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"Characterizing Belief-Free Review-Strategy Equilibrium Payoffs under Conditional Independence"

by

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Characterizing Belief-Free Review-Strategy Equilibrium Payoffs under Conditional Independence*

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Abstract

This paper proposes and studies a tractable subset of Nash equilibria, belief-free review-strategy equilibria, in repeated games with private monitoring. The payoff set of this class of equilibria is characterized in the limit as the discount factor converges to one for games where players observe statistically independent signals. As an application, we develop a simple sufficient condition for the existence of asymptotically efficient equilibria, and establish a folk theorem for N-player prisoner's dilemma. All these results are robust to a perturbation of the signal distribution, and hence remain true even under almost-independent monitoring.

Journal of Economic Literature Classification Numbers: C72, C73, D82.

Keywords: repeated game, private monitoring, conditional independence, belief-free review-strategy equilibrium, prisoner's dilemma.

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1 Introduction

Consider an oligopolistic market where firms sell to industrial buyers and interact repeatedly. Price and volume of transaction in such a market are typically determined by bilateral negotiation between a seller and a buyer. Therefore, both price and sales are private information. This is so-called "secret price-cutting" game of Stigler (1964) and is a typical example of *repeated games with imperfect private monitoring*, where players cannot observe the opponents' action directly but instead receive noisy private information. In fact, a firm's sales level can be viewed as a noisy information channel of price of the opponents, as it tends to be low if the opponents (secretly) undercut their price. Harrington and Skrzypacz (2011) point out that lysine and vitamin markets are recent examples of secret price-cutting games.[1]

The theory of repeated games with private monitoring has been an active research area for recent years, and many positive results have been obtained for the case where observations are nearly perfect or nearly public: Hörner and Olszewski (2006) and Hörner and Olszewski (2009) establish general folk theorems for these environments. On the other hand, for the case where observations are neither almost-perfect nor almost-public, attention has been restricted to games that have a simple structure. For example, assuming that players receive statistically independent signals conditional on an action profile, Matsushima (2004) establishes a folk theorem, but only for two-player prisoner's dilemma. Ely, Hörner, and Olszewski (2005, hereafter EHO) and Yamamoto (2007) consider a similar equilibrium construction, but their analyses are still confined to two-by-two games or to symmetric N-player prisoner's dilemma. The restriction to these simple games leaves out many potential applications; for example, none of these results apply to secret price-cutting games with more than two firms and with asymmetric payoff functions.

The present paper extends the key idea of Matsushima (2004), EHO, and Yamamoto (2007) to general N-player games, and shows that there often exist asymptotically efficient equilibria. For this, we first introduce the concept of *belief-free review-strategy equilibria*, which captures and generalizes the equilibrium strategies of these papers. Specifically, a strategy profile is a belief-free review-strategy equilibrium if (i) the infinite horizon is regarded as a sequence of *review phases* such that each player chooses a constant action throughout a review phase, and (ii) at the beginning of each review phase, a player's continuation strategy is a best reply *regardless of the history up to the present action*, i.e., regardless of the history in the past review phases and regardless of what pure action the opponents choose in the current phase. While the set of belief-free review-strategy equilibria is a subset of sequential equilibria, it has the following

[1] They characterize a stable collusive agreement in secret price-cutting games when players can communicate.

nice properties. First, it follows from condition (i) that if the length of each review phase is long enough, then players can make (almost) precise inferences about what the opponents did, using cumulative information within the review phase. This information aggregation allows players to punish the opponents efficiently, and as a result we can construct asymptotically efficient equilibria for some games. Also, condition (ii), which is called *strongly belief-free property* in this paper, ensures that a player's best reply does not depend on her beliefs about the opponents' history in the past review phases or about what action the opponents choose in the current review phase. Therefore, we do not need to track evolution of these beliefs when verifying incentive compatibility of a given strategy profile, which greatly simplifies our analysis.[2]

An important consequence of the strong belief-free property is that given a review phase, the set of optimal actions is independent of the history in the past review phase. This set of optimal actions is called a *regime*, and given a belief-free review-strategy equilibrium, let p be a probability distribution of regimes which measures how often each regime appears in the infinite horizon. Belief-free review-strategy equilibria can be classified in terms of a regime distribution p.

The main result of this paper is to precisely characterize the set of belief-free review-strategy equilibrium payoffs in the limit as the discount factor converges to one, assuming that players' signals are statistically independent conditional on an action profile. Specifically, we first find bounds on the payoff set of belief-free review-strategy equilibria given a regime distribution p. The lower bound of player i's equilibrium payoff is her minimax payoff when the opponents are restricted to choose actions from a regime which is randomly picked according to the distribution p. The upper bound of player i's equilibrium payoff is her "secured" payoff, that is, the payoff player i can obtain when the opponents maximize player i's worst payoffs (payoffs when player i take the worst action) and players are restricted to choose actions from a regime which is randomly picked according to the distribution p. Second, we show that these bounds are tight in the sense that given a p, the limit payoff set of belief-free review-strategy equilibrium is equal to (a feasible subset of) the product of the intervals between these upper and lower bounds.

The characterized payoff set is often a subset of the feasible and individually rational payoff set, but these two sets coincide in N-player prisoner's dilemma games, for which the folk theorem is established with arbitrary noise. Also, as an application of the main theorem, we develop a simple sufficient condition for the existence of asymptotically efficient equilibria. This sufficient condition is often satisfied in asymmetric

[2]Note that the strongly belief-free condition here is similar to but stronger than a requirement for being belief-free equilibria of EHO. In belief-free equilibria, a player's best reply is independent of the history up to the previous periods but can depend on the opponents' action today. On the other hand, the strong belief-free condition requires that a best reply be independent of the opponents' current action.

secret price-cutting games, so that cartel can be enforced even if a firm's price and sales are private information.

The interpretation of the bounds on the equilibrium payoff set is as follows. By definition, in each review phase of a belief-free review-strategy equilibrium, playing any action within the corresponding regime is optimal. Therefore, choosing the worst action within the regime in each review phase is optimal in the entire game. Since a player's equilibrium payoff cannot exceed the payoff yielded by this "worst" strategy, we obtain the upper bound stated above. An argument for the lower bound is standard; a player's equilibrium payoff must be at least her minimax payoff, as she plays an optimal action in a belief-free review-strategy equilibrium. Here, the opponents' actions in determining the minimax value are constrained by regimes, because in each review phase, actions not in the regime are suboptimal and should not be played on the equilibrium path. This gives the lower bound of equilibrium payoffs.

To prove that these bounds are tight, we substantially extend the equilibrium construction of Matsushima (2004), EHO, and Yamamoto (2007). In their analysis, attention is restricted to a simple class of belief-free review-strategy equilibria where each player independently chooses either "reward the opponent" or "punish the opponent" in each review phase; their main result is that this "bang-bang" strategy can often approximate efficiency in two-player games. However, if there are more than two players, this bang-bang strategy does not function well, because players $-i$ need to coordinate their play in order to reward or punish player i.[3] To deal with this problem, we borrow the idea of "informal communication" of Hörner and Olszewski (2006) and Yamamoto (2009), and constructs an equilibrium strategy such that some review phases are regarded as "communication stages" where players communicate through a choice of actions to coordinate a future play.

A main difference from the equilibrium construction of Hörner and Olszewski (2006) and Yamamoto (2009) is that we incorporate additional communication stages where players try to make a consensus about what happened in a previous communication stage. The role of this additional communication is roughly as follows. In our model, private signals are fully noisy, so that players i and j often make different inferences about what player l did in the previous communication stage, when player l deviated and did not choose a constant action. Then players i and j fail to coordinate a continuation play, which might yield better payoffs to the deviator l. The additional communication stages are useful to deter such a deviation; in the additional communication stage, players i and j communicate to make a consensus about player l's play so that they can avoid a miscoordination in a future play. Note that such a problem is not

[3] Yamamoto (2007) shows that the bang-bang strategy can approximate an efficient outcome in symmetric N-player prisoner's dilemma games, but this result rests on a strong assumption on the payoff function.

present in Hörner and Olszewski (2006) and Yamamoto (2009), because they assume almost-perfect monitoring so that players i and j have the same inference about player l's action with high probability.

One criticism of the past papers on belief-free review-strategy equilibria is that they assume conditional independence of signals, which is non-generic in the space of signal distributions. This paper addresses such a criticism by showing that a payoff vector in the limit equilibrium payoff set under conditionally-independent monitoring is also achievable under any nearby monitoring structure. In this sense, belief-free review-strategy equilibria work well as long as the signal distribution is almost (but not exactly) conditionally independent.

This robustness result is further extended by a subsequent work by Sugaya (2010). He modifies the equilibrium construction of this paper, and shows that the main theorem remains true for generic monitoring structure, if there are at least four players. That is, he shows that the limit set of belief-free review-strategy equilibrium payoffs is characterized by the formula identified by this paper, for any monitoring structure that satisfies a certain rank condition. His result gives a strong foundation to consideration of belief-free review-strategy equilibria. For example, in prisoner's dilemma games with more than three players, a folk theorem is obtained for generic monitoring structures, and hence patient players have less reason to play other sorts of equilibria, as far as equilibrium payoffs are concerned.

1.1 Literature Review

There is an extensive literature which studies repeated games with private monitoring. A pioneering work in this area is Sekiguchi (1997), who constructs a sort of trigger strategies to approximate an efficient outcome in prisoner's dilemma when monitoring approximates perfection. His equilibrium strategies are *belief-based* in that a player's best reply depends on her belief about the opponent's past history. Bhaskar and Obara (2002) extend this equilibrium construction to N-player prisoner's dilemma, and show that Pareto-efficiency can be approximated when observations are near perfect.

Meanwhile, Piccione (2002), Ely and Välimäki (2002), and EHO propose an alternative approach to the problem. They consider *belief-free strategies*, where a player's best reply is always independent of the past history and hence a player's belief about the past history is irrelevant to the incentive compatibility constraint. They show that these strategies often approximate Pareto-efficient outcomes in two-player games with almost-perfect monitoring. Yamamoto (2007) and Yamamoto (2009) extend their analysis to N-player games. Hörner and Olszewski (2006) further extend this approach, and show that the folk theorem holds for general games with almost-perfect monitoring. The analysis of this paper is close to this belief-free approach, since the concept of belief-free review-strategy equilibria is a combination of the idea of belief-free equilibria with review strategies of Radner (1985).

Recently, Fong, Gossner, Hörner, and Sannikov (2011) show an efficiency result in a repeated two-player prisoner's dilemma with fully-noisy and fully-private monitoring. They do not assume that players observe statistically independent signals, so that our folk theorem does not apply to their model. On the other hand, their analysis does not include ours either, since they do not have a full folk theorem and their equilibrium strategies cannot achieve some feasible and individually rational payoff vectors. Also, they assume the minimal informativeness, which requires that there be a signal that has a sufficiently high likelihood ratio to test the opponent's deviation. Such an assumption is not imposed in this paper.

For games where observations are almost public, Mailath and Morris (2002) and Mailath and Morris (2006) show that strict perfect public equilibria with bounded recall is robust to a perturbation of the monitoring structure. Also, Hörner and Olszewski (2009) show that a folk theorem obtains for games with almost-public monitoring.

Once outside cheap-talk communication is allowed, a folk theorem is restored for very general environments (Compte (1998), Kandori and Matsushima (1998), Fudenberg and Levine (2007), and Obara (2009)). Likewise, a folk theorem holds if players can acquire perfect information at a cost (Miyagawa, Miyahara, and Sekiguchi (2008)). For more detailed surveys, see Kandori (2002) and Mailath and Samuelson (2006). Also, see Lehrer (1990) for the case of no discounting, and Fudenberg and Levine (1991) for approximate equilibria with discounting.

2 Setup

2.1 The Model

The stage game is $\{I,(A_i,\Omega_i,g_i)_{i\in I},q\}$; $I=\{1,2,\cdots,N\}$ is the set of players, A_i is the finite set of player i's pure actions, Ω_i is the finite set of player i's private signals, $g_i : A_i\times\Omega_i \to \boldsymbol{R}$ is player i's profit function, and q is the probability distribution of the signals. Let $A=\times_{i\in I}A_i$ and $\Omega=\times_{i\in I}\Omega_i$.

In every stage game, players move simultaneously, and player $i\in I$ chooses an action $a_i\in A_i$ and then observes a noisy private signal $\omega_i\in\Omega_i$. The distribution of the signal profile $\omega=(\omega_1,\cdots,\omega_N)\in\Omega$ depends on the action profile $a=(a_1,\cdots,a_N)\in A$, and is denoted by $q(\cdot|a)\in\triangle\Omega$. Given an action a_i and a private signal ω_i, player i obtains payoff $g_i(a_i,\omega_i)$; note that in this setup, the payoff is not dependent on the opponents' actions and signals, and hence does not provide any extra information about the the opponents' private history.[4] Given an action profile $a \in A$, player i's expected payoff is $\pi_i(a) = \sum_{\omega \in \Omega} q(\omega|a) g_i(a_i, \omega_i)$. For each $a \in A$, let $\pi(a) = (\pi_i(a))_{i \in I}$.

Consider the infinitely repeated game with the discount factor $\delta \in (0,1)$. Let $(a_i^\tau, \omega_i^\tau)$ be the performed action and the observed signal in period τ, and let $h_i^t = (a_i^\tau, \omega_i^\tau)_{\tau=1}^t$ be player i's private history up to period $t \geq 1$. Let $h_i^0 = \emptyset$, and for each $t \geq 0$, let H_i^t be the set of all h_i^t. A strategy for player i is defined to be a mapping $s_i : \bigcup_{t=0}^\infty H_i^t \to \triangle A_i$. Let S_i be the set of all strategies of player i, and let $S = \times_{i \in I} S_i$. Let $w_i(s)$ denote player i's expected average payoff when players play a strategy profile $s \in S$, that is, $w_i(s) = (1-\delta) E[\sum_{t=1}^\infty \delta^{t-1} \pi_i(a^t)|s]$. For each strategy $s_i \in S_i$ and history $h_i^t \in H_i^t$, let $s_i|_{h_i^t}$ be player i's continuation strategy after h_i^t. Also, for each $s_i \in S_i$, $h_i^t \in H_i^t$ and $a_i \in A_i$, let $s_i|_{(h_i^t, a_i)}$ be player i's strategy $\tilde{s}_i \in S_i$ such that $\tilde{s}_i(h_i^0) = a_i$ and such that for any $h_i^1 \in H_i^1$, $\tilde{s}_i|_{h_i^1} = s_i|_{h_i^{t+1}}$ where $h_i^{t+1} = (h_i^t, h_i^1)$. In words, $s_i|_{(h_i^t, a_i)}$ denotes the continuation strategy after history h_i^t but the play in the first period is replaced with the pure action a_i.

As in Section 5 of EHO, we consider games with conditionally-independent monitoring, where players observe statistically independent signals conditional on actions played. Formally, we impose the following assumption:

Condition CI. There is $q_i : A \to \triangle \Omega_i$ for each i such that the following properties hold.

(i) For each $a \in A$ and $\omega \in \Omega$,

$$q(\omega|a) = \prod_{i \in I} q_i(\omega_i|a, \omega_0).$$

(ii) For each $i \in I$ and $a_i \in A_i$, $\text{rank} Q_i(a_i) = |A_{-i}|$ where $Q_i(a_i)$ is a matrix with rows $(q_i(\omega_i|a_i, a_{-i}))_{\omega_i \in \Omega_i}$ for all $a_{-i} \in A_{-i}$.

Clause (i) says that given an action profile a, players observe statistically independent signals. Clause (ii) is a version of individual full-rank condition of Fudenberg, Levine, and Maskin (1994); it requires that a player can statistically distinguish the opponents' actions. Clause (ii) is satisfied for generic monitoring structures, provided that the set of private signals is sufficiently rich so that $|\Omega_i| \geq |A_{-i}|$ for all i.

In addition to (CI), we assume the signal distribution to be full support:

Condition FS. The signal distribution q has *full support* in that $q(\omega|a) > 0$ for all $a \in A$ and $\omega \in \Omega$.

As Sekiguchi (1997) shows, (FS) assures that for any Nash equilibrium $s \in S$, there is a sequential equilibrium $\tilde{s} \in S$ that generates the same outcome distribution as for

[4] Here we follow the existing works and assume that payoffs are observable. However, this assumption is not necessary; all our results are valid even if payoffs are not observable and directly dependent on the opponents' private history.

s. Therefore, under (FS), the set of Nash equilibrium payoffs is identical with that of sequential equilibrium payoffs.

Remark 1. (CI) is a simple sufficient condition to obtain our main result, Theorem 1, but it is stronger than necessary and can be replaced with a weaker condition. In Appendix G, we show that Theorem 1 remains valid even if private signals are correlated through an unobservable common shock as in Matsushima (2004) and Yamamoto (2007).

2.2 Belief-Free Review-Strategy Equilibrium

This section introduces a notion of belief-free review-strategy equilibria, which captures and generalizes the idea of the equilibrium construction of Matsushima (2004), EHO, and Yamamoto (2007). In their equilibrium strategies, the infinite horizon is regarded as a sequence of *review phases* with length T, and players play constant actions in every review phase, i.e., once player i chooses an action a_i in the initial period of a review phase (say, period $nT+1$), then she continues to choose the same action a_i up to the end of the review phase (period $(n+1)T$). At the end of each review phase, players makes a statistical inference about the opponents' actions using the information pooled within the review phase. When T is sufficiently large, this statistical test has an arbitrarily high power so that players can obtain very precise information about what actions the opponents played.

The present paper considers a slightly broader class of review strategies where each review phase may have different length.

Definition 1. Let $(t_l)_{l=0}^{\infty}$ be a sequence of integers satisfying $t_0 = 0$ and $t_l > t_{l-1}$ for all $l \geq 1$. A strategy profile $s \in S$ is a *review strategy profile with sequence* $(t_l)_{l=0}^{\infty}$ if $s_i(h_i^t)[a_i^t] = 1$ for each $t \notin \{t_l | \forall l \geq 0\}$, for each $i \in I$, and for each $h_i^t = (a_i^{\tau}, \omega_i^{\tau})_{\tau=1}^{t} \in H_i^t$.

Intuitively, t_l denotes the last period of the lth review phase. For example, the above definition asserts that for each period $t \in \{2, \cdots, t_1\}$, a player has to choose the same action as in period one; thus the collection of the first t_1 periods is regarded as the first review phase. Likewise, the collection of the next $t_2 - t_1$ periods is the second review phase, and so forth. From the law of large numbers, players can obtain almost perfect information about the opponents' action in each review phase, if $t_l - t_{l-1}$ is sufficiently large for all $l \geq 1$.

