i)\}$ and $\#\widetilde{S} \geq l \cdot g(\#N)$. Because q is the l^{th} replica of p, $\#S_j = \frac{1}{l} \cdot \#\widetilde{S}$, for $S_j = \widetilde{S} \cap N_j$ and $j \in \{1,2,\ldots,l\}$. So, $\frac{\#S_j}{\#N} \geq \frac{g(l \cdot \#N)}{l \cdot \#N} > \frac{g(\#N)-1}{\#N}$ and therewith $\#S_j > g(\#N) - 1$. Hence, $\#S_j \geq g(\#N)$ which implies $(a,b) \in g(\#N)$-majority-relation(p). □

Lemma 3.3.2. *φ_g is strongly monotone.*

Proof. Consider profile p in $\mathbb{L}^N$ and any R in $\varphi_g(p)$. Consider any update q of p towards R such that for some agent j in N we have $p(i) = q(i)$ for all i in $N - \{j\}$ and, $q(j)$ is an elementary change of $p(j)$ in pair ab with $.a.b. = R$, $.ab. = q(j)$ and $.ba. = p(j)$. In view of Proposition 3.2.2 it is sufficient to prove that $R \in \varphi_g(q)$ and $\varphi_g(q) \subseteq \varphi_g(p)$. Now let $S \subseteq N$ with $\#S \geq g(\#N)$. Because of $R \in \varphi_g(p)$ it follows that $\cap\{p(i) : i \in S\} \subseteq R$ and as $.a.b. = R$ we have $(b,a) \notin \cap\{p(i) : i \in S\}$. So, by the choice of q

$$\begin{aligned}
\cap\{p(i) &: i \in S\} \subseteq \cap\{q(i) : i \in S\} \text{ and} \\
\cap\{q(i) &: i \in S\} \subseteq (\cap\{p(i) : i \in S\} \cup \{(a,b)\}).
\end{aligned}$$

As $.a.b. = R$ it follows that $\cap\{q(i) : i \in S\} \subseteq R$. Because this holds for arbitrary S, such that $\#S \geq g(\#N)$, we have that $R \in \varphi_g(q)$. To prove $\varphi_g(q) \subseteq \varphi_g(p)$ let $\overline{R} \in \varphi_g(q)$. It is sufficient to prove $\overline{R} \in \varphi_g(p)$. As $\cap\{p(i) : i \in S\} \subseteq \cap\{q(i) : i \in S\}$ for all $S \subseteq N$, with $\#S \geq g(\#N)$, we have by definition of φ_g that $\overline{R} \in \varphi_g(p)$. □

All of the above makes it clear that the rule φ_g is Pareto optimal, neutral, convex valued, replication invariant and strongly monotone. Next, we prove that only super majority rules satisfy these five conditions simultaneously. In the following two lemmas we provide some insight about coalitional power at profiles composed of two preferences which form an elementary change. Actually these profiles resemble collective decision situations between two alternatives, the ones on which the elementary change is based. In the following Lemmas we shall prove that on these profiles Pareto optimal, neutral, replication invariant, convex valued and strongly monotone rules φ are equal to a super majority rule φ_g for some function g. Thereafter, we show in Theorem 3.3.1 that this result expands to all profiles hence that $\varphi = \varphi_g$. Lemma 3.3.3 and 3.3.5 are based on the weak version of monotonicity and are also used in the following Section 3.4. Therefore, for now let rule φ be Pareto optimal, neutral, replication invariant, convex valued and monotone.

Lemma 3.3.3. *Let $\overline{R}$ be an elementary change of R in pair ab. Let $S \subseteq N_1$ and $T \subseteq N_2$ be coalitions such that $\frac{1}{2} \leq \frac{\#S}{\#N_1} \leq \frac{\#T}{\#N_2}$. Then:*

(a) $R \in \varphi(R^S, \overline{R}^{N_1-S})$ *(S is decisive; fraction $\frac{\#S}{\#N_1}$ is decisive);*

(b) $\varphi(R^S, \overline{R}^{N_1-S}) = \{R\}$ *implies* $\varphi(R^T, \overline{R}^{N_2-T}) = \{R\}$ *(if fraction $\frac{\#S}{\#N_1}$ is strictly decisive, then any greater fraction $\frac{\#T}{\#N_2}$ is also strictly decisive)*

(c) $\varphi(R^T, \overline{R}^{N_2-T}) = \{R, \overline{R}\}$ *implies* $\varphi(R^S, \overline{R}^{N_1-S}) = \{R, \overline{R}\}$ *(if fraction $\frac{\#T}{\#N_2}$ is not strictly decisive, then any smaller fraction $\frac{\#S}{\#N_1}$ larger than or equal to a half is not strictly decisive).*

Proof. (a) Let $2 \cdot \#S \geq \#N_1$. Let U be a coalition in N_1 such that $N_1 - S \subseteq U$ and $\#S = \#U$. Pareto optimality implies $\varphi(R^S, \overline{R}^{N_1-S}) \in \{\{R\}, \{\overline{R}\}, \{R, \overline{R}\}\}$. Suppose $\overline{R} \in \varphi(R^S, \overline{R}^{N_1-S})$. It is sufficient to prove that $R \in \varphi(R^S, \overline{R}^{N_1-S})$. Considering the permutation τ on A such that $\tau(x) = x$ for all $x \in A - \{a, b\}$, $\tau(a) = b$ and $\tau(b) = a$ it follows that $\tau R = \overline{R}$ and $\tau\overline{R} = R$. Now neutrality and $\overline{R} \in \varphi(R^S, \overline{R}^{N_1-S})$ implies $R = \tau\overline{R} \in \varphi(\tau R^S, \tau\overline{R}^{N_1-S}) = \varphi(\overline{R}^S, R^{N_1-S})$. So, monotonicity implies $R \in \varphi(\overline{R}^{N_1-U}, R^U)$. Now, anonymity which is implied by replication invariance, see Remark 3.2.1, yields $R \in \varphi(R^S, \overline{R}^{N_1-S})$.

(b) Let $\varphi(R^S, \overline{R}^{N_1-S}) = \{R\}$. Consider N_3 and V and W two subsets of N_3 such that $\#N_3 = \#N_1 \cdot \#N_2$, $\#V = \#N_2 \cdot \#S$ and $\#W = \#N_1 \cdot \#T$. Replication invariance now implies $\varphi(R^V, \overline{R}^{N_3-V}) = \{R\}$ and as $\#N_2 \cdot \#S \leq \#N_1 \cdot \#T$ monotonicity requires $R \in \varphi(R^W, \overline{R}^{N_3-W})$ and as $\overline{R} \notin \varphi(R^V, \overline{R}^{N_3-V})$ monotonicity and anonymity imply $\overline{R} \notin \varphi(R^W, \overline{R}^{N_3-W})$. So, $\varphi(R^W, \overline{R}^{N_3-W}) = \{R\}$ and replication invariance implies $\varphi(R^T, \overline{R}^{N_2-T}) = \{R\}$.

(c) Let $\varphi(R^T, \overline{R}^{N_2-T}) = \{R, \overline{R}\}$. Then part (b) implies $\varphi(R^S, \overline{R}^{N_1-S}) \neq \{R\}$. Hence Pareto optimality yields $\varphi(R^S, \overline{R}^{N_1-S}) = \{\overline{R}\}$ or $\varphi(R^S, \overline{R}^{N_1-S}) = \{R, \overline{R}\}$. Because of part (a) the latter part of this disjunction holds. □

Remark 3.3.1. *Note that in the previous Lemma 3.3.3 (part(a)) only anonymity, neutrality and monotonicity are needed. This means that under these conditions* $\varphi(R^S, \overline{R}^{N_1-S}) = \{R\}$ *implies* $\#S > \frac{n}{2}$. △

The following Lemma generalizes the decision power of minority coalitions as discussed at the beginning of this section.

Lemma 3.3.4. *Let* $\overline{R}$ *be an elementary change of* R *in pair* ab. *Let* $S \subseteq N$ *be a coalition such that* $\frac{1}{2} \leq \frac{\#S}{\#N} \leq \frac{m-1}{m}$. *Then* $\varphi(R^S, \overline{R}^{N-S}) = \left\{R, \overline{R}\right\}$.

Proof. First consider the special case where $\#N = m$ and $\#S = m-1$. So, $\frac{\#S}{\#N} = \frac{m-1}{m}$. Consider a numbering of the alternatives such that:

$$\begin{aligned}
a_1 a_2 a_3 a_4 \ldots a_t a_{t+1} \ldots a_m &= \widetilde{R}^1 = R \\
a_2 a_3 a_4 \ldots a_t a_{t+1} \ldots a_m a_1 &= \widetilde{R}^2 \\
a_3 a_4 \ldots a_t a_{t+1} \ldots a_m a_1 a_2 &= \widetilde{R}^3 \\
&\ldots \\
a_m a_1 a_2 \ldots a_t a_{t+1} \ldots a_{m-1} &= \widetilde{R}^m.
\end{aligned}$$

Without loss of generality we may assume that $a = a_t$, $b = a_{t+1}$ and then $a_1 a_2 \ldots a_{t-1} a_{t+1} a_t \ldots a_m = \overline{R}$. Let p be the Condorcet profile defined for all agents i in N by $p(i) = \widetilde{R}^i$. Next we prove $\varphi(p) = \mathbb{L}$.

Let τ be a permutation on A such that $\tau(a_t) = a_{t+1 \bmod m}$ for all t in $\{1, \ldots, m\}$. Then $\tau \widetilde{R}^i = \widetilde{R}^{i+1 \bmod m}$. Neutrality and replication invariance now imply that if $\widehat{R} \in \varphi(p)$, then $\tau^t \widehat{R} \in \varphi(p)$ for all $t \in \{1, \ldots, m\}$. For a given s there is a integer u such that $a_s \ldots. = \tau^u \widehat{R}$. So, $\cap\{\tau^t \widehat{R} : t \in \{1, \ldots, m\}\} = \emptyset$. Convex valuedness of φ and Proposition 3.2.1 now imply that $\varphi(p) = \mathbb{L}$.

Now $R = \widetilde{R}^1 \supseteq \widetilde{R}^t \cap \overline{R}$ for all $t \neq i+1$ and $\overline{R} \supseteq \widetilde{R}^{i+1} \cap \widetilde{R}^1$. As $\varphi(p) = \mathbb{L}$, monotonicity implies $\varphi((R^S, \overline{R}^{N-S}) \supseteq \left\{R, \overline{R}\right\}$. Pareto optimality now yields that $\varphi((R^S, \overline{R}^{N-S}) = \left\{R, \overline{R}\right\}$. Finally, for any S, N such that $\frac{1}{2} \leq \frac{\#S}{\#N} \leq \frac{m-1}{m}$, by Lemma 3.3.3, we have $\varphi(R^S, \overline{R}^{N-S}) = \{R, \overline{R}\}$. □

Remark 3.3.2. *Note that in the previous two Lemmas 3.3.3 and 3.3.4 only monotonicity, neutrality and anonymity is needed and not the strong version of monotonicity. This implies that* $\varphi(R^S, \overline{R}^{N-S}) = \left\{R, \overline{R}\right\}$ *for all preferences* R *and* $\overline{R}$ *which form an elementary change and all* S *such that*

$\frac{1}{m}\cdot\#N \le \#S \le \frac{m-1}{m}\cdot\#N$ and for all rules φ that are Pareto optimal, neutral, anonymous, convex valued rules and monotone. As it is easy to check that many well-known rules satisfy the former four and do not satisfy this result on elementary changes, we may conclude that many well-known rules are not monotone. For instance, any score rule is not monotone. Indeed, an elementary change involving different scores implies that the absolute majorities, instead of $\frac{m-1}{m}$ majorities, are decisive at such profiles. $\triangle$

Assume now, in addition, that φ is strongly monotone. In the sequel, we prove that φ is in fact a super majority rule. For such a rule φ, we define two logically different sets $\mathscr{B}_{all}$ and $\mathscr{B}_{some}$. In the latter we find those pairs of numbers (k,n) such that for some set of agents N, with $\#N = n$ and some coalition $S \subseteq N$, with $\#S = k$, and some pair of linear orders R and $\overline{R}$ forming a elementary change, such that both coalitions S and $N-S$ are decisive at profile $(R^S,\overline{R}^{N-S})$, that is $\varphi(R^S,\overline{R}^{N-S}) = \{R,\overline{R}\}$. Hence the outcome at those profiles equals the Pareto set of the profile. In $\mathscr{B}_{all}$ we find all pairs of numbers (k,n) for which this holds for all appropriate S, N, R and $\overline{R}$.

$$\mathscr{B}_{all} = \left\{\begin{array}{c} (k,n):\ \text{where } n-k \le k \le n-1 \text{ and } \varphi(R^S,\overline{R}^{N-S}) = \{R,\overline{R}\} \\ \text{for all } R,\overline{R} \text{ forming an elementary change} \\ \text{and all } S \subseteq N \text{ such that } \#S = k \text{ and } \#N = n\ \} \end{array}\right\}$$

$$\mathscr{B}_{some} = \left\{\begin{array}{c} (k,n):\ \text{where } n-k \le k \le n-1 \text{ and } \varphi(R^S,\overline{R}^{N-S}) = \{R,\overline{R}\} \\ \text{for some } R,\overline{R} \text{ forming an elementary change} \\ \text{and some } S \subseteq N \text{ such that } \#S = k \text{ and } \#N = n\ \} \end{array}\right\}$$

The two foregoing Lemmas imply that $\{(k,n) : \frac{1}{2} \le \frac{k}{n} \le \frac{m-1}{m}\} \subseteq \mathscr{B}_{all}$ and logically we have $\mathscr{B}_{all} \subseteq \mathscr{B}_{some}$. We shall prove that the latter subset relation is in fact an equation. Thereafter we refer to these sets by $\mathscr{B}$.

The following Lemma shows that for all pairs of sets of agents (S,N) such that $S \subset N$ and $(\#S,\#N) \in \mathscr{B}_{all}$ and all linear orders R and $\overline{R}$, not necessarily forming an elementary change, rule φ assigns the Pareto set to profile $(R^S,\overline{R}^{N-S})$.

Lemma 3.3.5. *Let $S \subseteq N$ be such that $(\#S,\#N) \in \mathscr{B}_{all}$. Let R and $\overline{R}$ be two linear orders. Then $\varphi(R^S,\overline{R}^{N-S}) = \mathbb{L}_{R\cap\overline{R}}$.*

Proof. By induction on $\delta(R,\overline{R})$.

Basis If $\delta(R,\overline{R})=0$, then it follows by Pareto optimality. Let $\delta(R,\overline{R})=1$. As $(\#S,\#N)\in\mathscr{B}_{all}$ it follows by definition that $\varphi(R^S,\overline{R}^{N-S})=\{R,\overline{R}\}=\mathbb{L}_{R\cap\overline{R}}$.

Induction Step Let $\delta(R,\overline{R})=t\geq 2$. Pareto optimality implies that $\varphi(R^S,\overline{R}^{N-S})\subseteq\mathbb{L}_{R\cap\overline{R}}$. Let $\widetilde{R}\in\mathbb{L}_{R\cap\overline{R}}-\{R,\overline{R}\}$. It follows that $\delta(R,\widetilde{R})<t$ and $\delta(\overline{R},\widetilde{R})<t$. So, by induction $\mathbb{L}_{R\cap\widetilde{R}}=\varphi(R^S,\widetilde{R}^{N-S})$ and $\mathbb{L}_{\overline{R}\cap\widetilde{R}}=\varphi(\widetilde{R}^S,\overline{R}^{N-S})$. If $R\in\varphi(R^S,\overline{R}^{N-S})$, then by strong monotonicity $\widetilde{R}\in\mathbb{L}_{R\cap\widetilde{R}}=\varphi(R^S,\widetilde{R}^{N-S})\subseteq\varphi(R^S,\overline{R}^{N-S})$. If $\overline{R}\in\varphi(R^S,\overline{R}^{N-S})$, then by strong monotonicity $\widetilde{R}\in\mathbb{L}_{\overline{R}\cap\widetilde{R}}=\varphi(\widetilde{R}^S,\overline{R}^{N-S})\subseteq\varphi(R^S,\overline{R}^{N-S})$. If $\widetilde{R}\in\varphi(R^S,\overline{R}^{N-S})$, then by strong monotonicity $R\in\mathbb{L}_{R\cap\widetilde{R}}=\varphi(R^S,\widetilde{R}^{N-S})\subseteq\varphi(R^S,\overline{R}^{N-S})$ and $\overline{R}\in\mathbb{L}_{\overline{R}\cap\widetilde{R}}=\varphi(\widetilde{R}^S,\overline{R}^{N-S})\subseteq\varphi(R^S,\overline{R}^{N-S})$.This then yields by convexity that $\mathbb{L}_{R\cap\overline{R}}\subseteq\varphi(R^S,\overline{R}^{N-S})$. Hence if $\widetilde{R}\in\varphi(R^S,\overline{R}^{N-S})$, then $\mathbb{L}_{R\cap\overline{R}}=\varphi(R^S,\overline{R}^{N-S})$. Since $\overline{R}\in\varphi(R^S,\overline{R}^{N-S})$, $\widetilde{R}\in\varphi(R^S,\overline{R}^{N-S})$, or $\widetilde{R}\in\varphi(R^S,\overline{R}^{N-S})$ for some $\widetilde{R}\in\mathbb{L}_{R\cap\overline{R}}-\{R,\overline{R}\}$, the above yields that $\varphi(R^S,\overline{R}^{N-S})=\mathbb{L}_{R\cap\overline{R}}$. □

Next lemma is about profiles polarized in elementary changes. We show that whenever some coalition of size k is not powerful enough to impose its preference uniquely at some profile of elementary changes, then any coalition of the same size is also not powerful enough to impose its preference uniquely at any profile of elementary changes. Since $\mathscr{B}_{all}\subseteq\mathscr{B}_{some}$, this boils down to say that $\mathscr{B}_{all}=\mathscr{B}_{some}$.

Lemma 3.3.6. *Let S be a subset of N such that $(\#S,\#N)\in\mathscr{B}_{some}$. Then $(\#S,\#N)\in\mathscr{B}_{all}$.*

Proof. Let $(\#S,\#N)\in\mathscr{B}_{some}$ such that $\#N-\#S\leq\#S\leq\#N-1$. Let R be an elementary change of $\overline{R}$ in pair ab at position l, such that $\varphi(R^S,\overline{R}^{N-S})=\{R,\overline{R}\}$. It is sufficient to prove that $(\#S,\#N)\in\mathscr{B}_{all}$. If $\frac{\#S}{\#N}\leq\frac{m-1}{m}$, then this holds by Lemma 3.3.4. Therefore we may assume that $\frac{\#S}{\#N}>\frac{m-1}{m}$. So, there are $T\subseteq S$ be such that $\#N-\#T\leq\#T\leq\frac{m-1}{m}\cdot\#N$ and by Lemma 3.3.4 $(\#T,\#N)\in\mathscr{B}_{all}$. Note that, by neutrality, $\varphi(R_l^S,\overline{R}_l^{N-S})=\{R_l,\overline{R}_l\}$ for all elementary changes in position l. First we will show that this also extends to all elementary changes in position $l+1$.

Let $.abc.=R$ and $.bac.=\overline{R}$. Consider $.abc.=R=R_1$, $.acb.=R_2$, $.cab.=R_3$, $.cba.=R_4$, $.bca.=R_5$ and $.bac.=R_6=\overline{R}$. It is sufficient to prove that

$\varphi(R_1^S, R_2^{N-S}) = \{R_1, R_2\}$ since R_1 and R_2 form an elementary change in position $l+1$.

Since $(\#T, \#N) \in \mathscr{B}_{all}$, by Lemma 3.3.5, $\varphi(R_5^{N-T}, R_2^T) = \{R_1, R_2, ..., R_6\}$. From this, monotonicity implies that $\{R_1, R_2\} \subseteq \varphi(R_6^{N-S}, R_1^{S-T}, R_2^T)$. Since $R_1 \in \varphi(R_6^{N-S}, R_1^{S-T}, R_2^T)$, by strong monotonicity we have that $\varphi(R_6^{N-S}, R_1^S) \subseteq \varphi(R_6^{N-S}, R_1^{S-T}, R_2^T)$. By assumption $\varphi(R_6^{N-S}, R_1^S) = \{R_6, R_1\}$, and therewith $\{R_1, R_2, R_6\} \subseteq \varphi(R_6^{N-S}, R_1^{S-T}, R_2^T)$. So, by Pareto optimality $\{R_1, R_2, R_6\} = \varphi(R_6^{N-S}, R_1^{S-T}, R_2^T)$. Since $(\#T, \#N) \in \mathscr{B}_{all}$, by Lemma 3.3.5 we have that $\varphi(R_6^{N-T}, R_3^T) = \{R_1, R_2, ..., R_6\}$. So, by monotonicity we have that $\{R_1, R_2, R_3\} \subseteq \varphi(R_6^{N-S}, R_1^{S-T}, R_3^T)$. As $R_2 \in \varphi(R_6^{N-S}, R_1^{S-T}, R_3^T)$, strong monotonicity implies $\varphi(R_6^{N-S}, R_1^{S-T}, R_3^T) \supseteq \varphi(R_6^{N-S}, R_1^{S-T}, R_2^T) = \{R_1, R_2, R_6\}$. Hence, we have that $\{R_1, R_2, R_3, R_6\} \subseteq \varphi(R_6^{N-S}, R_1^{S-T}, R_3^T)$ and convex valuedness now implies $\varphi(R_6^{N-S}, R_1^{S-T}, R_3^T) = \{R_1, R_2, ..., R_6\}$. By employing the permutation τ on A such that $\tau(x) = x$ for all $x \in A - \{a, b, c\}$, $\tau(a) = b$, $\tau(b) = c$ and $\tau(c) = a$ neutrality yields $\varphi(R_4^{N-S}, R_5^{S-T}, R_1^T) = \{R_1, R_2, ..., R_6\}$. Monotonicity and Pareto optimality finally yields that $\varphi(R_1^S, R_2^{N-S}) = \{R_1, R_2\}$. Then, by neutrality, for all elementary changes $(R_{l+1}^S, \overline{R}_{l+1}^{N-S})$ in position $l+1$, we have that $\varphi(R_{l+1}^S, \overline{R}_{l+1}^{N-S}) = \{R_{l+1}, \overline{R}_{l+1}\}$.

