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CHAOTIC PATTERNS IN COURNOT COMPETITION

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ABSTRACT

Chaos theory offers a new mode of analyzing the complexity of nonlinear (economic) dynamics. A growing list of applications is mainly focused on modeling macroeconomic (growth and business) cycles and dynamic (consumer's and firms') choice. This paper provides a nonlinear dynamic model of Cournot competition. The model improves upon Rand (1978) and Dana and Montrucchio (1986) by permitting monopoly output to be positive. The existence of chaotic regimes is proven and simulation experiments illustrate the implications.

1. CHAOTIC PATTERNS IN ECONOMICS

In the 1970s and 1980s chaos theory broke and still «breaks across the lines that separate scientific disciplines. Because it is a science of the global nature of systems it has brought together thinkers from fields that had been widely separated» (Gleick, 1987, p. 5). Gleick (1987, Chapter 2) does not hesitate to characterize the rise of chaos theory as a revolution, since it «has become not just a canon of belief but also a way of doing science. ... Some carry out their work explicitly denying that it is a revolution; others deliberately use Kuhn's language of paradigm shifts to describe the changes they witness» (pp. 38-39).

The essential notion of chaos theory is that (even simple) dynamic systems may generate seemingly random and chaotic patterns. Irregular

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and unpredictable time paths result from deterministic sources. Baumol and Quandt (1985) offer the illustrative, if imprecise, description that «[c]haos is defined as a fully deterministic behaviour pattern which is, in at least some respects, undistinguishable from a random process or, rather, a process perturbed by substantial random elements. It displays extreme sensitivity to changes in parameter values, and is characterized by an infinite number of equilibria each approached by (superimposed) cycles of different periodicities, and whose simultaneous presence is what gives the appearance of randomness to a time series generated by a deterministic process» (p. 3). Chaos theory reaches an analytical apparatus which has found application in many scientific disciplines.

This paper loosely defines chaos as to three features of dynamic trajectories: (i) sensitive dependence on initial conditions; (ii) existence of periodic orbits of all periods; and (iii) existence of an uncountable set of initial conditions that each give rise to (asymptotically) aperiodic time paths (Kelsey, 1988, p. 9). The point of departure is a first-order difference equation,

$$x_{t+1} = f(x_t), \tag{1}$$

which can be associated with chaotic trajectories if nonlinearity gives a hill-shaped function. The key point is that «[i]t cannot be too strongly emphasised that the process is generic to most functions [(1)] with a *hump of tunable steepness*» (May, 1976, p. 461, italics added). Particular specifications of equation (1) can give a sequence of bifurcations such that «the pattern never repeats» (May, 1976, p. 461). For the moment, this intuition suffices. An excellent, general review of the merits of nonlinear dynamics is May (1976), whereas Kelsey (1988) and Baumol and Benhabib (1989) offer nice introductions of chaos theory in economics.

In the late 1970s and early 1980s the methodology of nonlinear dynamics also entered the economic scenery. The most widespread use of chaos theory lies in the field of macroeconomic (business and growth) cycles (Stutzer, 1980; Benhabib and Day, 1980 and 1982; Day, 1982 and 1983; Dana and Malgrange, 1984; Day and Shafer, 1985 and 1987; Grandmont, 1985 and 1986; Boldrin and Montrucchio, 1986; Deneckere and Pelikan, 1986; and Julien, 1988). These models induce «the profession's growing awareness of the fact that, even in the absence of extraneous shocks, the internal (nonlinear) dynamics of an economy may generate quite complex periodic orbits or even nonexplosive 'chaotic' deterministic trajectories, that may be hard to distinguish from 'truly random' time series ... . Indeed, the recent approach to endogenous business cycles relies often on advances made lately in the mathematical theory of nonlinear dynamical systems, in particular the analysis of sudden qualitative changes displayed by their trajectories ('bifucations')» (Grandmont and Malgrange, 1986, p. 4) ([1]).

The second class of applications of chaos theory to economic frameworks are nonlinear models of (consumers' and firms') dynamic choice (Rand, 1978; Benhabib and Day, 1981; Baumol and Quandt, 1985; Dana and Montrucchio, 1986; Granovetter and Soong, 1986; Rasmussen and Mosekilde, 1988; and Iannaccone, 1989). These contributions «show that rational choice in a stationary environment can lead to erratic behaviour ... . We mean by erratic behaviour choice sequences that do not converge to a long-run stationary value or to any periodic pattern» (Benhabib and Day, 1981, p. 459). A particular type of nonlinear models of dynamic choice focuses on Cournot competition (Rand, 1978; and Dana and Montrucchio, 1986). This paper offers a constructive critique of the two existing nonlinear models of Cournot (duopoly) competition.

The paper is organized as follows. Section 2 describes the essential features of the two existing nonlinear models of Cournot (duopoly) competition. A basic flaw of these models is the (implicit) assumption that monopoly output is zero. Section 3 presents a model of Cournot duopoly competition which permits monopoly output to be positive. Section 4 illustrates the model's features with the help of the results of simulation experiments. Section 5 briefly indicates the applicability of the analytical apparatus of nonlinear dynamics to topics of theory of competition in industrial organization.

2. NONLINEAR MODELS OF COURNOT COMPETITION

Examples of the introduction of chaos theory in industrial organization are scarce. Dana and Montrucchio (1986) argue that «[t]he only exception is the seminal paper of Rand ..., which shows, in a very

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([1]) Of course, business and growth cycles with chaotic patterns may follow endogenously from agents' behavior, where the agents' decision making induces a chaotic sequence of choices [for instance, *via* agents' hill-shape offer curves in Grandmont (1985)].

abstract manner, that the Cournot tâtonnement in a duopoly model may display a complicated dynamical structure» (p. 41). The current state of the art is not much different. Only Dana and Montrucchio's (1986) treatment of Cournot duopoly models provides a further contribution to the application of nonlinear dynamics to topics in industrial organization ([2]). The nonlinear models of Cournot competition indicate that rivalry in a market can be associated with turbulent movements of the firms' quantities if the competitors' reaction functions are hill-shaped.