A belief-free review-strategy equilibrium, which we focus on in this paper, is a subset of review strategy profiles. For each i and s_{-i}, let $BR(s_{-i})$ denote the set of player i's best replies in the infinitely repeated game against s_{-i}. Also, let $\text{supp}\{s_{-i}(h_{-i}^t)\}$ denote the support of $s(h_{-i}^t)$; that is, $\text{supp}\{s_{-i}(h_{-i}^t)\}$ is the set of actions a_{-i} played with positive probability in period $t+1$ when players $-i$ follow the strategy s_{-i} and their past private history is h^t_{-i}.

Definition 2. A strategy profile $s \in S$ is *strongly belief-free in the lth review phase* if it is a review strategy profile with some sequence $(t_l)_{l=0}^{\infty}$, and if for all $i \in I$, $h^{t_{l-1}} \in H^{t_{l-1}}$, and $a_{-i} \in \text{supp}\{s_{-i}(h^{t_{l-1}}_{-i})\}$,

$$s_i|_{h_i^{t_{l-1}}} \in BR(s_{-i}|_{(h_{-i}^{t_{l-1}}, a_{-i})}). \tag{1}$$

A strategy profile s is a *belief-free review-strategy equilibrium with* $(t_l)_{l=0}^{\infty}$ if it is a review strategy profile with $(t_l)_{l=0}^{\infty}$ and is strongly belief-free in every review phase.

In words, a review strategy profile is strongly belief-free in the lth review phase if a player's continuation strategy from the lth review phase is a best reply independently of the past history and of what constant action the opponents pick in the lth review phase. By definition, playing a pure-strategy Nash equilibrium of the stage game in every period is a belief-free review-strategy equilibrium where each review phase has length one. On the other hand, playing a mixed-strategy equilibrium of the stage game in every period needs not be a belief-free review-strategy equilibrium, as it may not satisfy (1).

Note that the equilibrium strategies of Matsushima (2004), EHO, and Yamamoto (2007) are belief-free review-strategy equilibria. Note also that a belief-free review-strategy equilibrium needs not be a belief-free equilibrium of EHO; the reason is that in belief-free review-strategy equilibria, the belief-free condition is imposed only at the beginning of each review phase, while a belief-free equilibrium requires that a player's continuation strategy is a best reply independently of the past history *in every period*. Conversely, a belief-free equilibrium needs not be a belief-free review-strategy equilibrium, as a player's best reply might depend on the present action of the opponents in a belief-free equilibrium.

A study of belief-free review-strategy equilibria is motivated by its tractability. By definition, in this class of equilibria, a player's best reply does not depend on her beliefs about the opponents' history in the past review phases or about what action the opponents choose in the current review phase. Therefore, we do not need to calculate these beliefs at all when verifying incentive compatibility of a given strategy profile, which greatly simplifies our analysis. (As argued in the last paragraph, a belief-free review-strategy equilibrium is not a belief-free equilibrium of EHO; indeed, in a belief-free review-strategy equilibrium, a player's belief about what signals the opponents observed in the current review phase is relevant to her best reply. Under (CI), we can easily compute this belief, so that it does not cause a serious problem in our analysis.)

3 Characterizing the Limit Equilibrium Payoff Set

3.1 Main Theorem

This section presents the main result of this paper: The set of belief-free review-strategy equilibrium payoffs is characterized in the limit as the discount factor approaches one. To state the result, the following notation is useful. A non-empty subset $\mathscr{A}$ of A is a *regime generated from* A if $\mathscr{A}$ has a product structure, i.e., $\mathscr{A} = \times_{i\in I}\mathscr{A}_i$ and $\mathscr{A}_i \subseteq A_i$ for all i. Let $\mathscr{J}$ be the set of all regimes generated from A, and for each probability distribution $p \in \triangle\mathscr{J}$, let

$$V(p) \equiv \text{co}\left\{ \sum_{\mathscr{A}\in\mathscr{J}} p(\mathscr{A})\pi(a(\mathscr{A})) \,\middle|\, a(\mathscr{A}) \in \mathscr{A}, \quad \forall \mathscr{A} \in \mathscr{J} \right\}$$

where co$B$ stands for the convex hull of B. Intuitively, $V(p)$ is the *constrained feasible payoff set*, i.e., the set of feasible payoffs when a regime (or a "recommended action set") $\mathscr{A} \subseteq A$ is randomly picked according to the public randomization $p \in \triangle\mathscr{J}$, and players choose actions from this set. Letting $p^A \in \triangle\mathscr{J}$ be such that $p^A(\mathscr{A}) = 1$ for $\mathscr{A} = A$, the set $V(p^A)$ corresponds to the *feasible payoff set* of the repeated game. The feasible payoff set is *full dimensional* if $\dim V(p^A) = |I|$.

For each i and $\mathscr{A}$, let

$$\underline{v}_i(\mathscr{A}) \equiv \min_{a_{-i}\in\mathscr{A}_{-i}} \max_{a_i\in A_i} \pi_i(a) \quad \text{and} \quad \overline{v}_i(\mathscr{A}) \equiv \max_{a_{-i}\in\mathscr{A}_{-i}} \min_{a_i\in\mathscr{A}_i} \pi_i(a).$$

Also, for each i and $\mathscr{A}$, let $\underline{a}^i(\mathscr{A}) \in \mathscr{A}$ and $\overline{a}^i(\mathscr{A}) \in \mathscr{A}$ be such that $\underline{a}^i_{-i}(\mathscr{A})$ and $\overline{a}^i_{-i}(\mathscr{A})$ solve the above problems, that is,

$$\underline{v}_i(\mathscr{A}) = \max_{a_i\in A_i} \pi_i(a_i, \underline{a}^i_{-i}(\mathscr{A})) \quad \text{and} \quad \overline{v}_i(\mathscr{A}) = \min_{a_i\in\mathscr{A}_i} \pi_i(a_i, \overline{a}^i_{-i}(\mathscr{A})).$$

Note that this definition does not pose any constraint on the specification of $\underline{a}^i_i(\mathscr{A})$ and $\overline{a}^i_i(\mathscr{A})$; these actions can be arbitrarily chosen from the set $\mathscr{A}_i$. Intuitively, $\underline{v}_i(\mathscr{A})$ is the minimax payoff for player i when the opponents are restricted to play pure actions from the recommended set $\mathscr{A}_{-i} \subseteq A_{-i}$. In fact, player i cannot earn more than $\underline{v}_i(\mathscr{A})$ against $\underline{a}^i_{-i}(\mathscr{A})$. Likewise, $\overline{v}_i(\mathscr{A})$ is the secured payoff for player i when players are restricted to choose pure actions from the recommended set $\mathscr{A} \subseteq A$. Indeed, player i's payoff is at least $\overline{v}_i(\mathscr{A})$ against $\overline{a}^i_{-i}(\mathscr{A})$ as long as she chooses an action from $\mathscr{A}_i$.

For each i, let $\underline{v}_i$ be a column vector with the components $\underline{v}_i(\mathscr{A})$ for all $\mathscr{A} \in \mathscr{J}$, that is, $\underline{v}_i = {}^\top(\underline{v}_i(\mathscr{A}))_{\mathscr{A}\in\mathscr{J}}$. Also, let $\overline{v}_i = {}^\top(\overline{v}_i(\mathscr{A}))_{\mathscr{A}\in\mathscr{J}}$. Note that, for each distribution $p \in \triangle\mathscr{J}$, the product $p\underline{v}_i$ is equal to the weighted average of the minimax payoffs, $\sum_{\mathscr{A}\in\mathscr{J}} p(\mathscr{A})\underline{v}_i(\mathscr{A})$. Likewise, $p\overline{v}_i$ equals the weighted average of the secured payoffs, $\sum_{\mathscr{A}\in\mathscr{J}} p(\mathscr{A})\overline{v}_i(\mathscr{A})$.

The stage games are classified into four groups in the following way. Note that this classification depends only on the (expected) payoff function of the stage game.

- (positive case) For some $p \in \triangle \mathscr{J}$, the set $V(p) \bigcap \times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i]$ is N-dimensional.
- (empty case) For any $p \in \triangle \mathscr{J}$, the set $V(p) \bigcap \times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i]$ is empty.
- (negative case) The set $V(p) \bigcap \times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i]$ is a singleton or empty for all $p \in \triangle \mathscr{J}$, and there is $p \in \triangle \mathscr{J}$ such that the intersection of $V(p)$ and $\times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i]$ is a singleton.
- (abnormal case) The set $V(p) \bigcap \times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i]$ is not N-dimensional for all $p \in \triangle \mathscr{J}$, and there is $p \in \triangle \mathscr{J}$ such that the set $V(p) \bigcap \times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i]$ is neither empty nor a singleton.

Given a stage game and given a $\delta \in (0,1)$, let $E(\delta)$ be the set of belief-free review-strategy equilibrium payoffs. That is, for any payoff vector $v \in E(\delta)$, there is a belief-free review-strategy equilibrium with some sequence $(t_l)_{l=0}^{\infty}$ and with payoff v. The following is the main result of the paper, which characterizes the limit equilibrium payoff set for the positive, empty, and negative cases.

Theorem 1. *Suppose that (CI) and (FS) hold. Then,*

$$\lim_{\delta\to 1} E(\delta) = \bigcup_{p\in\triangle \mathscr{J}} (V(p) \cap \times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i]) \tag{2}$$

in the positive case; $E(\delta) = \emptyset$ *for every* $\delta \in (0,1)$ *in the empty case; and* $\lim_{\delta\to 1} E(\delta)$ *equals the convex hull of the set of pure-strategy Nash equilibrium payoffs of the stage game in the negative case.*

To interpret the statement of this theorem, we classify belief-free review-strategy equilibria in terms of a regime distribution p, where given a belief-free review-strategy equilibrium, p is a parameter which measures how often each regime appears in the infinite horizon. In the positive case, the theorem asserts that for a given p, the limit equilibrium payoff set equals (the feasible subset of) the product set $\times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i]$. That is, the lower bound of the equilibrium payoffs is equal to the minimax payoff $p\underline{v}_i$ while the upper bound is the secured payoff $p\bar{v}_i$. Yamamoto (2009) shows that the limit set of belief-free equilibrium payoffs is computed by a formula similar to (2); but note that players are allowed to choose mixed actions when determining the upper and lower bounds of the payoff set in Yamamoto (2009), while here players are constrained to play pure actions when calculating $\bar{v}_i$ and $\underline{v}_i$. This difference comes from the fact that belief-free review-strategy equilibria impose the strongly belief-free condition (1), while belief-free equilibria do not. Since (1) requires player i's continuation strategy to be optimal *after* mixture by players $-i$, her equilibrium payoff is at least the minimax payoff when players are restricted to pure actions. A similar argument applies to the upper bound.

As Yamamoto (2009) argues, if there are only two players, then the set $\times_{i\in I}[p\underline{v}_i, p\bar{v}_i]$ is a subset of $V(p)$ for each p, so that (2) reduces to

$$\lim_{\delta\to 1} E(\delta) = \bigcup_{p\in\triangle\mathscr{J}} \times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i].$$

This formula is exactly the same as that of Proposition 10 of EHO for two-by-two games, which means that Theorem 1 subsumes their result as a special case. Note also that our Theorem 1 encompasses Theorems 1 and 2 of Matsushima (2004) and Theorem 1 of Yamamoto (2007) as well; See Section 4 for more discussions. (Theorem 2 of Matsushima (2004) and Theorem 1 of Yamamoto (2007) allow that players' signals are correlated through a common shock. Our Theorem 1 extends to such a setting, as argued in Remark 1 and formally proved in Appendix G.)

As noted, a sufficient condition for the existence of belief-free review-strategy equilibria is that the stage game has a pure-strategy Nash equilibrium; indeed, playing a pure-strategy Nash equilibrium of the stage game in every period is a belief-free review-strategy equilibrium where each review phase has length one. On the other hand, if the stage game has only a mixed-strategy equilibrium, then belief-free review-strategy equilibria need not exist. For example, consider the following game:

	H	T
H	1, −2	−1, 1
T	−2, 1	1, −1

Player 1 chooses a row and player 2 chooses a column. Note that this stage game has no pure-strategy Nash equilibrium. In this game, it is easy to check that $\underline{v}_i(\mathscr{A}) = 1$ for all i and $\mathscr{A}$ so that $p\underline{v}_i = 1$ for all i and p. Since any feasible payoff vector is Pareto-dominated by $(p\underline{v}_1, p\underline{v}_2) = (1,1)$, the set $V(p)\bigcap\times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i]$ is empty for all p. Therefore, the game is classified to the empty case, and from Theorem 1, belief-free review-strategy equilibria do not exist for any discount factor δ.

Theorem 1 is a corollary of the next two propositions. Note that the statement of Proposition 1 is stronger than needed, as it does not assume (CI) or (FS). The proof of Proposition 1 is similar to that of Proposition 1 of Yamamoto (2009), and is provided in Appendix A for completeness. The proof of Proposition 2 is found in the following subsections.

Proposition 1. *In the positive case, $E(\delta)$ is a subset of the right-hand side of (2) for any $\delta\in(0,1)$. In the empty case, $E(\delta)=\emptyset$. In the negative case, $\lim_{\delta\to 1}E(\delta)$ is equal to the convex hull of the set of pure-strategy Nash equilibrium payoffs of the stage game.*

Proposition 2. *Suppose that (CI) and (FS) hold. Then, in the positive case,* $\lim_{\delta\to 1} E(\delta)$ *includes the right-hand side of (2).*

Theorem 1 gives a precise characterization of the limit equilibrium payoffs for the positive, empty, and negative cases, but it does not consider the abnormal case. In Appendix F, we show that for the abnormal case with generic payoff functions, the equilibrium payoff set is either empty or the convex hull of the set of pure-strategy Nash equilibrium payoffs of the stage game.

3.2 Proof of Proposition 2 with Two Players

3.2.1 Overview

To prove the proposition, it suffices to show that for any payoff vector v in the right-hand side of (2) and for a sufficiently large discount factor δ, there is a belief-free review-strategy equilibrium with payoff v. In this subsection, we explicitly construct such an equilibrium for two-player games. The analysis for three-or-more player games is more complex and will be presented in the next subsection.

Our equilibrium construction is based on EHO's for two-by-two games, so it will be helpful to explain how EHO's equilibrium strategies look like. The infinitely repeated game is regarded as a sequence of T-period review phases, and in each review phase, a player is either in *good state* G or in *bad state* B. When player $-i$ is in good state G, she chooses the action $\overline{a}^i_{-i}(\mathscr{A})$ for some $\mathscr{A}$ throughout the review phase to "reward" player i. (Recall that player i obtains at least the "secured" payoff $\overline{v}_i^{\mathscr{A}}$ against $\overline{a}^i_{-i}(\mathscr{A})$ if she chooses an action from the set $\mathscr{A}_i$.) On the other hand, when player $-i$ is in bad state B, she chooses the action $\underline{a}^i_{-i}(\mathscr{A})$ for some $\mathscr{A}$ throughout the review phase to "punish" player i. (Again, recall that player i's payoff against $\underline{a}^i_{-i}(\mathscr{A})$ is at most $\underline{v}_i^{\mathscr{A}}$.) After a T-period play, each player makes a statistical inference about the opponent's play using the private signals pooled within the review phase, and then decides which state to go to (either state G or B) for the next review phase. This transition rule between states G and B is judiciously chosen so that in every review phase, player i is indifferent between being in good state (i.e., playing $\overline{a}^i_{-i}(\mathscr{A})$ for T periods) and being in bad state (i.e., playing $\underline{a}^i_{-i}(\mathscr{A})$ for T periods), and is not willing to do other sorts of play. Therefore players' incentive compatibility is satisfied.

A difference between EHO's equilibrium and ours is a construction of statistical tests about actions. In EHO, the analysis is limited to two-by-two games, so that each player needs to distinguish only two actions of the opponent. For this, it is sufficient to consider a simple statistical test such that player $-i$ counts the number of observations of a particular signal ω_{-i} during a T-period play.

On the other hand, here we consider general two-player games, so that a player may have more than two possible actions. In such a case, a player's statistical inference must be based on two or more signals,[5] which often complicates the verification of a player's incentive compatibility. The contribution of this paper is to find an elaborate method of statistical inference which makes the verification of incentive compatibility constraints simple.

It may be noteworthy that (CI) plays an important role here. Under (CI), a player's signal has no information about the opponent's signal and hence no information about whether she is likely to pass the opponent's statistical test. Therefore, a player has no incentive to play a history-dependent strategy in a review phase, so that when verifying the incentive compatibility constraint of a given strategy profile, we can restrict attention to deviations to history-independent strategies.

3.2.2 Random Events

Here we introduce a notion of *random events*, which is used for statistical tests in our equilibrium construction. A random event ψ_i is defined as a function from $A_i \times \Omega_i$ to $[0,1]$, and ψ_i is *counted in period t* if $\psi_i(a_i^t, \omega_i^t) \geq z_i^t$, where (a_i^t, ω_i^t) denotes player i's action and signal in period t and z_i^t is randomly chosen by player i at the end of period t according to the uniform distribution on $[0,1]$. Put differently, $\psi_i(a_i^t, \omega_i^t)$ denotes the probability that the random event ψ_i is counted in period t conditional on (a_i^t, ω_i^t). A player may count multiple random events in a given period; for example, given an outcome $(a_i^t, \omega_i^t, z_i^t)$, both random events ψ_i and $\tilde{\psi}_i$ are counted in period t if $\psi_i(a_i^t, \omega_i^t) \geq z_i^t$ and if $\tilde{\psi}_i(a_i^t, \omega_i^t) \geq z_i^t$. With an abuse of notation, let h_i^t denote player i's private information up to period t, i.e., $h_i^t = (a_i^\tau, \omega_i^\tau, z_i^\tau)_{\tau=1}^t$. Let H_i^t be the set of all $h_i^t = (a_i^\tau, \omega_i^\tau, z_i^\tau)_{\tau=1}^t$.

For each $\psi_i : A_i \times \Omega_i \to [0,1]$, let $P(\psi_i|a)$ be the probability that the random event ψ_i is counted given an action profile $a \in A$, that is, $P(\psi_i|a) = \sum_{\omega \in \Omega} q(\omega|a)\psi_i(a_i, \omega_i)$. Let $\mathscr{J}_i$ be the set of non-empty subsets of A_i. For each $i \in I$ and $\mathscr{A}_{-i} \in \mathscr{J}_{-i}$, let $\psi_i(\mathscr{A}_{-i})$ be as in the following lemma.