By a similar approach it can be shown that for elementary changes R' and $\overline{R}'$ in position $l-1$ we have $\varphi(R'^S, \overline{R}'^{N-S}) = \{R', \overline{R}'\}$. Therefore we may conclude that $(\#S, \#N) \in \mathscr{B}_{all}$. □

Remark 3.3.3. *Now let $\mathscr{B} = \mathscr{B}_{all} = \mathscr{B}_{some}$. Consider the function f from $\mathbb{N} - \{0\}$ to $\mathbb{N} - \{0\}$ for an arbitrary number n by $f(n) = \min\{\#S : S \subseteq N$, with $\#N = n$, and $(\#S, \#N) \notin \mathscr{B}\}$. By definition and as $\{(k,n) : \frac{1}{2} \leq \frac{k}{n} \leq \frac{m-1}{m}\} \subseteq \mathscr{B}_{all} = \mathscr{B}$, it follows that $\frac{m-1}{m} < \frac{f(n)}{n} \leq 1$ for all n in $\mathbb{N}$ (first condition of the threshold function). By definition of f and since φ is replication invariant it follows that $k \cdot (f(n) - 1) < f(kn) \leq k \cdot f(n)$. Therefore it follows that $\frac{f(n)-1}{n} < \frac{f(kn)}{kn} \leq \frac{f(n)}{n}$ (second condition of the threshold function). Hence f is indeed a threshold function.* △

The following Theorem shows that φ equals super majority rule φ_g.

Theorem 3.3.1. *A rule φ is Pareto optimal, neutral, replication invariant, convex valued and strongly monotone if and only if $\varphi = \varphi_g$ for some threshold function g.*

Proof. (If part) Follows from the definition of φ_g and Lemmas 3.3.1 and 3.3.2.

(Only if-part) Let φ be Pareto optimal, neutral, replication invariant, convex valued and strongly monotone. By Lemmas above and Remark 3.3.3, we may find such a threshold function g such that for all elementary changes R and $\overline{R}$ and all coalitions $T \subseteq N$ we have that $\varphi_g(R^T, \overline{R}^{N-T}) = \varphi(R^T, \overline{R}^{N-T})$. Let p be any profile in $\mathbb{L}^N$, it remains to prove that $\varphi(p) = \varphi_g(p)$.

To show $\varphi(p) \subseteq \varphi_g(p)$, let $S \subseteq N$ be any coalition such that $\#S \geq g(\#N)$ and $.a.b. = p(i)$ for some alternatives a and b and for all i in S. It is sufficient to prove that $\varphi(p) \subseteq \mathbb{L}_{ab}$. To the contrary let $\widehat{R} \in \varphi(p) - \mathbb{L}_{ab} \subseteq \mathbb{L}_{ba}$. We distinguish two cases.

Case (a and b are consecutive in $\widehat{R}$): Let $.ba. = \widehat{R}$. Then order R' in $\mathbb{L}_{ab}$ being the elementary change of $\widehat{R}$ in ba, is between $\widehat{R}$ and $p(i)$ for all $i \in S$. Hence, monotonicity would yield that $\widehat{R} \in \varphi((R')^S, \widehat{R}^{N-S})$. So, $\varphi((R')^S, \widehat{R}^{N-S}) \neq \varphi_g((R')^S, \widehat{R}^{N-S}) = \{R'\}$ which cannot be as R' and $\widehat{R}$ form an elementary change in ba.

Case (a and b are not consecutive in $\widehat{R}$): Let $c_1, c_2...c_l$ be alternatives different from a and b such that $.bc_1c_2...c_la. = \widehat{R}$. We may take $\widehat{R}$ such that l is minimal. For $t \in \{1,2,...,l\}$ let $\widehat{R}^t$ be the linear order defined by $\widehat{R}^t|_{A-\{c_t\}} = \widehat{R}|_{A-\{c_t\}}$ and $.c_tb.a. = \widehat{R}^t$. Furthermore, let $\widehat{R}^0$ be the linear order defined by $\widehat{R}^0|_{A-\{c_l\}} = \widehat{R}|_{A-\{c_l\}}$ and $.bc_1c_2...c_{l-1}ac_l. = \widehat{R}^0$.Take any $i \in S$ and hence $p(i)$ in $\mathbb{L}_{ab}$. We prove that for each $p(i)$, there is a $t \in \{0,1,2,...,l\}$ such that $\widehat{R}^t$ is between $p(i)$ and $\widehat{R}$. Take s such that $(c_s, c_u) \in p(i)$ for all $u \in \{1,2,...,l\}$. Note that we either have $.c_s.a. = p(i)$ or $.a.c_s. = p(i)$. Assume $.c_s.a. = p(i)$. Hence, $.c_s.a.b. = p(i)$. To prove that $\widehat{R}^s$ is between $p(i)$ and $\widehat{R}$ let $(x,y) \in \widehat{R}^s - \widehat{R}$. It is sufficient to show that $(x,y) \in p(i)$. By construction of $\widehat{R}^s$ and $\widehat{R}$, we have that $x = c_s$ and either $y = b$ or $y = c_u$ for some $0 < u < s$. Because of $.c_s.a.b. = p(i)$ and $(c_s, c_u) \in p(i)$ for all $u \in \{1,2,...,l\}$, and by the choice of s,we have that $(x,y) \in p(i)$. Hence for the case $.c_s.a. = p(i)$, we have that $\widehat{R}^t$ is between $p(i)$ and $\widehat{R}$ for $t = s$. Now assume $.a.c_s. = p(i)$. So, by construction $.a.c_t. = p(i)$ for all $t \in \{1,2,...,l\}$. Then $\widehat{R}^0 \subseteq \widehat{R} \cup \{(a, c_l)\} \subseteq \widehat{R} \cup p(i)$. Therefore, $\widehat{R}^0$ is between $\widehat{R}$ and $p(i)$. Hence for the case $.a.c_s. = p(i)$, $\widehat{R}^t$ is between $p(i)$ and $\widehat{R}$ for $t = 0$. Therefore we may take $s_i \in \{0,1,...,l\}$ such that $\widehat{R}^{s_i}$ is between $p(i)$ and $\widehat{R}$ for all $i \in S$.

Next take $S_t = \{i \in S : t = s_i\}$, i.e., the set of individuals $i \in S$ which share $\widehat{R}^t$ as the preference between $\widehat{R}$ and their preference $p(i)$. Let S_v be such that $\#S_v \geq \#S_t$ for all $t \in \{0,1,...,l\}$. By the assumption on the minimality of l, we have that $\widehat{R}^v \notin \varphi(p)$ and as $\widehat{R}^v$ is between $\widehat{R}$ and $p(i)$ for all $i \in S_v$ it follows by strong monotonicity that $\widehat{R}^v \notin \varphi((\widehat{R}^v)^{S_v}, \widehat{R}^{N-S_v})$. But by construction of S_v we have that $\#S_v \geq \frac{1}{l+1} \cdot \frac{m-1}{m} \cdot \#N \geq \frac{1}{m-1} \cdot \frac{m-1}{m} \cdot \#N \geq \frac{\#N}{m}$. Therefore, $\#(N - S_v) \leq \frac{m-1}{m} \cdot \#N$. So, as $\{(k,n) : \frac{1}{2} \leq \frac{k}{n} \leq \frac{m-1}{m}\} \subseteq \mathscr{B}$ it follows that $\#(N - S_v) < g(\#N)$ and therewith that the contradiction $\widehat{R}^v \in \varphi((\widehat{R}^v)^{S_v}, \widehat{R}^{N-S_v})$.

To show $\varphi(p) \supseteq \varphi_g(p)$, let $R \in \varphi_g(p)$ and suppose to the contrary that $R \notin \varphi(p)$. As by the previous part $\varphi(p) \subseteq \varphi_g(p)$ and convex valuedness we may assume that there exist $\overline{R} \in \varphi(p) \subseteq \varphi_g(p)$ such that R and $\overline{R}$ form an elementary change in ab. Furthermore, without loss of generality, let $.ab. = R$ and $.ba. = \overline{R}$ and let $S = \{i \in N : p(i) \in \mathbb{L}_{ab}\}$. Strong monotonicity now implies that $R, \overline{R} \in \varphi_g(R^S, \overline{R}^{N-S}) \subseteq \varphi_g(p)$ and $\overline{R} \in \varphi(R^S, \overline{R}^{N-S}) \subseteq \varphi(p)$ and $R \notin \varphi(R^S, \overline{R}^{N-S})$. This however contradicts $\varphi_g(R^T, \overline{R}^{N-T}) = \varphi(R^T, \overline{R}^{N-T})$ for all coalitions $T \subseteq N$. This contradiction completes the proof. □

3.4 Dropping Convex Valuedness

Although super majority rules satisfy several nice conditions, they are not extremely resolute in the sense that they often assign a lot of outcomes to profiles due to lack of coalitional power. This is essentially caused by the lower bound $n \cdot \frac{m-1}{m}$ for k, a lower bound which is needed in order to avoid cyclical decisions. The discussion at the beginning of the previous section involving the Condorcet profile on three alternatives makes clear that an increase on the resolution of rules can most likely be achieved by dropping the convex valuedness condition. In general, we did not succeed in describing the class of all Pareto optimal, neutral, replication invariant and monotone rules. Therefore we fixed our attention to the well-known Kemeny rule belonging to this class. It can be seen as a natural application of the Kemeny distance, introduced by Kemeny and Snell (1962), to social choice theory. This rule assigns those ranking(s) to a profile which minimizes the sum of distances to each of the individual preference in that profile.

Definition 3.4.1. *Given a profile $p \in \mathbb{L}^N$, a preference relation R is a Kemeny* *ranking for p, if for all $R' \in \mathbb{L}$, we have:*

$$\sum_{i\in N} \delta(R,p(i)) \leq \sum_{i\in N} \delta(R',p(i)).$$

A rule which assigns all Kemeny rankings to each profile is called the Kemeny rule.

Definition 3.4.2. *Kemeny rule, denoted by φ_{Kemeny}, assigns to a profile $p \in \mathbb{L}^N$:*

$$\varphi_{Kemeny}(p) = \{R \in \mathbb{L} : R \textit{ is a Kemeny ranking for } p\}.$$

We characterize the Kemeny rule by Pareto optimality, neutrality, pairwiseness, consistency and monotonicity. For that we first prove that the Kemeny rule satisfies these conditions.

Lemma 3.4.1. *φ_{Kemeny} is Pareto optimal, neutral, pairwise, and consistent.*

To show Pareto optimality, take any profile $p \in \mathbb{L}^N$ such that for some $a,b \in A$ and for all $i \in N$, $p(i) = .a.b..$ Consider the permutation σ_{ab} on A such that $\sigma_{ab}(a) = b$, $\sigma_{ab}(b) = a$, and $\sigma_{ab}(c) = c$ for all $c \in A - \{a,b\}$. Then by Proposition 3.6.2 in the appendix, for any $i \in N$ and for any linear order $R^{ab} \in \mathbb{L}_{ab}$ and $R^{ba} = \sigma_{ab}R^{ab}$, we have that:

$$\delta(p(i),R^{ab}) < \delta(p(i),R^{ba}).$$

Hence,

$$\sum_{i\in N} \delta(p(i),R^{ab}) < \sum_{i\in N} \delta(p(i),R^{ba}).$$

Therefore, $\varphi_{Kemeny}(p) \cap \mathbb{L}_{ba} = \emptyset$ which implies $\varphi_{Kemeny}(p) \subseteq \mathbb{L}_{ab}$ and therewith it is Pareto optimal. It satisfies neutrality by definition and pairwiseness because $M(p) = M(q)$ implies

$$\begin{aligned}
\sum_{i\in N} \delta(R,p(i)) &= \sum_{(a,b)\in A\times A} \sum_{i\in N} (a,b) \in p(i) - R \\
&= \sum_{(a,b)\in A\times A-R} \sum_{i\in N} (a,b) \in p(i) \\
&= \sum_{(a,b)\in A\times A-R} M(p)_{(a,b)} \\
&= \sum_{(a,b)\in A\times A-R} M(q)_{(a,b)} \\
&= \sum_{i\in N} \delta(R,q(i)).
\end{aligned}$$

Hence, in that case $\sum_{i\in N}\delta(R,p(i))=\sum_{i\in N}\delta(R,q(i))$ for all profiles p and q and for all linear orders $R\in\mathbb{L}$. Consistency follows because for $R\in\mathbb{L}$ and profiles $p\in\mathbb{L}^{N_1}$ and $q\in\mathbb{L}^{N_2}$, where N_1 and N_2 are two disjoint finite and non-empty sets of agents,

$$\delta(R,(p,q))=\delta(R,p)+\delta(R,q).$$

The following lemma shows that the Kemeny rule is (strongly) monotone.

Lemma 3.4.2. *φ_{Kemeny} is strongly monotone.*

Proof. Consider profile p in $\mathbb{L}^N$, and any $R\in\varphi_{Kemeny}(p)$. In the view of Proposition 3.2.2 we can consider any update q of p towards R such that $p(i)=q(i)$ for all i in $N-\{j\}$ and for some agent j in N, $q(j)$ is an elementary change of $p(j)$ in pair ab with $.a.b.=R$, $.ab.=q(j)$ and $.ba.=p(j)$. It is sufficient to prove that $R\in\varphi_{Kemeny}(q)$ and $\varphi_{Kemeny}(q)\subseteq\varphi_{Kemeny}(p)$. First we prove $R\in\varphi_{Kemeny}(q)$. To the contrary suppose there exist $R'\in\mathbb{L}$ such that

$$\sum_{i\in N}\delta(R,q(i))>\sum_{i\in N}\delta(R',q(i)).$$

Hence,

$$\sum_{i\in N}\delta(R,q(i))\geq\sum_{i\in N}\delta(R',q(i))+1.$$

Since R is a Kemeny ranking for p, we have $R\in\varphi_{Kemeny}(p)$ therefore,

$$\sum_{i\in N}\delta(R,p(i))\leq\sum_{i\in N}\delta(R',p(i)).$$

But then we have

$$\begin{aligned}
\delta(R',p(j))+\sum_{i\in N-\{j\}}\delta(R',p(i)) &= \sum_{i\in N}\delta(R',p(i))\\
&\geq \sum_{i\in N}\delta(R,p(i))\\
&= \delta(R,p(j))+\sum_{i\in N-\{j\}}\delta(R,p(i))\\
&= \delta(R,q(j))+1+\sum_{i\in N-\{j\}}\delta(R,q(i))\\
&= \sum_{i\in N}\delta(R,q(i))+1\\
&\geq \sum_{i\in N}\delta(R',q(i))+2\\
&= \delta(R',q(j))+\sum_{i\in N-\{j\}}\delta(R',q(i))+2\\
&= \delta(R',q(j))+\sum_{i\in N-\{j\}}\delta(R',p(i))+2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\delta(R',p(j)) &\geq \delta(R',q(j))+2\\
&= \delta(R',q(j))+\delta(q(j),p(j))+1,
\end{aligned}$$

which, by the triangle inequality yields the contradiction

$$\delta(R',p(j))\geq\delta(R',p(j))+1.$$

This completes the proof of $R\in\varphi_{Kemeny}(q)$. Next we show $\varphi_{Kemeny}(q)\subseteq\varphi_{Kemeny}(p)$. To the contrary, suppose there exist $R'\in\varphi_{Kemeny}(q)-\varphi_{Kemeny}(p)$. Since $R\in\varphi_{Kemeny}(p)$ and $R'\notin\varphi_{Kemeny}(p)$ we have

$$\sum_{i\in N}\delta(R,p(i))+1\leq\sum_{i\in N}\delta(R',p(i)). \tag{1}$$

Now $R\in\varphi_{Kemeny}(q)$ and $R'\in\varphi_{Kemeny}(q)$ imply

$$\sum_{i\in N}\delta(R,q(i))=\sum_{i\in N}\delta(R',q(i)).$$

Hence,

$$\delta(R,q(j))+\sum_{\substack{i\in N\\ i\neq j}}\delta(R,p(i))=\delta(R',q(j))+\sum_{\substack{i\in N\\ i\neq j}}\delta(R',p(i)).$$

So,

$$\delta(R,p(j))-1+\sum_{\substack{i\in N\\ i\neq j}}\delta(R,p(i))=\delta(R',q(j))+\sum_{\substack{i\in N\\ i\neq j}}\delta(R',p(i)).$$

That is

$$\sum_{i\in N}\delta(R,p(i))-1=\sum_{\substack{i\in N\\ i\neq j}}\delta(R',p(i))+\delta(R',q(j)).$$

Using inequality (1) this yields

$$\sum_{i\in N}\delta(R',p(i))-2\geq\sum_{\substack{i\in N\\ i\neq j}}\delta(R',p(i))+\delta(R',q(j)).$$

Hence,

$$\delta(R',p(j))-2\geq\delta(R',q(j)).$$

So,

$$\begin{array}{rcl}\delta(R',p(j)) & > & \delta(R',q(j))+1\\ & = & \delta(R',q(j))+\delta(p(j),q(j))\\ & \geq & \delta(R',p(j)).\end{array}$$

This contradiction ends the proof □

In order to prove that Kemeny rule is the only rule satisfying these five conditions simultaneously, let φ be a rule that is Pareto optimal, neutral, pairwise, consistent, and monotone. The following five Lemmas prepare the proof of this characterization.

First, we prove that equality holds between φ and φ_{kemeny} for *maximal conflicts*. These are profiles p at which two equal groups of agents have totally opposing preferences over all pairs of alternatives. So, there are $2k$ agents and a preference R in $\mathbb{L}$ such that k agents have preference $p(i)=R$ and the remaining k agents have preference $p(i)=-R$, where $-R=\{(b,a):(a,b)\in R\}$.

Lemma 3.4.3. *Let p be a maximal conflict. Then $\varphi(p)=\mathbb{L}=\varphi_{Kemeny}(p)$.*

Proof. It is straightforward to show that $\varphi_{Kemeny} = \mathbb{L}$. For alternatives a and b consider the permutation σ_{ab} on A such that $\sigma_{ab}(a) = b$, $\sigma_{ab}(b) = a$ and $\sigma_{ab}(c) = c$ for all $c \in A - \{a,b\}$. Neutrality and pairwiseness imply that $\sigma_{ab} \circ \varphi(p) = \varphi(\sigma_{ab}p) = \varphi(p)$. This means that $\varphi(p)$ is closed under any permutation of A which implies that $\varphi(p) = \mathbb{L}$. □

Next, we prove that the equality of φ and φ_{Kemeny} extends to *almost maximal conflicts*. These are profiles p at which equal groups of agents have totally opposing preferences over all but one pair of alternatives. So, there are $2k$ agents, two alternatives a and b and a preference R in $\mathbb{L}$ such that $ab... = R$, meaning that a is best and b is second best at R, k agents have preference $p(i) = R$ and the remaining k agents have preference $p(i) = \sigma_{ab}(-R)$. Let us denote these profiles by $\pi(k,ab)$.