Recall that the Cournot (1838) duopoly model implies that a firm i chooses a supply quantity (q^i) so as to maximize a profit (π^i) function, conditional upon the quantity offered by the rival $j \neq$ (q^j),

$$\max_{q^i} \pi^i = p(q^i + q^j) \cdot q^i - c^i(q^i), \tag{2}$$

where p denotes the inverse demand function and so price and c the cost of production. Solving the maximand (2) gives the first-order condition

$$p(q^i + q^j) + q^i \cdot dp/d(q^i + q^j) \cdot (1 + dq^j/dq^i) - dc^i/dq^i = 0. \tag{3}$$

From condition (3) the firm's reaction function follows:

$$q^i = r^i(q^j), \tag{4}$$

where $i, j = 1, 2$ and $i \neq j$ with the Nash assumption that a firm expects a passive reaction of the rival upon its quantity strategy ($dq^j/dq^i = 0$). In the standard Cournot duopoly models (Tirole, 1988) chaotic patterns cannot emerge, since the rivals' reaction functions (4) are assumed to be linear or insufficiently nonlinear (*i.e.*, without a hill-shape).

The introduction of nonlinear dynamics in a Cournot duopoly model requires that at least one of the rivals' reaction functions is hill-shaped. The reason is straightforward. Suppose that neither of the reaction functions takes a hill-shaped form; that is $dr^i/dq^j \leq$ (or $\geq$) 0 for $q^j \geq 0$, where $i, j = 1, 2$ and $i \neq j$. Assume that rival 1 and 2 react according

([2]) A further exception is perhaps Baumol and Quandt's (1985) nonlinear model of advertizing.

to the time sequence ..., $t-1$, t, $t+1$, ... For example, if rival 1 offers a quantity q^1_{t-1} at time $t-1$, then rival 2 reacts by supplying q^2_t at t, which provokes rival 1's reaction q^1_{t+1} at $t+1$, *etcetera*. Then, $q^i_{t+1} = r^i(q^j_t)$ so that $q^i_{t+1} = r^i[r^j(q^i_{t-1})] = k^i(q^i_{t-1})$. Now, $dk^i/dq^i_{t-1} = dr^i/dq^j_t \cdot dr^j/dq^i_{t-1} \leq$ (or $\geq$) 0. Hence, the absence of hill-shaped reaction functions implies that the second-order difference equation of a rival's quantities shows not even a single hump.

Rand's (1978) approach to the Cournot duopoly model is very abstract indeed by directly postulating unspecified reaction functions with sufficient nonlinearity. Rand treats an example of an analytical hill-shaped function and a nonanalytical tent map. Analytical hill-shaped first-order difference equations can have chaotic regimes (Section 1). Besides, nonanalytical first-order difference equations can also give chaotic patterns for particular ranges of parameter values (May, 1976, p. 465). Dana and Montrucchio (1986) supplement Rand's treatment of chaotic behavior in Cournot duopoly models by, among other things, providing five specified examples. Given the desired hill-shaped specification of the reaction function(s), they derive (an) associated specification of profit function(s).

Both Rand's and Dana and Montrucchio's reaction curves have the shape which is depicted in Exhibit 1 (for the analytical case).

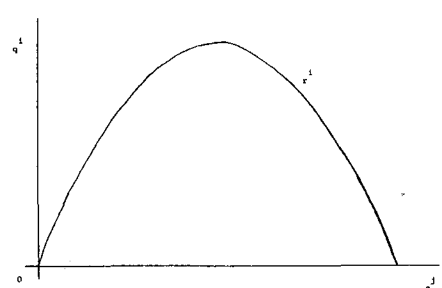

Exhibit 1. Hill-shaped Cournot reaction curve with zero monopoly output.

The *ad hoc* assumption of hill-shaped reaction functions in Rand's and Dana and Montrucchio's analyses leaves an essential question unanswered: Can an economic rationale be provided for the (very complicated) nonlinear shape of the profit functions? Kelsey (1988) points out that the «shapes of the reaction functions [Rand] assumes are very extreme indeed. It does not look like they could be generated by plausible demand and cost functions» (p. 19).

One observation immediately strikes the eye. The usual illustration of a hill-shaped curve implies that $r^i(0) = q^i = 0$. The reaction curves in Rand (1978) and Dana and Montrucchio (1986) all show this feature. This means that the (implicit) assumption is imposed that firm i offers a *zero* output in response to firm j's zero production: that is, monopoly output is taken to be zero! However, this assumption is not very realistic. This extreme case can be bypassed by introducing a $q^i_M = r^i(0) > 0$, where q^i_M represents firm i's monopoly output (which, for example, can follow from the standard maximization procedure of a monopolist). Exhibit 2 illustrates the shape of this reaction function.

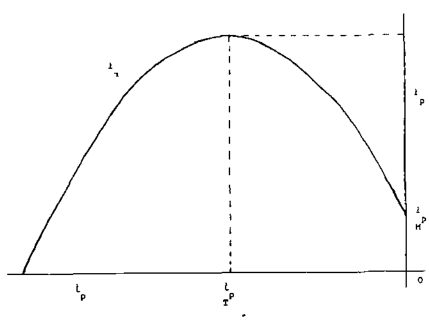

Exhibit 2. Hill-shaped Cournot reaction curve with positive monopoly output.

The hill-shaped Cournot reaction curve with positive monopoly output induces a further question: Can a proof of the existence of chaotic regimes still be provided? Section 3 goes on to examine both questions of economic interpretation and proof of existence.

3. COURNOT REACTION CURVES WITH POSITIVE MONOPOLY OUTPUT

3.1. *Economic Rationale*

3.1.1. *Asymmetric Reaction Pattern*

The critical implication of the hill-shape of a reaction function is that a firm shows an *asymmetric* reaction pattern. For $q^j < q^j_T$ (Exhibit

2) firm i and j's supplies are *positively* correlated, whereas $q^j > q^j_T$ is associated with a *negative* relationship between q^i and q^j. Hence, for $0 \leq q^j < q^j_T$ firm i acts as a *follower* or *imitator*. If firm j expands output, so does firm i. Whenever firm j contracts supply, firm i too introduces a decrease of the quantity offered. However, if firm j expands its output beyond q^j_T, then firm i starts to act as a *fringe competitor* or *accommodator*. On the one hand, whenever firm j expands output, firm i simply adapts to reduced residual demand. On the other hand, if firm j contracts output, then firm i adopts an aggressive strategy by expanding its supply.