Lemma 1. *Suppose that (CI) holds. Then, for some q_1 and q_2 satisfying $0 < q_1 < q_2 < 1$, there is a random event $\psi_i(\mathscr{A}_{-i}) : A_i \times \Omega_i \to [0,1]$ for all i and $\mathscr{A}_{-i} \in \mathscr{J}_{-i}$ such that for all $a \in A$,*

$$P(\psi_i(\mathscr{A}_{-i})|a) = \begin{cases} q_2 & \text{if} \quad a_{-i} \in \mathscr{A}_{-i} \\ q_1 & \text{otherwise} \end{cases} . \tag{3}$$

[5] To see this, suppose that player $-i$ observes a signal ω_{-i} with 0.8 against a_i, with 0.5 against a_i', and with 0.2 against a_i''. If player $-i$ tries to infer player i's action only from ω_{-i}, she cannot distinguish whether player i plays a_i' or mixes a_i and a_i'' with fifty-fifty.

Proof. Analogous to Lemma 1 of Yamamoto (2007). *Q.E.D.*

The condition (3) implies that a player can statistically distinguish the opponent's action using these random events. For example, the random event $\psi_i(\{a_{-i}\})$ is counted with high probability q_2 if and only if player $-i$ chooses the action a_{-i}. Hence, player i can conjecture that the action a_{-i} is played if $\psi_i(\{a_{-i}\})$ is counted many times within a review phase.

Let $F(\tau,T,r)$ denote the probability that $\psi_i(\{a_{-i}\})$ is counted exactly r times out of T periods when player $-i$ chooses some $\tilde{a}_{-i} \neq a_{-i}$ in the first τ periods and then a_{-i} in the remaining $T-\tau$ periods. As Matsushima (2004) shows, there is a sequence of integers $(Z_T)_{T=1}^{\infty}$ such that

$$\lim_{T\to\infty} \sum_{r>Z_T} F(0,T,r) = 1, \tag{4}$$

$$\lim_{T\to\infty} \sum_{r>Z_T} F(T,T,r) = 0, \tag{5}$$

and

$$\lim_{T\to\infty} TF(0,T-1,Z_T) = \infty. \tag{6}$$

3.2.3 Equilibrium Construction with Two Players

Let $v=(v_1,v_2)$ be a payoff vector in the interior of the right-hand side of (2). In what follows, we construct a belief-free review-strategy equilibrium with payoff v for sufficiently large δ. To simplify the notation, we write $\underline{a}_{-i}^{\mathscr{A}}$ and $\overline{a}_{-i}^{\mathscr{A}}$ for $\underline{a}_{-i}^{i}(\mathscr{A})$ and $\overline{a}_{-i}^{i}(\mathscr{A})$, respectively.

As Yamamoto (2009) shows, given such a v, there is $p \in \triangle\mathscr{J}$ such that v is an element of the interior of the set $V(p)\cap \times_{i\in I}[p\underline{v}_i,\ p\overline{v}_i]$. Assume that players can observe a public signal y from the set $\mathscr{J}$ according to the distribution p in every period. This assumption greatly simplifies the equilibrium construction, and does not cause loss of generality; indeed, such a public randomization device is dispensable, as EHO argue in the online appendix.

For each i and $\mathscr{A}$, let $(a_i^{B,\mathscr{A},l})_{l=1}^{|A_i|}$ be an ordering of all elements of A_i such that $\pi_i(a_i^{B,\mathscr{A},l-1},\underline{a}_{-i}^{\mathscr{A}}) \geq \pi_i(a_i^{B,\mathscr{A},l},\underline{a}_{-i}^{\mathscr{A}})$ for each $l\geq 2$; that is, $(a_i^{B,\mathscr{A},l})_{l=1}^{|A_i|}$ is an ordering of actions in terms of payoffs against $\underline{a}_{-i}^{\mathscr{A}}$. Then, for each $\mathscr{A}$ and $l\in\{1,\cdots,|A_i|\}$, let $1_i^{B,\mathscr{A},l}: H_{-i}^T \to \{0,1\}$ be an indicator function such that $1_i^{B,\mathscr{A},l}(h_{-i}^T)=1$ if and only if the random event $\psi_{-i}(\{a_i^{B,\mathscr{A},\tilde{l}} | l\leq\tilde{l}\leq|A_i|\})$ is counted more than Z_T times within a T-period history h_{-i}^T.

To see how this indicator function works, fix $l\in\{1,\cdots,|A_i|\}$, and consider the random event $\psi_{-i}(\{a_i^{B,\mathscr{A},\tilde{l}} | l\leq\tilde{l}\leq|A_i|\})$. From (3), this random event is counted with high probability q_2 if player i chooses an action a_i from the set $\{a_i^{B,\mathscr{A},\tilde{l}} | l \leq \tilde{l} \leq |A_i|\}$ against $\underline{a}_{-i}^{\mathscr{A}}$, and is counted with low probability q_1 if player i chooses other actions. Then it follows from (4) and (5) that the indicator function $1_i^{B,\mathscr{A},l}(h_{-i}^T)$ almost surely takes 1 in the former case, while it almost surely takes 0 in the latter case. This property implies that player $-i$ can test whether player i took an action from the set $\{a_i^{B,\mathscr{A},\tilde{l}} | l \leq \tilde{l} \leq |A_i|\}$ or not, by referring to the indicator function $1_i^{B,\mathscr{A},l}$. Thus, by checking all the indicator functions $(1_i^{B,\mathscr{A},l})_{l=1}^{|A_i|}$, player $-i$ can obtain almost perfect information about player i's action in a T-period interval.

Likewise, for each i and $\mathscr{A}$, let $(a_i^{G,\mathscr{A},l})_{l=1}^{|\mathscr{A}_i|}$ be an ordering of all elements of $\mathscr{A}_i$ such that $\pi_i(a_i^{G,\mathscr{A},l-1}, \overline{a}_{-i}^{\mathscr{A}}) \geq \pi_i(a_i^{G,\mathscr{A},l}, \overline{a}_{-i}^{\mathscr{A}})$ for each $l \geq 2$. Then, for each $\mathscr{A}$ and $l \in \{1, \cdots, |\mathscr{A}_i|\}$, let $1_i^{G,\mathscr{A},l} : H_{-i}^T \to \{0,1\}$ be an indicator function such that $1_i^{G,\mathscr{A},l}(h_{-i}^T) = 1$ if and only if the random event $\psi_{-i}(\{a_i^{G,\mathscr{A},\tilde{l}} | l \leq \tilde{l} \leq |\mathscr{A}_i|\})$ is counted more than Z_T times during T periods, according to a T-period history h_{-i}^T. Again, using this indicator function $1_i^{G,\mathscr{A},l}$, player $-i$ can test whether player i chooses her action from the set $\{a_i^{G,\mathscr{A},\tilde{l}} | l \leq \tilde{l} \leq |\mathscr{A}_i|\}$.

Let η be such that $0 < \eta < p\overline{v}_i - v_i$ for all i, and let C be such that $C > \max_{a_i \in A_i} \pi_i(a_i, \overline{a}_{-i}^{\mathscr{A}}) - \overline{v}_i(\mathscr{A})$ for all i and $\mathscr{A}$. For notational convenience, let

$$\lambda_i^{B,\mathscr{A},l} = \begin{cases} 0 & \text{if } \ l = 1 \\ \dfrac{\pi_i(a_i^{B,\mathscr{A},l-1}, \underline{a}_{-i}^{\mathscr{A}}) - \pi_i(a_i^{B,\mathscr{A},l}, \underline{a}_{-i}^{\mathscr{A}})}{\sum_{r>Z_T} F(0,T,r) - \sum_{r>Z_T} F(T,T,r)} & \text{if } \ l \in \{2, \cdots, |A_i|\} \end{cases} \tag{7}$$

for each $\mathscr{A} \in \mathscr{J}$. Also, let

$$\lambda_i^{G,\mathscr{A},l} = \begin{cases} \dfrac{C + \overline{v}_i(\mathscr{A}) - \pi_i(a_i^{G,\mathscr{A},1}, \overline{a}_{-i}^{\mathscr{A}})}{\sum_{r>Z_T} F(0,T,r) - \sum_{r>Z_T} F(T,T,r)} & \text{if } \ l = 1 \\ \\ \dfrac{\pi_i(a_i^{G,\mathscr{A},l-1}, \overline{a}_{-i}^{\mathscr{A}}) - \pi_i(a_i^{G,\mathscr{A},l}, \overline{a}_{-i}^{\mathscr{A}})}{\sum_{r>Z_T} F(0,T,r) - \sum_{r>Z_T} F(T,T,r)} & \text{if } \ l \in \{2, \cdots, |\mathscr{A}_i|\} \end{cases} \tag{8}$$

for each $\mathscr{A} \in \mathscr{J}$. For each i, let

$$\underline{w}_i = \sum_{\mathscr{A} \in \mathscr{J}} p(\mathscr{A}) \left[\underline{v}_i(\mathscr{A}) + \sum_{r>Z_T} F(T,T,r) \sum_{l \geq 1} \lambda_i^{B,\mathscr{A},l} \right]$$

and

$$\overline{w}_i = \sum_{\mathscr{A} \in \mathscr{J}} p(\mathscr{A}) \left[\overline{v}_i(\mathscr{A}) - \eta + \sum_{r>Z_T} F(T,T,r) \sum_{l \geq 1} \lambda_i^{G,\mathscr{A},l} \right].$$

It follows from (4) and (5) that for all i,

$$\lim_{T \to \infty} \underline{w}_i = p\underline{v}_i < p\overline{v}_i - \eta = \lim_{T \to \infty} \overline{w}_i. \tag{9}$$

In what follows, we show that any interior point v^* of the set $\times_{i\in I}[p\underline{v}_i, p\overline{v}_i - \eta]$ can be achieved by belief-free review-strategy equilibria for sufficiently large T and δ. This completes the proof, as $p\underline{v}_i < v_i < p\overline{v}_i - \eta$.

In our equilibrium strategies, players play a strategy profile in which the infinite horizon is regarded as a sequence of review phases with length T. Specifically, for each i, player $-i$'s strategy is described by the following automaton with initial state v_i^*.

State $w_i \in [\underline{w}_i, \overline{w}_i]$: Go to phase B with probability α_{-i}, and go to phase G with probability $1-\alpha_{-i}$ where α_{-i} solves $w_i = \alpha_{-i}\underline{w}_i + (1-\alpha_{-i})\overline{w}_i$.

Phase B : Play the action $\underline{a}_{-i}^{\mathscr{A}}$ for T periods, where $\mathscr{A}$ is the outcome of the public randomization p in the initial period of the phase (say, period $nT+1$). After that, go to state $w_i = \underline{w}_i + (1-\delta)U_i^{B,\mathscr{A}}(h_{-i}^T)$, where h_{-i}^T is the recent T-period private history and the function $U_i^{B,\mathscr{A}} : H_{-i}^T \to \boldsymbol{R}$ is defined to be

$$U_i^{B,\mathscr{A}}(h_{-i}^T) = \frac{1-\delta^T}{\delta^T(1-\delta)}\sum_{l=1}^{|A_i|} 1_i^{B,\mathscr{A},l}(h_{-i}^T)\lambda_i^{B,\mathscr{A},l}.$$

Phase G : Play the action $\overline{a}_{-i}^{\mathscr{A}}$ for T periods, where $\mathscr{A}$ is the outcome of the public randomization p in the initial period of the phase (say, period $nT+1$). After that, go to state $w_i = \overline{w}_i + (1-\delta)U_i^{G,\mathscr{A}}(h_{-i}^T)$, where h_{-i}^T is the recent T-period private history, and the function $U_i^{G,\mathscr{A}} : H_{-i}^T \to \boldsymbol{R}$ is defined to be

$$U_i^{G,\mathscr{A}}(h_{-i}^T) = \frac{1-\delta^T}{\delta^T(1-\delta)}\left[-C-\eta+\sum_{l=1}^{|\mathscr{A}_i|} 1_i^{G,\mathscr{A},l}(h_{-i}^T)\lambda_i^{G,\mathscr{A},l}\right].$$

The idea of this automaton is as follows. In each review phase, player $-i$ is either in state G or in state B. Player $-i$ in state G chooses $\overline{a}_{-i}^{\mathscr{A}}$ to reward the opponent, while in state B, she chooses $\underline{a}_{-i}^{\mathscr{A}}$ to punish the opponent. The functions $U_i^{B,\mathscr{A}}$ and $U_i^{G,\mathscr{A}}$ determine the transition probability between states G and B at the end of each review phase; roughly, the larger $U_i^{B,\mathscr{A}}(h_{-i}^T)$ (or $U_i^{G,\mathscr{A}}(h_{-i}^T)$) is, the more likely player $-i$ moves to "reward state" G. In this sense, the functions $U_i^{B,\mathscr{A}}$ (and $U_i^{G,\mathscr{A}}$) can be viewed as a reward function to player i, i.e., an increase in $U_i^{B,\mathscr{A}}(h_{-i}^T)$ means more continuation payoffs of player i.

The reward functions $U_i^{B,\mathscr{A}}$ and $U_i^{G,\mathscr{A}}$ are defined in such a way that player i receives a "bonus" $\lambda_i^{B,\mathscr{A},l}$ (or $\lambda_i^{G,\mathscr{A},l}$) if player $-i$ counts the random event $\psi_{-i}(\{a_i^{B,\mathscr{A},\tilde{l}} | l \leq \tilde{l} \leq |A_i|\})$ (or the random event $\psi_{-i}(\{a_i^{G,\mathscr{A},\tilde{l}} | l \leq \tilde{l} \leq |\mathscr{A}_i|\})$) more than Z_T times during the T-period review phase so that the corresponding indicator function takes one. Here the values $\lambda_i^{B,\mathscr{A},l}$ and $\lambda_i^{G,\mathscr{A},l}$ are carefully chosen so that player i is indifferent among all constant actions $a_i \in \mathscr{A}$. To see this, suppose that player $-i$ is in state B so that she chooses $\underline{a}^{\mathscr{A}}_{-i}$ in the current review phase. By definition, playing $a_i^{B,\mathscr{A},l-1}$ against $\underline{a}^{\mathscr{A}}_{-i}$ yields more stage-game payoffs to player i than playing $a_i^{B,\mathscr{A},l}$. However, playing $a_i^{B,\mathscr{A},l-1}$ decreases the expected value of the reward function $U_i^{B,\mathscr{A}}$, as player $-i$ counts the event $\psi_{-i}(\{a_i^{B,\mathscr{A},\tilde{l}}|l \leq \tilde{l} \leq |A_i|\})$ less likely; thus player i receives the bonus $\lambda_i^{B,\mathscr{A},l}$ less likely if she chooses $a_i^{B,\mathscr{A},l-1}$. The value $\lambda_i^{B,\mathscr{A},l}$ is chosen to offset these two effects, and as a consequence, player i is indifferent between playing $a_i^{B,\mathscr{A},l-1}$ for T periods and $a_i^{B,\mathscr{A},l}$ for T periods. In this way, we can make player i indifferent over all constant actions $a_i \in \mathscr{A}_i$.

In addition, as in Matsushima (2004), the threshold value Z_T is carefully chosen so that mixing two or more actions in a T-period review phase is suboptimal. Therefore, the above automaton constitutes a belief-free review-strategy equilibrium with payoff v^* for sufficiently large T and δ, as desired. The formal proof is found in Appendix B.

3.3 Proof of Proposition 2 with Three or More Players

3.3.1 Notation and Overview

If there are more than two players, the equilibrium construction presented in Section 3.2 does not work. The reason is as follows. In the equilibrium strategies of Section 3.2, player $-i$ transits between states B and G to punish or reward player i, and provides appropriate incentives. However, if there are more than two players, players $-i$ have to coordinate their play in order to punish or reward player i. This poses a new difficulty, as players do not share any common information under private monitoring, and it is not obvious whether players $-i$ can coordinate their play to reward or punish player i.

Taking this problem into account, we will provide an alternative equilibrium construction for games with three or more players. The key is to extend the idea of "coordination through informal communication" of Hörner and Olszewski (2006) and Yamamoto (2009) to our setting.

Throughout the proof, let "player $i-1$" refer to player $i-1$ for each $i \in \{2, \cdots, N\}$, and to player N for $i=1$. Likewise, let "player $i+1$" refer to player $i+1$ for each $i \in \{1, \cdots, N-1\}$, and to player 1 for $i=N$. Let $X_i = \{G, B\}$, and $X = \times_{i \in I} X_i$. As explained later, X_i will be interpreted as player i's message space; G is called *good message*, and B is *bad message*. For each i, pick two elements of A_i arbitrarily, and call each of them a_i^G and a_i^B, respectively.

In the equilibrium construction below, the infinite repeated game is regarded as a sequence of *block games* with length T_b. In each block game, a player is either in state G or in state B. Player i with state G plays a T_b-period repeated game strategy s_i^G during a block game, while player i with state B plays a strategy s_i^B. These block-game strategies s_i^G and s_i^B are chosen in such a way that player i's block-game payoff is high if player $i-1$ is in state G, and is low if player $i-1$ is in state B. At the end of each block game, a player transits over two states G and B. Here players' transition rule is carefully chosen so that (i) in each block game, the strategies s_i^G and s_i^B are best replies for player i, regardless of players $-i$'s current state; (ii) for each $j \neq i-1$, player j's current state (either G or B) is irrelevant to player i's continuation payoff; and (iii) player i's continuation payoff is high if player $(i-1)$'s current state is G and it is low if player $(i-1)$'s current state is B. From (i), the constructed strategy profile is an equilibrium. Also, (ii) and (iii) imply that player $i-1$ can solely control player i's continuation payoff through a choice of states; that is, players $i-1$ needs not coordinate a choice of states with other players to reward or punish player i. Specifically, player $i-1$ chooses state G if she wants to reward player i, and chooses state B if she wants to punish.

So far the idea is very similar to Hörner and Olszewski (2006) and in particular Yamamoto (2009). However, the block game considered here has a more complex structure than theirs. A block game with length T_b is divided into *rounds*, and each round is further divided into review phases. Specifically, each block game consists of a *signaling round*, a *confirmation round*, K pairs of a *main round* and a *supplemental round*, and a *report round*. See Figure 1.