Lemma 3.4.4. *Let p be an almost maximal conflict, say $p = \pi(k,ab)$. Then $\varphi(p) = \mathbb{L}_{ab} = \varphi_{Kemeny}(p)$.*

Proof. It is straightforward to show that $\mathbb{L}_{ab} = \varphi_{Kemeny}(p)$. Pareto optimality implies that $\varphi(p) \subseteq \mathbb{L}_{ab}$. In order to prove the reverse inclusion consider R in $\mathbb{L}_{ab}$. Let q be the maximal conflict on the same set of $2k$ agents such that the agents at p, which have preference R, have preference R at q and those having preference $\sigma_{ab}(-R)$ at p have preference $-R$ at q. Monotonicity and Lemma 3.4.3 imply that R is in $\varphi(p)$. As R is arbitrarily chosen in $\mathbb{L}_{ab}$, it follows that $\mathbb{L}_{ab} \subseteq \varphi(p)$. So, $\mathbb{L}_{ab} = \varphi(p)$. □

In the following lemma we show that φ and φ_{Kemeny} are strongly pairwise, i.e., the outcome only depends on the net pairwise comparisons of alternatives. For an arbitrary profile p let $D(p)$ denote the $m \times m$ difference matrix defined for every cell (a,b) by

$$D(p)_{ab} = \max\{0, M(p)_{ab} - M(p)_{ba}\}.$$

It is easy to see that for any profile p we have $\varphi_{Kemeny}(p) = \{R : \lambda(p,R) \leq \lambda(p,R') \text{ for all } R' \in \mathbb{L}\}$ where $\lambda(p,R) = \sum_{ab \notin R} D(p)_{ab}$. Indeed for all linear orders $R \in \mathbb{L}$, $\delta(p,R) = \sum_{ab \notin R} M(p)_{ab} = \sum_{ab \notin R} D(p)_{ab} + \min\{M(p)_{ab}, M(p)_{ba}\} = \sum_{ab \notin R} D(p)_{ab} + \sum_{ab \notin R} \min\{M(p)_{ab}, M(p)_{ba}\}$, where the former term equals $\lambda(p,R)$ and the latter term is constant.

Consider two profiles p in $\mathbb{L}^N$ and p' in $\mathbb{L}^{N'}$ such that $\#N = \#N'$, N and N' are disjoint and for all $i \in N$ there is a unique $i' \in N'$ such that $p(i) = p'(i')$. Since $M(p) = M(p')$ it follows that $\varphi(p) = \varphi(p')$. So, consistency implies that $\varphi(p) = \varphi(p') = \varphi(p, p')$. Therefore with a little abuse of notation we will write profile (p, p) instead of (p, p').

Lemma 3.4.5. *Let p and q be profiles such that $D(p) = kD(q)$ for some integer $k \geq 1$. Then $\varphi(p) = \varphi(q)$ and $\varphi_{Kemeny}(p) = \varphi_{Kemeny}(q)$.*

Proof. We only prove $\varphi(p) = \varphi(q)$. The proof of the second equation follows similarly from the properties: Pareto optimality, neutrality, consistency, monotonicity, and pairwiseness. Joining almost maximal conflict profiles for each pair (a, b) with $D(p)_{ab} > 0$ and possibly adding maximal conflicts yields a profile r such that $M(r) = 2M(p) + 2s\widehat{E} = 2kM(q) + 2t\widehat{E}$ for some positive integers s and t, where $\widehat{E}$ is the $m \times m$ matrix with all cells equal to one except $\widehat{E}_{aa} = 0$ for all a[5]. Replicating p once and q for $2k$ times yields profiles $p' = (p, p)$ and $q' = (q, q, ..., q)$ such that $M(r) = M(p') + 2s\widehat{E}$ and $M(r) = M(q') + 2t\widehat{E}$. Note that maximal conflicts have the pairwise matrix equal to an even multiple of $\widehat{E}$. Therefore pairwiseness, consistency and Lemma 3.4.3 imply that $\varphi(p') = \varphi(r)$ and $\varphi(q') = \varphi(r)$. Furthermore consistency implies $\varphi(p') = \varphi(p)$ and $\varphi(q') = \varphi(q)$. So, the desired result $\varphi(p) = \varphi(q)$ follows. □

Remark 3.4.1. *Note that Lemmas 3.4.3 and 3.4.4 follow immediately from the Condorcet condition as stated in Young and Levenglick (1978). This condition in our notation means that for all profiles p and for all alternatives a such that $D(p)_{ax} \geq 0$ for all alternatives x different from a (hence alternative a is a weak Condorcet-winner at p):*

1. *If $D(p)_{ay} = 0$ for some alternative y different from a, then $R \in \varphi(p) \Leftrightarrow R' \in \varphi(p)$ for all R and R' in $\mathbb{L}$ forming an elementary change in ay.*
2. *If $D(p)_{ay} > 0$ for some alternative y different from a, then $R \notin \varphi(p)$ for $R \in \mathbb{L}$ with $.ya. = R$.*

[5] Note that $\widehat{E}$ can be considered as the pairwise matrix of any maximal conflict. Furthermore, for each unit of $D(p)_{ab}$, one almost maximal conflict, $\pi(1, ab)$ is added to build r. As $M(\pi(1, ab))_{ab} = 2$ we have $M(r) = 2M(p) + 2s\widehat{E}$.

Although under different assumptions, Lemma 3.4.5 is also deduced as an intermediate result in Young and Levenglick (1978, Lemma 1). $\triangle$

The following step is to prove that φ and φ_{Kemeny} coincide on profiles where the pairwise majority relation is non cyclic. Roughly speaking, this boils down to proving that φ is Condorcet like.

Lemma 3.4.6. *Let p be a profile in $\mathbb{L}^N$ and a_1, a_2,..., a_m a numbering of the alternatives such that for all $1 \leq i < j \leq m$*

$$M(p)_{a_i a_j} \geq M(p)_{a_j a_i}.$$

Then $\varphi_{Kemeny}(p) = \cap\{\mathbb{L}_{a_i a_j} : 1 \leq i < j \leq m$ such that $M(p)_{a_i a_j} > M(p)_{a_j a_i}\} = \varphi(p)$.

Proof. Note that the intersection at the right hand side is not empty so the right equality follows because of Lemma 3.4.4, Lemma 3.4.5 and consistency. Now φ_{Kemeny} is consistent. This and Lemma 3.4.4 and Lemma 3.4.5 yields the left equality. $\square$

Lemma 3.4.7. *φ is strongly monotone. Moreover, for all non-empty finite subsets N of $\mathcal{N}$, all agents i in N, all alternatives a and b in A and all profiles p and q in $\mathbb{L}^N$, such that $\mathbb{L}_{ab} \cap \varphi(p) \neq \emptyset$, $(b,a) \in p(i)$, $(a,b) \in q(i)$, $\delta(p(i), q(i)) = 1$ and $p|_{N-\{i\}} = q|_{N-\{i\}}$,*

$$\varphi(q) = \mathbb{L}_{ab} \cap \varphi(p).$$

Proof. Let N, i, a, b, p and q as in the formulation of the Lemma. Without loss of generalization suppose that $\{1,2\} \cap N = \emptyset$. Consider $r \in \mathbb{L}^{\{1,2\}}$ such that $r = \pi(1, ab)$. Lemma 3.4.4 yields that $\varphi(r) = \mathbb{L}_{ab}$. Also we have $D(q) = D(p,r)$. So, as $\mathbb{L}_{ab} \cap \varphi(p) \neq \emptyset$ Lemma 3.4.5 and consistency imply $\varphi(q) = \varphi(p,r) = \varphi(r) \cap \varphi(p) = \mathbb{L}_{ab} \cap \varphi(p)$. $\square$

Now we are able to prove the characterization of Kemeny rule.

Theorem 3.4.1. *The Kemeny rule is the only correspondence which is simultaneously Pareto optimal, neutral, pairwise, consistent and monotone.*

Proof. (*If part*) Lemma 3.4.1 and 3.4.2 show that the Kemeny rule satisfies these five conditions.

(*Only if part*) Let φ be a Pareto optimal, neutral, pairwise, consistent and monotone rule. Let p be a profile. For $\lambda(p,R) = \sum_{ab \notin R} D(p)_{ab}$, let $\lambda(p,\mathbb{L}) = \min\{\lambda(p,R) : R \in \mathbb{L}\}$. In the sequel we will prove by induction on $\lambda(p,\mathbb{L})$ that $\varphi(p) = \varphi_{Kemeny}(p)$. In view of Lemma 3.4.5 we may assume that p consists of almost maximal conflicts only.

(*Induction basis*) Let $\lambda(p,\mathbb{L}) = 0$. In that case there exist linear orders R such that $D(p)_{ab} = 0$ for all $ab \notin R$. So there are no pairwise majority cycles at p. Hence, by Lemma 3.4.6 $\varphi(p) = \varphi_{Kemeny}(p)$.

(*Induction step*) Let $\lambda(p,\mathbb{L}) = k+1$. Define the set of unanimous pairs at p by $U(p) = \{(x,y) \in A \times A : .x.y. = p(i) \text{ for all agents } i \in N\}$.

Claim Let $(a,b) \notin U(p)$ and $\mathbb{L}_{ab} \cap \varphi(p) \neq \emptyset$ and $\mathbb{L}_{ab} \cap \varphi_{Kemeny}(p) \neq \emptyset$. Then $\mathbb{L}_{ab} \cap \varphi(p) = \mathbb{L}_{ab} \cap \varphi_{Kemeny}(p)$.

Proof of the claim Let $R^K \in \mathbb{L}_{ab} \cap \varphi_{Kemeny}(p)$ and $R \in \mathbb{L}_{ab} \cap \varphi(p)$. Then it follows by the definition of φ_{Kemeny} that $\lambda(p,R^K) = k+1$. As p consists of almost maximal conflicts we may assume that there exists an update say q of p towards R and R^K such that:

$$\begin{aligned} q(i) &= (p(i) - \{(b,a)\}) \cup \{(a,b)\} \text{ for some } i \in N \\ q(j) &= p(j) \text{ for all } j \in N - \{i\}. \end{aligned}$$

Lemma 3.4.7 implies that $\mathbb{L}_{ab} \cap \varphi_{Kemeny}(p) = \varphi_{Kemeny}(q)$ and $\mathbb{L}_{ab} \cap \varphi(p) = \varphi(q)$. As $\lambda(q,R^K) = k$, the induction hypothesis implies $\varphi_{Kemeny}(q) = \varphi(q)$. Hence, $\mathbb{L}_{ab} \cap \varphi(p) = \mathbb{L}_{ab} \cap \varphi_{Kemeny}(p)$.

End of proof of claim

Next we distinguish two cases:

Case: $\#\varphi(p) \geq 2$ and $\#\varphi_{Kemeny}(p) \geq 2$. Since $\#\varphi(p) \geq 2$ there are alternatives a and b such that $(a,b) \notin U(p)$ and $\mathbb{L}_{ab} \cap \varphi(p) \neq \emptyset$, $\mathbb{L}_{ba} \cap \varphi(p) \neq \emptyset$ and $\mathbb{L}_{ab} \cap \varphi_{Kemeny}(p) \neq \emptyset$. Hence, by the previous claim $\mathbb{L}_{ab} \cap \varphi(p) = \mathbb{L}_{ab} \cap \varphi_{Kemeny}(p)$. In case $\mathbb{L}_{ba} \cap \varphi_{Kemeny}(p) \neq \emptyset$ the previous claim implies that also $\mathbb{L}_{ba} \cap \varphi(p) = \mathbb{L}_{ba} \cap \varphi_{Kemeny}(p)$. So, then $\varphi(p) = \varphi_{Kemeny}(p)$. If $\mathbb{L}_{ba} \cap \varphi_{Kemeny}(p) = \emptyset$, then $\varphi_{Kemeny}(p) \subseteq \varphi(p)$. So, $\#\varphi(p) \geq 2$ implies $\varphi_{Kemeny}(p) \subseteq \varphi(p)$. Similarly $\#\varphi_{Kemeny}(p) \geq 2$ implies $\varphi_{Kemeny}(p) \supseteq \varphi(p)$. Hence, in this case $\varphi_{Kemeny}(p) = \varphi(p)$.

Case: There is a renaming φ_1 and φ_2 of φ and φ_{Kemeny} and there are linear orders $R^1$and R^2 such that $\varphi_1(p) = \{R^1\}$ and $R^2 \in \varphi_2(p)$. Note that we are done if $\varphi_2(p) = \{R^1\}$. So, therefore to the contrary suppose that $R^2 \neq R^1$. We end the proof by showing that this assumption leads to a contradiction.

Pareto optimality implies that $U(p) \subseteq R^1 \cap R^2$. Next we prove that $R^1 \cap R^2 \subseteq U(p)$. Suppose to the contrary that $(R^1 \cap R^2) - U(p) \neq \emptyset$ that is for some pair of alternatives $(a,b) \in (R^1 \cap R^2) - U(p)$. Then, by the previous claim, it follows that $\mathbb{L}_{ab} \cap \varphi_1(p) = \mathbb{L}_{ab} \cap \varphi_2(p)$. Since $(a,b) \in R^2$ and $(a,b) \in R^1$, we have that $R^2 \in \mathbb{L}_{ab} \cap \varphi_2(p) = \mathbb{L}_{ab} \cap \varphi_1(p) = \{R^1\}$. This would yield the contradiction $R^2 = R^1$. Therefore, we may assume that $(R^1 \cap R^2) - U(p) = \emptyset$ and as $R^1 \cap R^2 \subseteq U(p)$ we have $(R^1 \cap R^2) = U(p)$.

For numbers i and j such that $\{i, j\} = \{1,2\}$ and different alternatives x and y we call the ordered pair (x, y) "free" at R^i if $.xy. = R^i$ and $.y.x. = R^j$. So, x and y are consecutively ordered x above y at R^i and reversely ordered y above x at R^j. Therefore the pair (x, y) is in $U(p)$. Furthermore, let $N_{xy} = \{i \in N : .x.y. = p(i)\}$ and let $n_{xy} = \#N_{xy}$. Because of $R^1 \neq R^2$ we may choose alternatives a and b such that (a,b) is free at R^1 and for all (x, y) free at R^1, we either have $x = a$ or $.a.x. = R^1$. That is (a,b) is the highest ordered free pair in R^1. As (a,b) is free at R^1, we have $.b.a. = R^2$. Since $.b.a. = R^2$ and $.a.b. = R^1$, we may take (y, x) free in R^2, such that $b = y$ or $.b.y. = R^2$, $x = a$ or $.x.a. = R^2$ and for all free (c,d) in R^2 if $d \neq x$, then either $.d.x. = R^2$ or $.a.d. = R^2$. So, (y, x) is the lowest free pair in R^2 just above a in R^2. Where we may find this pair ordered between b and a in R^2.

Now if $y = a$ or $.y.a.b. = R^1$, then there is a free pair between x and y in R^1 and as $.x.y. = R^1$ this violates the assumption that (a,b) is the highest ordered free pair at R^1. So, $b = y$ or $.b.y. = R^1$, that is b is weakly preferred to y at R^1. If $x = b$ or $.ab.x. = R^1$, then there is a free pair between x and a at R^2 contradicting that (y, x) is the lowest free pair in R^j just before a in R^j. Therefore $x = a$ or $.x.ab. = R^1$, which means that x is weakly preferred to a at R^1. Therefore at R^1, we have: x is weakly preferred to a, and a strictly to b, and b weakly to y, whereas, by the choice of (y, x), at R^2, we have: b is weakly preferred to y, and y strictly to x, and x weakly to a. Now since $U(p) = (R^1 \cap R^2)$, we have either $b = y$ or $N_{by} = N$ and either $x = a$ or $N_{xa} = N$. Therefore $N_{ab} \subseteq N_{ay} \subseteq N_{xy}$ and $N_{yx} \subseteq N_{bx} \subseteq N_{ba}$. Consider $R^1_{ba} = (R^1 - \{(a,b)\}) \cup \{(b,a)\}$ and profile q such that $q(i) = R^1$ if $i \in N_{ab}$ and

$q(i) = R^1_{ba}$ if $i \in N_{ba}$. Now strong monotonicity which follows from Lemma 3.4.7, the fact that $q(i)$ is between $p(i)$ and R^1 for all $i \in N$ and $\varphi_1(p) = \{R^1\}$ yield that $\varphi_1(q) = \{R^1\}$. Hence, by Remark 3.3.1, it follows that $n_{ab} > n_{ba}$. Similarly by considering a preference $R^2_{xy} = (R^2 - \{(y,x)\}) \cup \{(x,y)\}$ it follows that $n_{yx} \geq n_{xy}$. But then $N_{ab} \subseteq N_{ay} \subseteq N_{xy}$, $n_{yx} \geq n_{xy}$ and $N_{yx} \subseteq N_{bx} \subseteq N_{ba}$ imply $n_{ab} \leq n_{ay} \leq n_{xy} \leq n_{yx} \leq n_{bx} \leq n_{ba}$. Hence, we have the contradiction $n_{ab} \leq n_{ba}$, which ends the proof. □

3.5 Discussion and Further Research

Update monotonicity, as also discussed in the introduction, is essentially a monotonicity condition for rules. We investigated preference correspondences and their reaction to an increase in support for a collective preference. Roughly speaking, we have analyzed monotone rules both in case the rules are convex valued and in case they are not. In the former, our finding is a class of rules that do not involve well-known convex valued rules such as scoring rules. In the latter, we end up with a new characterization of the Kemeny rule based on this monotonicity condition.

In following subsection we show, respectively, the independence of our conditions, some logical relations with the conditions in Young & Levenglick characterization, some possible variations of update monotonicity and finally, we consider this condition in relation with single valuedness.

3.5.1 Independence of Characterizing Conditions

Below we provide some rules showing the independence of the characterizing conditions of Theorems 3.3.1 and 3.4.1.

Super majority rule (φ_g): Defined in Section 3.3.

As shown in Theorem 3.3.1, φ_g is Pareto optimal, neutral, convex valued, replication invariant, and strongly monotone. Rule φ_g is pairwise because the k-majority-relation, on which it is solely based, is pairwise. To see that φ_g is not consistent consider $A = \{a,b,c\}$, $p(1) = p(2) = p(3) = q(5) = abc$ and $p(4) = p(6) = cba$. Then $\varphi_g(p) = \{abc\}$ and $\varphi_g(q) = \mathbb{L}$ in case $g(4) = 3$. But $\varphi_g(p,q) = \mathbb{L} \neq \varphi_g(p) \cap \varphi_g(q)$.

Biased super majority rule ($\widehat{\varphi}_g$) Define the rule $\widehat{\varphi}_g$ for an arbitrary profile p as follows: $\widehat{\varphi}_g(p) = \{R, \overline{R}\}$ whenever $p = (R^S, \overline{R}^{N-S})$ for some non-empty and finite subsets S, N of $\mathscr{N}$ such that $S \subsetneq N$ and R and $\overline{R}$ are in $\mathbb{L}$ forming an elementary change. In all other cases $\widehat{\varphi}_g(p) = \varphi_g(p)$.

Clearly, $\widehat{\varphi}_g$ is Pareto optimal, neutral, convex valued, replication invariant and monotone. It is not consistent because φ_g is not consistent. Consider profiles $q = (R_1^T, R_2^{N-T})$ for arbitrary preferences R_1 and R_2, such that $\#T \geq g(\#N)$. It follows that $\widehat{\varphi}_g$ is not strongly monotone, because for an elementary change $\overline{R}_1$ between R_1 and R_2, we have $\widehat{\varphi}_g(R_1^T, \overline{R}_1^{N-T}) = \{R_1, \overline{R}_1\}$ $\nsubseteq \widehat{\varphi}_g(R_1^T, R_2^{N-T}) = \{R_1\}$. This rule shows that the stronger monotonicity condition in Theorem 3.3.1 is logically essential.

Kemeny rule (φ_{Kemeny}): Defined in Section 3.4.

As discussed in Section 3.4, φ_{Kemeny} is Pareto optimal, neutral, pairwise, consistent and strongly monotone. By Remark 3.2.1, it is also replication invariant. The Kemeny rule, however, is not convex valued, e.g., for a Condorcet profile, it assigns the profile itself as Kemeny rankings.

Selective Kemeny rule (ψ_{Kemeny}): A rule φ is called a selective Kemeny rule whenever for some enumeration of all linear orders, for instance $\mathbb{L} = \{R_1, R_2, \ldots, R_{m!}\}$, $\psi_{Kemeny}(p) = \{R_i : i \text{ is minimal for all } R_i \text{ in } \varphi_{Kemeny}(p)\}$, i.e., ψ_{Kemeny} assigns to each profile the Kemeny ranking with the minimal predefined index.

Note that, since for any profile p, $\psi_{Kemeny}(p) \subseteq \varphi_{Kemeny}(p)$, selective Kemeny rule also satisfies Pareto optimality. It is also pairwise since for two profiles, say p and q, with identical pairwise matrices, $\varphi_{Kemeny}(p) = \varphi_{Kemeny}(q)$ therefore $R_i \in \varphi_{Kemeny}(p)$ with minimal i is the same as $R_i \in \varphi_{Kemeny}(q)$ with minimal i. To show that the rule is consistent let p, q be two profiles such that $\psi_{Kemeny}(p) \cap \psi_{Kemeny}(q) = \{R_j\}$. Then j is the smallest number among R_i in both $\varphi_{Kemeny}(p)$ and $\varphi_{Kemeny}(q)$. Hence, by consistency, R_j is in $\varphi_{Kemeny}(p,q)$ and R_j has the smallest index among those R_i which are in $\varphi_{Kemeny}(p) \cap \varphi_{Kemeny}(q) = \varphi_{Kemeny}(p,q)$. So, $\psi_{Kemeny}(p,q) = \{R_j\}$. Monotonicity is straightforward by strong monotonicity of Kemeny rule. It is also trivially convex valued as it always assigns a single outcome to each profile, yet it is not neutral by construction.

Trivial rule (φ_T): A rule φ is called the trivial rule if for all profiles p in

$\mathbb{L}^N$, $\varphi(p) = \mathbb{L}$.