The asymmetric reaction pattern follows from the switch in sign of the first-order derivative of the reaction curve. A firm with an asymmetric reaction pattern can be called a *dualist*: that is, the firm's reply can be to imitate as well as to accommodate, depending on the scale of the rival's output. The reaction curve of a dualist is hill-shaped.

3.1.2. *Strategic Substitutes and Complements*

Types of reaction patterns can be distinguished as to the features of the cross-partial derivatives of the firm's marginal profit with respect to its opponents' action (Bulow *et al.*, 1986, pp. 491-497; and Tirole, 1988, p. 208). Here it suffices to note that «with strategic substitutes *B*'s optimal response to more aggressive play by *A* is to be *less* aggressive ... . With strategic complements *B* responds to more aggressive play with *more* aggressive play» (Bulow *et al.*, 1986, p. 494). In terms of Cournot competition this means that strategic substitutes predict $dr^i/dq^j < 0$, whereas strategic complements indicate $dr^i/dq^j > 0$.

Hence, the substitute or complement nature of the firm's reaction pattern is reflected in the sign of the reaction curve's slope. Both cases are depicted in Exhibit 3 (Tirole, 1988, p. 208).

The reaction function which follows from strategic *complements* (curves I), describes the reaction pattern of an *imitator*, whereas an *accommodator's* responses are reflected in the reaction curve with strategic *substitutes* (curves II).

3.1.3. *Idle Capacity*

Bulow *et al.* (1985, p. 180, Fig. 1) provide an answer to the first question by offering an economic interpretation of hill-shaped reaction

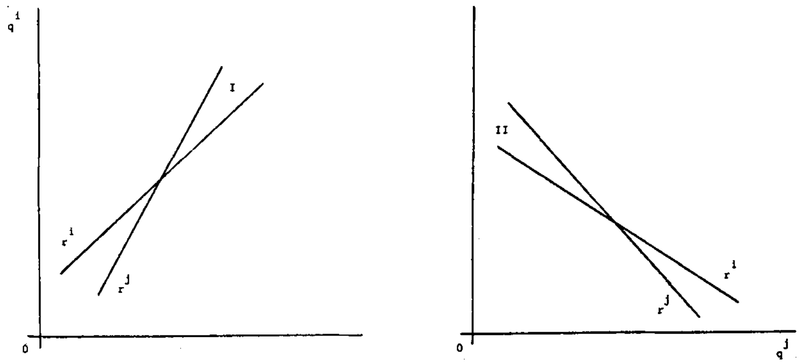

Exhibit 3. Strategic substitutes and complements.

curves with positive monopoly output in terms of strategic substitutes and complements. Here the following brief intuition suffices. Starting from monopoly output a firm is willing to *increase* supply in reaction upon entry, which contradicts the downward slope of standard Cournot reaction curves: that is, starting from monopoly output the firm regards outputs as strategic complements. This assumption follows from the literature on entry deterrence. Two arguments offer a case in point. First, the aggressive strategy after entry is described in the literature on idle capacity as an entry-deterring instrument (Spence, 1977 and 1979; and Ware, 1985). Second, the post-entry expansion policy can be grounded in long-run reputation arguments, even if this strategy is not profit-maximizing from a short-run perspective (Milgrom and Roberts, 1982 and 1987; and Arvan, 1986).

However, the expansion policy does not pay if the rival's scale moves beyond a particular point. After a certain scale of expansion (q_T) the benefit of accommodation starts to dominate over the advantage of the aggressive strategy, which implies that the standard downward slope of the Cournot reaction curve sets in: that is, the firms consider outputs to be strategic substitutes. The hill-shape of Cournot reaction curves can follow from demand specifics. The key point is that the «assumption that each firm's marginal revenue is always decreasing in the other's output ... is quite a restrictive assumption. For example it is *never* satisfied in the relevant range for economists' second-favourite demand curve – constant elasticity demand» (Bulow *et al.*, 1985, p. 178).

3.2. ***Proof of Existence***

3.2.1. *Li and Yorke's Theorem*

Granovetter and Soong (1986, pp. 92-93) provide a graphical intuition which suggests that chaotic regimes can occur in hill-shaped functions with a positive intercept (in a model of nonlinear consumers' choice) without, however, offering a proof. Li and Yorke (1975) provide however a theorem which can be used to *prove* the existence of chaotic regimes. This paper employs an abbreviate version of this theorem.

Theorem 1: The first-order difference equation (1) gives chaotic regimes if there exists a value of x_t such that

$$x_{t+3} \leq x_t < x_{t+1} < x_{t+2} \text{ (3)}. \tag{5}$$

Following Day (1982 and 1983) Li and Yorke's theorem can be re-expressed as

$$f(x^m) \leq x^c < x^* < x^m, \tag{6}$$

where

$$x^m = f(x^*) = \max f(x) > 0 \qquad \text{and} \qquad f(x^c) = x^*. \tag{7}$$

If the function $f(x)$ is hill-shaped, the inequality $x^* < x^m$ is equivalent to $x^c < x^*$ and the inequality $f(x^m) \leq x^c$ can be formulated as $f^2(x^m) \leq x^*$, where $f^2 = f[f(.)]$. Then, the theorem of Li and Yorke can be expressed as

$$f^2(x^m) \leq x^* < x^m. \tag{8}$$

Form (8) will be used to prove that there exists an uncountable set of initial conditions that give rise to chaotic time paths for a significant class of hill-shaped reaction functions with positive monopoly output. Three cases are considered: (i) one firm (re)acts as a dualist, whereas

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(3) Hence, if there exists a three-period cycle, then there are chaotic regimes as well. This result is related to Sarkovskii's (1964) theorem (Kelsey, 1988, p. 5).

the rival is an imitator (Subsection 3.2.2); (ii) one firm is a dualist, while the rival responds as an accommodator (Subsection 3.2.3); and (iii) both rivals behave as dualists (Subsection 3.2.4). It appears that all three cases can be associated with chaotic reaction patterns.