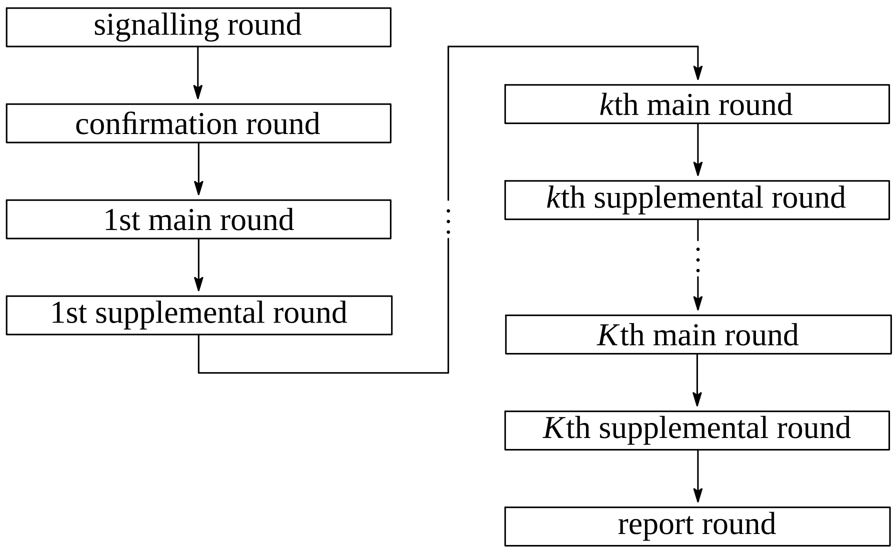

Figure 1: Block Game

Signaling, confirmation, supplemental, and report rounds are regarded as "communication stages," where players disclose their private information via a choice of actions. Unlike cheap-talk games, actions in these communication stages are payoff-relevant. However, the length of the communication stages is much shorter than that of the main rounds, so that payoffs in the communication stages are almost negligible.

The following is a brief explanation of the role of each round.

Signaling Round : This round is used for communication, and each player reveals whether she is in state G or in state B. Specifically, player i's message space is $X_i = \{G, B\}$, and we say that *player i sends message* $x_i \in X_i$ if she chooses action $a_i^{x_i}$ constantly in the signaling round. The length of the signaling round is of order T, and hence for sufficiently large T, communication is almost perfect; that is, each player can receive her opponents' messages correctly with very high probability.

Confirmation Round : This round is also used for communication, and players try to make sure what happened in the signaling round. Specifically, each player i reports (i) what she did and (ii) what her neighbors (players $i-1$ and $i+1$) did in the signaling round. The length of the confirmation round is of order T, so that for sufficiently large T, the communication here is almost perfect.

Main Rounds and Supplemental Rounds : Players' play in the main rounds is dependent on communication in the confirmation round. Roughly, if players agreed in the confirmation round that the message profile in the signaling round was $x = (x_i)_{i \in I} \in X$, then in the main rounds, they play actions such that (i) for each i with $x_{i-1} = G$, player i's payoff is high and (ii) for each i with $x_{i-1} = B$, player i's payoff is low. This ensures that player i's expected block-game payoff is high if player $(i-1)$'s current state is G, and is low if player $(i-1)$'s state is B.

In each supplemental round, every player reports whether or not her neighbors deviated in the previous main round. That is, in the kth supplemental round (here $k \in \{1, \cdots, K\}$), each player i reports whether or not player $i-1$ or player $i+1$ deviated in the kth main round. If both players $j-1$ and $j+1$ report in the kth supplemental round that player j has deviated, then players regard player j as a deviator and change their continuation play accordingly.

Report Round : This round is also used for communication, and each player reports her private history in the confirmation and supplemental rounds. The information revealed in the report round is utilized to determine the transition probability between states G and B for the next block game.

As described above, in the confirmation and supplemental rounds, players communicate to make a consensus about what happened in previous rounds. This is a new feature compared to Yamamoto (2009); in the block game of Yamamoto (2009) there is no confirmation round or supplemental round.

Communicating in the confirmation round plays an important role to deter deviations in the signaling round. In the signaling round, players are asked to send message

G or B on the equilibrium path, and if they take any other sort of play, it must be punished. However, under private monitoring, players may have different inferences about what actions were played in the signaling round, and may fail to coordinate to punish a deviator. For example, suppose that player i deviated in the signaling round by playing a_i^G almost all the time but a_i^B in a few periods. Since this play is almost the same as sending message G, it is hard to distinguish these two, and as a result, it is often the case that some of the opponents notice player i's deviation while others do not. To resolve such a conflict, we ask players to communicate again in the confirmation round to make a consensus about what happened in the signaling round. This enables players to coordinate their continuation play, and ensures that deviating in the signaling round does not deliver big gains to the deviator.

Likewise, communicating in the supplemental rounds is important for punishing a player who deviated in the main rounds. As in the signaling round, often times players have different inferences about what actions were played in the kth main round, which may cause a coordination failure in later periods. To avoid such a miscoordination, players communicate in the kth supplemental round and to make a consensus about what happened in the kth main round.

The confirmation and supplemental rounds are dispensable in Yamamoto (2009), because almost-perfect monitoring is assumed there. Under almost-perfect monitoring, players' inferences about past play within a block game are (almost) common information, so that players can almost surely coordinate their play without communication. More discussions on the confirmation and supplemental rounds are given in Section 3.3.6.

3.3.2 Actions, Regimes, and Payoffs

Let $v = (v_1, \cdots, v_N)$ be an interior point of the right-hand side of (2). We will construct a belief-free review-strategy equilibrium with payoff v for sufficiently large δ.

The following notation is used throughout the proof. Let $p \in \triangle \mathscr{J}$ be such that v is included in the interior of $V(p) \cap \times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i]$. Let $(\underline{w}_i)_{i\in I}$ and $(\overline{w}_i)_{i\in I}$ be such that $\underline{w}_i < v_i < \overline{w}_i$ for all $i \in I$, and such that the hyper-rectangle $\times_{i\in I}[\underline{w}_i,\ \overline{w}_i]$ is included in the interior of $V(p) \cap \times_{i\in I}[p\underline{v}_i,\ p\bar{v}_i]$. Then, as Yamamoto (2009) shows, there is a natural number $\tilde{K}$, a sequence $(\mathscr{A}^1, \cdots, \mathscr{A}^{\tilde{K}})$ of regimes, and 2^N sequences $(a^{x,1}, \cdots, a^{x,\tilde{K}})_{x\in X}$

of action profiles such that

$$\frac{1}{\tilde{K}}\sum_{k=1}^{\tilde{K}} \underline{v}_i(\mathscr{A}^k) < \underline{w}_i < v_i < \overline{w}_i < \frac{1}{\tilde{K}}\sum_{k=1}^{\tilde{K}} \overline{v}_i(\mathscr{A}^k), \quad \forall i \in I \tag{10}$$

$$a^{x,k} \in \mathscr{A}^k, \quad \forall x \in X \forall k \in \{1,\cdots,\tilde{K}\}, \tag{11}$$

$$\frac{1}{\tilde{K}}\sum_{k=1}^{\tilde{K}} \pi_i(a^{x,k}) = \begin{cases} < \underline{w}_i & \text{if} \quad x_{i-1} = B \\ > \overline{w}_i & \text{if} \quad x_{i-1} = G \end{cases}, \quad \forall i \in I. \tag{12}$$

To interpret (10), recall that $\underline{v}_i(\mathscr{A})$ is player i's minimax payoff when players $-i$ are restricted to play pure actions from $\mathscr{A}_{-i}$. The first inequality of (10) implies that player i's time-average minimax payoff for $\tilde{K}$ periods is less than $\underline{w}_i$ given the regime sequence $(\mathscr{A}^1,\cdots,\mathscr{A}^{\tilde{K}})$. Likewise, the last inequality of (10) says that player i's time-average reward payoff is greater than $\overline{w}_i$. (11) says that for each x, the action sequence $(a^{x,1},\cdots,a^{x,\tilde{K}})$ is consistent with the regime sequence $(\mathscr{A}^1,\cdots,\mathscr{A}^{\tilde{K}})$, in the sense that each component of the action sequence is an element of the corresponding regime. (12) implies that player i's time-average payoff of the sequence $(a^{x,1},\cdots,a^{x,\tilde{K}})$ is high if $x_{i-1} = G$ and low if $x_{i-1} = B$. Figure 2 shows how to pick $\underline{w}_i$, $\overline{w}_i$, and $(a^{x,1},\cdots,a^{x,\tilde{K}})$ for two player games.

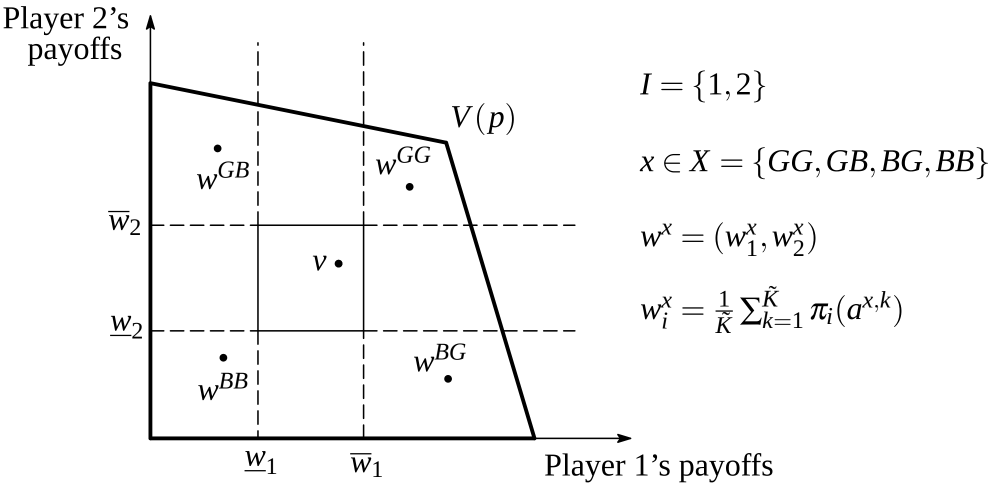

Figure 2: Actions

Given a natural number K, let $(\mathscr{A}^1,\cdots,\mathscr{A}^K)$ be a cyclic sequence of $(\mathscr{A}^1,\cdots,\mathscr{A}^{\tilde{K}})$ with length K, that is, $\mathscr{A}^{k+n\tilde{K}} = \mathscr{A}^k$ for all $k \in \{1,\cdots,\tilde{K}\}$ and $n \geq 0$. Likewise, for each $x \in X$, let $(a^{x,1},\cdots,a^{x,K})$ be a cyclic sequence of $(a^{x,1},\cdots,a^{x,\tilde{K}})$.

3.3.3 Blocks and Rounds

Given integers T and K, let $T_b = NT + 6NT + K^2T + 2KNT + 2N^2(3+K)T$. In what follows, the infinite periods are regarded as a series of *block games* with length T_b. Integers T and K are to be specified.

Each block game is further divided into several *rounds*. The collection of the first NT periods of a block game is called a *signaling round*, and the collection of the next $6NT$ periods is a *confirmation round*. Then, a pair of a *main round* (KT periods) and a *supplemental round* ($2NT$ periods) appears K times, and the collection of the remaining $2N^2(3+K)T$ periods is a *report round*. See Figure 1 in Section 3.3.1.

The confirmation, supplemental, and report rounds are regarded as a series of review phases with length T. The signaling and main rounds themselves are regarded as review phases. Let $(t_l)_{l=0}^{\infty}$ be the sequence of integers such that $t_0 = 0$ and t_l denotes the last period of the lth review phase for each $l \geq 1$. For example, $t_1 = NT$ (because the first review phase of the infinite horizon game is the signaling round of the first block game, which consists of NT periods) and $t_2 = NT + T$, (because the second review phase of the infinite horizon game is the first review phase of the confirmation round, which consists of T periods).

The following is a detailed description of each round.

Signaling Round : This round is used for communication, and each player i reveals her current state (G or B) by choosing message x_i from $X_i = \{G, B\}$. We say that *player i sends message $x_i \in X_i$* if she chooses the action $a_i^{x_i}$ constantly (i.e., playing $a_i^{x_i}$ in every period of the signaling round).

Confirmation Round : In this round, each player i reports what she did and what players $i-1$ and $i+1$ did in the signaling round. Specifically, player i chooses a message $m_i^0 = (m_{i,i-1}^0, m_{i,i}^0, m_{i,i+1}^0)$ from the message space $M_i^0 \equiv \{G, B, E\}^3$, where the component $m_{i,j}^0$ denotes player i's inference about what player j did. Roughly speaking, player i chooses $m_{i,j}^0 = G$ if she believes that player j sent message G, $m_{i,j}^0 = B$ if she believes that player j sent message B, and $m_{i,j}^0 = E$ if she is uncertain about what player j did. Let M^0 denote the set of all message profiles, that is, $M^0 \equiv \times_{i \in I} M_i^0$.

In the confirmation round, players send their messages sequentially, i.e., player 1 sends her message m_1^0 first, then does player 2, and so forth. Each player spends $2T$ periods sending each component of her message; that is, player i sends $m_{i,j}^0$ using her actions in the $(6(i-1)+2(j-i+2)-1)$st review phase and the $(6(i-1)+2(j-i+2))$nd review phase of the confirmation round. Player i sends $m_{i,j}^0 = G$ by choosing a_i^G constantly in both of these phases; she sends $m_{i,j}^0 = B$ by choosing a_i^B constantly in both of these phase; and she sends $m_{i,j}^0 = E$ by choosing a_i^G constantly in the first review phase and then a_i^B constantly in the second review phase.

Given a message profile $m^0 \in M^0$, we say that *$x_i = G$ is confirmed by players* if either (i) $(m_{i-1,i}^0, m_{i+1,i}^0) = (G, G)$, (ii) $(m_{i-1,i}^0, m_{i+1,i}^0) = (G, E)$ or $(m_{i-1,i}^0, m_{i+1,i}^0) = (E, G)$, or (iii) $(m_{i-1,i}^0, m_{i+1,i}^0, m_{i,i}^0) = (G, B, G)$ or $(m_{i-1,i}^0, m_{i+1,i}^0, m_{i,i}^0) = (B, G, G)$. That is, players confirm that player i's message in the signaling round was G if either (i) both players

$i-1$ and $i+1$ claim that player i's message was G; (ii) one of these two players claims that player i's message was G and the other player says that she is uncertain about player i's message; or (iii) one of these players claims that player i's message was G, the other player claims that player i's message was B, and player i claims that her message was G. On the other hand, given a $m^0 \in M^0$, $x_i = B$ *is confirmed by players* if $x_i = G$ is not confirmed. Note that player i's report about her own play, $m^0_{i,i}$, is relevant only when players $i-1$ and $i+1$ have different inferences about player i's play (i.e., only when $(m^0_{i-1,i}, m^0_{i+1,i}) = (G,B)$ or $(m^0_{i-1,i}, m^0_{i+1,i}) = (B,G)$). Otherwise, player i's report $m^0_{i,i}$ is ignored, and $x_i = B$ or $x_i = G$ is confirmed contingently on the reports from players $i-1$ and $i+1$ only. Given a message profile m^0, $x = (x_i)_{i \in I} \in X$ *is confirmed by players* if each component of x is confirmed. Let $M^0(x)$ denote the set of all $m^0 \in M^0$ such that x is confirmed.

Main Rounds : Players' behavior in the main rounds is contingent on what happened in the confirmation round. Roughly, if $x \in X$ is confirmed in the confirmation round, then players follow the sequence $(a^{x,1}, \cdots, a^{x,K})$ of action profiles in the main rounds, i.e., the action profile $a^{x,k}$ is played in the kth main round for each $k \in \{1, \cdots, K\}$. However, if someone unilaterally deviates from this prescribed rule and if the deviation is reported in the subsequent supplemental round, then they switch their play. Details are stated later.

Supplemental Rounds : The kth supplemental round is used for communication, and each player reports whether or not her neighbors deviated from the prescribed action profile $a^{x,k}$ in the kth main round. Specifically, in the kth supplemental round, player i chooses a message m^k_i from the message space $M^k_i = \{i-1, i+1, 0\}$. Roughly, player i chooses $m^k_i = i-1$ if she believes that player $i-1$ deviated from $a^{x,k}$ in the kth main round, $m^k_i = i+1$ if she believes that player $i+1$ deviated, and $m^k_i = 0$ otherwise. Let M^k be the set of all message profiles $m^k = (m^k_1, \cdots, m^k_N)$.

For each $k \in \{1, \cdots, K\}$ and $i \in I$, let $M^k(i)$ be the set of all message profiles $m^k \in M^k$ such that $m^k_{i-1} = m^k_{i+1} = i$ and such that for each $j \in \{1, \cdots, i-1\}$, either $m^k_{j-1} \neq j$ or $m^k_{j+1} \neq j$. In words, $M^k(i)$ is the set of message profiles m^k such that both players $i-1$ and $i+1$ report player i's deviation, and for each $j \in \{1, \cdots, i-1\}$, player j's deviation is not reported by player $j-1$ or player $j+1$. Let $M^k(0)$ be the set of all $m^k \in M^k$ such that $m^k \notin M^k(i)$ for all $i \in I$.

As in the confirmation round, players send their messages sequentially in each supplemental round. Specifically, player i sends a message $m^k_i \in M^k_i$ using her actions in the $(2i-1)$st and $2i$th review phases of the kth supplemental round. Player i sends $m^k_i = i-1$ by choosing a^G_i constantly in both of these phases; she sends $m^k_i = i+1$ by choosing a^B_i constantly in both of these phases; and she sends $m^k_i = 0$ by choosing a^G_i

constantly in the first review phase and then a_i^B constantly in the second review phase.

Report Round : This round is also regarded as a communication stage, and each player reports what message profiles were sent in the confirmation round and the supplemental rounds. Thus the message space for player i is $M^0 \times M^1 \times \cdots \times M^K$. As in the confirmation and supplemental rounds, players report their messages sequentially; player 1 sends her message in the first $2N(3+K)$ review phases of the report round, then does player 2, and so forth. How to send a message is analogous to that in the confirmation and supplemental rounds.

Observe that the ratio of the total length of the main rounds to that of the block game is $\frac{K^2T}{T_b}$, which approaches one as $K \to \infty$. Therefore, for sufficiently large K, a player's average payoff in the block game is approximated by that in the main rounds. In other words, payoffs during the communication stages are almost negligible.