By construction, the trivial rule always assign the same set of linear orders, $\mathbb{L}$, therefore it is neutral, pairwise, convex valued, consistent, replication invariant, and monotone. Obviously it is not Pareto optimal.

Dictatorial rule ($\varphi_{dictatorial}$): A rule φ is called a dictatorial rule if there exists an individual $d \in N$ such that for all profiles p, $\varphi(p) = \{p(d)\}$.

Dictatorial rule $\varphi_{dictatorial}$ is known to be Pareto optimal. To show consistency, for any predefined order over individuals in $\mathcal{N}$, to choose the dictator for each subset of individuals $N \subseteq \mathcal{N}$, consider two profiles and two dictators in each society. If two profiles agree on an outcome, when the two profiles merge, one dictator will remain as the dictator, according to the predefined order, of the merged profile hence the outcome will be the same. It is also trivially convex valued as it always assigns a single outcome to each profile. It is neutral and monotone by construction. It is not anonymous; hence, it also fails to be pairwise and replication invariant.

Borda rule (φ_{Borda}): Defined in Example 3.2.1 of Section 3.2, where the score vector $\overrightarrow{s} = (m, m-1, m-2, ..., 1)$.

Borda is known to be Pareto optimal, neutral, pairwise and consistent and therewith replication invariant. By construction it is convex valued like all score rules. It fails to be monotone; hence, it is also not strongly monotone.

The table below summarizes the findings discussed above. It shows the logical independence of the characterizing conditions in both Theorem 3.3.1 and Theorem 3.4.1. We denote φ_{Kemeny} and ψ_{Kemeny}, by φ_K and ψ_K respectively.

	φ_g	φ_K	ψ_K	φ_T	$\varphi_{dictatorial}$	φ_{Borda}	$\widehat{\varphi}_g$
Pareto optimal	Y	Y	Y	N	Y	Y	Y
Neutral	Y	Y	N	Y	Y	Y	Y
Consistent	N	Y	Y	Y	Y	Y	N
Convex valued	Y	N	Y	Y	Y	Y	Y
Replication invariant	Y	Y	Y	Y	N	Y	Y
Pairwise	Y	Y	Y	Y	N	Y	Y
Monotone	Y	Y	Y	Y	Y	N	Y
Strongly monotone	Y	Y	Y	Y	Y	N	N

3.5.2 Logical Relations Regarding Young & Levenglick Characterization of the Kemeny Rule

In this subsection we show that the two sets of characterizing conditions of the Kemeny can be deduced from one another directly. First, in Lemma 3.5.1 and 3.5.2, it is proved that the conditions of Young and Levenglick imply the characterizing conditions of Theorem 3.4.1.

Lemma 3.5.1. *Strong monotonicity is implied by neutrality, consistency and the Condorcet condition.*

Proof. Let φ be such a rule. Let p and q be profiles and i an agent such that $p|_{N-\{i\}} = q|_{N-\{i\}}$ and $q(i)$ is an elementary change of $p(i)$ in ab. It is sufficient to prove that $\varphi(q) = \mathbb{L}_{ab} \cap \varphi(p)$ in case $\mathbb{L}_{ab} \cap \varphi(p) \neq \emptyset$. Let $r = (p, p, \pi(1, ab))$, then $D(r) = 2D(q)$. Also the Condorcet condition implies that $\varphi(\pi(1, ab)) = \mathbb{L}_{ab}$. Consistency implies $\varphi(r) = \varphi(p) \cap \mathbb{L}_{ab}$ as $\mathbb{L}_{ab} \cap \varphi(p) \neq \emptyset$. In view of Remark 3.4.1 , $\varphi(q) = \varphi(r) = \varphi(p) \cap \mathbb{L}_{ab}$. □

Lemma 3.5.2. *Pareto optimality is implied by neutrality, consistency and the Condorcet condition.*

Proof. Let φ be a neutral and consistent rule which satisfies the Condorcet condition. Let p be a profile in $\mathbb{L}^N$ and let a and b be two different alternatives such that $.a.b. = p(i)$ for all agents i in N. And, to the contrary, let $R \in \varphi(p)$ with $.b.a. = R$. First, we prove that in this case we may take

p such that $.ab. = p(i)$ for all agents i in N. Let $.ax_1x_2...x_tb. = p(j)$ for some agent j. Let s be the smallest number such that $.x_s.a. = R$ assuming that there are x_u such that $.x_u.a. = R$. Consider profile q such that $.x_sax_1x_2...x_{s-1}x_{s+1}...x_tb. = q(j)$ and $q(i) = p(i)$ for $i \in N - \{j\}$. Clearly $q(j) \subseteq p(j) \cup R$, hence by monotonicity, which holds by Lemma 3.5.1, we have that $R \in \varphi(q)$ where $.a.b. = q(i)$ for all agents i in N. In case for all x_u we have that $.a.x_u. = R$. we may take $q(j) = .ax_1x_2...x_{t-1}bx_t.$ and have the same result. Therefore, by the finiteness of the set of alternatives, we may assume that $.ab. = p(i)$ for all agents i in N. Next, consider the permutation σ on A such that $\sigma(a) = b$, $\sigma(b) = a$ and $\sigma(x) = x$ for all alternatives $x \in A - \{a,b\}$. Note that $\sigma p(i) \subseteq p(i) \cup R$ for all agents i in N. Hence, similar as in Lemma 3.5.1 we find that $\varphi(\sigma p) = \varphi(p) \cap \mathbb{L}_{ba}$. Hence, $\varphi(\sigma p) \subseteq \mathbb{L}_{ba}$. But neutrality implies that $\varphi(\sigma p) = \sigma(\varphi(p))$ and as $R \in \varphi(p)$ this yields the contradiction that $\sigma R \in \varphi(\sigma p)$ that is $\varphi(\sigma p) \cap \mathbb{L}_{ab} \neq \emptyset$. □

Note that Lemma 1 in Young and Levenglick (1978) shows that neutrality, consistency and the Condorcet condition implies pairwiseness. So, all together, the conditions of Young and Levenglick -neutrality, consistency and the Condorcet condition- imply the set of conditions: Pareto optimality, neutrality, pairwiseness, consistency and monotonicity. The latter set is the set of characterizing conditions used in Theorem 3.4.1.

Next, we show that this latter set of conditions implies the former, which actually boils down to the following Lemma.

Lemma 3.5.3. *Pareto optimality, neutrality, pairwiseness, consistency and monotonicity imply the Condorcet condition.*

Proof. Let φ be a Pareto optimal, neutral, consistent, pairwise and monotone rule. Let p be a profile in $\mathbb{L}^N$. Let c and d be two different alternatives and R in $\mathbb{L}$ such that $.cd. = R$ and $R \in \varphi(p)$.

Claim: There is a profile q, an update of p towards R, such that $R \in \varphi(q)$, $M(p)_{cd} = M(q)_{cd}$ and $.cd. = q(i)$ or $.dc. = q(i)$ for all agents i in N.

Proof of Claim Take agent j arbitrarily. We distinguish two cases.

Case $.cx_1x_2...x_td. = p(j)$. Define profile r by $r(i) = p(i)$ for $i \in N - \{j\}$. In case there is a smallest number s such that $.x_s.cd. = R$ defined $r(j) = .x_scx_1x_2...x_{s-1}x_{s+1}...x_td.$. If R is such that $.cd.x_s. = R$ for all s take $r(j) =$

$.cx_1x_2...x_{t-1}dx_t.$.Then r is an update of p towards R such that $M(p)_{cd} = M(r)_{cd}$, $M(p)_{cd} = M(r)_{cd}$ and by monotonicity we have $R \in \varphi(r)$.

Case $.dx_1x_2...x_tc. = p(j)$. Define profile r by $r(i) = p(i)$ for $i \in N - \{j\}$. In case there is a smallest number s such that $.x_s.cd. = R$ defined $r(j) = .x_sdx_1x_2...x_{s-1}x_{s+1}...x_tc.$. If R is such that $.cd.x_s. = R$ for all s take $r(j) = .dx_1x_2...x_{t-1}cx_t.$. Then r is an update of p towards R such that $M(p)_{cd} = M(r)_{cd}$, $M(p)_{cd} = M(r)_{cd}$ and by monotonicity we have $R \in \varphi(r)$.

Repeating these cases a finite number of times yields the desired result.

End of proof of Claim:

Let a be an alternative such that $D(p)_{ax} \geq 0$ for all alternatives x in $A - \{a\}$. In order to prove the second part of the Condorcet condition let $D(p)_{ay} = 0$ for some alternative y in $A - \{a\}$. Let $\{c,d\} = \{a,y\}$ and $.cd. = R$ for some linear order in $\varphi(p)$.We have to prove that $R' = .dc.$ which forms with R an elementary change in c and d is in $\varphi(p)$.Consider the permutation σ on A such that $\sigma(a) = y$, $\sigma(y) = a$ and $\sigma(z) = z$ for all $z \in A - \{a,y\}$. Take q as in the foregoing claim. Then $D(q)_{ay} = D(q)_{ya} = 0$ and strong monotonicity (which follows from Lemma 3.4.7) implies $R \in \varphi(q) \subseteq \varphi(p)$. Neutrality now implies $R' \in \varphi(\sigma q)$. But as for all agents i either $.ay. = q(i)$ or $.ya. = q(i)$ and $D(q)_{ay} = D(q)_{ya} = 0$ it follows that $M(q) = M(\sigma q)$. Hence, pairwiseness implies $R' \in \varphi(q) \subseteq \varphi(p)$, which proves the second part of the Condorcet condition.

In order to prove the first part of the Condorcet condition let $D(p)_{ay} > 0$ and let to the contrary $.ya. = R''$ be in $\varphi(p)$. Consider a profile r which consists of two times p and almost maximal conflict profile $\pi(D(p)_{ay}, ya)$.So, $r = (p, p, \pi(D(p)_{ay}, ya))$. Then consistency and Lemma 3.4.4 imply $\varphi(r) = \varphi(p) \cap \mathbb{L}_{ya}$. Hence, $R'' \in \varphi(r) \subseteq \mathbb{L}_{ya}$. But $D(r)_{ax} = 2D(p)_{ax} \geq 0$ for all $x \in A - \{a,y\}$ and $D(r)_{ay} = D(r)_{ya} = 0$. Hence, as $R'' \in \varphi(r)$ the second part of the Condorcet condition implies $\varphi(r) \cap \mathbb{L}_{ay} \neq \emptyset$ which contradicts $\varphi(r) \subseteq \mathbb{L}_{ya}$. □

3.5.3 Possible Variations in the Concept of Update and Monotonicity

By different variations in the concept of an update, one may acquire various conditions in a similar manner. We define *Kemeny-update monotonicity* as the ability of a rule to preserve the collective preferences whenever the preference profile is updated towards the collective preference in terms of a decrease in Kemeny[6] distance of each individual preference to the collective preference. Similarly, *extreme-update monotonicity* requires that a rule preserves the collective preference whenever some (possibly all) individuals identically imitate the same collective preference while others remain unchanged. Note that Kemeny-update monotonicity is the strongest of all variations mentioned whereas extreme-update monotonicity is the weakest.

Kemeny-update monotonicity: A rule is *(strongly) Kemeny-update monotone* if for all profiles $p \in \mathbb{L}^N$, and for all $R \in \varphi(p)$, and for all $q \in \mathbb{L}^N$ such that $\delta(q(i),R) \leq \delta(p(i),R)$ for all $i \in N$,

$$\begin{array}{rcl} R & \in & \varphi(q) \\ \text{(and in addition } \varphi(q) & \subseteq & \varphi(p)). \end{array}$$

Note that in the regular update monotonicity condition, the betweenness requirement, already implied a decrease in the distance between the individual preference and the outcome. Dropping this leads to a strenghtening of the monotonicity condition to the extent that even the Kemeny rule is not Kemeny-update monotone. In Can and Storcken (2011a), it is shown that strong Kemeny-update monotonicity leads to so called impossibility theorems.

One can also only consider transformations of profiles in which some agents identically copy the outcome and the rest remain the same. In such extreme updates of collective preference, we have a much more weaker update monotonicity condition:

Extreme-update monotonicity: A rule is *extreme-update monotone* if for all profiles $p \in \mathbb{L}^N$, and for all $R \in \varphi(p)$, and for all $q \in \mathbb{L}^N$ such that $q(i) \in \{R, p(i)\}$ for all $i \in N$,

$$R \in \varphi(q).$$

Extreme-update monotonicity is the least demanding of all since it requires that a rule preserves the outcome only when some agents copy the outcome while the rest remains as they are. In that context, it resembles the simple monotonicity condition in collective choice rules. However

[6]Kemeny distance counts the number of ordered pairs on which two binary relations i.e., strict preferences, are different. Note that one may formulate distance based conformity with respect to other metrics, however for our model we restrict our attention to Kemeny metric.

weak extreme-update monotonicity may be; most rules, including Borda and other score rules, still fail to satisfy it. For instance, let $A = \{a,b,c\}$, and $n = 4$ and consider a profile p, such that individuals respectively have the following preferences: abc, bac, acb, cab. Consider the Borda rule, $\varphi_{Borda}(p) = \{abc, acb\}$. Consider, now, the extreme update q of p towards acb, where individuals respectively have, abc, bac, acb, acb, i.e. 4^{th} individual imitated one of the outcomes. It is easy to see, however, that $\varphi(q) = \{abc\}$ is the unique collective preference at the updated profile.

3.5.4 Update Monotone Welfare Functions

In this subsection we consider single valued rules, which are also known as welfare functions. Because a singleton set is convex, such rules are convex valued. Furthermore, it is easy to see that monotone single valued rules are strongly monotone. The selective Kemeny rule is an example of a non trivial monotone welfare function. Similarly, selective super majority rules ψ_g can be defined by assigning to a profile p that order in $\varphi_g(p)$ having the smallest index. It is straightforward to prove that selective super majority rules are Pareto optimal, replication invariant, pairwise and (strongly) monotone. They violate, however, the conditions of neutrality and consistency. The condition of neutrality is also violated by selective Kemeny rules. In view of Lemma 3.3.4, it is straightforward that rules satisfying the conditions in Theorem 3.3.1 cannot be single valued. Similarly, welfare functions that are Pareto optimal, neutral and monotone cannot be replication invariant. We leave the study of welfare functions satisfying those conditions, except replication invariance, for future research.

As hinted in the discussion, there is a strong logical relationship between strategy-proofness and monotonicity conditions in the choice rules framework. Bossert and Storcken (1992) analyze strategy-proofness for single valued preference correspondences with the following result. Let A contain at least 4 alternatives and fix N, such that $\#N$ is even: then, there does not exist a welfare function which is simutaneously, nonimposed, coalitional strategy-proof and weakly extrema independent. Nonimposition, which is in fact implied by Pareto optimality, means that the function is surjective, i.e. the range of the welfare function is the complete set of linear order. Weak extrema independence is formally introduced below:

Definition 3.5.1. *A welfare function ψ is weakly extrema independent if for any two profiles p and q orders R_1, R_2, $\widehat{R}_1$and $\widehat{R}_2$ and disjoint coalitions S and T such that:*

$$\begin{aligned}
\#S &= \#T \\
p(i) &= R_1,\ q(i) = \widehat{R}_1 \text{ for } i \in S \\
p(i) &= R_2,\ q(i) = \widehat{R}_2 \text{ for } i \in T \\
p(i) &= q(i) \notin \{R_1, R_2, \widehat{R}_1, \widehat{R}_2\} \text{ for } i \in N-(S\cup T), \\
p(N), q(N) &\subseteq \mathbb{L}_{R_1\cap R_2} = \mathbb{L}_{\widehat{R}_1\cap \widehat{R}_2}.
\end{aligned}$$

we have $\psi(p) = \psi(q)$.

Proposition 3.5.1. *Selective Kemeny rules are weakly extrema independent.*

Proof. It is sufficient to prove that for two profiles as defined in Definition 3.5.1, $\varphi_{Kemeny}(p) = \varphi_{Kemeny}(q)$. Since Kemeny rule is pairwise it is sufficient to show that $M(p) = M(q)$. Note that $\mathbb{L}_{R_1\cap R_2} = \mathbb{L}_{\widehat{R}_1\cap \widehat{R}_2}$ is equivalent to $R_1 \cap R_2 = \widehat{R}_1 \cap \widehat{R}_2$. Furthermore, as $p|_{(S\cup T)} = (R_1^S, R_2^T)$ and $q|_{(S\cup T)} = (\widehat{R}_1^S, \widehat{R}_2^T)$, $R_1 \cap R_2 = \widehat{R}_1 \cap \widehat{R}_2$ and $\#S = \#T$ it follows, although a bit cumbersome, that $M(p|_{(S\cup T)}) = M(q|_{(S\cup T)})$. Of course $M(p|_{N-(S\cup T)}) = M(q|_{N-(S\cup T)})$. So, $M(p) = M(p|_{(S\cup T)}) + M(p|_{N-(S\cup T)}) = M(q|_{(S\cup T)}) + M(q|_{N-(S\cup T)}) = M(q)$. □

Remark 3.5.1. *Note that the condition $\#S = \#T$ is essential in proving that selective Kemeny rules are pairwise. Indeed the more demanding condition of extrema independence, at which this equality is not required, is not implied by pairwiseness. Selective Kemeny rules are not extrema independent which can be deduced by considering for instance maximal conflicts $p = (R_1^S, -R_1^{N-S})$ and $q = (\widehat{R}_1^S, -\widehat{R}_1^{N-S})$ where $\#S > \#N - S$. Then $\psi_{Kemeny}(p) = \varphi_{Kemeny}(p) = \{R_1\}$ and $\psi_{Kemeny}(q) = \varphi_{Kemeny}(q) = \{\widehat{R}_1\}$.* △

As selective Kemeny rules are Pareto, hence nonimposed, and weakly extreme independent, it follows that they are not coalitional strategy-proof. Indeed consider $N = \{1,2\}$, $A = \{a,b,c\}$ and profiles p, q such that $p(1) = R_6 = bac$, $p(2) = q(2) = R_4 = cba$, $q(1) = R_1 = abc$ (although redundant for this example may take further $R_2 = acb$, $R_3 = cab$ and $R_4 = cba$). Then based on the indexation of the linear orders $\psi_{Kemeny}(p) = \{R_4\}$ and $\psi_{Kemeny}(q) = \{R_1\}$. As R_1 is strictly closer to $p(1)$ than R_4 it follows that ψ_{Kemeny} is not strategy-proof and therewith not coalitional strategy-proof. Similarly, selective super majority rules ψ_g are weakly extrema independent, non strategy-proof and not extrema independent.

Most probably, under some kind of *non-bossiness* condition, strategy-proofness implies update monotonicity. We stop here, however, as we hope that the above has convinced the reader that this subject is at least for the time being interesting enough to investigate further.

3.6 Appendix

The following Proposition shows that intersections of subsets, being sufficiently large, are not empty.

Proposition 3.6.1. *Let T^1 up to T^l be a collection of l subsets of finite and non-empty set N such that $\#T^j > (l-1)\cdot \#N/l$. Then $\cap\{T^j : j \in \{1,\dots,l\}\} \neq \emptyset$.*

Proof. To prove the contra position suppose $\cap\{T^j : j \in \{1,\dots,l\}\} = \emptyset$. Now we may take T^j such that for all i in N there are precisely $l-1$ sets say T^{i_1} up to $T^{i_{l-1}}$ such that i is in each of these. So, as

$$\sum_{j=1}^{l} \#T^j = \sum_{i\in N} \#\{T^j : i \in T^j\}$$

it follows that $\sum_{j=1}^{l} \#T^j = \#N\cdot(l-1)$. Let $\#T^1 \leq \#T^j$ for all $j \in \{1,2,\dots,l\}$. Then $\#T^1\cdot l \leq \sum_{j=1}^{l} \#T^j = \#N\cdot(l-1)$. But then $\#T^1 \leq (l-1)\cdot \#N/l$, which proves the contra position. □

Next we discuss a result on linear orders concerning the Kemeny distance.

Proposition 3.6.2. *Let R and R^{ab} be two linear orders in $\mathbb{L}_{ab}$. Let $R^{ba} = \sigma_{ab}R^{ab}$, where σ_{ab} is the permutation on A such that $\sigma_{ab}(a) = b$, $\sigma_{ab}(b) = a$ and $\sigma_{ab}(c) = c$ for all $c \in A - \{a,b\}$. Then $\delta(R,R^{ab}) < \delta(R,R^{ba})$.*

Proof. Let $\overline{R}^{ba}, \overline{R}^{ab} \in \mathbb{L}$ be two linear orders that are between R^{ba} and R such that $\overline{R}^{ba}$ and $\overline{R}^{ab}$ form an elementary change in ba. Then it is straightforward to see that $\delta(\overline{R}^{ba}, R^{ba}) = \delta(\overline{R}^{ab}, R^{ab})$. Furthermore by betweenness, $\delta(R, R^{ba}) = \delta(R, \overline{R}^{ab}) + \delta(\overline{R}^{ab}, \overline{R}^{ba}) + \delta(\overline{R}^{ba}, R^{ba})$. By triangular inequality, $\delta(R, R^{ab}) \leq \delta(R, \overline{R}^{ab}) + \delta(\overline{R}^{ab}, R^{ab}) = \delta(R, \overline{R}^{ab}) + \delta(\overline{R}^{ba}, R^{ba})$. Hence $\delta(R, R^{ab}) < \delta(R, R^{ba})$. □

Chapter 4

Weighted Distances Between Preferences

4.1 Introduction

It is plausible to measure the degree of disagreement by the number of ordered pairs that are ranked oppositely. The well-known Kemeny distance (Kemeny, 1959) is commonly used to that end. Given a strict preference $R_1 = abc$, which is interpreted as: a is preferred to b, b to c, and by transitivity a to c, the Kemeny distance between R_1 and another strict preference $R_2 = bac$ is 1, because the two preferences only disagree on how to order a and b. For $R_3 = acb$, the distance between R_1 and R_3 would be again 1 (the disagreement is on how to order b and c). Consider a situation where R_1 is perceived to be closer to R_3 than it is to R_2 because the disagreement between R_1 and R_3 only concerns the bottom two alternatives in these preferences, i.e., b and c, whereas the disagreement between R_1 and R_2 concerns the top two alternatives in these preferences, i.e., a and b.