3.2.2. *Dualist Against Imitator*

The reaction function of firm i, $r^i(q^j)$, is assumed to be hill-shaped (with positive monopoly output), whereas the reaction function of firm j is supposed to resemble $r^j(q^i) = q^i$. This scenario describes competition between a dualist and a perfect imitator. Rival i and j react according to the time sequence ..., $t-1$, t, $t+1$, ... This means that

$$q^i_{t+2} = r^i(q^j_{t+1}) = r^i[r^j(q^i_t)] = r^i(q^i_t). \tag{9}$$

With doubled lengths of the time intervals equation (9) has the same form as the first-order nonlinear difference equation (1). Exhibit 4 illustrates the applicability of Theorem 1 to a first-order difference equation which resembles Exhibit 2's hill-shaped Cournot reaction curve with positive monopoly output (the dualist, curve I) and the 45°-line (the perfect imitator, curve II).

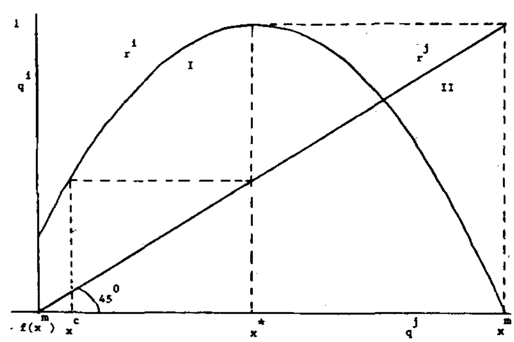

Exhibit 4. Dualist against imitator.

This paper uses the specification

$$q^i = r^i(q^j) = 1 - \alpha \cdot (q^j - 1 + 1/\sqrt{\alpha})^2 \tag{10}$$

for the dualist's hill-shaped reaction curve.

For a significant class of second-degree reaction functions r^i with positive monopoly output and assuming that firm j acts as a perfect imitator, the existence of chaotic regimes can be proven algebraically with the use of condition (8). Following Rand (1978) and Dana and Montrucchio (1986) output is scaled to q^i, $q^j \in [0, 1]$. If $r^i(1) = 0$ and the maximum of r^i is 1, then condition (8) indicates that a necessary condition for proving Li and Yorke's chaos is $r^i(0) \leq x^*$. That is, the monopoly output is restricted by the upperbound x^* (*i.e.*, the location of the maximum).

Proposition 1: If $q^i = r^i(q^j) = 1 - \alpha \cdot (q^j - 1 + 1/\sqrt{\alpha})^2$ and $q^j = r^j(q^i) = q^i$, there exists an uncountable set of initial conditions with chaotic (asymptotically aperiodic) time paths and for *every* natural number k there exists a time path with period k for $3.0795... \leq \alpha < 4$.

The proof of Proposition 1 is offered in Appendix A. Proposition 1 indicates that Cournot duopoly competition can be associated with chaotic trajectories if a dualist (that is, a firm with a hill-shaped reaction function) competes against a perfect imitator, even when monopoly output is assumed to be positive.

Proposition 1 is robust as regards to modifications of the assumption that rival j (re)acts as a perfect imitator. First, take the case where competitor j only imitates imperfectly.

Proposition 2: If the reaction function of firm i has the parabolic form as indicated in Proposition 1 but with $3.0795... < \alpha < 4$, then the case where the reaction function of rival j reflects imperfect imitation also gives rise to chaotic time paths.

The proof of Proposition 2 is given in Appendix B. The key point is that the reaction function of firm j is turned into $r^j(q^i) = q^i + \delta(q^i)$, where $\delta(q^i)$ indicates a small disturbance. The composed reaction function $r^i(r^j)$ then has the same shape as the one in the proof of Proposition 1, except for a small disturbance term, so that Li and Yorke's condition can still be satisfied if the disturbance is small enough.

3.2.3. *Dualist Against Accommodator*

The assumption that rival j (re)acts as an imitator, can be dropped in favor of the well-established case which assumes a downward sloping

Cournot reaction curve. That is, the dualist i (reaction curve I) faces an accommodator j (reaction function II). This scenario is illustrated in Exhibit 5. A for perfect accommodation. This means that the accommodator j (re)acts according to $r^j(q^i) = 1 - q^i = q^j$.

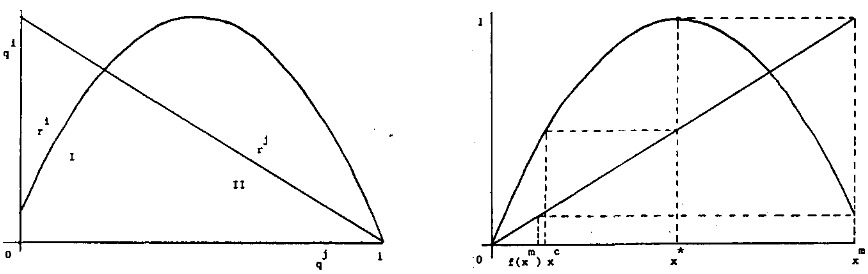

Exhibit 5. A. Dualist against accommodator. B. Composed reaction function.

Exhibit 5 shows graphically that Li and Yorke's theorem can be applied. The composed reaction function (Exhibit 5.B) can be used to illustrate that condition (6) does hold. By making use of Proposition 1 this result can be proven algebraically.

Proposition 3: If the reaction function of firm i has the parabolic form as indicated in Proposition 1 with $3.6708\ldots \leq \alpha < 4$, then the case where the reaction function of rival j reflects perfect accommodation is also associated with chaotic time paths.

Proposition 3's proof is presented in Appendix C. Proposition 3 indicates that Cournot accommodation can also give chaotic time patterns if one of the two firms decides on the basis of a hill-shaped reaction function with a positive intercept (*i.e.*, if one of the rivals is a dualist with positive monopoly output).

3.2.4. *Dualist Against Dualist*

The third case describes the scenario where both firms have the same hill-shaped reaction function with positive monopoly output. That is, the two rivals behave as dualists. This case is depicted in Exhibit 6.A.