3.3.4 Block Strategies under Perfect Monitoring

Let $S_i^{T_b}$ be the set of player i's strategies in the T_b-period block game. Also, let $\mathscr{S}_i^{T_b}$ be the set of all strategies $s_i^{T_b} \in S_i^{T_b}$ such that player i plays a constant action (i.e., she does not mix two or more actions) in each review phase and such that for each $k \in \{1, \cdots, K\}$, player i chooses an action from the set $\mathscr{A}_i^k$ in the kth main round. Intuitively, $\mathscr{A}_i^k$ is the set of "recommended actions" for the kth main round, and $\mathscr{S}_i^{T_b}$ is the set of strategies which follow this recommendation; so we will call it the set of recommended strategies. Given a strategy profile $s^{T_b} \in S^{T_b}$, let $w_i^P(s^{T_b})$ denote player i's average payoff in the block game with perfect monitoring where payoffs in the periods other than the main rounds are replaced with zero. Note that $w_i^P(s^{T_b})$ approximates the average payoff in the block game with perfect monitoring, as payoffs in the communication stages are almost negligible.

In our equilibrium, player i with state G plays a "good" block strategy $s_i^G \in S_i^{T_b}$, and player i with state B plays a "bad" block strategy $s_i^B \in S_i^{T_b}$. In this section, we specify these two block strategies under perfect monitoring, and then specifies the parameter K, which determines the number of main and supplemental rounds within a block game. As mentioned earlier, these strategies will be constructed in such a way that player i's payoff is high if player $(i-1)$'s current state is G (so that she plays s_i^G), and it is low if player $(i-1)$'s current state is B (so that she plays s_i^B).

To define s_i^G and s_i^B, the following notation is useful. For each $i, j \in I$, $t \geq NT$, and $h_i^t = (a^\tau)_{\tau=1}^t \in H_i^t$ (here h_i^t is represented by a sequence of action profiles, since monitoring is perfect), let

$$\hat{x}_j(h_i^t) = \begin{cases} G & \text{if} \quad (a_j^1, \cdots, a_j^{NT}) = (a_j^G, \cdots, a_j^G) \\ B & \text{if} \quad (a_j^1, \cdots, a_j^{NT}) = (a_j^B, \cdots, a_j^B) \\ E & \text{otherwise} \end{cases} .$$

Intuitively, $\hat{x}_j(h_i^t)$ denotes player i's inference on player j's message x_j in the signaling round, given her private history h_i^t: Note that $\hat{x}_j(h_i^t) = G$ if and only if player j sent message G, and $\hat{x}_j(h_i^t) = B$ if and only if player j sent message B. Otherwise $\hat{x}_j(h_i^t) = E$, which means that the history is *erroneous*. Given a history $h_i^t \in H_i^t$, let $\hat{x}(h_i^t) = (x_j(h_i^t))_{j \in I}$, that is, $\hat{x}(h_i^t)$ is player i's inference on the message profile in the signaling round.

Likewise, given a history h_i^t, let $\hat{m}_j^0(h_i^t) = (\hat{m}_{j,j-1}^0(h_i^t), \hat{m}_{j,j}^0(h_i^t), \hat{m}_{j,j+1}^0(h_i^t))$ denote player i's inference on player j's message m_j^0 in the confirmation round. Specifically, for each $l = j-1, j, j+1$, let $\hat{m}_{j,l}^0(h_i^t) = G$ if player j sent $m_{j,l}^0 = G$; $\hat{m}_{j,l}^0(h_i^t) = B$ if player j sent $m_{j,l}^0 = B$; and $\hat{m}_{j,l}^0(h_i^t) = E$ otherwise. Let $\hat{m}^0(h_i^t)$ denote player i's inference on the message profile in the confirmation round, that is, $\hat{m}^0(h_i^t) = (\hat{m}_j^0(h_i^t))_{j \in I}$.

Also, for each k and h_i^t, let $\hat{m}_j^k(h_i^t)$ denote player i's inference on player j's message m_j^k in the kth supplemental round. Specifically, let $\hat{m}_j^k(h_i^t) = j-1$ if player j sent $m_j^k = j-1$; $\hat{m}_j^k(h_i^t) = j+1$ if player j sent $m_j^k = j+1$; and $\hat{m}_j^k(h_i^t) = 0$ otherwise. Let $\hat{m}^k(h_i^t)$ denote player i's inference on the message profile in the kth supplemental round, that is, $\hat{m}^k(h_i^t) = (\hat{m}_j^k(h_i^t))_{j \in I}$.

Under perfect monitoring, the block strategies s_i^G and s_i^B are defined as follows. In the signaling round, s_i^G sends message G and s_i^B sends message B. In the confirmation round and in the report round, both s_i^G and s_i^B tell the truth; i.e., both strategies send the message $m_i^0 = \hat{x}(h_i^t)$ in the confirmation round and send the message $(\hat{m}^0(h_i^t), \cdots, \hat{m}^K(h_i^t))$ in the report round. (See the previous section for how to send these messages.) Both strategies play the action a_i^G in periods where player $j \neq i$ sends a message.

Players' play in the main rounds are contingent on the outcome in the past communication; that is, for each $k \in \{1, \cdots, K\}$, player i's action in the kth main round is dependent on $(\hat{m}^0(h_i^t), \cdots, \hat{m}^{k-1}(h_i^t))$. If $(\hat{m}^0(h_i^t), \cdots, \hat{m}^{k-1}(h_i^t))$ is an element of $M^0(x) \times M^1(0) \times \cdots \times M^{k-1}(0)$ for some $x \in X$, then both strategies say to play the action $a_i^{x,k}$ constantly in the kth main round. If $(\hat{m}^0(h_i^t), \cdots, \hat{m}^{\tilde{k}}(h_i^t))$ is an element of $M^0(x) \times M^1(0) \times \cdots \times M^{\tilde{k}-1}(0) \times M^{\tilde{k}}(j)$ for some $j \in I$, $\tilde{k} \in \{1, \cdots, k-1\}$, and $x \in X$ satisfying $x_{j-1} = B$, then both strategies say to play the action $\underline{a}_i^j(\mathscr{A}^k)$ constantly in the kth main round. Likewise, if $(\hat{m}^0(h_i^t), \cdots, \hat{m}^{\tilde{k}}(h_i^t))$ is an element of $M^0(x) \times M^1(0) \times \cdots \times M^{\tilde{k}-1}(0) \times M^{\tilde{k}}(j)$ for some $j \in I$, $\tilde{k} \in \{1, \cdots, k-1\}$, and $x \in X$ satisfying $x_{j-1} = G$, then both strategies say to play the action $\overline{a}_i^j(\mathscr{A}^k)$ constantly in the kth main round. In words, if x was confirmed by players in the confirmation round, and if no deviation was reported in the past supplemental rounds, then players play $a^{x,k}$ in the kth main round. On the other hand, if player j's deviation is reported in some supplemental round, then players change their behavior thereafter, depending on the profile x confirmed in the confirmation round; they play $\underline{a}^j(\mathscr{A}^k)$ if $x_{j-1} = B$, and play $\overline{a}^j(\mathscr{A}^k)$ if $x_{j-1} = G$.

A play in the kth supplemental round is dependent on the outcome in the past communication and on what happened in the kth main round. If $(\hat{m}^0(h_i^t), \cdots, \hat{m}^{k-1}(h_i^t))$ is an element of $M^0(x) \times M^1(0) \times \cdots \times M^{k-1}(0)$ and if all players but player $i-1$ play the action profile $a^{x,k}$ constantly in the kth main round, then both strategies say to send $m_i^k = i-1$ in the kth supplemental round. Likewise, if $(\hat{m}^0(h_i^t), \cdots, \hat{m}^{k-1}(h_i^t))$ is an element of $M^0(x) \times M^1(0) \times \cdots \times M^{k-1}(0)$ and if all players but player $i+1$ play the action profile $a^{x,k}$, then both strategies say to send $m_i^k = i+1$. Otherwise, both strategies say to send $m_i^k = 0$. In words, player i reports a deviation by player $i-1$ or $i+1$ only when it is the first deviation in the current block game. For periods during which player $j \neq i$ sends a message, both strategies say to play the action a_i^G.

Figure 3 is a flowchart of the block-game strategy s^x. Note that both s_i^G and s_i^B are in the set $\mathscr{S}_i^{T_b}$ of recommended strategies, since (11) holds.

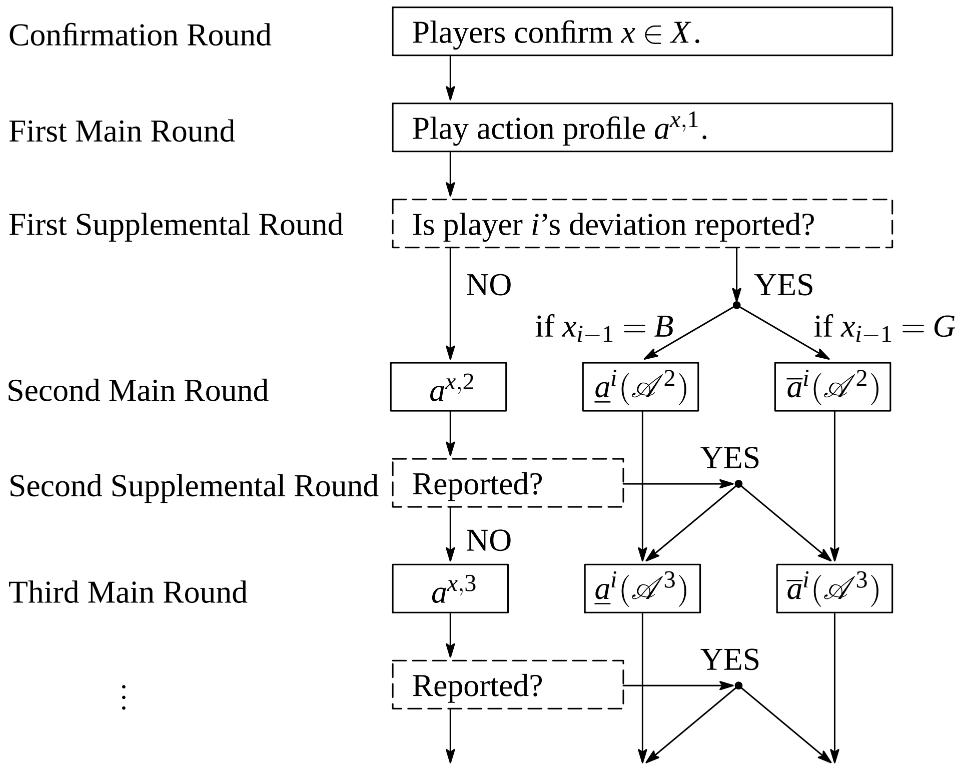

Figure 3: Block-Game Strategy Profile s^x

The following lemma gives bound on player i's block-game payoffs when players $-i$ follow $s_{-i}^{x_{-i}} = (s_j^{x_j})_{j \neq i}$. It shows that if player $i-1$ chooses the bad strategy s_{i-1}^B, then player i's block-game payoff is less than $\underline{w}_i$ no matter what player i does. On the other hand, if player $i-1$ chooses the good strategy s_{i-1}^G and if player i chooses her strategy from the recommended set $\mathscr{S}_i^{T_b}$, then player i's block-game payoff is greater than $\overline{w}_i$.

Lemma 2. *There is $\overline{K}$ such that for all $K > \overline{K}$ and for all T, there is $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$, $i \in I$, $x_{-i} \in X_{-i}$ with $x_{i-1} = B$, $\tilde{x}_{-i} \in X_{-i}$ with $\tilde{x}_{i-1} = G$, $s_i^{T_b} \in S_i^{T_b}$, and $\tilde{s}_i^{T_b} \in \mathscr{S}_i^{T_b}$,*

$$w_i^P(s_i^{T_b}, s_{-i}^{x_{-i}}) < \underline{w}_i < \overline{w}_i < w_i^P(\tilde{s}_i^{T_b}, s_{-i}^{\tilde{x}_{-i}}).$$

Proof. The statement here is very similar to (9) of Yamamoto (2009); a difference is that the block game of this paper contains the confirmation and supplemental rounds, which are not present in the block game of Yamamoto (2009). To prove the lemma, note that player i's messages in the confirmation and supplemental rounds are irrelevant under perfect monitoring, in the sense that these messages never affect the opponents' continuation play, as long as players $-i$ follow $s_{-i}^{x_{-i}}$. Thus, players' play in the block game becomes very similar to that of Yamamoto (2009), and hence the result follows. *Q.E.D.*

This lemma guarantees that there is a natural number K such that for any natural number T, there is $\overline{\delta} \in (0,1)$ such that

$$\underline{w}_i - \max_{s_i^{T_b} \in S_i^{T_b}} w_i^P(s_i^{T_b}, s_{-i}^{x_{-i}}) > \left(1 - \frac{K^2 T}{T_b}\right) 3\overline{u}_i \tag{13}$$

for all $\delta \in (\overline{\delta}, 1]$, $i \in I$, and $x_{-i} \in X_{-i}$ with $x_{i-1} = B$, and

$$\min_{s_i^{T_b} \in \mathscr{S}_i^{T_b}} w_i^P(s_i^{T_b}, s_{-i}^{x_{-i}}) - \overline{w}_i > \left(1 - \frac{K^2 T}{T_b}\right) (|A_i| + 2)\overline{u}_i \tag{14}$$

for all $\delta \in (\overline{\delta}, 1]$, $i \in I$, and $x_{-i} \in X_{-i}$ with $x_{i-1} = G$. Indeed, both (13) and (14) are satisfied for sufficiently large K, as the left-hand sides of these inequalities are positive from Lemma 2, while the term $\frac{K^2T}{T_b}$, which denotes the ratio of the length of the main rounds to that of the block game, approaches one as $K \to \infty$.

The specification of K here will be maintained in the following sections.

3.3.5 Block Strategies under Private Monitoring

Now, consider the case with private monitoring. As in the two player games, players use random events $\psi_i(\mathscr{A}_j)$ and $\psi_i(a_{i+1}, a_{i-1})$, to perform statistical tests about the opponents' actions. Specifically, we consider the random events specified in the next lemma. (See Section 3.2 for the interpretation of random events).

Lemma 3. *Suppose that (CI) holds. Then, for some q_1, q_2, and q_3 satisfying $0 < q_1 < q_2 < q_3 < 1$, there are random events $\psi_i(\mathscr{A}_j) : A_i \times \Omega_i \to [0,1]$ and $\psi_i(a_{i+1}, a_{i-1}) : A_i \times \Omega_i \to [0,1]$ for all i, j, $a \in A$, and $\mathscr{A}_j \in \mathscr{J}_j$ such that for all $\tilde{a} \in A$,*

$$P(\psi_i(\mathscr{A}_j)|\tilde{a}) = \begin{cases} q_3 & \text{if} \quad a_j \in \mathscr{A}_j \\ q_2 & \text{otherwise} \end{cases},$$

$$P(\psi_i(a_{i+1}, a_{i-1})|\tilde{a}) = \begin{cases} q_1 & \text{if} \quad a_{i-1} = \tilde{a}_{i-1} \text{ and } a_{i+1} \neq \tilde{a}_{i+1} \\ q_3 & \text{if} \quad a_{i+1} = \tilde{a}_{i+1} \text{ and } a_{i-1} \neq \tilde{a}_{i-1} \\ q_2 & \text{otherwise} \end{cases}.$$

Proof. Analogous to that of Lemma 1 of Yamamoto (2007). *Q.E.D.*

Let $F_1(\tau, T, r)$ be the probability that $\psi_i(\{a_j\})$ is counted r times out of T periods when player j chooses some $\tilde{a}_j \neq a_j$ in the first τ periods and then chooses a_j in the remaining $T - \tau$ periods. Let $F_2(\tau, T, r)$ be the probability that $\psi_i(a_{i+1}, a_{i-1})$ is counted r times out of T periods when player $i+1$ chooses $\tilde{a}_{i+1} \neq a_{i+1}$ in the first τ periods and then chooses a_{i+1} in the remaining $T - \tau$ periods, while player $i-1$ chooses a_{i-1} constantly. Let $(Z_T)_{T=1}^{\infty}$, $(Z'_T)_{T=1}^{\infty}$, and $(Z''_T)_{T=1}^{\infty}$ be sequences of integers such that

$$Z''_T < q_2 T < Z'_T, \tag{15}$$

$$Z_T \leq q_3 T, \tag{16}$$

$$\lim_{T\to\infty} \sum_{r=Z''_T+1}^{Z'_T} F_1(T,T,r) = \lim_{T\to\infty} \sum_{r=Z''_T+1}^{Z'_T} F_2(0,T,r) = \lim_{T\to\infty} \sum_{r>Z_T} F_1(0,T,r) = 1, \tag{17}$$

$$\lim_{T\to\infty} \left| \frac{Z''_T}{T} - q_2 \right| = \lim_{T\to\infty} \left| \frac{Z'_T}{T} - q_2 \right| = \lim_{T\to\infty} \left| \frac{Z_T}{T} - q_3 \right| = 0, \tag{18}$$

and

$$\lim_{T\to\infty} T F_1(0, T-1, Z_T) = \infty. \tag{19}$$

Note that the specification of Z_T here is the same as in the two player case, and that the existence of Z'_T and Z''_T is guaranteed because of the law of large numbers.

As in the perfect monitoring case, we denote by $\hat{x}(h_i^t) = (\hat{x}_j(h_i^t))_{j\in I}$ player i's inference on the message profile in the signaling round. The specification of $\hat{x}_i(h_i^t)$ here is the same as in the perfect monitoring case, as player i knows what she did in the signaling round. However, player i cannot observe the opponents' action directly, so that for each $j \neq i$, the specification of $\hat{x}_j(h_i^t)$ must be modified in the following way. Given any real number r, let $[r]$ denote the integer part of r. Let $\hat{x}_j(h_i^t) = G$ if the random event $\psi_i(\{a_j^G\})$ is counted more than $[\frac{q_2+2q_3}{3}T]$ times in the jth T-period interval of the signaling round (i.e., the T-period interval from period $(j-1)T+1$ to period jT of the block game); let $\hat{x}_j(h_i^t) = B$ if $\psi_i(\{a_j^G\})$ is counted at most $[\frac{2q_2+q_3}{3}T]$ times during this T-period interval; and let $\hat{x}_j(h_i^t) = E$ for other h_i^t.

Note that player i's inference $\hat{x}_j(h_i^t)$ is almost perfect information about player j's play in the signaling round for sufficiently large T. Indeed, if player j sends message G by choosing a_j^G, then the random event $\psi_i(\{a_j^G\})$ is counted around q_3T times during the T-period interval, which means that $\hat{x}_j(h_i^t) = G$. Likewise, if she sends message B by choosing a_j^B, then the random event $\psi_i(\{a_j^G\})$ is likely to be counted round q_2T times during the T-period interval, which means that $\hat{x}_j(h_i^t) = B$. Note that the probability of the erroneous histories ($\hat{x}_j(h_i^t) = E$) approximates zero unless player j deviates and mixes a_i^G and a_i^B in the T-period interval.