Assigning weights to the position of differences in preferences might be useful in many applications. For instance, consider three search engines, $(G)oogle$, $(Y)ahoo$ and $(B)ing$. Given a word search, assume these engines give a strict ranking of the same millions of alternatives, i.e., links to websites that are relevant to the search term. All three engines rank ten links per page. Suppose that G and Y provide identical results in the first three pages and differ in the remaining millions of websites. Suppose also that G differs from B in the first three pages but is identical to B for the remaining hundreds of thousands of pages. Nevertheless, it is natural to argue that G is closer to Y than it is to B, even if G and Y disagree on how to rank the remaining millions of links after the third page. This is because what matters most, for internet users, is the first two-three pages (BBC[1], 2006), i.e., the first $20-30$ links that are ranked.

Note that the distribution of weights may not always follow a monotonically decreasing pattern. In fact, in cases where certain positions in preferences are critical, the distance caused by a change in those positions might be more important than changes in other positions. An example would be the ranking of football teams in a league, where the first f teams of the last week's ranking are to be promoted, e.g., to join the European Champions' League. In such cases, a swap in positions f and $f+1$ might be much more critical, hence influential in the distance, than a swap between the top two football teams. Therefore, it makes sense to assign more weight to a change at those critical positions.

In this chapter, we propose distance functions in a similar spirit as that of the Kemeny distance, i.e., respectful to the number of disagreements, but we also allow variation in the treatment of different pairs of disagreements. We provide two conditions that essentially characterize a class of distance functions, which we call the "weighted distance functions". The first one, "positional neutrality" is a neutrality condition towards the position of disagreement between two *adjacent* preferences, i.e., preferences which have only one disagreement. The second one, "decomposability" is a condition which requires that the distance between any two preferences is equal to the sum of distances of pairs of adjacent preferences which establish a *path* between the two.

In Section 4.2, we introduce the notation and basic conditions for distance functions over strict preferences together with the two new conditions we introduce. In Section 4.3 we introduce the class of weighted distance functions and discuss some members of this class: the Kemeny distance, the

[1]http://news.bbc.co.uk/2/hi/technology/4900742.stm.

Lehmer distance, the inverse Lehmer distance, and the path-minimizing distance. Section 4.4 shows the logical relations between these distances and the effect of the weight distribution on the triangular inequality condition. It is shown that only for one weighted distance function, the path-minimizing distance, the variation in the weights does not affect the triangular inequality condition. Section 4.5 concludes the chapter and points to possibilities for further research.

4.2 The Model

4.2.1 Notation

Let A be the set of alternatives with cardinality $m \geq 3$. Strict preferences are modeled by linear orders[2] over A, and the set of all linear orders is denoted by $\mathscr{L}$. Given $R \in \mathscr{L}$, $a\,R\,b$ is interpreted as a is strictly preferred to b, i.e., the ordered pair $(a,b) \in R$. We sometimes write $R = \ldots a \ldots b \ldots$ if $a\,R\,b$, and $R = \ldots ab \ldots$ if $a R\,b$ and there exists no $c \in A \setminus \{a,b\}$ such that $a R\,c$ and $c R\,b$, i.e., a and b are adjacent in R. Given any $a \in A$, $UC(a,R) = \{b \in A \mid b\,R\,a\}$ is the "upper contour set" of a in R, i.e., the set of alternatives that are ranked above a in the linear order R. Correspondingly, $LC(a,R) = \{b \in A \mid a\,R\,b\}$ is the "lower contour set" of a in R.

For $l = 1,2,\ldots,m$, $R(l)$ denotes the alternative in the l^{th} position in R, and we use $rank(a,R)$ to denote the position of alternative a in R. Given a linear order $R \in \mathscr{L}$ and some subset of alternatives $B \subseteq A$, $R|_B$ denotes the preference reduced to B, i.e., $R|_B = R \cap (B \times B)$. Given any two linear orders $R,R' \in \mathscr{L}$, the set difference $R \setminus R'$ denotes the set of ordered pairs that exist in R and not in R', i.e., $\{(x,y) \in A \times A \mid x\,R\,y \text{ and } y\,R'\,x\}$. Two linear orders $(R,R') \in \mathscr{L}^2$ form an *elementary change*[3] in position k whenever $R(k) = R'(k+1)$, $R'(k) = R(k+1)$ and for all $t \not\in \{k,k+1\}$, $R(t) = R'(t)$, i.e. $|R \setminus R'| = 1$. Given any two distinct linear orders $R,R' \in \mathscr{L}$, a vector of linear orders $\rho = (R_0,R_1,\ldots,R_k)$ is called a *path* between R and R' if $k = |R \setminus R'|$, $R_0 = R$, $R_k = R'$ and for all $i = 1,2,\ldots k$, (R_{i-1},R_i) forms an

[2] Complete, transitive and antisymmetric binary relations.

[3] We omit the paranthesis whenever it is clear and write R,R' instead.

elementary change. For the special case where $R = R'$, we denote the unique path as $\rho = (R,R)$.

A bijection $\pi : \{1,2,\dots,m\} \to \{1,2,\dots,m\}$ is called a *permutation* and the set of all permutations is denoted by Π. We use $\pi(R)$ (or $\pi \cdot R$) to denote the permutation of the linear order R by π, i.e., $\pi(R) = R'$ if and only if $R(i) = R'(\pi(i))$ for all $i = 1,2,\dots,m$. Given $R,R' \in \mathscr{L}$, a permutation $\pi \in \Pi$ is called the *corresponding permutation*[4] for R,R', if $\pi(R) = R'$. We denote the *conjugate* of a permutation π by $\tilde{\pi} \in \Pi$, i.e., $\tilde{\pi}(R') = R$ if and only if $\pi(R) = R'$. A permutation that swaps the k^{th} alternative of a linear order with the $(k+1)^{th}$ is called an *elementary permutation* and is denoted by σ_k. Hence, σ_k is the corresponding permutation for any $R,R' \in \mathscr{L}$ which form an elementary change in position k. The set of all elementary permutations is denoted by $S = \{\sigma_1,\sigma_2,\dots,\sigma_{m-1}\} \subseteq \Pi$. The identity permutation is denoted by σ_0.

Note that the set of all permutations Π over the set of alternatives A forms a symmetric group (also known as a permutation group) with the group operator "$\cdot$" which implies any permutation $\pi \in \Pi$ can be obtained by composition of some other permutations with the group operator, e.g., $\pi'' \cdot \pi' \cdot R = \pi \cdot R$ refers to the situation where R is first permuted via π' and then π'', and $\pi'' \cdot \pi' = \pi$. Note, however, that unless $m \leq 2$, the group fails commutativeness, e.g., for $R = abc$; note that $\sigma_1 \cdot \sigma_2 \cdot R = cab$ whereas $\sigma_2 \cdot \sigma_1 \cdot R = bca$.

In this chapter, we will especially make use of compositions of permutations via elementary permutations in S. Since Π is a permutation group it has S, as the generator set, which means every permutation $\pi \in \Pi$, including the identity permutation σ_0, can be expressed by some composition of elements of S. Let $I(\pi)$ denote the size of π, which is the minimal number of elementary permutations required to obtain π via the group operator. For instance, let π be a permutation over $\{1,2,3\}$ such that $\pi(1) = 3$, $\pi(2) = 1$ and $\pi(3) = 2$. Obviously $\pi = \sigma_2 \cdot \sigma_1$, i.e., applying σ_1, and σ_2 respectively yields the same result as applying π. Therefore $I(\pi) = 2$. Note that for the identity permutation, therefore, we have $I(\sigma_0) = 0$. Note also that for two linear orders $R,R' \in \mathscr{L}$ with the corresponding permutation $\pi \in \Pi$, we have that

[4]We omit this expression whenever it is clear which permutation we employ.

$I(\pi) = |R \setminus R'|$. Next we define *factorizations*, i.e., compositions of permutations via elementary/identity permutations.

Definition 4.2.1. *Given $\pi \in \Pi$ and some positive integer $r \geq I(\pi)$, a vector of elementary / identity permutations $f = (f(1), f(2), \ldots, f(r)) \in (S \cup \{\sigma_0\})^r$ is called a* factorization *of π whenever for all $i = 1, 2, \ldots r-1$:*

a) $f(r) \cdot f(r-1) \cdot \ldots \cdot f(1) = \pi$,

b) $f(i) \neq f(i+1)$.

Next we define the minimal factorizations, i.e., the compositions that require the fewest possible elementary permutations. These factorizations are also known as *reduced factorizations*.

Definition 4.2.2. *Given $\pi \in \Pi$, a factorization d of π is called a* decomposition *of π whenever:*

a) $d = \sigma_0$ if $I(\pi) = 0$,

b) $d = (d(1), d(2), \ldots, d(I(\pi)))$ if $I(\pi) > 0$.

For any factorization $f = (f(1), f(2), \ldots, f(r))$ of π, if $r > I(\pi)$, i.e., the number of inversions is not minimal in the factorization f, then f is called a *non-reduced factorization* of π.

We denote the set of all decompositions of a permutation π by D_π. In case there are many permutations under consideration, we distinguish decompositions by using the permutations as superscript, e.g., d^π for π and $d^{\hat{\pi}}$ for $\hat{\pi}$. Note that factorizations, as well as decompositions, of a permutation are not necessarily unique, e.g., for $R = abc$ and $R' = cba$, the corresponding permutation π can be decomposed by $d = (\sigma_1, \sigma_2, \sigma_1)$ as well as by $d' = (\sigma_2, \sigma_1, \sigma_2)$. However, once a decomposition is given, then there is an *induced path*, i.e., a sequence of linear orders, starting from R and ending at R' via elementary changes.

Definition 4.2.3. *Given R, R' and $\pi \in \Pi$, let $d = (d(1), d(2), \ldots, d(k)) \in D_\pi$ be a decomposition of π. A vector of linear orders $\rho_d = (\rho_d(1), \rho_d(2), \ldots, \rho_d(k+1))$ is called the* path induced by *d between R, R' whenever:*

a) $\rho_d(1) = R$ and $\rho_d(k+1) = R'$, i.e., the sequence starts with R and ends with R', *b)* $\rho_d(i+1) = d(i) \cdot \rho_d(i)$ *for all* $i = 1,2,\ldots,k$*, i.e., all consecutive linear orders in the path form elementary changes in the positions induced by the decomposition (or* $\rho_d = (R,R)$ *in case* $\pi = \sigma_0$*).*

Remark 4.2.1. *Given* $R,R' \in \mathscr{L}$*, let* π *be the corresponding permutation, then* ρ *is a path between* R *and* R' *if and only if there exists a decomposition* $d \in D_\pi$ *such that* $\rho = \rho_d$*, i.e.,* ρ *is a path induced by some decomposition* d *of the corresponding permutation. See Appendix 4.6.4 for a visualization of this correspondence between decompositions and paths.* △

For simplicity, we also refer to a path between R,R' induced by $d \in D_\pi$ as $\rho_d = (R_0, R_1, \ldots, R_{I(\pi)})$ whenever $R_0 = R$, $R_{I(\pi)} = R'$ and for all $i = 1,2,\ldots I(\pi)$, $R_i = \rho_d(i) = d(i) \cdot d(i-1) \cdots d(1) \cdot R$. Note that the path induced by a decomposition is unique as long as the initial start point (or the end point) is defined. Similarly every path also induces a unique decomposition which is a sequence of elementary permutations that correspond to the positions of the elementary changes on the path. For the special case $R = R'$, i.e., $d^\pi = \sigma_0$, we write the induced path as $\rho_d = (R,R)$ to avoid unnecessary complication.

Example 4.2.1. *Consider the linear orders* $R = abc$ *and* $R' = cba$*, and the corresponding permutation* π*. Let* d *be a decomposition of* π *such that* $d = (\sigma_1, \sigma_2, \sigma_1)$*. Let* $R_0 = R$ *and* $R_3 = R'$ *and consider the vertical rearrangement of these linear orders below. Then the path induced by d, denoted by* ρ_d*, is as follows:*

R_0	R_1	R_2	R_3
a	b	b	c
b	a	c	b
c	c	a	a

Note that the decomposition d *is not unique, in fact it is easy to see that* $d' = (\sigma_2, \sigma_1, \sigma_2)$ *is also a decomposition of* π *leading to a different path.* ⋄

R_0	R_1	R_2	R_3	R_4	R_5
a	a	a	(d)	d	d
b	b	(d)	a	a	(c)
c	(d)	b	b	(c)	a
(d)	c	c	(c)	b	b

4.2.2 Distance Functions and Properties

In the social choice literature, distance (also known as dissimilarity) functions and metric functions are often interchangeably used. We, however, follow the convention developed in Deza and Deza (2006), which is also employed in a similar study by García-Lapresta and Pérez-Román (2008). The main distinction is that a distance does not need to satisfy the triangular inequality whereas a metric is a distance function which also satisfies the triangular inequality. We follow this convention to later allow more variety in the set of weighted distance functions in the next section. Below are the conditions for distance functions:

Definition 4.2.4. *A function $\delta : \mathscr{L} \times \mathscr{L} \to \mathbb{R}$ is a distance function on the set of linear orders if it satisfies the following conditions:*

a) Non-negativity: $\delta(R,R') \geq 0$ for all $R,R' \in \mathscr{L}$,

b) Identity of indiscernibles: $\delta(R,R') = 0$ if and only if $R = R'$ for all $R,R' \in \mathscr{L}$,

c) Symmetry: $\delta(R,R') = \delta(R,R')$ for all $R,R' \in \mathscr{L}$.

A distance function δ is a metric if, in addition to the aforementioned conditions, it satisfies the triangular inequality condition, i.e., $\delta(R,R'') \leq \delta(R,R') + \delta(R',R'')$ for all $R,R',R'' \in \mathscr{L}$. We discuss the triangular inequality condition further in Section 4.4.3.

Next, we introduce two new conditions for distance functions. The first condition, *positional neutrality*, ensures that the elementary changes in the same positions are treated impartially. Hence, a distance function should assign the same distance to any two pairs of linear orders that form elementary changes in the same position. Therefore, the distance is neutral, in the sense that, as long as the swaps in alternatives happens at the same position, it remains unchanged.

Definition 4.2.5. ***Positional Neutrality:*** *A distance function δ satisfies positional neutrality if for all $k < m$ and for all elementary changes (R,R') and $(\bar{R},\bar{R}')$ in position k:*

$$\delta(R,R') = \delta(\bar{R},\bar{R}').$$

Note that this definition is equivalent to the following: "A distance function δ satisfies positional neutrality if for all $k < m$ and for all $R, \bar{R} \in \mathscr{L}$, $\delta(R, \sigma_k(R)) = \delta(\bar{R}, \sigma_k(\bar{R}))$". This equivalence is due to the fact that every elementary change is associated with a unique elementary permutation.

The second condition is presented in twofold. *Decomposability*, requires that the distance between two linear orders is equal to the sum of distances assigned to each elementary change on *some* path between these linear orders. *Strong decomposability*, however, requires that this statement holds for *all* paths between these linear orders. We present both conditions below where quantifiers in parentheses are for the strong version.

Definition 4.2.6. ***(Strong) Decomposability:*** *A distance function δ satisfies (strong) decomposability if for all R, R' and for (all) some path(s) $\rho = (R_0, R_1, \ldots, R_k)$ between R and R':*

$$\delta(R, R') = \textstyle\sum_{i=1}^{k} \delta(R_{i-1}, R_i).$$

Remark 4.2.2. *As hinted in Remark 4.2.1, the elementary changes on a path are defined on the basis of a decomposition of the corresponding permutation. Therefore an equivalent definition would be as follows: A distance function δ satisfies (strong) decomposability if for all R, R' and π, and for (all) some $d = (d(1), d(2), \ldots, d(k)) \in D_\pi$,*

$$\delta(R, R') = \sum_{i=1}^{k} \delta(\rho_d(i), \rho_d(i+1)).$$

△

The strong decomposability essentially implies that all paths between two linear orders lead to same distance. We show in Section 4.3 that strong decomposability is too demanding and it does not leave a lot of room to define distance functions.

Remark 4.2.3. *Note that decomposability imposes a choice of decomposition for any given permutation π. This means that any two pairs of linear orders, which share π as their corresponding permutation, will have the same decomposition. Then, the sequence of "positions of elementary changes" along* *their paths will also be the same. Together with positional neutrality this implies that the distances among linear orders in each pair will be identical. Formally; for any $\pi \in \Pi$ and for any four linear orders $R, R', \bar{R}, \bar{R}' \in \mathscr{L}$ such that $\pi(R) = R'$ and $\pi(\bar{R}) = \bar{R}'$, we have that $\delta(R, R') = \delta(\bar{R}, \bar{R}')$, i.e., the distance between linear orders is the same for all pairs of linear orders that are permuted in the same way.* △

4.3 Weighted Distance Functions

We define the class of weighted distance functions on the basis of the two conditions; positional neutrality and decomposability. The former ensures that the distance functions are sensitive and neutral to the positions of elementary changes. The latter requires that when two linear orders (not necessarily elementary changes) are considered, the distance between them should be decomposable into that of elementary changes between them so that the essence of positional neutrality can be extended to any two linear orders. We first introduce some functional forms to define weighted distance functions, then dwell on their properties.

Given an $(m-1)$−dimensional non-degenerate[5] weight vector such as $\omega = (\omega_1, \omega_2, \ldots, \omega_{m-1}) \in \mathbb{R}^{m-1}_{++}$, let $g_\omega : S \cup \{\sigma_0\} \to R^{m-1}_{+}$ be the associated weight function on the set of generators S of Π and the identity permutation such that:

$$g_\omega(\sigma_x) = \begin{cases} \omega_x & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases} \tag{4.1}$$

For any permutation $\pi \in \Pi$ and a factorization (possibly a decomposition) $f^\pi = (f(1), f(2), \ldots, f(t))$ of π, we make use of the weight function for the factorizations as well by setting $g(f^\pi) = \sum_{i=1}^{t} g(f^\pi(i))$. Next we define the class of weighted distance functions:

Definition 4.3.1. *A distance function $\delta : \mathscr{L} \times \mathscr{L} \to \mathbb{R}$ is called a* weighted distance function *if there exists a weight vector ω such that for all R, R' with corresponding permutation π, and for some decomposition $d \in D_\pi$:*

$$\delta(R, R') = g_\omega(d^\pi).$$

[5]All elements in the weight vector are positive.

Given a weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_{m-1}) \in \mathbb{R}^{m-1}_{++}$, we denote an associated weighted distance function by δ_ω, and the class of all weighted distance functions associated with ω by Δ_ω.

Note that it is almost straightforward to see that the class of weighted distance functions is defined by two conditions; positional neutrality and decomposability. In fact, the class of all distance functions that satisfy these two conditions correspond to the class of weighted distance functions.

Proposition 4.3.1. *Let δ be a distance function. δ satisfies positional neutrality and decomposability if and only if $\delta = \delta_\omega$ for some $\omega \in \Omega$, i.e., δ is a weighted distance function.*

Proof. If part follows immediately from the functional form g_ω. To show the only if part let δ satisfy the two conditions. By definition of distance functions and positional neutrality, for each $i = 1, 2, \ldots, m-1$ and for all elementary changes (R_i, R_{i+1}) in position i, there exists $c_i = \delta(R_i, R_{i+1}) > 0$. Now let $R, R' \in \mathscr{L}$ and $\pi \in \Pi$. By decomposability, there exists a path $\rho = (R_0, R_1, \ldots, R_k)$ between R and R' such that $\delta(R, R') = \sum_{i=1}^{k} \delta(R_{i-1}, R_i)$. By Remark 4.2.1, there exists $d \in D_\pi$ which induces ρ, i.e., $\rho = \rho_d$. Then for all elementary changes $(\rho_d(i), \rho_d(i+1))$ on the path ρ_d, the distance is c_x if and only $d(i) = \sigma_x$. Then letting $\omega = c = (c_1, c_2, \ldots, c_{m-1})$, we have that $g_\omega(d) = \sum_{i=1}^{k} \delta(R_{i-1}, R_i)$. □

Next, we focus on variation in weight vectors. We distinguish between classes of weight vectors such as $\bar{\Omega} \subsetneq \mathbb{R}^{m-1}_{+}$ denoting the class of monotonically decreasing weight vectors, i.e., for all $\omega \in \bar{\Omega}$, $\omega_i \geq \omega_j$ if and only if $i \leq j$. Similarly we denote the class of monotonically increasing weight vectors by $\underline{\Omega}$. The set of all possible weight vectors is denoted by Ω, i.e., all possible $(m-1)$-dimensional vectors of nonnegative real numbers.