Exhibit 6.B graphically proves the existence of Li and Yorke's chaos. The analytical proof follows from Proposition 1.

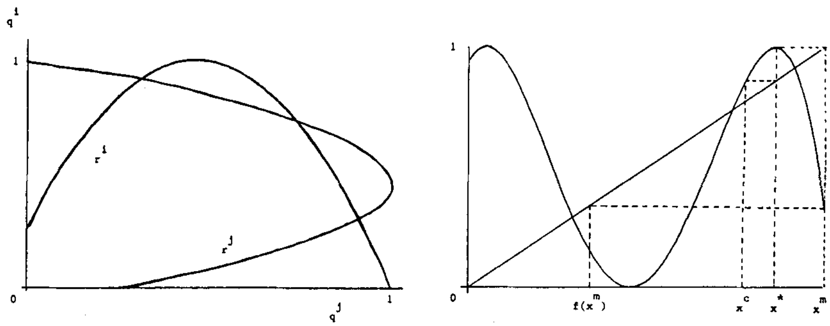

Exhibit 6. A. Dualist against dualist. B. Composed reaction function.

Proposition 4: If the reaction functions of firm i and j have identical hill-shaped forms, chaotic regimes can be derived.

Proposition 4 is proven in Appendix D. Proposition 4 predicts that Cournot competition between two rivals which are making use of equivalent hill-shaped reaction functions with positive monopoly output, can be associated with chaotic trajectories of output.

4. SIMULATION EXAMPLES

4.1. *Functional Specifications*

The implications of hill-shaped Cournot reaction curves can be illustrated through simulation of competition for a series of (counter-)moves (4). The simulation experiments cover 120 moves (or periods $t = 1, \ldots, 120$): that is, both rivals act and react 60 times. Rival i sets supply in odd-numbered periods ($t = 1, 3, 5, \ldots, 119$), whereas rival j's replies are effectuated at even-numbered dates ($t = 2, 4, 6, \ldots, 120$). Two initializations dictate the simulation results. First, by varying the value of the parameter (α) the steepness of the hill-shaped reaction function (10) can be tuned. Second, variation of the first move (q^i_1) manipulates the initial competitive condition.

The experiments simulate the dualist against imitator rivalry (Subsection 3.2.2). Firm i is the dualist $[q^i = r^i(q^j) = 1 - \alpha \cdot (q^j - 1 + 1/\sqrt{\alpha})^2]$ and

(4) For example, Baumol and Quandt (1985) an Baumol (1986) also offer interesting simulation examples, whereas Sterman (1989) presents the results of an experimental study.

firm j acts as a perfect imitator $[q^j = r^j(q^i) = q^i]$. Since the results are similar for both rivals, this section presents only firm i's outputs (at odd-numbered periods). Exhibit 7 indicates the initial values of the simulation experiments.

Simulation experiment	Initial monopoly output	Steepness parameter	Exhibit
I	0.310	3.35	8
II	0.300	3.35	9
III	0.310	3.34	10 and 12
IV	0.998	3.35	11

Exhibit 7. Initial values simulation experiments.

The simulation experiments reveal three properties of complex dynamics: (i) chaotic regimes for particular parameter values (Subsection 4.2); (ii) sensitive dependence on initial conditions (Subsection 4.3); and (iii) sudden breaks in qualitative patterns (Subsection 4.4). These features can pose serious problems to econometric estimation (Subsection 4.5).

4.2. *Chaotic Trajectories*

The first consequence of nonlinear dynamics can of course be the occurrence of chaotic trajectories. If rival firms are engaged in Cournot competition while at least one competitor is making supply decisions on the basis of a sufficiently steep, hill-shaped reaction function, the time pattern of both rivals' quantities mimics a random walk. The first simulation experiment (I) illustrates this point. Exhibit 8 depicts the series of supplies of firm i [the Lyapunov exponent (L) is 0.48] (5).

In period $t = 1$ firm i starts to supply close to monopoly output ($q^i_1 = 0.31$). The subsequent reactions of firm i reveal a chaotic trajectory. The series of firm i's supplies fails to show a systematic (periodic) pattern: history does not repeat. For example, the pattern of quantities from period $t = 105$ to $t = 109$ differs qualitatively from the trajectories

(5) The Lyapunov exponent indicates chaos for $L > 0$ (Lorenz, 1989, pp. 186-191).

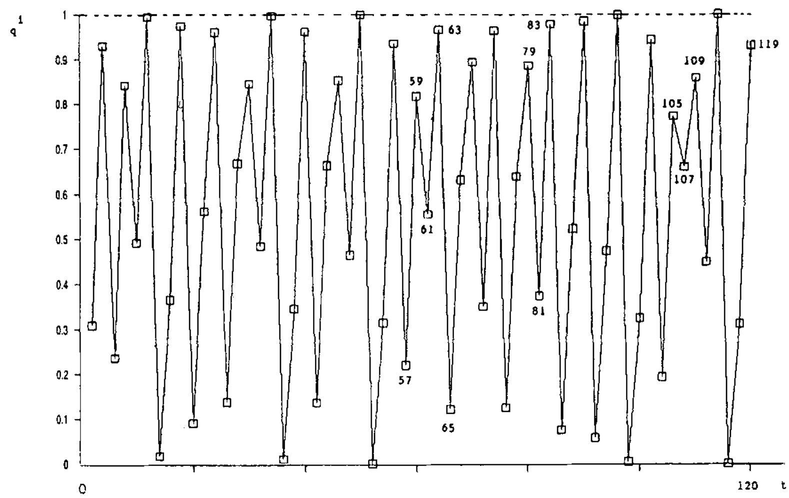

Exhibit 8. Chaotic trajectory.

in both history and future. The time pattern of firm *i*'s outputs mimics a random walk.

4.3. *Sensitive Dependencies*

The second property of complex dynamics can be illustrated by assuming a small change in the initial conditions. The second simulation (II) assumes monopoly output to be slightly below the first simulation's level. Exhibit 9 shows that the trajectory of rival *i*'s quantities changes dramatically ($L = 0.49$). This means that history matters.