Similarly, for each $j \neq i$, the specification of player i's inference on what player j reported in the confirmation round, which is denoted by $\hat{m}_j^0(h_i^t) = (\hat{m}_{j,j-1}^0(h_i^t), \hat{m}_{j,j}^0(h_i^t), \hat{m}_{j,j+1}^0(h_i^t))$, must be modified in the following way. Recall that, for each $l = j-1, j, j+1$, player j sends $m_{j,l}^0 \in \{G,B,E\}$ using actions in the $(6(j-1)+2(l-j+2)-1)$st and $(6(j-1)+2(l-j+2))$nd review phases of the confirmation round. Let $\hat{m}_{j,l}^0(h_i^t) = G$ if the random event $\psi_i(\{a_j^G\})$ is counted at least $[\frac{q_2+q_3}{2}T]$ times in each of these review phases; let $\hat{m}_{j,l}^0(h_i^t) = B$ if $\psi_i(\{a_j^G\})$ is counted less than $[\frac{q_2+q_3}{2}T]$ times in each of these review phases; and let $\hat{m}_{j,l}^0(h_i^t) = E$ otherwise. Again, this statistical inference is almost perfect, in the sense that the probability that $\hat{m}_j^0(h_i^t)$ coincides with player j's message approximates one for large T.

For each $k \in \{1, \cdots, K\}$ and $j \neq i$, the specification of player i's inference on what player j reported in the kth supplemental round, which is denoted by $\hat{m}_j^k(h_i^t)$, is modified as follows. Recall that player j sends her message $m_j^k \in M_j^k$ using actions in the $(2(j-1)+1)$st and $2j$th T-period review phase of the kth supplemental round. Let $\hat{m}_j^k(h_i^t) = j-1$ if the random event $\psi_i(\{a_j^G\})$ is counted at least $[\frac{q_2+q_3}{2}T]$ times in each of these review phases; let $\hat{m}_j^k(h_i^t) = j+1$ if $\psi_i(\{a_j^G\})$ is counted less than $[\frac{q_2+q_3}{2}T]$ times in each of these review phases; and let $\hat{m}_j^k(h_i^t) = 0$ otherwise. Once again, these statistical inferences are almost perfect.

Now we are ready to define the block strategies s_i^G and s_i^B under private monitoring. A play in the signaling round, the confirmation round, the main rounds, and the report round is almost the same as in the perfect monitoring case; the difference is only that the specification of $(\hat{x}, \hat{m}^0, \cdots, \hat{m}^K)$ is modified as stated above.

So what remains is to specify a play in the supplemental rounds. The idea is very similar to the perfect monitoring case; in the kth supplemental round, each player i reports whether or not her neighbors deviated in the kth main round. To test what the neighbors did in the kth main round, player i uses the random event $\psi_i(a_{i+1}^{x,k}, a_{i-1}^{x,k})$. (Recall that the random event $\psi_i(a_{i+1}^{x,k}, a_{i-1}^{x,k})$ is counted with probability q_1 if player $i+1$ deviated, with probability q_3 if player $i-1$ deviated, and with probability q_2 if nobody deviated from $a^{x,k}$. Therefore, player i can statistically distinguish whether or not her neighbors deviated from this random event.) Specifically, if $(\hat{m}^0(h_i^t), \cdots, \hat{m}^{k-1}(h_i^t))$ is an element of $M^0(x) \times M^1(0) \times \cdots \times M^{k-1}(0)$ for some $x \in X$ and if the random event $\psi_i(a_{i+1}^{x,k}, a_{i-1}^{x,k})$ is counted more than Z'_{KT} times in the kth main round, then both strategies send $m_i^k = i-1$ in the kth supplemental round. If $(\hat{m}^0(h_i^t), \cdots, \hat{m}^{k-1}(h_i^t))$ is an element of $M^0(x) \times M^1(0) \times \cdots \times M^{k-1}(0)$ for some $x \in X$ and if $\psi_i(a_{i+1}^{x,k}, a_{i-1}^{x,k})$ is counted at most Z''_{KT} times in the kth main round, then both strategies send $m_i^k = i+1$. Otherwise, both strategies send $m_i^k = 0$. For periods where player $j \neq i$ sends a message, both strategies play the action a_i^G.

3.3.6 Comments on the Role of Additional Communication Stages

As mentioned, the block game considered here is different from that of Yamamoto (2009), since there are additional communication stages, the confirmation and supplemental rounds. Hence it will be useful to see how these additional communication stages work in our setting.

The purpose of communication in the confirmation round is to let players make a consensus about what happened in the signaling round, which allows players to coordinate their continuation play. That is, players can make an agreement about what the state profile x is, so that they can choose the appropriate action profile $a^{x,1}$ in the first main round. In particular, the majority rule in the confirmation round is carefully constructed so that players can make such a consensus with high probability even if player i unilaterally deviates in the signaling round or in the confirmation round.

To see this, suppose first that player $i-1$ sent message x_{i-1} in the signaling round. Then in the confirmation round, both players $i-2$ and $i-1$ report that player $i-1$ sent message x_{i-1}; thus other players confirm that player $(i-1)$'s message was x_{i-1} without referring to what player i says in the confirmation round. A similar argument shows that no matter what player i says in the confirmation round, players can make a consensus about what player j reported in the signaling round for each $j \neq i$.

Also, the same is true for the consensus about what player i reported in the signaling round, no matter what player i does in the signaling and confirmation rounds. To check this, recall that player i can become a pivotal voter in the confirmation round only when players $i-1$ and $i+1$ have opposite opinions about player i's play in the signaling round and send messages such that $(m_{i-1,i}^0, m_{i+1,i}^0) = (G,B)$ or (B,G); in other cases, players make a consensus without referring to what player i says in the confirmation round. But, from the law of large numbers, the event that $(m_{i-1,i}^0, m_{i+1,i}^0) = (G,B)$ or (B,G) is less likely, no matter what player i does in the signaling round (in particular, even if player i mixes a_i^G and a_i^B in the signaling round). Therefore we can conclude that player can make a consensus about player i's play in the signaling round no matter what player i does in the signaling and confirmation rounds.

In sum, players can make a consensus in the confirmation round with high probability and can coordinate a continuation play, no matter what player i does in the signaling and confirmation rounds. This implies that player i has less reason to deviate in the signaling and confirmation rounds, since stage-game payoffs of these rounds are almost negligible. This property plays a key role in the proofs of Lemmas 4 and 5.

Likewise, communication in the kth supplemental round enables players to make a consensus about whether someone unilaterally deviated in the kth main round with high probability, no matter what player i says. This allows players to coordinate to switch their behavior in the $(k+1)$st main rounds. (Recall that players stop playing $a^{x,k}$ and switch to choosing $\underline{a}^i(\mathscr{A})$ or $\overline{a}^i(\mathscr{A})$ once someone's deviation is reported in the supplemental round.) Again, this property implies that player i has less reason to deviate in the kth supplemental round, which is a key in the proof of Lemmas 4 and 5.

3.3.7 Block Game with Transfers

Before going to the analysis of infinitely repeated games, it is convenient to consider the following T_b-period repeated game with transfers, as in Fudenberg and Levine (1994) and Hörner and Olszewski (2006). Let $U_i : H_{i-1}^{T_b} \to \boldsymbol{R}$, and suppose that player i receives a transfer $U_i(h_{i-1}^{T_b})$ after the T_b-period block game. Note that U_i is a function of $h_{i-1}^{T_b}$, that is, the value of the transfer depends only on player $(i-1)$'s block history. Let $w_i^A(s^{T_b}, U_i)$ denote player i's average payoff in this *auxiliary scenario* given a block strategy profile $s^{T_b} \in S^{T_b}$, that is,

$$w_i^A(s^{T_b}, U_i) \equiv \frac{1-\delta}{1-\delta^{T_b}} \left[\sum_{t=1}^{T_b} \delta^{t-1} E\left[\pi_i(a^t)|s^{T_b}\right] + \delta^{T_b} E\left[U_i(h_{i-1}^{T_b})|s^{T_b}\right] \right].$$

Let $s_i^{T_b}|_{h_i^t}$ denote player i's continuation strategy after history $h_i^t \in H_i^t$ induced by $s_i^{T_b} \in S_i^{T_b}$. Also, let $BR^A(s_{-i}^{T_b}|_{h_{-i}^t}, U_i)$ be the set of player i's best replies in the auxiliary-scenario continuation game after history, given that the opponents play $s_{-i}^{T_b} \in S_i^{T_b}$ in the block game and their past history was h_{-i}^t.

The following lemma shows that there is a transfer U_i^B which can be regarded as a subsidy to offset the difference between player i's actual payoff of the block game and the target payoff $\underline{w}_i$ and to give right incentives to player i. This is an extension of Lemma 4(a) of Hörner and Olszewski (2006) and Lemma 1 of Yamamoto (2009).

Lemma 4. *Suppose that (CI) and (FS) hold. Then, there is $\overline{T}$ such that for all $T > \overline{T}$, there is $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$ and for all $i \in I$, there is $U_i^B : H_{i-1}^{T_b} \to \boldsymbol{R}$*

such that for all $l \geq 0$, $h^{t_l} \in H^{t_l}$, $h_{i-1}^{T_b} \in H_{i-1}^{T_b}$, *and* $x \in X$ *with* $x_{i-1} = B$,

$$s_i^{x_i}|_{h_i^{t_l}} \in BR^A(s_{-i}^{x_{-i}}|_{h_{-i}^{t_l}}, U_i^B), \tag{20}$$

$$w_i^A(s^x, U_i^B) = \underline{w}_i, \tag{21}$$

and

$$0 < U_i^B(h_{i-1}^{T_b}) < \frac{\overline{w}_i - \underline{w}_i}{1-\delta}. \tag{22}$$

Proof. The outline of the proof is similar to that of Lemma 1 of Yamamoto (2009), and what is new here is how to provide the truth-telling incentives for the confirmation and supplemental rounds, which are not present in the block game of Yamamoto (2009). As explained in 3.3.6, the event that player *i* becomes a pivotal voter in the confirmation and supplemental rounds is less likely. This, together with the fact that the stage-game payoffs for these rounds are almost negligible, implies that player *i* is almost indifferent over all messages in these rounds. Therefore, by giving player *i* a small transfer depending on her message, one can make player *i* exactly indifferent over all messages. The formal proof is found in Appendix C. *Q.E.D.*

Likewise, the next lemma shows that there is a transfer U_i^G which can be regarded as a fine to offset the difference between player *i*'s actual payoff of the block game and the target payoff $\overline{w}_i$ and to give right incentives to player *i*.

Lemma 5. *Suppose that (CI) and (FS) hold. Then, there is* $\overline{T}$ *such that for all* $T > \overline{T}$, *there is* $\overline{\delta} \in (0,1)$ *such that for all* $\delta \in (\overline{\delta}, 1)$ *and for all* $i \in I$, *there is* $U_i^G : H_{i-1}^{T_b} \to \mathbf{R}$ *such that for all* $l \geq 0$, $h^{t_l} \in H^{t_l}$, $h_{i-1}^{T_b} \in H_{i-1}^{T_b}$, *and* $x \in X$ *with* $x_{i-1} = G$,

$$s_i^{x_i}|_{h_i^{t_l}} \in BR^A(s_{-i}^{x_{-i}}|_{h_{-i}^{t_l}}, U_i^G), \tag{23}$$

$$w_i^A(s^x, U_i^G) = \overline{w}_i, \tag{24}$$

and

$$-\frac{\overline{w}_i - \underline{w}_i}{1-\delta} < U_i^G(h_{i-1}^{T_b}) < 0. \tag{25}$$

Proof. See Appendix D. The basic idea is similar to Lemma 4. *Q.E.D.*

Note that the information transmitted in the report round plays a crucial role in the construction of U_i^B and U_i^G. To see this, note that (20) and (23) require that player *i*'s continuation play be optimal independently of the opponents' past history $h_{-i}^{t_l}$. For this to be the case, the amount of the transfers U_i^B and U_i^G should be adjusted contingent on the realization of $h_{-i}^{t_l}$. However the transfers U_i^B and U_i^G cannot directly depend on $h_{-i}^{t_l}$, as they are functions of player $(i-1)$'s private history only. To overcome this problem, player $i-1$ adjust the amount of the transfers U_i^B and U_i^G contingent on information obtained in the report round; in the report round, each player reports what happened in the past communication stages, so that player $i-1$ can get precise information about $h_{-i}^{t_l}$. This idea is very similar to Yamamoto (2009).

3.3.8 Equilibrium Construction

Now we consider the infinitely repeated game, and show that for any payoff vector $v^* \in \times_{i\in I}[\underline{w}_i, \overline{w}_i]$, there is a belief-free review-strategy equilibrium with payoff v^*. This completes the proof of Proposition 2, as v is included in $\times_{i\in I}[\underline{w}_i, \overline{w}_i]$.

Fix a target payoff vector $v^* = (v_i^*)_{i\in I}$ from the set $\times_{i\in I}[\underline{w}_i, \overline{w}_i]$ arbitrarily. Let U_i^B and U_i^G be as in Lemmas 4 and 5. For each i, player $(i-1)$'s strategy in the infinitely repeated game is specified by the following automaton with initial state $v_i^* \in [\underline{w}_i, \overline{w}_i]$.

State w_i (for $w_i \in [\underline{w}_i, \overline{w}_i]$) : Go to phase B with probability α_{i-1}, and go to phase G with probability $1-\alpha_{i-1}$ where α_{i-1} satisfies $w_i = \alpha_{i-1}\underline{w}_i + (1-\alpha_{i-1})\overline{w}_i$.

Phase B : Play the block strategy s_{i-1}^B for T_b periods. After that, go to state w_i given by $w_i = \underline{w}_i + (1-\delta)U_i^B(h_{i-1}^{T_b})$ where $h_{i-1}^{T_b}$ is her recent T_b-period history.

Phase G : Play the block strategy s_{i-1}^G for T_b periods. After that, go to state w_i given by $w_i = \overline{w}_i + (1-\delta)U_i^G(h_{i-1}^{T_b})$.

It follows from (22) and (25) that for any history $h_{i-1}^{T_b} \in H_{i-1}^{T_b}$, both $\underline{w}_i + (1-\delta)U_i^B(h_{i-1}^{T_b})$ and $\overline{w}_i + (1-\delta)U_i^G(h_{i-1}^{T_b})$ lie in the interval $[\underline{w}_i, \overline{w}_i]$, and hence the above automaton is well-defined. Also, from (20), (21), (23), (24), and the one-shot deviation principle, the constructed strategy profile is a Nash equilibrium with payoff v^*. Moreover, this strategy profile is a belief-free review-strategy equilibrium since (20) and (23) hold and the block game strategy $s_i^{x_i}$ never mixes actions after every history.

Remark 2. In the above equilibrium construction, each review phase has different length; the signaling and main rounds have longer review phases than those in the other rounds. However, considering review phases with different length is not essential, and one can construct an equilibrium with the same payoff such that each review phase has length T. (For this, it suffices to show that there are U_i^B and U_i^G that satisfies incentive compatibility condition (20) and (23) for every T-period interval of the block game. The proof is omitted, as it requires a longer and more complex argument.) Therefore, Theorem 1 remains true even if we restrict attention to review strategies where each review phase has equal length.

4 Sufficient Conditions for Efficient Equilibria

Theorem 1 in the previous section characterizes the limit set of belief-free review-strategy equilibrium payoffs for general games. In this section, we apply this result and show that efficiency is often approximated by belief-free review-strategy equilibria. Specifically, we obtain the following proposition:

Proposition 3. *Suppose that the feasible payoff set is full dimensional, and that there are action profiles a^* and a^{**} such that $\max_{a_i \in A_i} \pi_i(a_i, a^{**}_{-i}) < \pi_i(a^*) \leq \pi_i(a^{**}_i, a^*_{-i})$ for all $i \in I$. Then the stage game is classified to the positive case and the payoff vector $\pi(a^*)$ is an element of the right-hand side of (2). Therefore, if (CI) and (FS) hold, then $\pi(a^*) \in \lim_{\delta \to 1} E(\delta)$.*

The proof of the proposition is provided in Appendix E. Letting a^* be an efficient action profile, this proposition gives a sufficient condition for the existence of asymptotically efficient equilibria; that is, the efficient payoff vector $\pi(a^*)$ can be achieved in the limit if there is $a^{**} \in A$ such that $\max_{a_i \in A_i} \pi_i(a_i, a^{**}_{-i}) < \pi_i(a^*) \leq \pi_i(a^{**}_i, a^*_{-i})$ for all $i \in I$. An example that satisfies this sufficient condition is prisoner's dilemma. Also, this sufficient condition is often satisfied in price-setting oligopoly markets. To see this, let a^*_i be cartel price and a^{**}_i be "cheating" price. The above condition is satisfied if (i) a firm's profit from cartel is higher than its profit when all the opponents cheat, and (ii) a firm can earn more profits by cheating than by choosing cartel price when the opponents choose cartel price. Proposition 3 asserts that under this condition, cartel is self-enforced even if firms cannot communicate each other.

It may be noteworthy that the sufficient condition here is much weaker than the one provided by Yamamoto (2007). Theorem 1 of Yamamoto (2007) assumes the payoff function to be almost symmetric, that is, players who choose the same action obtain similar stage-game payoffs. Proposition 3 does not impose such a symmetry assumption, so that it can apply to oligopoly markets where firms have different market shares and/or different production functions. Also, Yamamoto (2007) imposes several assumptions on a player's stage-game payoffs when some of the opponents choose a^* and others choose a^{**}; Proposition 3 show that these assumptions are not necessary to approximate efficiency.[6]

The next proposition shows that even the folk theorem is established if more assumptions are imposed on the payoff function. This is a generalization of Theorem 2 of Matsushima (2004) to N-player games. The stage game is an *N-player prisoner's dilemma* if $|I| = N$; $A_i = \{C_i, D_i\}$ for all $i \in I$; $\pi_i(D_i, a_{-i}) \geq \pi_i(C_i, a_{-i})$ for all $i \in I$ and $a_{-i} \in A_{-i}$; $\pi_i(C_j, a_{-j}) \geq \pi_i(D_j, a_{-j})$ for all $i \in I$, $j \neq i$ and $a_{-j} \in A_{-j}$; and

[6] Note also that the sufficient condition provided by Proposition 3 is weaker than that of Theorem 1 of Matsushima (2004).