Note that, given a weight vector ω and the associated weight function g_ω, positional sensitivity is not sufficient to express the distance between any two linear orders. This leaves a lot of variation in the class of weighted distance functions Δ_ω. In fact, from the weight function g_ω, the only information one can infer is that elementary changes in the k^{th} position are assigned a distance of ω_k. Decomposability furthermore makes sure that every linear order can be dissolved into elementary changes in some way so that positional neutrality is still reflected in those linear orders. Hence, distance functions are defined by the choice of the decomposition of the corresponding permutation of the linear orders.

In the proposition below we show that the strong version of decomposability leaves no room for variation in the class of weighted distance functions. In fact, the class of weighted distance functions Δ_ω can be nonempty if and only if the weight vector is constant. Formally:

Proposition 4.3.2. *Let δ_ω be a weighted distance function. δ_ω satisfies strong decomposability if and only if ω is a constant weight vector, i.e., $\omega = (c, c, \ldots, c)$ for some positive real number $c \in \mathbb{R}_{++}$.*

Proof. The if part is straightforward. To show the only if part, let δ_ω satisfy strong decomposability and suppose $\omega_i \neq \omega_{i+1}$ for some $i \in \{1, 2, \ldots, m-1\}$. Then consider some R, R' with corresponding permutation π such that $d = (\sigma_i, \sigma_{i+1}, \sigma_i) \in D_\pi$ is a decomposition of π. It is easy to see that there exists another decomposition[6] $d' \in D_\pi$ such that $d' = (\sigma_{i+1}, \sigma_i, \sigma_{i+1})$ (see Example 4.2.1). Then by strong decomposability, $\delta(R, R') = g_\omega(d) = g_\omega(d')$ which contradicts $\omega_i \neq \omega_{i+1}$. □

In the following subsections we focus on four examples within the class of weighted distance functions. The first one is the well-known Kemeny distance (Kemeny, 1959), which is defined only for the constant weight vector $\omega = (1, 1, \ldots, 1)$. Then, we introduce the Lehmer distance, and the inverse Lehmer distance which -regardless of the weight vector- are based on well-defined ex ante choices of decompositions for each permutation. Finally, we introduce the path-minimizing distance which chooses, ex post, a decomposition depending on the distribution in the weight vector.

4.3.1 Kemeny Distance

Kemeny (1959) introduced a distance function which can be used to model the concept of ideological distances between strict preferences, i.e., linear

[6]This is due to the fact that Π is a permutation group. In group theory, certain groups have the property that $\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}$ where σ_x is a generator and "·" is the group operator.

orders. Interestingly, the very same idea has numerous applications in other disciplines such as computer sciences, information theory, group theory etc. Other names for the same concept include: the Kendall tau distance (Kendall, 1938), bubble sort distance, swap distance, inversion metric, word metric, permutation swap, the Damerau-Levenshtein distance (Damerau, 1964; Levenshtein, 1966), the Hamming distance (Hamming, 1950), and so on and so forth. In fact, prior to Kemeny (1959), the use of this distance can even be traced back to Cramer (1750). Formally:

Definition 4.3.2. ***(Kemeny distance)*** *Given $R, R' \in \mathscr{L}$ and a corresponding permutation π, the Kemeny distance δ^K between R, R' is:*

$$\delta^K(R,R') = I(\pi) = |R \setminus R'|.$$

It is easy to see that the Kemeny distance is a weighted distance. It assigns a weight of 1 to each elementary change. It satisfies strong decomposability since the sum of 1's assigned to each elementary change on a path induced by any decomposition equals the size of the permutation $I(\pi)$. Together with Proposition 4.3.2, this implies that any weighted distance function δ_ω which satisfies strong decomposability is a multiple of the Kemeny distance, i.e., $c \times \delta^K$, where $c = \omega_i$ for all $i = 1, 2, \ldots, m-1$.

Remark 4.3.1. *Note that, as shown in Proposition 4.3.2, imposing strong decomposability restricts the class of weighted distance functions to only those with a constant weight vector. As we have explained in the introduction, our main motivation, however, is to find distance functions that possibly assign different weights to elementary changes in different positions. Hence we use the regular decomposability condition instead of the strong version.* △

4.3.2 Lehmer Distance

The inverse of a permutation, according to Knuth (1998) was first defined by Rothe (for a historical account, see Muir, 1906). By using the diagram Rothe introduced, a list of numbers (also known, now, as the Lehmer code developed by Lehmer, 1960) can be obtained for each permutation (See Example 4.6.1 in the appendix). The Lehmer code essentially was used to generate all possible permutations of any number of objects. In this work, it corresponds to a particular way of decomposing permutations.

Given any R,R' and a corresponding permutation π, a decomposition $d_L \in D_\pi$ is the *winners' decomposition* if it permutes R such that $R'(1)$ is carried to the 1^{st} position, then $R'(2)$ is carried to the 2^{nd} position and so forth. Iteratively R' will be achieved. We call the path induced by this decomposition the *winners' path* and denote it by ρ_L. We illustrate the winners' decomposition and the induced path below with an example. For the formal description of the winners' decomposition, see Appendix 4.6.2.

Example 4.3.1. *Let $R = abcd$ and $R' = dcab$. Then the winners' decomposition first permutes the alternative d to the top, thereafter c and so on. The induced path will look like:*

R_0	R_1	R_2	R_3	R_4	R_5
a	a	a	(d)	d	d
b	b	(d)	a	a	(c)
c	(d)	b	b	(c)	a
(d)	c	c	(c)	b	b

The winners' decomposition, therefore, is $d_L^\pi = (\sigma_3,\sigma_2,\sigma_1,\sigma_2,\sigma_1)$. This decomposition is well-defined for any two linear orders and so is the path $\rho_L = (R_0,R_1,\ldots,R_5)$. ◇

Definition 4.3.3. *Given any R,R' with π and any weight vector ω, the Lehmer distance is:*

$$\delta^L_\omega(R,R') = g_\omega(d^\pi_L). \tag{4.2}$$

Note that by construction, the Lehmer distance satisfies the identity of indiscernibles and nonnegativity conditions. We show symmetry in Proposition 4.6.1 in Appendix 4.6.2. Therefore, it is a distance function and by Equation 4.2, it is a weighted distance. Finally, by Proposition 4.3.1 it satisfies positional neutrality and decomposability. The triangular inequality is satisfied if $\omega \in \bar{\Omega}$, i.e., ω is a decreasing weight vector. We discuss this condition further in Section 4.4.3.

4.3.3 Inverse Lehmer Distance

An immediate dual of the winners' decomposition is the *losers' decomposition*. Given R,R' and a corresponding permutation π, a decomposition d_{IL}

is the *losers' decomposition* if it permutes R such that $R'(m)$ is carried to the m^{th} position, then $R'(m-1)$ is carried to the $(m-1)^{th}$ position and so forth. Iteratively R' will be achieved. We illustrate this decomposition and the induced *losers' path* below, denoted as ρ_{IL}, via the same linear orders as in Example 4.3.1. For the formal description of the losers' decomposition, see Appendix 4.6.3.

Example 4.3.2. *Let $R = abcd$ and $R' = dcab$. Then the losers' decomposition first permutes the alternative b to the bottom, thereafter a and so on. The induced path will look like:*

R_0	R_1	R_2	R_3	R_4	R_5
a	a	(a)	c	(c)	d
(b)	c	c	(a)	d	(c)
c	(b)	d	d	(a)	a
d	d	(b)	b	b	b

The losers' decomposition, therefore, is $d^{\pi}_{IL} = (\sigma_2, \sigma_3, \sigma_1, \sigma_2, \sigma_1)$. This decomposition is also well-defined for any two linear orders and so is the path $\rho_{IL} = (R_0, R_1, \ldots, R_5)$. ◇

Definition 4.3.4. *Given any R, R' with π, and any weight vector ω, the inverse Lehmer distance is:*

$$\delta^{IL}_{\omega}(R, R') = g_{\omega}(d^{\pi}_{IL}). \tag{4.3}$$

Note that by construction, the inverse Lehmer distance satisfies the identity of indiscernibles and nonnegativity conditions. We show symmetry in Proposition 4.6.2 in Appendix 4.6.3. Therefore it is a distance function and by Equation 4.3, it is a weighted distance. Finally by Proposition 4.3.1 it satisfies positional neutrality and decomposability. The triangular inequality is satisfied if $\omega \in \underline{\Omega}$, i.e., ω is a increasing weight vector. We discuss this condition further in Section 4.4.3.

Most of the aforementioned conditions, which the inverse Lehmer distance satisfies, are shown by a duality argument between the Lehmer distance and the inverse Lehmer distance. We discuss this argument, in detail, in Section 4.4.1.

4.3.4 Path-Minimizing Distance

As explained in the beginning of the section, the path minimizing distance does not choose a decomposition for each permutation ex ante. Instead, depending on the distribution of the weights in ω, it chooses a decomposition that induces a path with elementary changes that has the minimal sum of distances. Formally:

Definition 4.3.5. *Given any R,R' with π, and any weight vector ω, the path-minimizing distance is:*

$$\delta_{\omega}^{PM}(R,R') = \min_{d \in D_{\pi}}\{g_{\omega}(d)\}. \tag{4.4}$$

Note that for each weight vector ω, the path-minimizing distance chooses a decomposition that minimizes the weight function $g_{\omega}(d)$. We call such decompositions, *minimal decompositions* with respect to ω. By Remark 4.2.1, there exists a uniquely induced path for each of these decompositions. We denote such paths by ρ^{PM} and call them *minimal paths* between R and R' with respect to ω. Below we remark that the minimality of these paths are, in fact, preserved between any two point within the same path.

Remark 4.3.2. *Given a weight vector ω, consider $R,R' \in \mathscr{L}$ with π and a minimal path $\rho^{PM} = (R_0,R_1,R_2,\ldots,R_{k-1},R_k)$ with respect to ω between $R = R_0$ and $R' = R_k$. Then any portion of this minimal path is also a minimal path between its beginning and its end with respect to the same ω, i.e., the "subpath" $\rho = (R_i,R_{i+1},\ldots,R_{j-1},R_j)$, for $i,j \in \mathbb{N}$ such that $0 \leq i < j < k$, is a minimal path between R_i and R_j. Otherwise ρ^{PM} is not a minimal path between R and R', since it could have followed a different subpath between R_i to R_j.* △

4.4 Results

We now present our results about the class of weighted distances. We first provide some observations regarding the Lehmer distance and the inverse Lehmer distance. These two distances are shown to be, in a sense, the dual of one another. Thereafter we study the logical relations between each of the distances mention in the previous section. We finally study the triangular inequality under varying distributions of the weights in ω.

4.4.1 Duality Argument

Let us now dwell upon the duality between the Lehmer distance and the inverse Lehmer distance. The relationship between the two distances is not only the naming but further. In fact the winners' decomposition of some permutations looks quite similar to anothers' losers' decomposition. Given any linear order $R \in \mathscr{L}$, let $\hat{R}$ denote the inverse linear order, e.g., for $R = abcd$, $\hat{R} = dcba$. Consider the linear orders $R = abcd$ and $R' = dcab$ in Example 4.3.2 and the inverse linear orders $\hat{R}, \hat{R}'$. Let us call the corresponding permutation of these inverse linear orders as: the *dual of* π and denote by $\hat{\pi}$ (not to be confused by $\tilde{\pi}$, i.e., the conjugate of π). Below is the losers' path of π and the winners' path of $\hat{\pi}$.

$$\underbrace{\begin{array}{c|c|c|c|c|c} R_0 & R_1 & R_2 & R_3 & R_4 & R_5 \\ a & a & (\mathbf{a}) & c & (\mathbf{c}) & d \\ (\mathbf{b}) & c & c & (a) & d & (c) \\ c & (b) & d & d & (a) & a \\ d & d & (b) & b & b & b \end{array}}_{\text{Losers' path } \rho^{\pi}_{IL} \text{ for } \pi} \qquad \underbrace{\begin{array}{c|c|c|c|c|c} \hat{R}_0 & \hat{R}_1 & \hat{R}_2 & \hat{R}_3 & \hat{R}_4 & \hat{R}_5 \\ d & d & (b) & b & b & b \\ c & (b) & d & d & (a) & a \\ (\mathbf{b}) & c & c & (a) & d & (c) \\ a & a & (\mathbf{a}) & c & (\mathbf{c}) & d \end{array}}_{\text{Winners' path } \rho^{\hat{\pi}}_{L} \text{ for } \hat{\pi}}$$

As observed in the figure above, for each linear order R_i in the losers' path for π its inverse linear order $\hat{R}_i$ occurs at the exact same point in the winners' path for $\hat{\pi}$. This duality is observed in all decompositions of π and $\hat{\pi}$. We state this formally in a remark.

Remark 4.4.1. *Given any $d \in D_\pi$ there exists a dual decomposition $\hat{d} \in D_{\hat{\pi}}$ of d such that for all $x = 1,2,\ldots,m-1$ and for all $i = 1,2,\ldots,I(\pi)$, $\sigma_x = d(i)$ if and only if $\sigma_{m-x} = \hat{d}(i)$.* △

Next we formalize the observation that the dual of a losers' decomposition for π is the winners' decomposition in $\hat{\pi}$. As the dual permutation of $\hat{\pi}$ will be $\hat{\hat{\pi}} = \pi$, i.e., inverting linear orders twice will result in the original linear order, we can also conclude that the dual of a winners' decomposition for π, is the losers' decomposition for $\hat{\pi}$.

Proposition 4.4.1. *Given $R, R' \in \mathscr{L}$, and the corresponding permutation $\pi \in \Pi$, let $\hat{R}, \hat{R}'$ denote the inverse linear orders for R, R' and let $\hat{\pi} \in \Pi$ be the corre-*

sponding permutation of these inverse linear orders. If $d^{\pi}_{IL} = (\sigma_{a_1}, \sigma_{a_2}, \ldots, \sigma_{a_k})$ *denote the losers' decomposition of* $\pi \in \Pi$*, then:*

$$d^{\hat{\pi}}_{L} = (\sigma_{m-a_1}, \sigma_{m-a_2}, \ldots, \sigma_{m-a_k}).$$

Proof. Note that for each R_i on the losers' path of π, there exists $\hat{R}_i$ in the winners' path of $\hat{\pi}$. Therefore for any i, if R_i, R_{i+1} is an elementary change in position k, then $\hat{R}_i, \hat{R}_{i+1}$ is an elementary change in position $m-k$, since the latter two is the inverse linear orders of the former two. Then, the relevant elementary permutation in $d^{\pi}_{IL}(i+1) = \sigma_k$ for the former, whereas the relevant elementary permutation in $d^{\hat{\pi}}_{L}(i+1) = \sigma_{m-k}$ for the latter. □

Now, given a weight vector $\omega \in \Omega$, let $\hat{\omega} \in \Omega$ be such that for all $i = 1, 2, \ldots, m-1$, $\omega_i = \hat{\omega}_{m-i}$, i.e., the vector $\hat{\omega}$ is the "inverse weight vector" of ω. Next, we remark about the connection between the duality in decompositions and the weight vectors.

Remark 4.4.2. *Given a weight vector* ω*, an associated weight function* g_{ω}*, the total sum of weights for a decomposition* $d \in D_{\pi}$ *is equivalent to that of the dual decomposition* $\hat{d} \in D_{\hat{\pi}}$ *under the inverse weight vector* $\hat{\omega}$*. Formally, given any weight vector* $\omega \in \Omega$*, and its inverse* $\hat{\omega}$*, consider a decomposition* $d \in D_{\pi}$ *and the dual decomposition* $\hat{d} \in D_{\hat{\pi}}$*. We have the following relation:*

$$g_{\omega}(d) = g_{\hat{\omega}}(\hat{d}).$$

△

Considering the remark above, an immediate corollary to Proposition 4.4.1 is about the duality between the Lehmer and the inverse Lehmer distances:

Corollary 4.4.1. $g_{\omega}(d^{\pi}_{IL}) = g_{\hat{\omega}}(d^{\hat{\pi}}_{L})$*, and* $g_{\hat{\omega}}(d^{\hat{\pi}}_{IL}) = g_{\omega}(d^{\pi}_{L})$*.*

The significance of Proposition 4.4.1 and Corollary 4.4.1 is that we can carry most of the results for the Lehmer distance to the inverse Lehmer distance by means of the corollary above. Furthermore, since the Lehmer code is more often used in the computer sciences literature, it also enables one to easily calculate the inverse of it by using the Lehmer code for another permutation.

4.4.2 Logical Relations Between δ_ω^L, δ_ω^{IL} and δ_ω^{PM}

Now we focus on the effects of variation in the weight distribution. It turns out that for particular classes of weight distributions, some weighted distances are equivalent. There is a varying degree of computational complexity in calculating the weighted distances. For instance given a weight distribution finding the path that minimizes the sum of weights on all possible paths between two linear orders is equivalent to a short-path problem in graph theory. However finding the winners'/losers' decompositions is much less complex. Therefore showing equivalence between these distances for certain weight distributions can actually be useful in terms of computational complexity. We mention this briefly in the conclusion.

First, we show that if the weight vector is monotonically decreasing then the Lehmer distance and the path-minimizing distance are equal.

Proposition 4.4.2. $\delta_\omega^L(R,R') = \delta_\omega^{PM}(R,R')$ *for all* $R,R' \in \mathscr{L}$ *if and only if* $\omega \in \bar{\Omega}$*, i.e., the Lehmer distance equals the path-minimizing distance if and only the weight vector is decreasing.*

Proof. **(If part)** Let ω be a decreasing weight vector. We will show by induction on the size of difference between $R,R' \in \mathscr{L}$, i.e., $k = |R \setminus R'|$, that for all decreasing weight vectors $\omega \in \bar{\Omega}$ and for all $R,R' \in \mathscr{L}$ with corresponding permutation π, $\delta_\omega^L(R,R') = \delta_\omega^{PM}(R,R')$.

(Induction basis:) Take any $R,R' \in \mathscr{L}$ with π such that $|R \setminus R'| = 1$. Then there exists a unique decomposition $\{d\} = D_\pi$ of π such that $d \in S$. Then, by positional neutrality, $\delta_\omega^L(R,R') = \delta_\omega^{PM}(R,R') = g_\omega(d)$.

(Induction hypothesis:) Take any $R,R' \in \mathscr{L}$ with π such that $|R \setminus R'| = k$. Assume $\delta_\omega^L(R,R') = \delta_\omega^{PM}(R,R')$.

(Induction step:) Take any $R,R' \in \mathscr{L}$ with π such that $|R \setminus R'| = k+1$. Let R and R' be denoted by R_0 and R_{k+1}. Let $\rho^{PM} = (R_0,R_1,\ldots,R_k,R_{k+1})$ be a minimal path induced by a minimal decomposition $d^{PM} \in D_\pi$ of the path-minimizing distance $\delta_\omega^{PM}(R_0,R_{k+1})$. Suppose the elementary permutation in the last switch of the decomposition d^{PM} is in the i^{th} position of R_k, i.e., $d^{PM}(k+1) = \sigma_i$ and $\sigma_i(R_k) = R_{k+1}$. Note that by the induction hypothesis, we have that $\delta_\omega^{PM}(R_0,R_k) = \delta_\omega^L(R_0,R_k)$. By Remark 4.3.2 and decomposability of δ_ω^{PM}, we have $\delta_\omega^{PM}(R_0,R_{k+1}) = \delta_\omega^{PM}(R_0,R_k) + \omega_i = \delta_\omega^L(R_0,R_k) + \omega_i$.