Period $t = 1$'s monopoly output is slightly below the first simulation's level (q_1^i is decreased from 0.31 to 0.30). The trajectory of firm *i*'s outputs in simulation experiment II is completely different from experiment I's pattern. For example, any resemblance between simulation I and II's output trajectory in period $t = 57$ to $t = 65$ and $t = 79$ to $t = 83$ is absent. This illustrates the observation that the pattern of quantities is extremely sensitive to minor changes (here a 0.01 reduction) in the level of initial monopoly output. Exhibit 10 shows that the same is true for small variations in the value of parameter $\alpha (L = 0.49)$.

The third simulation (III) retains monopoly output at experiment I's level, but introduces a minor reduction in α (α is reduced from 3.35

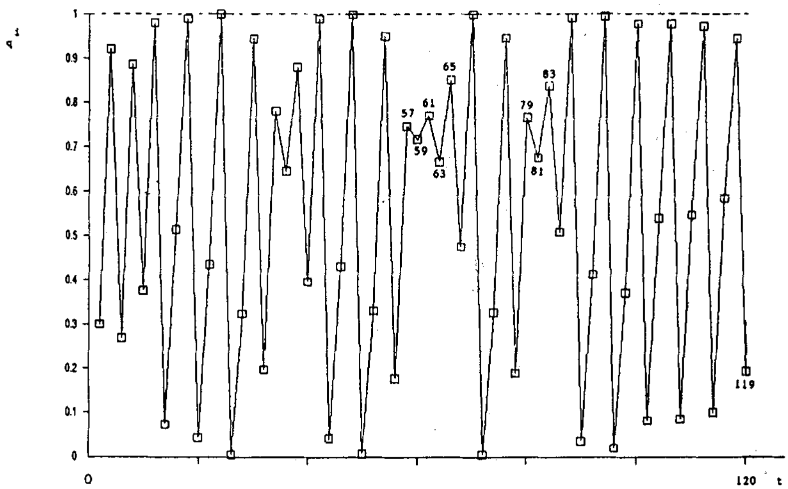

Exhibit 9. Sensitive dependence on q^i_1.

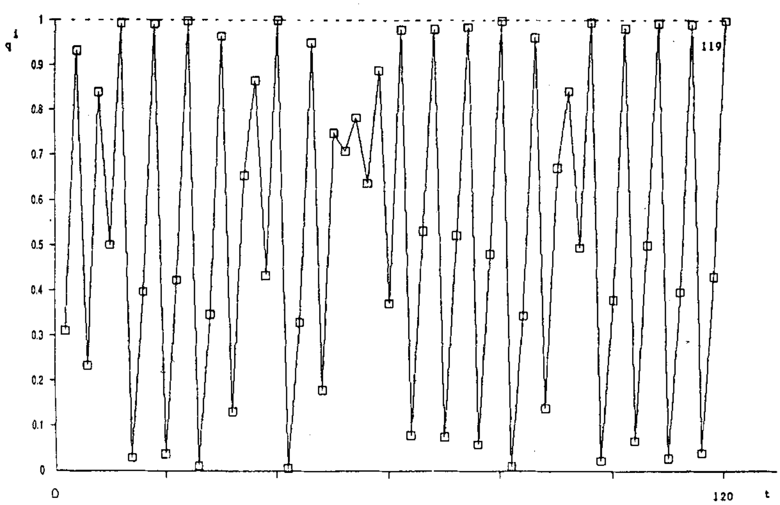

Exhibit 10. Sensitive dependence on α.

to 3.34). A sidelong glance at Exhibit 8 and 10 reveals that a 0.01 variation of parameter α induces a radical transformation of firm i's output pattern.

4.4. *Qualitative Breaks*

A peculiar feature of complex dynamics is that a chaotic trajectory is associated with sudden breaks in the qualitative pattern. The fourth simulation experiment (IV) reveals this feature as supply suddenly shows a regularity for two significant time intervals. Exhibit 11 presents the result ($L = 0.48$). For two (short) periods of time the pattern suggests that history repeats.

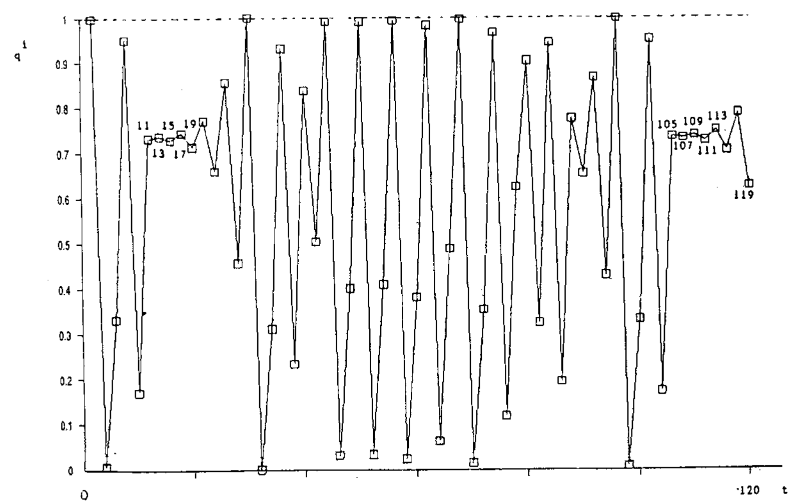

Exhibit 11. Qualitative breaks.

Experiment IV retains α's value at simulation I and II's level, but assumes monopoly output to increase from $q_1^i = 0.300$ to $q_1^i = 0.998$. From period $t = 11$ to $t = 19$ an $t = 105$ to $t = 113$ firm i's supply remains almost constant, which suggests convergence to a single equilibrium point. However, after period $t = 21$ respectively $t = 115$ the pattern breaks down again.

4.5. *Econometric Dilemma*

Deterministic chaos poses serious problems to econometric estimation (Baumol and Benhabib, 1989). On the one hand, a time trajectory which is extremely sensitive on initial conditions, is difficult to predict. On the other hand, it is problematic to distinguish deterministic chaos from stochastic randomness. This is even more relevant if one recognizes Kelsey's (1988, p. 12) observation that imposing a random error term on a hill-shaped function implies that chaos becomes more common. However, (at least) three arguments can be put forward to modify this claim.