$\pi_i(C_1,\cdots,C_N) > \pi_i(D_1,\cdots,D_N)$ for all $i \in I$. In words, defection weakly dominates cooperation, cooperation weakly increases opponents' profits, and mutual cooperation yields higher payoffs than mutual defection.

Proposition 4. *Suppose that the stage game is an N-player prisoner's dilemma, and the feasible payoff set is full dimensional. Suppose also that (CI) and (FS) hold. Then,* $\lim_{\delta\to 1} E(\delta)$ *exactly equals the feasible and individually rational payoff set.*

The proof is similar to Proposition 3 of Yamamoto (2009), and hence omitted.

5 Almost-Independent Monitoring

This section demonstrates that the limit characterization result is robust to a perturbation of the monitoring structure, i.e., Theorem 1 remains valid even under almost-independent monitoring.

To formalize the concept of "almost-independent monitoring," we introduce a measure of closeness between distinct signal distributions. The following notion is attributed to Mailath and Morris (2006): For a fixed $(I,(A_i,\pi_i,\Omega_i)_{i\in I})$, the signal distribution $q: A \to \triangle\Omega$ is *ε-close* to a conditionally-independent signal distribution $(q_i)_{i\in I}$ if

$$\left| q(\omega|a) - \prod_{i\in I} q_i(\omega_i|a,\omega_0) \right| < \varepsilon$$

for all a and ω. The following proposition shows that if there is a belief-free review strategy equilibrium with payoff v under (CI), then v can be achieved even if the monitoring structure is slightly perturbed so that the monitoring is almost independent.

Proposition 5. *Suppose that a stage game* $(I,(A_i,\Omega_i,\pi_i,q_i)_{i\in I})$ *satisfies (CI) and (FS). Suppose also that this game is classified to the positive case. Then, for any payoff vector* v *in the interior of the right-hand side of (2), there are* $\overline{\delta} \in (0,1)$ *and* $\varepsilon > 0$ *such that for any* $\delta \in (\overline{\delta},1)$ *and for any signal distribution* ε*-close to* $(q_i)_{i\in I}$*, there is a belief-free review-strategy equilibrium with payoff* v*.*

The intuition behind this result is as follows. As shown in Section 3.2, under (CI), a player's private signal has no information about the opponents' signals. and thus no feedback on what the opponents will do in a continuation play. Therefore, players have no incentive to deviate to a history-dependent strategy within a review phase, which is a key element in the proof of Theorem 1. When (CI) is violated, a player's private signal contains some information about the opponents' signals so that players may want to play history-dependent strategies; however, if the signal distribution is almost independent (i.e., taking ε close to zero), then a player's private signal has almost no information about the opponents' signals. Therefore, given a T, when we take ε sufficiently close to zero, playing a history-dependent strategy becomes suboptimal and we can construct an equilibrium as in the case of conditionally-independent monitoring. The formal proof is omitted, as it is straightforward.

Appendix A: Proof of Lemma 4

Consider the positive case, and let $s \in S$ be a belief-free review-strategy equilibrium with sequence $(t_l)_{l=0}^{\infty}$. Let $\mathscr{A}_i(l)$ denote the set of player i's actions taken with positive probability in the initial period of the lth review phase for some history. That is, $\mathscr{A}_i(l)$ is the union of the support of $s_i(h_i^{t_{l-1}})$ over all $h_i^{t_{l-1}} \in H_i^{t_{l-1}}$.

Since s is strongly belief-free in the lth review phase, player i's continuation payoff after history $h^{t_{l-1}} \in H^{t_{l-1}}$ is independent of her own private history $h_i^{t_{l-1}}$. So let us denote this continuation payoff by $w_i(h_{-i}^{t_{l-1}})$. Likewise, for each $h_{-i}^{t_{l-1}} \in H_{-i}^{t_{l-1}}$ and $a_{-i} \in \text{supp}\{s_{-i}(h_{-i}^{t_{l-1}})\}$, let $w_i(h_{-i}^{t_{l-1}}, a_{-i})$ denote player i's continuation payoff from the lth review phase when the opponents' history in the past review phases is $h_{-i}^{t_{l-1}}$ and they play the constant action a_{-i} in the lth review phase. Then

$$w_i(h_{-i}^{t_{l-1}}) = \sum_{a_{-i} \in A_{-i}} s_{-i}(h_{-i}^{t_{l-1}})[a_{-i}] w_i(h_{-i}^{t_{l-1}}, a_{-i}) \tag{26}$$

for all $h_{-i}^{t_{l-1}} \in H_{-i}^{t_{l-1}}$. Also, since s is strongly belief-free in the lth review phase,

$$w_i(h_{-i}^{t_{l-1}}, a_{-i}) \geq (1-\delta^{t_l - t_{l-1}})\pi_i(a) + \delta^{t_l - t_{l-1}} \sum_{h_{-i}^{t_l} \in H_{-i}^{t_l}} \Pr(h_{-i}^{t_l} | h_{-i}^{t_{l-1}}, a) w_i(h_{-i}^{t_l}) \tag{27}$$

for all $h_{-i}^{t_{l-1}} \in H_{-i}^{t_{l-1}}$, $a_{-i} \in \text{supp}\{s_{-i}(h_{-i}^{t_{l-1}})\}$, and $a_i \in A_i$ with equality if $a_i \in \mathscr{A}_i(l)$. Here, the term $\Pr(h_{-i}^{t_l} | h_{-i}^{t_{l-1}}, a)$ denotes the probability of the realization of $h_{-i}^{t_l}$ given that the history up to the end of the $(l-1)$th review phase is $h_{-i}^{t_{l-1}}$ and players choose the action profile $a \in A$ constantly in the lth review phase.

For each $l \geq 1$, let $\overline{w}_i^l$ be player i's best continuation payoff from the lth review phase, i.e, $\overline{w}_i^l$ is the maximum of $w_i(h_{-i}^{t_{l-1}})$ over all $h_{-i}^{t_{l-1}} \in H_{-i}^{t_{l-1}}$. Using (27) and $w_i(h_{-i}^{t_l}) \leq \overline{w}_i^{l+1}$,

$$w_i(h_{-i}^{t_{l-1}}, a_{-i}) \leq (1-\delta^{t_l - t_{l-1}}) \min_{a_i \in \mathscr{A}_i(l)} \pi_i(a) + \delta^{t_l - t_{l-1}} \overline{w}_i^{l+1}$$

for all $h_{-i}^{t_{l-1}} \in H_{-i}^{t_{l-1}}$ and $a_{-i} \in \text{supp}\{s_{-i}(h_{-i}^{t_{l-1}})\}$. Then, from (26) and the definition of $\overline{v}_i(\mathscr{A})$,

$$w_i(h_{-i}^{t_{l-1}}) \leq (1-\delta^{t_l - t_{l-1}})\overline{v}_i(\mathscr{A}(l)) + \delta^{t_l - t_{l-1}} \overline{w}_i^{l+1}$$

for all $h_{-i}^{t_{l-1}} \in H_{-i}^{t_{l-1}}$. This implies that

$$\overline{w}_i^l \leq (1-\delta^{t_l - t_{l-1}})\overline{v}_i(\mathscr{A}(l)) + \delta^{t_l - t_{l-1}} \overline{w}_i^{l+1}. \tag{28}$$

Likewise, let $\underline{w}_i^l$ be the minimum of $w_i(h_{-i}^{t_{l-1}})$ over all $h_{-i}^{t_{l-1}} \in H_{-i}^{t_{l-1}}$. It follows from (27) and $w_i(h_{-i}^{t_l}) \geq \underline{w}_i^{l+1}$ that

$$w_i(h_{-i}^{t_{l-1}}, a_{-i}) \geq (1-\delta^{t_l - t_{l-1}}) \max_{a_i \in A_i} \pi_i(a) + \delta^{t_l - t_{l-1}} \underline{w}_i^{l+1}$$

for all $h_{-i}^{t_{l-1}} \in H_{-i}^{t_{l-1}}$ and $a_{-i} \in \text{supp}\{s_{-i}(h_{-i}^{t_{l-1}})\}$. Then, as in the above argument,

$$\underline{w}_i^l \geq (1-\delta^{t_l - t_{l-1}})\underline{v}_i(\mathscr{A}(l)) + \delta^{t_l - t_{l-1}}\underline{w}_i^{l+1}. \tag{29}$$

Iterating (28) and (29) and using (26), it turns out that $w_i(h_{-i}^{t_{l-1}})$ is in the interval $[p^l\underline{v}_i, p^l\overline{v}_i]$ for all $l \geq 1$ and $h_{-i}^{t_{l-1}} \in H_{-i}^{t_{l-1}}$, where $p^l \in \triangle\mathscr{J}$ is defined to be

$$p^l(\mathscr{A}) = \sum_{\{k \geq l \mid \mathscr{A}(k) = \mathscr{A}\}} \delta^{t_{k-1} - t_{l-1}}(1-\delta^{t_k - t_{k-1}}) \tag{30}$$

for all $\mathscr{A} \in \mathscr{J}$. Therefore the equilibrium payoff vector $(w_i(s))_{i\in I}$ is in the product set $\times_{i\in I}[p^1\underline{v}_i, p^1\overline{v}_i]$. On the other hand, from the feasibility constraint, $(w_i(s))_{i\in I} \in V(p^1)$. Taken together, $(w_i(s))_{i\in I}$ is in the intersection of $V(p^1)$ and $\times_{i\in I}[p^1\underline{v}_i, p^1\overline{v}_i]$. This proves that the right-hand side of (2) includes $E(\delta)$ in the positive case.

Next, consider the empty case. Suppose that there is a belief-free review-strategy equilibrium $s \in S$. Then as in the positive case, the equilibrium payoff $w_i(s)$ must be in the interval $[p^1\underline{v}_i, p^1\overline{v}_i]$ for all $i \in I$. However, since this is the empty case, there is $i \in I$ such that $p^1\underline{v}_i > p^1\overline{v}_i$, that is, the interval $[p^1\underline{v}_i, p^1\overline{v}_i]$ is empty. This is a contradiction, and hence there is no belief-free review-strategy equilibrium.

Finally, consider the negative case. Since playing pure-strategy Nash equilibria in every period is a belief-free review-strategy equilibrium, $\lim_{\delta\to 1} E(\delta)$ includes the convex hull of the set of pure-strategy Nash equilibrium payoffs of the stage game. Hence, it suffices to show that $E(\delta)$ is included in the convex hull of the set of pure-strategy Nash equilibrium payoffs for every $\delta \in (0,1)$.

Let $s \in S$ be a belief-free review-strategy equilibrium. As in the positive case, for each $i \in I$ and $l \geq 1$, $w_i(h_{-i}^{t_{l-1}})$ is included in the interval $[p^l\underline{v}_i, p^l\overline{v}_i]$, which must be a singleton in the negative case. This implies that no dynamic incentive is provided in this equilibrium, and hence in every review phase, player i's action must be a static best reply to any outcome of the opponents' mixture (here, the optimality *after* the mixture is required, since s is strongly belief-free in every review phase). Thus a pure-strategy Nash equilibrium is played in every period, which completes the proof.

Appendix B: Incentive Compatibility for Two-Player Games

In this appendix, we show that the strategy profile presented in Section 3.2.3 constitutes a belief-free review-strategy equilibrium with payoff v^*.

The following lemma asserts that the automaton is well-defined if T is large enough and δ is close to one.

Lemma 6. *There is $\overline{T}$ such that for all $T > \overline{T}$, there is $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$, (i) v_i^* is in the state space $[\underline{w}_i, \overline{w}_i]$ for all $i \in I$, and (ii) both $\underline{w}_i + (1-\delta)U_i^{B,\mathscr{A}}(h_{-i}^T)$*

and $\overline{w}_i + (1-\delta)U_i^{G,\mathscr{A}}(h_{-i}^T)$ *are in the state space* $[\underline{w}_i, \overline{w}_i]$ *for all* $i \in I$, $\mathscr{A} \in \mathscr{J}$, *and* $h_{-i}^T \in H_{-i}^T$.

Proof. Part (i). Since v^* is an interior point of $\times_{i\in I}[p\underline{v}_i, p\overline{v}_i - \eta]$ and (9) holds, v_i^* is in the state space $[\underline{w}_i, \overline{w}_i]$ if T is large enough. This proves (i).

Part (ii). Using (4) and (5), it follows that $\lim_{T\to\infty}\lambda_i^{B,\mathscr{A},l} \geq 0$ for all $\mathscr{A}$ and l. Likewise, from (4), (5), and $\pi_i(a_i^{G,\mathscr{A},|\mathscr{A}_i|}, \overline{a}_{-i}^{\mathscr{A}}) = \overline{v}_i(\mathscr{A})$, one can check that $\lim_{T\to\infty}\lambda_i^{G,\mathscr{A},l} \geq 0$ and $\lim_{T\to\infty}\sum_{l=1}^{|\mathscr{A}_i|}\lambda_i^{G,\mathscr{A},l} = C$. These observations, together with (9), completes the proof. *Q.E.D.*

The next step is to show that the specified strategy profile constitutes an equilibrium and achieves v^*, assuming that players are constrained to a constant action in every T-period review phase. Suppose that player $-i$ chooses $\underline{a}_{-i}^{\mathscr{A}}$ in the current review phase. Suppose also that player i's continuation payoff from the next review phase is $\underline{w}_i + (1-\delta)U_i^{B,\mathscr{A}}(h_{-i}^T)$ where h_{-i}^T is the history in the current review phase. (Note that this is the same as assuming that player i's continuation payoff is $\underline{w}_i$ when the opponent is in phase B, and her continuation payoff is $\overline{w}_i$ when the opponent is in phase G.) Then, for each $l \geq 1$, player i's payoff to playing $a_i^{B,\mathscr{A},l}$ is

$$(1-\delta^T)\pi_i(a_i^{B,\mathscr{A},l}, \underline{a}_{-i}^{\mathscr{A}}) + \delta^T\left[\underline{w}_i + \frac{1-\delta^T}{\delta^T}\left(\sum_{\tilde{l}\leq l}\sum_{r>Z_T}F(0,T,r)\lambda_i^{B,\mathscr{A},\tilde{l}} + \sum_{\tilde{l}>l}\sum_{r>Z_T}F(T,T,r)\lambda_i^{B,\mathscr{A},\tilde{l}}\right)\right].$$

Using (7) and $\pi_i(a_i^{B,\mathscr{A},1}, \underline{a}_{-i}^{\mathscr{A}}) = \underline{v}_i(\mathscr{A})$, this payoff is rewritten as

$$(1-\delta^T)\left[\underline{v}_i(\mathscr{A}) + \sum_{\tilde{l}\geq 1}\sum_{r>Z_T}F(T,T,r)\lambda_i^{B,\mathscr{A},\tilde{l}}\right] + \delta^T\underline{w}_i,$$

which does not depend on l. Therefore we can conclude that player i is indifferent over all actions against $\underline{a}_{-i}^{\mathscr{A}}$. Also, multiplying by $p(\mathscr{A})$ and summing over all $\mathscr{A} \in \mathscr{J}$ show that when player $-i$ is in phase B, player i's expected payoff is indeed $\underline{w}_i$.

Suppose next that player $-i$ chooses $\overline{a}_{-i}^{\mathscr{A}}$ in the current review phase, and that player i's continuation payoff from the next review phase is given by $\overline{w}_i + (1-\delta)U_i^{G,\mathscr{A}}(h_{-i}^T)$. Using (8), player i's payoff to playing $a_i^{G,\mathscr{A},l}$ is

$$(1-\delta^T)\left[\overline{v}_i(\mathscr{A}) - \eta + \sum_{\tilde{l}\geq 1}\sum_{r>Z_T}F(T,T,r)\lambda_i^{G,\mathscr{A},\tilde{l}}\right] + \delta^T\overline{w}_i,$$

while the payoff to playing $a_i \notin \mathscr{A}_i$ is

$$(1-\delta^T)\left[\pi_i(a_i, \overline{a}_{-i}^{\mathscr{A}}) - C - \eta + \sum_{\tilde{l}\geq 1}\sum_{r>Z_T}F(T,T,r)\lambda_i^{G,\mathscr{A},\tilde{l}}\right] + \delta^T\overline{w}_i.$$

Note that the payoff from $a_i^{G,\mathscr{A},l}$ does not depend on l, and hence player i is indifferent over all actions $a_i \in \mathscr{A}_i$ Also, since $C > \max_{a_i \in A_i} \pi_i(a_i, \overline{a}_{-i}^{\mathscr{A}}) - \overline{v}_i(\mathscr{A})$, the payoff from $a_i \notin \mathscr{A}_i$ is less than that from $a_i^{G,\mathscr{A},l}$. Hence, playing $a_i \in \mathscr{A}_i$ is a best reply against $\overline{a}_{-i}^{\mathscr{A}}$. Also, multiplying the payoff by $p(\mathscr{A})$ and summing over all $\mathscr{A} \in \mathscr{J}$ show that when player $-i$ is in phase G, player i's expected payoff is $\overline{w}_i$. Therefore, the strategy profile specified by the above automaton with the initial state (v_2^*, v_1^*) is an equilibrium in the constrained game, and yields the target payoff v^*.

What remains is to establish that this strategy profile is an equilibrium even if players are not restricted to a constant action. Recall that under (CI), player i's signal ω_i has no information about the opponent' signal, so that player i cannot be better off by conditioning her play on observed signals. Hence, it suffices to show that player i cannot profit by deviating to any sequence of actions with length T.