Let $R_k = a_1 a_2 \dots a_i a_{i+1} \dots a_m$. Then, by construction we have that $R_{k+1} = a_1 a_2 \dots a_{i+1} a_i \dots a_m$. Then, let $\bar{\pi}$ be such that $\bar{\pi}(R_0) = R_k$. By construction, the Lehmer codes (See Definition 4.6.1 in the appendix) for each of the permutations, π and $\bar{\pi}$ are equal except for the i^{th} and $(i+1)^{th}$ components. Namely, for all $t \in \{1,2,\dots,m\} \setminus \{i, i+1\}$, $|L(\bar{\pi})_t| = |L(\pi)_t|$ and therefore, for the same values of t, we also have $d^{\bar{\pi}}_{L_t} = d^{\pi}_{L_t} = (\sigma_{t+|L(\pi)_t|-1}, \sigma_{t+|L(\pi)_t|-2}, \dots, \sigma_t)$. This implies $g(d^{\bar{\pi}}_{L_t}) = g(d^{\pi}_{L_t})$. Note that $L(\pi)_i = \{(x,y) \in R \setminus R' \mid y = R'(i)\} = R \setminus R' \cap [UC(R'(i),R) \times R'(i)]$. Therefore, $|L(\bar{\pi})_i| = |L(\pi)_{i+1}|$ and $|L(\bar{\pi})_{i+1}| = |L(\pi)_i| - 1$. Consider the Lehmer distances between R_0, R_k and R_0, R_{k+1} which are as follows:

$$\delta^L_\omega(R_0,R_{k+1}) = g_\omega(d^\pi_L) = g_\omega(d^\pi_{L_1}) + g_\omega(d^\pi_{L_2}) + \dots + g_\omega(d^\pi_{L_m}),$$

$$\delta^L_\omega(R_0,R_k) = g_\omega(d^{\bar{\pi}}_L) = g_\omega(d^{\bar{\pi}}_{L_1}) + g_\omega(d^{\bar{\pi}}_{L_2}) + \dots + g_\omega(d^{\bar{\pi}}_{L_m}).$$

Expanding these two expressions and subtracting the latter from the former by inserting $|L(\pi)_{i+1}|$ instead of $|L(\bar{\pi})_i|$, and $|L(\pi)_i| - 1$ instead of $|L(\bar{\pi})_{i+1}|$, we end up with:

$$\delta^L_\omega(R_0,R_{k+1}) - \delta^L_\omega(R_0,R_k) = g_\omega(\sigma_{|L(\pi)_{i+1}|+i}).$$

This implies that $\delta^L_\omega(R_0,R_{k+1}) = \delta^L_\omega(R_0,R_k) + \omega_{|L(\pi)_{i+1}|+i}$. Remember that $\delta^{PM}_\omega(R_0,R_{k+1}) = \delta^L_\omega(R_0,R_k) + \omega_i$. Furthermore, since ω is decreasing, we have that $\omega_{|L(\pi)_{i+1}|+i} \le \omega_i$. Therefore $\delta^L_\omega(R_0,R_{k+1}) \le \delta^{PM}_\omega(R_0,R_{k+1})$. Hence, by the definition of the path-minimizing distance, $\delta^L_\omega(R_0,R_{k+1}) = \delta^{PM}_\omega(R_0,R_{k+1})$.

(**Only if part**) Let $\delta^L_\omega(R,R') = \delta^{PM}_\omega(R,R')$ for all $R,R' \in \mathscr{L}$. Suppose for a contradiction ω is not decreasing, i.e., for some $i = 1,2,\dots,m-1$, $\omega_i < \omega_{i+1}$. Let $\omega_i = x$ and $\omega_{i+1} = x + \epsilon$ for some $x, \epsilon > 0$. Consider $R_1,R_2,R_3,R_4,R_5,R_6 \in \mathscr{L}$ such that $R_1 = a_1 a_2 \dots a_i a_{i+1} a_{i+2} \dots a_m$ and all six linear orders are identically ranked except for a_i, a_{i+1} and a_{i+2}:

$$\begin{aligned}
R_1 &= \dots a_i a_{i+1} a_{i+2} \dots, \\
R_2 &= \dots a_i a_{i+2} a_{i+1} \dots = \sigma_{i+1} \cdot R_1, \\
R_3 &= \dots a_{i+2} a_i a_{i+1} \dots = \sigma_i \cdot R_2, \\
R_4 &= \dots a_{i+2} a_{i+1} a_i \dots = \sigma_{i+1} \cdot R_3, \\
R_5 &= \dots a_{i+1} a_{i+2} a_i \dots = \sigma_i \cdot R_4, \\
R_6 &= \dots a_{i+1} a_i a_{i+2} \dots = \sigma_{i+1} \cdot R_5.
\end{aligned}$$

Note that $|A| = m \geq 3$, therefore $|\mathscr{L}| \geq 6$ and the aforementioned linear orders always exist in $\mathscr{L}$. Then, consider the winners' path ρ_L for the Lehmer distance $\delta^L_\omega(R_1,R_4)$:

$$\rho_L = (R_1,R_2,R_3,R_4)$$

Then the Lehmer distance is: $\delta^L_\omega(R_1,R_4) = \omega_{i+1} + \omega_i + \omega_{i+1}$. Note, however, that there exists another path between R_1 and R_4 (in fact, the losers' path, $\rho_{IL} = (R_1,R_6,R_5,R_4)$ which results in a distance of $\omega_i + \omega_{i+1} + \omega_i$ (in fact, the inverse Lehmer distance). Obviously the latter path induces a smaller distance $3x+\epsilon$ whereas the Lehmer distance is $3x+2\epsilon$. Hence $\delta^L_\omega(R_1,R_4) > \delta^{PM}_\omega(R_1,R_4)$, which is a contradiction. □

Considering the duality between the inverse Lehmer and the Lehmer distance, it is very intuitive, after Proposition 4.4.2, that the inverse Lehmer distance should also be related to the path-minimizing distance in case the weight vector is inverted. We show in the following proposition that if the weight vector is monotonically increasing then the inverse Lehmer distance and the path-minimizing distance are equivalent.

Proposition 4.4.3. $\delta^{IL}_\omega(R,R') = \delta^{PM}_\omega(R,R')$ *for all* $R,R' \in \mathscr{L}$ *if and only if* $\omega \in \underline{\Omega}$, *i.e., the inverse Lehmer distance is the minimal weighted distance if and only the weight vector is increasing.*

Proof. (**If part**) Let ω be a increasing weight vector. Let $R,R' \in \mathscr{L}$ with π. Suppose for a contradiction, $g_\omega(d^\pi_{IL}) > g_\omega(d^\pi_{PM})$. Consider the inverse weight vector $\hat{\omega}$, and the dual permutation $\hat{\pi}$ of π, and the dual decompositions $d^{\hat{\pi}}_L$ and $d^{\hat{\pi}}_{PM}$ of (respectively) d^π_{IL} and d^π_{PM}. By the duality argument in Section 4.4.1, and Corollary 4.4.1, we have that $g_{\hat{\omega}}(d^{\hat{\pi}}_L) > g_{\hat{\omega}}(d^{\hat{\pi}}_{PM})$. Note that $\hat{\omega} \in \bar{\Omega}$, i.e., $\hat{\omega}$ is a decreasing weight vector which is a contradiction to Proposition 4.4.2.

(**Only if part**) Similar duality argument as in Proposition 4.4.2 follows. □

4.4.3 Triangular Inequality

Next we introduce a lemma which is crucial for showing the triangular inequality of the path-minimizing distance. The lemma argues that, given a permutation, if in a sequence of elementary changes (induced by a non-reduced factorization), two adjacent alternatives are swapped twice then there can be a shorter factorization for the same permutation.

Lemma 4.4.1. *Let $R = \ldots xy \ldots$ with $rank(x,R) = i$ and $R' = \ldots xy \ldots$ with $rank(x,R') = j$ such that $i \neq j$ and let π be the corresponding permutation. Let $f = (f(1), f(2), \ldots, f(t))$ be a factorization of π such that $f(1) = \sigma_i$ and $f(t) = \sigma_j$. Then $\tilde{f} = (f(2), f(3), \ldots, f(t-1))$ is also a factorization of π.*

Proof. As $f(1) = \sigma_i$ and $f(t) = \sigma_j$, the induced path starting from R ending at R', i.e., $\rho_f = (R_0, R_1, \ldots, R_k, \ldots, R_{t-1}, R_t)$, with $R_0 = R$ and $R_t = R'$, looks like:

$$\begin{array}{lcl}
R_0 & = & \ldots\ldots xy \ldots\ldots \\
R_1 = & = & \ldots\ldots yx \ldots\ldots \\
\vdots & & \\
R_k & = & \ldots y \ldots x \ldots\ldots \\
\vdots & & \\
R_{t-1} & = & \ldots yx \ldots\ldots\ldots \\
R_t & = & \ldots xy \ldots\ldots\ldots
\end{array}$$

Then, for R_1 and R_{t-1} with corresponding permutation $\bar{\pi}$, we have $\bar{f} = (f(2), f(3), \ldots, f(t-1))$ as a factorization of $\bar{\pi}$. Now, for each linear order R_l for $l = 1, 2, \ldots, t-1$ in the subpath $\rho_{\bar{\pi}}$, consider a renaming of alternatives and write x instead of y and y instead of x. Let these new linear orders be denoted as R_l^{xy} for $l = 1, 2, \ldots, t-1$. Note that this changes neither the corresponding permutation $\bar{\pi}$ nor the factorization $\bar{f}$. Hence also for R_1^{xy} and R_{t-1}^{xy}, $\bar{\pi}$ is the corresponding permutation and $\bar{f}$ is a factorization of $\bar{\pi}$. As $R_1^{xy} = R_0$ and $R_{t-1}^{xy} = R_t$, and the corresponding permutations are unique, we conclude $\bar{\pi} \equiv \pi$. Therefore $\bar{f}$ is a factorization of π.

$\square$

Now we prove that the only weighted distance that always satisfies the triangular inequality condition, regardless of the distribution of weights, is the path-minimizing distance. This also implies that the only weighted generalization of the Kemeny distance within the class of weighted distances is the path-minimizing distance.

Theorem 4.4.1. *Given any weight vector $\omega \in \Omega$, a weighted distance function δ_ω satisfies the triangular inequality condition if and only if $\delta_\omega = \delta_\omega^{PM}$, i.e., δ_ω^{PM} is the only weighted distance function that satisfies the triangular inequality for any $\omega \in \Omega$.*

Proof. (**If part**) Take any $\omega \in \Omega$. Let $\delta_\omega = \delta_\omega^{PM}$. Take any R, R', R'' and let π, π', π'' be such that $\pi(R) = R'$, $\pi'(R) = R''$, and $\pi''(R') = R''$.

Let $d_1 \in D_\pi$, $d_2 \in D_{\pi''}$, and $d_3 \in D_{\pi'}$ be the path minimizing decompositions for respective permutations. Let $k_1, k_2, k_3 \in \mathbb{N}$ be the size of the these decompositions. We want to show that $\delta_\omega^{PM}(R, R'') \leq \delta_\omega^{PM}(R, R') + \delta_\omega^{PM}(R', R'')$, i.e., $g_\omega(d_3) \leq g_\omega(d_1) + g_\omega(d_2)$. Now let $f = (d_1, d_2)$ be a sequence of elementary permutations joining d_1 and d_2, consecutively. Note that f is a factorization of π', i.e., $f(k_1+k_2) \cdot f(k_1+k_2-1) \cdot \ldots \cdot f(1) \cdot R = R''$. Furthermore $g_\omega(f) = g_\omega(d_1) + g_\omega(d_2)$.

Case 1: If $k_1 + k_2 = k_3$ then f is in fact a decomposition of π' and R' is on the path $\rho_f = (R = R_0, R_1, \ldots, R_{k_1+k_2} = R'')$ induced by this decomposition, in particular $R' = R_{k_1} = \rho_f(k_1 + 1)$. Then, by definition of δ_ω^{PM}, $g_\omega(d_3) \leq g_\omega(f) = g_\omega(d_1) + g_\omega(d_2)$.

Case 2: If $k_1 + k_2 > k_3$ then f is a non-reduced factorization of π' and there exists some pair of alternatives $xy \in R \cap R''$ which is (unnecessarily) inverted on the path ρ_f at least twice. Then there exists i, j with $0 \leq i < i + 1 < j < j + 1 \leq k_1 + k_2$ such that:

$$
\begin{array}{lcl}
R_i & = & \ldots\ldots xy \ldots\ldots \\
R_{i+1} = & = & \ldots\ldots yx \ldots\ldots \\
\vdots & & \\
R_j & = & \ldots yx \ldots\ldots\ldots \\
R_{j+1} & = & \ldots xy \ldots\ldots\ldots
\end{array}
$$

Lemma 4.4.1 applies and there exists a reduction of f to f^{xy} and obviously $g_\omega(f^{xy}) < g_\omega(f)$. Applying Lemma 4.4.1 repeatedly for all such pairs eventually leads to a reduced factorization $f^* = f^{x_1y_1, \ldots x_l y_l}$ which is a decomposition of π'. Note that $g_\omega(f^*) < g_\omega(f)$. Then, by definition of δ_ω^{PM}, $g_\omega(d_3) \leq g_\omega(f^*) < g_\omega(f) = g_\omega(d_1) + g_\omega(d_2)$.

(**Only if part**)

Let R, R', and π. We will show by induction on the size of π, i.e., $I(\pi)$.

(Induction basis) For $I(\pi)=1$, by positional neutrality $\delta_\omega=\delta_\omega^{PM}$.

(Induction hypothesis) For $I(\pi)=k$, assume $\delta_\omega=\delta_\omega^{PM}$.

(Induction step) Let $I(\pi)=k+1$. Let $d\in D_\pi$ be a path minimizing decomposition (a minimal decomposition) of π under δ_ω^{PM}. Let $R=R_0$ and $R'=R_{k+1}$ and let $\rho_d=(R_0,R_1,\ldots,R_k,R_{k+1})$ be the induced path. Consider R_0,R_k and the corresponding permutation $\bar{\pi}$. Let $\bar{d}\in D_{\bar{\pi}}$ be the decomposition such that $\rho_{\bar{d}}=(R_0,R_1,\ldots,R_k)$. By the induction hypothesis, $\delta_\omega^{PM}(R_0,R_k)=\delta_\omega(R_0,R_k)$, by positional neutrality $\delta_\omega^{PM}(R_k,R_{k+1})=\delta_\omega(R_k,R_{k+1})$. By the triangular inequality, $\delta_\omega(R_0,R_k)+\delta_\omega(R_k,R_{k+1})\geq\delta_\omega(R_0,R_{k+1})$. Then by Remark 4.3.2 this implies $\delta_\omega(R_0,R_k)+\delta_\omega(R_k,R_{k+1})=\delta_\omega^{PM}(R,R')\geq\delta_\omega(R,R')=\delta_\omega(R_0,R_{k+1})$. By definition of δ_ω^{PM}, we conclude $\delta_\omega^{PM}(R,R')=\delta_\omega(R,R')$.

□

By Proposition 4.4.2 and Theorem 4.4.1 we have the following triangular inequality result for the Lehmer distance:

Corollary 4.4.2. *The Lehmer distance satisfies the triangular inequality if and only if $\omega\in\bar{\Omega}$, i.e., ω is a decreasing weight vector.*

By Proposition 4.4.3 and Theorem 4.4.1 we have the following triangular inequality result for the inverse Lehmer distance:

Corollary 4.4.3. *The inverse Lehmer distance satisfies the triangular inequality if and only if $\omega\in\underline{\Omega}$, i.e., ω is an increasing weight vector.*

4.5 Conclusion

We have described a class of distance functions over linear orders that are sensitive to the positions of elementary changes and decomposable into sums of distances between elementary changes. Note that both of these properties are essential elements of the Kemeny distance. We have shown that only the path-minimizing distance satisfies the triangular inequality condition for all possible weight vectors.

We have shown that if weights are monotonically decreasing (or increasing) from the upper parts of a ranking to the lower parts monotonically, then the Lehmer distance (the inverse Lehmer distance) satisfies the triangular inequality condition and is equivalent to the path-minimizing distance.

Note that finding the path-minimizing distance is not trivial. This is equivalent to the shortest-path problem which requires implementation of nonlinear algorithms such as the algorithm of Dijkstra (1959), or finding out all possible paths between two linear orders and calculating the sums for each elementary change on these paths to obtain the minimal one. However, our results show that if the weights have a monotonic pattern (increasing or decreasing), there is an easy way out. By calculating the Lehmer code (or a dual code for the inverse Lehmer distance), we can immediately conclude that the distance of the winners' (or losers') path is the minimal one. Note that finding the Lehmer code of a permutation requires an easier algorithm[7] than the path-minimizing distance. Since most scenarios impose a monotonic pattern on the weights and by the equivalence results, we can conclude that the path-minimizing distance can also be found easily in such scenarios.

The class of weighted distance functions may also be implementable as collective preference rules, in the same fashion as the Kemeny-Young rule which assigns outcomes (strict preferences) that minimizes the Kemeny distance to a group of individual preferences. Since we have shown that the Kemeny distance is a particular example of the path-minimizing distance, it would be interesting to see the properties of a preference rule that uses the latter distance for the minimization.

Another possible line of research is to study what other conditions the class of weighted distances satisfies. Bogart (1973) introduced several conditions for distance functions, among them, an additivity condition. This condition requires that for any three preferences R_1, R_2, and R_3 if R_2 is on some path between R_1 and R_3 then $\delta(R_1,R_2)+\delta(R_2,R_3)=\delta(R_1,R_3)$. It is straightforward to see that this condition can be satisfied by all weighted distances only if weights are constant. However, one may restrict the betweenness requirement such that R_2 is required to be on a path that gives the distance between R_1 and R_3, e.g., the minimal path for the path-minimizing distance or the winners' path for the Lehmer distance.

Finally, we would like to point out some work on distances over choice

[7] In terms of computational complexity finding the winners'/losers' decompositions is similar to finding the smallest number in an unsorted array.

functions. Klamler (2008) discusses distances in the framework of individual choice functions, i.e., functions that choose from each possible subsets of alternatives. These choice functions and distances defined on them turn out to be related to the Kemeny distance for preferences. It would be interesting to extend the results therein to see a correspondence between some class of distances on choice functions and the class of weighted distances on preferences.

4.6 Appendix

4.6.1 Appendix A: Notation for Permutation and Group Theory

Given any $R, R' \in \mathscr{L}$ with π, consider M^{π}, the $m \times m$ matrix form of π where $M^{\pi}_{ij} = 1$ if and only if $\pi(j) = i$ and $M^{\pi}_{ij} = 0$ otherwise. The matrix M^{π} has entries of 1 in the intersection of the i^{th} row and the j^{th} column since the j^{th} alternative in R is equal to the i^{th} alternative in R'. Note that this particular notation has its own advantages, e.g., when the linear orders R, R' are written as a $m \times 1$ column vectors, we have $R' = M^{\pi} \cdot R$.

Given a permutation matrix M^{π}, let us define a $m \times m$ diagram by replacing all M^{π}_{kl} with crosses for all (k,l) with $k < \pi(l)$ and $l < j$ for j such that $\pi(j) = k$. Note that such (k,l)'s are the indices with zeros that come before an entry of 1 in a row and also before an entry of 1 in a column. Furthermore replace all entries with 1 by some dots. The established diagram is called the Rothe diagram[8] and denoted by $\Gamma(\pi)$ where each crossed index refers to an inversion that is necessary to permute R to R'.

Example 4.6.1. *Consider the same linear orders in Example 4.3.1, $R = abcd$ and $R' = dcab$ and π such that $\pi(1) = 3,\ \pi(2) = 4,\ \pi(3) = 2,\ \pi(4) = 1$:*

$$M^{\pi} = \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \qquad \Gamma(\pi) = \begin{array}{cccc|c} x & x & x & \cdot & 3 \\ x & x & \cdot & 0 & 2 \\ \cdot & 0 & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 & 0 \\ \hline 2 & 2 & 1 & 0 & \end{array}$$

[8]For the application of this diagram see Muir (1906), and Knuth (1998)

◇

Definition 4.6.1. ***Lehmer Partition and Inverse Partition***: *For any $k,j = 1,2,\ldots m$, let $L(\pi)_k$ denote the set of crossed indices (k,j) in the k^{th} row of the Rothe diagram $\Gamma(\pi)$. Then we call $L(\pi) = (L(\pi)_k)_{k=1}^m$ the Lehmer partition. Similary let $IL(\pi)_j$ denote the set of crossed indices (k,j) in the j^{th} column of the Rothe diagram $\Gamma(\pi)$. Then we call $IL(\pi) = (IL(\pi)_k)_{k=1}^m$ the inversion partition. The vector composed of cardinalities of each component of the Lehmer partition, i.e., $(|L(\pi)_1|,|L(\pi)_2|,\ldots,|L(\pi)_m|)$, is known as the* Lehmer code. *The vector composed of the cardinalities of each component of the inversion partition, i.e., $(|IL(\pi)_1|,|IL(\pi)_2|,\ldots,|IL(\pi)_m|)$, is known as the* inversion list.

The Lehmer code for π in Example 4.6.1 is the column on the right hand side of the Rothe diagram, i.e., $(3,2,0,0)$ whereas the inversion list is the row just below the Rothe Diagram, i.e., $(2,2,1,0)$. The interpretation of the Lehmer code is that the alternative $R'(1)$ has to be raised 3 times, and $R'(2)$ has to be raised 2 times to achieve R' from R by elementary changes. The interpretation of the inversion list is that the alternative $R(1)$ has to be lowered 2 times, and $R(2)$ has to be lowered 2 times, and $R(3)$ has to be lowered 1 times to achieve R' from R.