First, chaotic trajectories are associated with (long) periods of regularity. This follows from the feature that sudden regularities characterize the qualitative pattern. Second, new econometric techniques have been (and are) developed to test whether deterministic chaos or stochastic randomness (predominantly) underlies a particular time series (Brock, 1986). Third, an additional argument follows from the specifics of this paper's application. The fact that *individual* firms can offer a chaotic series of quantities, does not necessarily mean that the trajectory of *market* supply is dictated by chaos as well.

First, if a dualist faces an imitator, the firms' chaotic output trajectories are replicated at the market level. To be precise, if market supply follows from the summation of two subsequent moves (*i.e.*, an output decision of both rivals), market output is determined by $2 \cdot q^i + \delta(q^i)$. Perfect imitation of a dualist $[\delta(q^i) = 0]$ implies that one firm's chaotic output trajectory is duplicated at the market level. Second, if a dualist and perfect accommodator are engaged in duopoly Cournot competition, the results is reversed. The firms' chaotic trajectories are not observed at the market level as market output follows from $q^i + 1 - q^i = 1$. That is, perfect accommodation induces stationary market supply.

The implications for market output are not so obvious if two dualists compete over quantities. Summation of two subsequent output levels in the four simulation experiments mimics market supply (Q) in a dynamic dualist against dualist game with doubled period lengths (T). Hence, $Q_T = q^i_{2T-1} + q^j_{2T}$: that is, $Q_1 = q^i_1 + q^j_2$, $Q_2 = q^i_3 + q^j_4$, *etcetera*. The four simulation experiments give the same result: the chaotic output trajectories of both rivals seem to induce chaotic patterns of market supply. Bearing in mind that 120 moves give market supply for 60 periods ($T = 1, \ldots, 60$), Exhibit 12 illustrates this result for simulation III.

5. COMPETITION AND NONLINEAR DYNAMICS

This paper improves upon Rand's (1978) and Dana and Montrucchio's (1986) models of Cournot duopoly competition by permitting monopoly

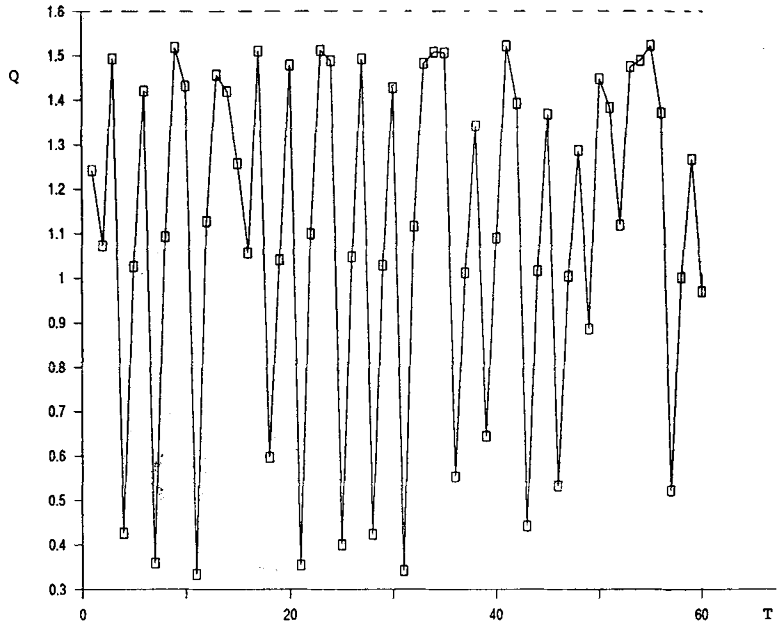

Exhibit 12. Chaotic market supply.

output to be positive. Bulow *et al.*'s (1985) argument indicates the economic plausibility of hill-shaped Cournot reaction curves with positive monopoly profit. Future research can be directed to at least two topics. First, this paper ignores the dilemma of the specification of cost and demand functions, as hill-shaped reaction curves [Exhibit 2 and Proposition 1] are postulated on the basis of *a priori* arguments. This raises the question whether there is a (large) class of economically plausible demand and/or cost functions which predicts such asymmetries. Second, other models of competition can be analyzed as to the (non)existence of complex dynamics, where the quest for chaotic regimes in models of competition is not to be restricted to one-shot Cournot games.

APPENDICES

A. *Dualist Against Imitator*

Proof of Proposition 1. Firm j acts as a perfect imitator, which gives (with doubled length of the time intervals) the following difference equation for q^i:

$$q^i_{t+1} = r^i(q^i_t). \tag{A1}$$

For the sake of convenience, q^i is replaced by x and r^i by f. Second-degree polynomials with the following properties are considered:

$$\max_{0 \leq x \leq 1} f(x) = f(x^*) = x^m = 1 \quad \text{with} \quad 0 < x^* < 1, \quad \text{and} \tag{A2.i}$$

$$f(1) = 0. \tag{A2.ii}$$

The parabola

$$f_\alpha(x) = 1 - \alpha \cdot (x - 1 + 1/\sqrt{\alpha})^2 \quad \text{with} \quad \alpha > 1 \tag{A3}$$

satisfies the conditions (A2.i) and (A2.ii) (then $x^* = 1 - 1/\sqrt{\alpha}$). Applying Li an Yorke's condition – $f^2(x^m)[=f(0)] \leq x^* < x^m(=1)$ – with the further restriction $f(0) > 0$ (positive monopoly output) to the parabola (A3) gives the inequalities

$$0 < 1 - \alpha \cdot (1/\sqrt{\alpha} - 1)^2 \leq 1 - 1/\sqrt{\alpha}. \tag{A4}$$