Note that $(k,j) \in L(\pi)_k$ if and only if $(k,j) \in IL(\pi)_j$. Furthermore, each crossed index in $(k,j) \in \Gamma(\pi)$ corresponds to a pair in $R \setminus R'$ that is to be inverted. In particular, the crossed index $(k,j) \in \Gamma(\pi)$ corresponds to the pair $(R(j),R'(k)) \in R \setminus R'$. Therefore $L(\pi)_k = R \setminus R' \cap \{UC(R'(k),R) \times R'(k)\} = \{(x,y) \in R \setminus R' \mid y = R'(k)\}$, i.e., the k^{th} component of the Lehmer partition contains the pairs $(a,b) \in R \setminus R'$ where b is the alternative in k^th position of R' and $R = .a.b.$, but $R' = .b.a.$. Similarly, $IL(\pi)_j = R \setminus R' \cap \{R(j) \times LC(R(j),R)\}$, i.e., the j^{th} component of the inversion partition contains the pairs $(a,b) \in R \setminus R'$ where a is the alternative in the j^{th} position of R and $R = .a.b.$, but $R' = .b.a.$.

Remark 4.6.1. *Note that $M^{\pi} = (M^{\tilde{\pi}})^T$, i.e., the permutation matrices of π and its conjugate (the inverse matrix) $\tilde{\pi}$ are the transpose of each other, hence for the Rothe diagrams, $\Gamma(\pi) = (\Gamma(\tilde{\pi}))^T$. This implies the following relation between the Lehmer partition and the inverse partition: $(k,j) \in L(\pi)_k$ if and only if $(j,k) \in IL(\tilde{\pi})_j$.* △

4.6.2 Appendix B: Lehmer Distance and the Winners' Decomposition

The winners' decomposition introduced in Section 4.3.2 is also known as "canonical factorization" in Garsia (2002). It is visualized with the help of a diagram in Kassel et al. (2000), a similar diagram to that of Rothe according to Muir (1906). Remember that the winners' decomposition first raises the alternative in R that should be at the top of R'. This means the inversions in $L(\pi)_1 = \{(x,y) \in R \setminus R' \mid y = R'(1)\}$ are made beforehand. Let π_{L_1} denote the permutation that raise $R'(1)$ from its position in R, i.e., $rank(R'(1),R)$, to the top of R. Obviously the unique decomposition, call it $\{d_{L_1}\} = D_{\pi_{L_1}}$, makes only the inversions in $L(\pi)_1$ and looks like $d_{L_1} = (\sigma_{|L(\pi)_1|}, \sigma_{|L(\pi)_1|-1}, \dots, \sigma_2, \sigma_1)$. Formally, for each component of the Lehmer partition $L(\pi)_k$, let π_{L_k} denote the permutation that makes the inversions in $L(\pi)_k$. Then consider the decomposition $d_{L_k} \in D_{\pi_{L_k}}$ such that $d_{L_k} = (\sigma_{k+|L(\pi)_k|-1}, \sigma_{k+|L(\pi)_k|-2}, \dots, \sigma_k)$ if $|L(\pi)_k| > 0$ and $d_{L_k} = (\sigma_0)$ otherwise. As $\pi = \pi_{L_m} \cdot \pi_{L_{m-1}} \cdot \dots \cdot \pi_{L_1}$, then $d_L = (d_{L_1}, d_{L_2}, \dots, d_{L_m}) \in D_\pi$ is a well-defined decomposition of the permutation π.

Note that each inversion (k,l) in the Lehmer partition $L(\pi)_k$ is assigned an elementary change in some position via the k^{th} component of the winners' decomposition d_{L_k} depending on the number k and the amount of crosses that occur in $\Gamma(\pi)$ on the same row before (k,l). In particular, given $d_{L_k} = (\sigma_{k+|L(\pi)_k|-1}, \sigma_{k+|L(\pi)_k|-2}, \dots, \sigma_{k+1}, \sigma_k)$, the first cross on the k^{th} row is assigned σ_k, the second cross on the k^{th} row is assigned σ_{k+1} and so on. Note, however, that this does not necessarily imply (k,l) is assigned σ_{k+l}. In general a crossed entry $(k,l) \in \Gamma(\pi)$, is inverted by an elementary permutation of σ_{k+m}, whenever there are m crosses in the k^{th} row of $\Gamma(\pi)$ before (k,l).

Definition 4.6.2. *A decomposition $d_L^\pi \in D_\pi$ is called the "winners' decomposition" if it has the form: $d_L^\pi = (d_{L_1}, d_{L_2}, \dots, d_{L_m})$ for all permutations $\pi \in \Pi \setminus \{\sigma_0\}$ and $d_L^\pi = \sigma_0$ for $\pi = \sigma_0$.*

By using the values in Example 4.6.1, the elementary permutations that occur in the winners' decomposition can be visualized by the Rothe diagram as follows:

$$\begin{array}{cccc|c} x & x & x & \cdot & 3 \\ x & x & \cdot & 0 & 2 \\ \cdot & 0 & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 & 0 \\ \hline 2 & 2 & 1 & 0 & \end{array} \longrightarrow \begin{array}{cccc|c} \sigma_1 & \sigma_2 & \sigma_3 & \cdot & 3 \\ \sigma_2 & \sigma_3 & \cdot & 0 & 2 \\ \cdot & 0 & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 & 0 \\ \hline 2 & 2 & 1 & 0 & \end{array}$$

Proposition 4.6.1. δ^L_ω *is symmetric.*

Proof. Take any $R, R' \in \mathscr{L}$. Let π be the corresponding permutation and $\tilde{\pi}$ be the conjugate of π, i.e., $\pi(R) = R'$ and $\tilde{\pi}(R') = R$. We want to show that $\delta^L_\omega(R, R') = \delta^L_\omega(R', R)$, i.e.,

$$g_\omega(d^\pi_L) = g_\omega(d^{\tilde{\pi}}_L).$$

As $M^\pi = (M^{\tilde{\pi}})^T$, the Rothe diagrams of each permutation are also the transpose of each other. Then for any crossed index in $(k,l) \in \Gamma(\pi)$, there exists a crossed index $(l,k) \in \Gamma(\tilde{\pi})$. As these crosses refer to an inversion in respective winners' decompositions, d^π_L and $d^{\tilde{\pi}}_L$, it is sufficient to show that each of such crossed indices correspond to an elementary change in the same position. Consider now the k^{th} component of the winners' decomposition $d^\pi_{L_k}$, and the l^{th} component of the winners' decomposition $d^{\tilde{\pi}}_{L_l}$ which invert respectively $(k,l) \in \Gamma(\pi)$ and $(l,k) \in \Gamma$. Let $K^\pi = \{(x,y) \mid M^\pi_{xy} = 1 \text{ and } x < k \text{ and } y < l\}$ and $K^{\tilde{\pi}} = \{(x,y) \mid M^{\tilde{\pi}}_{xy} = 1 \text{ and } x < l \text{ and } y < k\}$. As $M^\pi = (M^{\tilde{\pi}})^T$, we have $|K^\pi| = |K^{\tilde{\pi}}|$. Note also that for each $(x,y) \in K^\pi$ there will be one less cross in the k^{th} row before $(k,l) \in \Gamma(\pi)$ and one less cross in the l^{th} row before $(l,k) \in \Gamma(\tilde{\pi})$. Therefore $(k,l) \in \Gamma(\pi)$ will have an elementary permutation of $\sigma_{k+(l-1)-|K^\pi|}$. Similarly $(l,k) \in \Gamma(\tilde{\pi})$ will have an elementary permutation of $\sigma_{l+(k-1)-|K^{\tilde{\pi}}|}$. As $k+(l-1)-|K^\pi| = l+(k-1)-|K^{\tilde{\pi}}|$ and the choice of (k,l) is arbitrary, this completes the proof.

□

4.6.3 Appendix C: Inverse Lehmer Distance and the Losers' Decomposition

Let $r(j) = rank(R'(j), R)$ denote the position, of the j^{th} alternative of R', in R. Remember that the losers' decomposition first lowers the alternative in R that should be at the bottom of R'. This means the inversions in $IL(\pi)_{r(m)}$

are made beforehand. Let $\pi_{IL_{r(m)}}$ denote the permutation that lowers $R'(m)$ from its position in R, i.e. $r(m) = rank(R'(m), R)$, to the bottom of R. Obviously the unique decomposition, call it $\{d_{IL_{r(m)}}\}$, makes only the inversions in $IL(\pi)_{r(m)}$ and looks like $d_{IL_{r(m)}} = (\sigma_{r(m)}, \sigma_{r(m)+1}, \ldots, \sigma_{m-2}, \sigma_{m-1})$. Formally for each component of the inversion partition $IL(\pi)_{r(k)}$, let $\pi_{IL_{r(k)}}$ denote the permutation that makes inversions in $IL(\pi)_{r(k)}$. Then consider the decomposition $d_{IL_{r(k)}} \in D_{\pi_{IL_{r(k)}}}$ such that $d_{IL_{r(k)}} = (\sigma_{k-|IL(\pi)_{r(k)}|}, \sigma_{k-|IL(\pi)_{r(k)}|+1}, \ldots, \sigma_{k-1})$ if $|IL(\pi)_{r(k)}| > 0$ and $d_{IL_{r(k)}} = (\sigma_0)$ otherwise. As $\pi = \pi_{IL_{r(1)}} \cdot \pi_{IL_{r(2)}} \cdot \ldots \cdot \pi_{IL_{r(m-1)}} \cdot \pi_{IL_{r(m)}}$, then $d_{IL} = (d_{IL_{r(m)}}, d_{IL_{r(m-1)}}, \ldots, d_{IL_{r(1)}}) \in D_\pi$ is a well-defined composition of the permutation π.

Definition 4.6.3. *A decomposition $d^\pi_{IL} \in D_\pi$ is called the "losers' decomposition" if it has the form: $d^\pi_{IL} = (d_{IL_{r(m)}}, d_{IL_{r(m-1)}}, \ldots, d_{IL_{r(1)}})$ for all permutations $\pi \in \Pi \setminus \{\sigma_0\}$ and $d^\pi_{IL} = \sigma_0$ for $\pi = \sigma_0$.*

Proposition 4.6.2. *δ^{IL}_ω is symmetric.*

Proof. Suppose for a contradiction δ^{IL}_ω is not symmetric, i.e., for some $R, R' \in \mathscr{L}$, we have $\delta^{IL}_\omega(R, R') \neq \delta^{IL}_\omega(R', R)$. Then by duality argument in Section 4.4.1 and Corollary 4.4.1, we have that $\delta^L_{\hat{\omega}}(\hat{R}, \hat{R}') \neq \delta^L_{\hat{\omega}}(\hat{R}', \hat{R})$ which contradicts the symmetry of the Lehmer distance in Proposition 4.6.1. □

4.6.4 Appendix D: A Visualization for Paths and Decompositions

Since Π is a permutation group on the set of alternatives A, it is well-known that when the number of alternatives is $m = |A|$, a visualization of all linear orders $\mathscr{L}$ over A is possible with an geometric object known as permutahedron (Santmyer, 2007) in an $(m-1)$−dimensional space. For instance, the set of all linear orders over $A = \{a, b, c\}$ can be visualized in $\mathbb{R}^2_+$ as a hexagon:

Example 4.6.2. *Consider the graph in Figure 4.1 where each vertex corresponds to some R_i and two vertices are connected by an edge if and only if they form an elementary change.* ◇

$R_1 = abc$
$R_2 = acb$
$R_3 = cab$
$R_4 = bac$
$R_5 = bca$
$R_6 = cba$

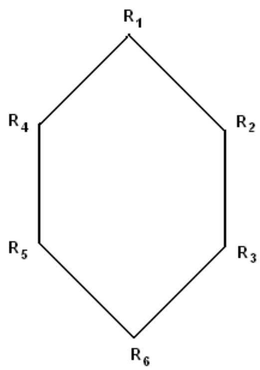

Figure 4.1: A graph for linear orders when $m = 3$.

Note that for two linear orders R_1 and R_3, the sequence between them (R_1, R_2, R_3) is a path and induced by the decomposition $d = (\sigma_2, \sigma_1)$ whereas the sequence $(R_1, R_4, R_5, R_6, R_3)$ is not a path because it is induced by a non-reduced factorization $f = (\sigma_1, \sigma_2, \sigma_1, \sigma_2)$.

Example 4.6.3. *For $A = \{a, b, c, d\}$, a visualization of the set of all possible linear orders $\mathscr{L}$ can be achieved by a three dimensional permutohedron, known as truncated octahedron. In Figure 4.2 we provide a two dimensional reduction of the truncated octahedron as a graph.* ◇

$R_1 = abcd$	$R_5 = adbc$	$R_9 = acdb$	$R_{13} = bcda$	$R_{17} = dbac$	$R_{21} = dcab$
$R_2 = abdc$	$R_6 = badc$	$R_{10} = adcb$	$R_{14} = cbad$	$R_{18} = bdca$	$R_{22} = dbca$
$R_3 = bacd$	$R_7 = bcad$	$R_{11} = dabc$	$R_{15} = cadb$	$R_{19} = cbda$	$R_{23} = cdba$
$R_4 = acbd$	$R_8 = cabd$	$R_{12} = bdac$	$R_{16} = dacb$	$R_{20} = cdab$	$R_{24} = dcba$

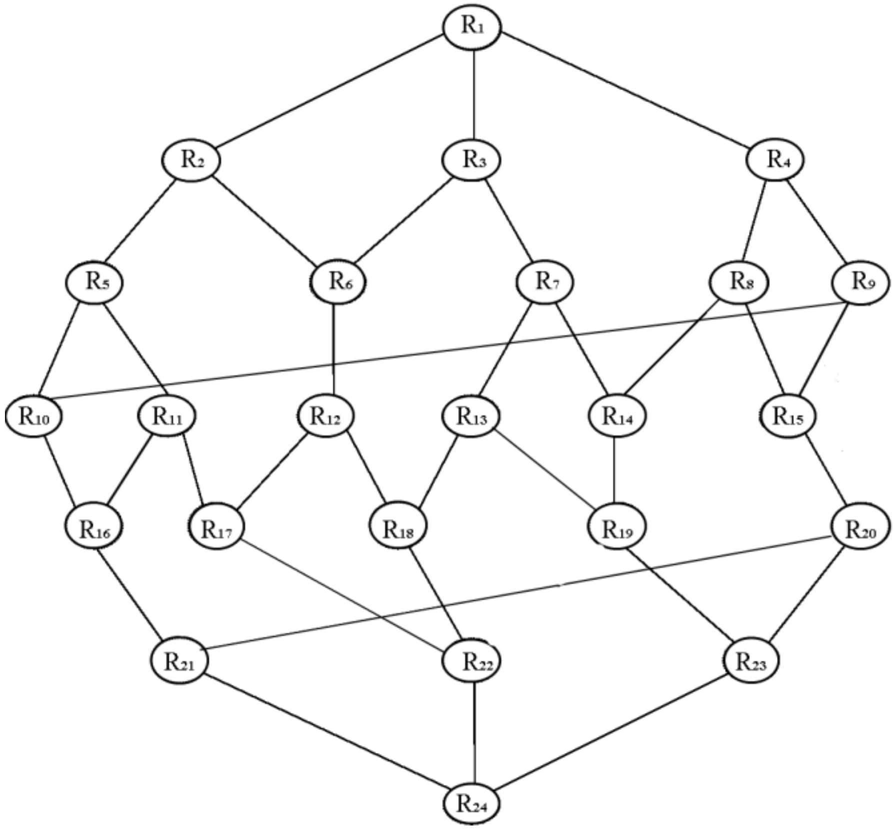

Figure 4.2: A graph for linear orders when $m = 4$. The lines which connect two linear orders by passing through fewest possible edges are called paths.

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Nederlandse samenvatting (Dutch summary)

Sinds Samuelson (1947) is de micro-economische theorie van een normatief naar een positief kader verschoven. Aangezien niemand meer betwist of een stelling handig is, kan dit gezien worden als een positieve ontwikkeling. De discussie vindt nu juist plaats over de aannames die de theorie zijn bestaansrecht geven. Het introduceren van wiskundige kracht en sterk bewijs voor stellingen wordt door velen gezien als een goede zaak binnen de economie, terwijl anderen dit als een geheime zonde, specifiek aan de economische wetenschap zien (McCloskey, 2002). Dit proefschrift bestaat uit drie op zichzelf staande hoofdstukken die als duidelijke voorbeelden van deze zonden fungeren.

Aan deze drie hoofdstukken, waarin drie verschillende vragen worden beantwoord, ligt hetzelfde beginsel van economische analyse ter grondslag. In ieder hoofdstuk, fungeren individuele voorkeuren als uitgangspunt van het analytisch kader. We onderzoeken de wisselwerking tussen individuele voorkeuren binnen de volgende drie scenario's; i) individuen worden, rekening houdend met hun voorkeuren, gekoppeld om een kamer te delen, ii) individuen krijgen bepaalde voorkeuren toegewezen die hen collectief representeren, iii) ideologische afstanden tussen individuen worden gemeten door de verschillen in hun voorkeuren.

Hoofdstuk twee en drie zijn voorbeelden van twee onderzoeksgebieden binnen de moderne Samuelsonse micro-economische theorie: matching theorie en sociale keuze theorie. Hoofdstuk vier, echter, onderzoekt het uitgangspunt dat ten grondslag ligt aan het theoretisch kader, namelijk de individuele voorkeuren. In dit hoofdstuk, introduceren we een ongelijkheidsfunctie met betrekking tot deze voorkeuren.

In hoofdstuk twee maken we gebruik van individuele voorkeuren in het kader van matching theorie. We bestuderen de zogenaamde “kamergenootmarkt” geïntroduceerd door Gale en Shapley (1962). Deze markt wordt gekenmerkt door een-op- een matching waarbij mensen of in paren gekoppeld worden om een kamer te delen of alleen blijven. In het bijzonder analyseren we het effect van variabele populatiegroottes, waarbij personen zowel als consumenten als als productiemiddel worden gezien. Klaus (2011) introduceerde twee nieuwe “bevolkingsgevoeligheid” eigenschappen die het effect van nieuwkomers op de bestaande populatie beschrijven: “concurrentie gevoeligheid” en “productiemiddel gevoeligheid”. We karakteriseren de core met behulp van de eigenschappen gerelateerd aan bevolkingsgevoeligheid en zwakke unanimiteit en consistentie. We krijgen twee bijbehorende onmogelijkheden als resultaat.

In hoofdstuk drie worden de individuele voorkeuren over een bepaalde set van alternatieven beschouwd als een collectief besluitvormingsprobleem. Door middel van voorkeursregels worden de collectieve besluiten gemodelleerd. We richten ons op een nieuwe voorwaarde: “update-monotonie” voor voorkeursregels. Deze voorwaarde vereist, over het algemeen, dat wanneer individuele voorkeuren ten gunste van de uitkomst van een voorkeursregel veranderen, deze uitkomst ongeacht het nieuwe voorkeursprofiel van de individuen, nog steeds zou moeten worden uitgekozen door de voorkeursregel. Hoewel veel onmogelijkheidsstellingen voor keuzeregels gebaseerd zijn op, of gerelateerd zijn aan voorwaarden van monotonie, voldoen verschillende belangrijke voorkeursregels aan deze voorwaarde. In geval van paarsgewijze, Pareto optimale, neutrale, en consistente regels, is de Kemeny-Young regel de enige regel die hier aan voldoet. In geval van convex gewaardeerde, Pareto optimale, neutrale en replicatie invariante regels, houdt sterke update-monotonie in, dat de regel gelijk staat aan de vereniging van alle voorkeuren waarbij de voorkeursparen bij k personen unaniem overeenkomen, waarbij k gerelateerd is aan het aantal alternatieven en personen. In beide gevallen vinden we een karakterisering van deze regels.

In hoofdstuk vier, ligt de nadruk op de voorkeuren alleen. De bekende Kemeny afstand, bijvoorbeeld, berekent de paarsgewijze verschillen tussen twee individuele voorkeuren. In deze context, dragen we een klasse van gewogen afstandsfuncties aan, die gebaseerd is op een bepaalde verdeling van gewichten over de posities van paren waarover onenigheid bestaat. Binnen deze klasse vallen bijvoorbeeld de Kemeny afstand, de Lehmer afstand, de inverse Lehmer afstand, en de "route-minimalisatie" afstand. We analyseren implicaties van het veranderen van deze gewichten op de structuur van deze afstandsfuncties. Het blijkt dat de "route-minimalisatie" afstand de gewogen veralgemenisering is van de Kemeny afstand, in die zin dat het de enige afstandsfunctie is die voor elke strikt positieve gewichtsvector aan de driehoeksongelijkheid voldoet. Verder laten we zien dat deze afstand gemakkelijk kan worden berekend wanneer de gewichten over de posities van de meningsverschillen monotoon toe- of afnemen.

Short Curriculum Vitae

Burak Can was born in April, 1982 in Adana, Turkey. Between 1988-1997, he attended primary school at Cebesoy İ.Ö.O, then entered to Adana Anatolian High School (as ranked 39^{th} nationwide). He finished Adana Fen Lisesi, (Science College) with Honors degree in 2000. He studied Economics at Boğaziçi University, İstanbul between 2000-2005. He won the 4^{th} Boğaziçi Debate Tournament in 2001 and he also taught sculpting at Fine Arts Club between 2001-2005. After Boğaziçi University, he attained a M.Sc. degree in Economics at İstanbul Bilgi University in İstanbul in 2007. Thereafter he started his doctoral studies at Maastricht University, the Netherlands between 2007-2011.

Burak's research focuses on collective decision making, game theory and positive political economy.