The inequality on the left hand side can be solve analytically and gives

$$0 < \alpha < 4. \tag{A5}$$

Numerically solving the inequality $1 - \alpha. (1/\sqrt{\alpha} - 1)^2 \leq 1 - 1/\sqrt{\alpha}$ (which is equivalent to $\alpha \cdot \sqrt{\alpha} + \sqrt{\alpha} - 2 \cdot \alpha - 1 \geq 0$) imposes a second restriction on the parameter α:

$$\alpha \geq 3.0795 \ \ldots\ . \tag{A6}$$

Combining (A3), (A4), (A5) and (A6) now gives the result that the class of parabola's

$$f_\alpha(x) = 1 - \alpha \cdot (x - 1 + 1/\sqrt{\alpha})^2 \quad \text{with} \quad 0 \leq x \leq 1 \quad \text{and} \quad 3.0795\ldots \leq \alpha < 4 \tag{A7}$$

has the properties (A2.i) and (A2.ii), $f(0) > 0$ and satisfies Li and

Yorke's condition. Therefore, the difference equation (A1) gives rise to chaotic regimes. *Q.E.D.*

B. *Small Disturbances*

Proof of Proposition 2. The reaction functions

$$r^i(q^j) = 1 - \alpha \cdot (q^j - 1 + 1/\sqrt{\alpha})^2 \quad \text{with} \quad 3.0795\ldots \leq \alpha < 4, \quad \text{and} \tag{B1.i}$$

$$r^j(q^i) = q^i + \delta(q^i) \quad \text{with} \quad \delta(0) > 0 \tag{B1.ii}$$

are assumed. With doubled length of the time intervals the output of firm i at «time $t+1$» is

$$\begin{aligned} q^i_{t+1} &= r^i[r^j(q^i_t)] = 1 - \alpha \cdot [q^i_t + \delta(q^i_t) - 1 + 1/\sqrt{\alpha}]^2 = \\ &= 1 - \alpha \cdot (q^i_t - 1 + 1/\sqrt{\alpha})^2 - \alpha \cdot [\delta(q^i_t)]^2 + 2 \cdot \alpha \cdot (1 - 1/\sqrt{\alpha} - \\ &\quad - q^i_t) \cdot \delta(q^i_t). \end{aligned} \tag{B2}$$

If again q^i is replaced by x, substitution of the function $f_\alpha(x)$, as indicated by (A3) in Appendix A, gives

$$x_{t+1} = f_\alpha(x_t) + \tau(x_t), \quad \text{and} \tag{B3.i}$$

$$\tau(x_t) = -\alpha \cdot [\delta(x_t)]^2 + 2 \cdot \alpha \cdot (1 - 1/\sqrt{\alpha} - x_t) \cdot \delta(x_t). \tag{B3.ii}$$

where τ is a «disturbance term». If the function f_α satisfies Li and Yorke's condition, then this condition can still be satisfied when f_α is disturbed by a small τ. So, if δ (the disturbance of the linear reaction function r^j) is «small enough», the conditions for the existence of chaotic time paths can still be satisfied. *Q.E.D.*

C. *Dualist Against Accommodator*

Proof of Proposition 3. Firm j acts a perfect accommodator, which gives (with doubled length of time intervals) the following difference equation for q^i:

$$q^i_{t+1} = r^i(1 - q^i_t). \tag{C1}$$

If $r^i(q^j) = 1 - \alpha \cdot (q^j - 1 + 1/\sqrt{\alpha})^2$ with $0 < \alpha < 4$ (reaction curve of a dualist with positive monopoly output), (C1) gives

$$q^i_{t+1} = 1 - \alpha \cdot (q^i_t - 1/\sqrt{\alpha})^2 \quad \text{with} \quad 0 < \alpha < 4. \tag{C2}$$

If again, for the sake of convenience, q^i is replaced by x, (C2) can be rewritten as

$$x_{t+1} = f(x_t) \quad \text{with} \quad f(x) = 1 - \alpha \cdot (x - 1/\sqrt{\alpha})^2 \quad \text{and} \quad 0 < \alpha < 4. \tag{C3}$$

The function $f(x)$ is a second-degree polynomial with maximum location $x^* = 1/\sqrt{\alpha}$ and maximum value $x^m = 1$. Applying Li and Yorke's condition – $f^2(x^m) \leq x^* < x^m(= 1)$ – to the parabola (C3) gives the inequalities

$$1 - \alpha \cdot (-\alpha + 2 \cdot \sqrt{\alpha} - 1/\sqrt{\alpha})^2 \leq 1/\sqrt{\alpha} < 1. \tag{C4}$$

Combining the inequality on the right hand side with $0 < \alpha < 4$ reveals

$$1 < \alpha < 4. \tag{C5}$$

Numerically solving the inequality on the left hand side (which is equivalent to $\alpha^3 \cdot \sqrt{\alpha} - 4 \cdot \alpha^3 + 4 \cdot \alpha^2 \cdot \sqrt{\alpha} + 2 \cdot \alpha^2 - 4 \cdot \alpha \cdot \sqrt{\alpha} + 1 \geq 0$) imposes a second restriction on the parameter α:

$$\alpha \geq 3.6708\ldots\,. \tag{C6}$$

Combining (C3), (C5) and (C6) provides the result that the class of parabola's (composed of a hill-shaped reaction function with positive monopoly output and the reaction function of a perfect accommodator)

$$f_\alpha(x) = 1 - \alpha \cdot (x - 1/\sqrt{\alpha})^2 \quad \text{with} \quad 0 \leq x \leq 1 \quad \text{and} \quad 3.6708\ldots \leq \alpha < 4. \tag{C7}$$

satisfies Li and Yorke's condition. Therefore, the difference equation (C1) gives rise to chaotic time paths. *Q.E.D.*

D. *Dualist Against Dualist*

Proof of Proposition 4. With reference to Appendix A this proof can be brief. Appendix A proves that the function $f_\alpha(x)$ in the difference equation gives rise to chaotic time paths. Two dualists firm i and i react according to the same reaction function (with doubled length of the time intervals). So,

$$q^i_{t+1} = f_\alpha[f_\alpha(q^i_t)] \tag{D1}$$

and with q^i replaced by x

$$x_{t+1} = f_\alpha[f_\alpha(x_t)]. \tag{D2}$$

The function $f_\alpha(f_\alpha)$ gives rise to (asymptotically aperiodic) time paths, because a time path of $f_\alpha(f_\alpha)$ can be derived by skipping the «odd terms» in a time path of f_α. Because f_α generates time paths with period k (Proposition 1), where k can be every natural number, the function $f_\alpha(f_\alpha)$, gives the same result. *Q.E.D.*

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