First, consider the case where player $-i$ chooses $\underline{a}_{-i}^{\mathscr{A}}$ in the current review phase. As mentioned, for any $a_i^* \in A_i$ and $a_i^{**} \neq a_i^*$, player i is indifferent between playing a_i^* for T periods and playing a_i^{**} for T periods. In what follows, we show that player i prefers playing a_i^* for T periods to mixing two actions a_i^* and a_i^{**} (i.e., playing a_i^* for τ periods and playing a_i^{**} for $T-\tau$ periods). For notational convenience, let $\Delta\pi_i = \pi_i(a_i^{**}, \underline{a}_{-i}^{\mathscr{A}}) - \pi_i(a_i^*, \underline{a}_{-i}^{\mathscr{A}})$. Let l^* be the integer l satisfying $a_i^{B,\mathscr{A},l} = a_i^*$ and let l^{**} be the integer l satisfying $a_i^{B,\mathscr{A},l} = a_i^{**}$. Without loss of generality, assume $l^* > l^{**}$, so that $\Delta\pi_i \geq 0$. For each $\tau \in \{0, \cdots, T\}$, let $W_i(\tau)$ denote player i's (unnormalized) payoff to playing a_i^{**} in the first τ periods and a_i^* in the remaining $T-\tau$ periods. Then,

$$W_i(\tau) - W_i(0) = \frac{1-\delta^\tau}{1-\delta}\Delta\pi_i + \frac{1-\delta^T}{1-\delta}\sum_{r>Z_T}(F(\tau,T,r) - F(0,T,r))\sum_{l=l^{**}+1}^{l^*}\lambda_i^{B,\mathscr{A},l}.$$

Arranging,

$$W_i(\tau) - W_i(0) = \Delta\pi_i \frac{1-\delta^T}{1-\delta}\left(\frac{1-\delta^\tau}{1-\delta^T} - g(\tau)\right) \tag{31}$$

where

$$g(\tau) = \frac{\sum_{r>Z_T} F(0,T,r) - \sum_{r>Z_T} F(\tau,T,r)}{\sum_{r>Z_T} F(0,T,r) - \sum_{r>Z_T} F(T,T,r)}.$$

Lemma 7. *Let $h(\tau) = \lim_{\delta\to 1}(\frac{1-\delta^\tau}{1-\delta^T}) - g(\tau)$. Then, there is $\overline{T}$ such that for every $T > \overline{T}$, $h(\tau)$ is negative for all $\tau \in \{1, \cdots, T-1\}$.*

Proof. This directly follows from (28) of Ely, Hörner, and Olszewski (2005). Note that (6) is used here. *Q.E.D.*

Suppose first that $\Delta\pi_i > 0$. Applying this lemma to (31), it follows that in the limit as $\delta \to 1$, player i strictly prefers playing a_i^* constantly to mixing a_i^* and a_i^{**}. By continuity, she strictly prefers a constant action a_i^* even if δ is slightly less than one.

Suppose next that $\Delta\pi_i = 0$. Then (31) implies that player i weakly prefers playing a_i^* constantly to mixing a_i^* and a_i^{**} for any $\delta \in (0,1)$. Thus, in both cases, player i prefers playing a constant action to mixing two actions.

By a similar argument, player i prefers mixing n actions to mixing $n+1$ actions.[7] Hence, playing an action $a_i \in A_i$ for T periods is a best reply against $\underline{a}_{-i}^{\mathscr{A}}$. Likewise, one can show that playing an action $a_i \in \mathscr{A}_i$ constantly is a best reply against $\overline{a}_{-i}^{\mathscr{A}}$.

In summary, if T is sufficiently large so that the conditions in Lemmas 6 and 7 are satisfied, then there is $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta},1)$, the strategy profile specified by the automaton constitutes a belief-free review-strategy equilibrium, and achieves v^*. This completes the proof.

Appendix C: Proof of Lemma 4

Let $u_i : A_{i-1} \times \Omega_{i-1} \to \boldsymbol{R}$ be such that $\pi_i(a) + E[u_i(a_{i-1}, \omega_{i-1})|a] = 0$ for all $a \in A$. The existence of such u_i is guaranteed from the full rank condition. Let $\overline{u}_i$ denote the maximum of $|u_i(a_{i-1}, \omega_{i-1})|$ over all $a_{i-1} \in A_{i-1}$ and $\omega_{i-1} \in \Omega_{i-1}$.

For each $k \in \{1, \cdots, K\}$, let $h_i^{[k]}$ denote player i's private history up to the end of the kth supplemental round. Also, let $h_i^{[k,m]}$ be player i's history up to the end of the kth main round. Let $h_i^{[0]}$ be player i's history up to the end of the confirmation round, and $h_i^{[-1]}$ be player i's history up to the end of the signaling round. For each k, let $H_i^{[k]}$ be the set of all $h_i^{[k]}$, and $H_i^{[k,m]}$ be the set of all $h_i^{[k,m]}$.

Recall that a player's message space in the report round is $M^0 \times M^1 \times \cdots \times M^K$, i.e., a player reports what happened in the past communication stages. Given player $(i-1)$'s block history $h_{i-1}^{T_b} \in H_{i-1}^{T_b}$, let $I_{-i} \in (M^0 \times \cdots \times M^K)^{N-1}$ denote player $(i-1)$'s inference on the messages from players $-i$ in the report round. Notice that player i's actions in the report round cannot affect the realization of I_{-i}, since player $i-1$ makes her inference on player j's message using the random event $\psi_{i-1}(\{a_j^G\})$ and Lemma 3 asserts that player i cannot manipulate the realizations of this random event. For each $k \in \{0, \cdots, K\}$, let I_{-i}^k be the projection of I_{-i} onto $(M^0 \times M^1 \times \cdots \times M^k)^{N-1}$. That is, I_{-i}^k denotes player $(i-1)$'s inference on the messages from players $-i$ corresponding to the history up to the kth supplemental round.

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[7] The formal proof is as follows. With an abuse of notation, let $W_i(\tau)$ denote player i's payoff to playing a_i^{**} for τ periods, playing a_i^* for $\tilde{T} - \tau$ periods, and playing a $(T - \tilde{T})$-length sequence of actions consisting of $\{a_i^{B,\mathscr{A},l^*+1}, \cdots, a_i^{B,\mathscr{A},|A_i|}\}$, where $0 \leq \tau \leq \tilde{T} < T$. Then,

$$W_i(\tau) - W_i(0) = \Delta\pi_i \frac{1-\delta^T}{1-\delta}\left(\frac{1-\delta^\tau}{1-\delta^T} - g(\tau)\right).$$

Using Lemma 7, one can confirm that player i is (weakly) better off by playing a_i^* for $\tilde{T}$ periods rather than playing a_i^{**} for τ periods and a_i^* for $\tilde{T} - \tau$ periods.

Without loss of generality, consider a particular $i \in I$. We consider U_i^B which is decomposable into real-valued functions $(\theta^{-1}, \cdots, \theta^{K+1})$ as follows;

$$U_i^B(h_{i-1}^{T_b}) = \frac{1}{\delta^{T_b}} \left[\begin{array}{l} \delta^{NT} \theta^{-1}(h_{i-1}^{[-1]}) + \delta^{7NT} \theta^0(h_{i-1}^{[0]}) \\ + \sum_{k=1}^{K} \delta^{7NT+k(K+2N)T} \theta^k(h_{i-1}^{[k]}, I_{-i}^{k-1}) + \delta^{T_b} \theta^{K+1}(h_{i-1}^{T_b}) \end{array} \right].$$

Intuitively, player i receives a transfer θ^{-1} after the signaling round, θ^0 after the confirmation round, θ^k after the kth supplemental round for each $k \in \{1, \cdots, K\}$, and θ^{K+1} after the report round.

In this transfer scheme, the transfers for the past rounds are irrelevant to player i's incentive compatibility. For example, consider the report round. Note that the transfer θ^{-1} is a function of $(h_{i-1}^{[-1]})$, and hence does not depend on the history in the report round. Likewise, the transfer θ^0 does not depend on the history in the report round. Moreover, for each $k \in \{1, \cdots, K\}$, θ^k depends on the history in the report round only through I_{-i}^k, and player i's action in the report round cannot affect the realization of I_{-i}^k. Therefore, the transfers $(\theta^{-1}, \cdots, \theta^K)$ are irrelevant to player i's incentive compatibility in the report round, i.e., player i maximizes the sum of the stage game payoffs and the transfer θ^{K+1} in the report round. Likewise, one can check that the transfers $(\theta^{-1}, \cdots, \theta^{k-1})$ are irrelevant to player i's incentive compatibility in the continuation game from the kth main round.

In what follows, we show that there are transfers $(\theta^{-1}, \cdots, \theta^{K+1})$ satisfying (20) through (22). To simplify the notation, let X^B denote the set of all $x \in X$ satisfying $x_{i-1} = B$. Likewise, let X_{-i}^B be the set of all $x_{-i} \in X_{-i}$ satisfying $x_{i-1} = B$.

C.1 Constructing θ^{K+1}

Note first that the transfer θ^{-1} is a function of $(h_{i-1}^{[-1]})$, and hence does not depend on the history in the report round. Likewise, the transfer θ^0 does not depend on the history in the report round. Moreover, for each $k \in \{1, \cdots, K\}$, θ^k depends on the history in the report round only through I_{-i}^k, and player i's action in the report round cannot affect the realization of I_{-i}^k. Therefore, the transfers $(\theta^{-1}, \cdots, \theta^K)$ are irrelevant to player i's incentive compatibility in the report round, i.e., player i maximizes the sum of the stage game payoffs and the transfer θ^{K+1}.

Let

$$\theta^{K+1}(h_{i-1}^{T_b}) = \sum_{t=1}^{2N^2T(3+K)} \frac{u_i(a_{i-1}^t, \omega_{i-1}^t)}{\delta^{2N^2T(3+K)+1-t}}$$

where $(a_{i-1}^t, \omega_{i-1}^t)$ is player $(i-1)$'s action and signal in the tth period of the report round. Recall that $\pi_i(a) + \sum_\omega q(\omega|a) u_i(a_{i-1}, \omega_{i-1}) = 0$ for all $a \in A$. Thus the term $u_i(a_{i-1}^t, \omega_{i-1}^t)$ offsets the stage game payoff in the tth period of the report round, and hence player i is indifferent among all actions in every period of the report round, regardless of the past history. This shows that (20) holds for all $l \geq 1+6N+K(1+2N)$, $h^{t_l} \in H^{t_l}$, and $x \in X^B$.

Also, since the term $\frac{\overline{w}_i - \underline{w}_i}{(K+3)(1-\delta)}$ goes to infinity as $\delta \to 1$, it follows that

$$-2N^2T(3+K)\overline{u}_i < \theta^{K+1}(h_{i-1}^{T_b}) < \frac{\overline{w}_i - \underline{w}_i}{(K+3)(1-\delta)} \tag{32}$$

for all $h_{i-1}^{T_b} \in H_{i-1}^{T_b}$, provided that δ is close to one.

C.2 Constructing θ^k for all $k \in \{1,\ldots,K\}$

In this step, the following notation is useful. For each $x \in X$, let $H_{-i}^{[0]}(x)$ be the set of $h_{-i}^{[0]} \in H_{-i}^{[0]}$ such that for each $j \neq i$ and $\tilde{h}_j^{[0]} \in H_j^{[0]}$,

$$(\hat{m}_{-i}^0(h_j^{[0]}), \hat{m}_i^0(\tilde{h}_j^{[0]})) \in M^0(x)$$

where $\hat{m}_{-i}^0(h_j^{[0]}) = (\hat{m}_l^0(h_j^{[0]}))_{l \neq i}$, In words, $H_{-i}^{[0]}(x)$ is the set of all histories up to the end of the confirmation round such that player $j \neq i$ will play $a_j^{x,1}$ in the first main round (recall that $s_j^G(h_j^{[0]}) = s_j^B(h_j^{[0]}) = a_j^{x,1}$ if $\hat{m}^0(h_j^{[0]}) \in M^0(x)$) even if her inference on the message from player i in the confirmation round is replaced with any other information $\hat{m}_i^0(\tilde{h}_j^{[0]})$. Thus $h_{-i}^{[0]} \in H_{-i}^{[0]}(x)$ implies that player i's message in the confirmation round is irrelevant to players $-i$'s continuation play.

Note that $h_{-i}^{[0]} \in H_{-i}^{[0]}(x)$ is a "regular history" when nobody deviates from the block strategy profile s^x. For example, when players play s^x under perfect monitoring, one can check that $\hat{m}_{-i}^0(h_j^{[0]}) = (x, \cdots, x)$ for all $j \neq i$, so that $h_{-i}^{[0]} \in H_{-i}^{[0]}(x)$. Also, when players play s^x under imperfect private monitoring, one can check that the probability of $h_{-i}^{[0]} \in H_{-i}^{[0]}(x)$ approximates one as $T \to \infty$. (See the discussion in Section 3.3.6.)

Likewise, for each $k \in \{1, \cdots, K-1\}$ and $x \in X$, let $H_{-i}^{[k]}(x)$ be the set of $h_{-i}^{[k]} \in H_{-i}^{[k]}$ such that for each $j \neq i$ and $\tilde{h}_j^{[k]} \in H_j^{[k]}$,

$$(\hat{m}^0(h_j^{[k]}), \hat{m}^1(h_j^{[k]}), \cdots, \hat{m}^{k-1}(h_j^{[k]})) \in M^0(x) \times M^1(0) \times \cdots \times M^{k-1}(0),$$

and

$$\left(\hat{m}_{-i}^k(h_j^{[k]}), \hat{m}_i^k(\tilde{h}_j^{[k]})\right) \in M^k(0)$$

where $\hat{m}_{-i}^k(h_j^{[k]}) = (\hat{m}_l^k(h_j^{[k]}))_{l \neq i}$. That is, $H_{-i}^{[k]}(x)$ is the set of all histories up to the kth supplemental round such that player $j \neq i$ will play $a_j^{x,k+1}$ in the $(k+1)$st main round even if the inference on player i's message in the kth supplemental round is replaced with any other information. Thus $h_{-i}^{[k]} \in H_{-i}^{[k]}(x)$ implies that player i's message in the

kth supplemental round is irrelevant to players $-i$'s continuation play. Again, one can check that these are "regular histories" when nobody deviates from the block strategy profile s^x.

Also, for each $k \in \{1, \cdots, K\}$ let $H_{-i}^{[k]}(x,i)$ be the set of $h_{-i}^{[k]} \in H_{-i}^{[k]}$ such that there is $n \in \{1, \cdots, k\}$ such that for each $j \neq i$ and $\tilde{h}_j^{[k]} \in H_j^{[k]}$,

$$(\hat{m}^0(h_j^{[k]}), \hat{m}^1(h_j^{[k]}), \cdots, \hat{m}^{n-1}(h_j^{[k]})) \in M^0(x) \times M^1(0) \times \cdots \times M^{n-1}(0),$$

and

$$\left(\hat{m}_{-i}^n(h_j^{[k]}), \hat{m}_i^n(\tilde{h}_j^{[k]})\right) \in M^n(i).$$

Note that if $M^0(x) \times M^1(0) \times \cdots \times M^{n-1}(0)$ and $\hat{m}^n(h_j^{[k]}) \in M^n(i)$ for some $n \leq k$, then player j chooses $\underline{a}_j^i(\mathscr{A}^{k+1})$ in the $(k+1)$st main round, irrespective of the history after the nth supplemental round. Thus $H_{-i}^{[k]}(x,i)$ is the set of histories up to the kth supplemental round such that player i's message in the kth supplemental round cannot affect the opponents' continuation play and they will play $\underline{a}^i(\mathscr{A})$ or $\overline{a}^i(\mathscr{A})$. Roughly speaking, these are histories reachable by player i's unilateral deviation. To see this, suppose that monitoring is perfect and that players follow s^x but player i unilaterally deviates from $a^{x,n}$ in the nth main round. Then both players $i-1$ and $i+1$ detect this deviation and send the messages $m_{i-1}^n = i$ and $m_{i+1}^n = i$ in the nth supplemental round, while player $j \neq i-1, i, i+1$ sends $m_j^n = 0$. In this case, player i's action in the nth supplemental round cannot affect player $-i$'s continuation play, so that the history is an element of $H_{-i}^{[n]}(x,i)$.

For notational convenience, let $\overline{H}_{-i}^{[0]}$ denote a union of $H_{-i}^{[0]}(x)$ over all $x \in X^B$. Also, for each $k \in \{1, \cdots, K\}$, let $\overline{H}_{-i}^{[k]}$ be the union of $(H_{-i}^{[k]}(x) \cup H_{-i}^{[k]}(x,i))$ over all $x \in X^B$.

In what follows, the transfers $(\theta^1, \cdots, \theta^K)$ are specified by backward induction. To define θ^k, assume that $(\theta^{k+1}, \cdots, \theta^{K+1})$ have already been determined so that player i's continuation payoff after history $h_{-i}^{[k]} \in H_{-i}^{[k]}$, augmented by $(\theta^{k+1}, \cdots, \theta^{K+1})$, is equal to $V_i(h_{-i}^{[k]})$, and that (20) holds for all $l \geq 1 + 6N + k(1+2N)$, $h^{t_l} \in H^{t_l}$, and $x \in X^B$. Here, for each $k \in \{1, \cdots, K\}$ and $h_{-i}^{[k]} \in \overline{H}_{-i}^{[k]}$, the value $V_i(h_{-i}^{[k]})$ is defined to be the maximum of player i's actual continuation payoff (i.e., the discounted sum of stage game payoffs) after history $h_{-i}^{[k]}$ over all her continuation strategies, subject to the constraints that monitoring is perfect and that payoffs in the communication stages are replaced with zero. For each $h_{-i}^{[0]} \in \overline{H}_{-i}^{[0]}$, the value $V_i(h_{-i}^{[0]})$ is defined to be the maximum of player i's actual continuation payoff after history $\tilde{h}_{-i}^{[0]}$ over all all $\tilde{h}_{-i}^{[0]} \in \overline{H}_{-i}^{[0]}$ and over all her continuation strategies, subject to the constraints that monitoring is perfect and that payoffs in the communication stages are replaced with zero. For each $k \in \{0, \cdots, K\}$ and $h_{-i}^{[k]} \notin \overline{H}_{-i}^{[k]}$, the value $V_i(h_{-i}^{[k]})$ is defined to be player i's actual continuation payoff when she earns

$\max_{a\in A}\pi_i(a)$ in periods of the main rounds and zero in other periods. Notice that the transfers $(\theta^1,\cdots,\theta^K)$ are specified in such a way that player i's continuation payoff $V_i(h_{-i}^{[k]})$ from the kth main round is high and is the same for all histories $h_{-i}^{[k]}\notin\overline{H}_{-i}^{[k]}$. As explained later, this "constant continuation payoff" property is used to show that player i has a truth-telling incentive in the $(k-1)$th supplemental round, even when $h_{-i}^{[k]}\notin\overline{H}_{-i}^{[k]}$ so that player i's message in that round can affect the opponents' continuation play.

In what follows, we show that there is θ^k such that player i's continuation payoff after $h_{-i}^{[k-1]}\in H_{-i}^{[k-1]}$ augmented by $(\theta^k,\cdots,\theta^{K+1})$ is equal to $V_i(h_{-i}^{[k-1]})$, and such that (20) holds for all $l\geq 1+6N+(k-1)(1+2N)$, $h^{t_l}\in H^{t_l}$, and $x\in X^B$. Iterating this argument determines the transfers $(\theta^1,\cdots,\theta^K)$ so that player i's continuation payoff after $h_{-i}^{[k-1]}\in H_{-i}^{[k-1]}$ is equal to $V_i(h_{-i}^{[k-1]})$ for all $k\in\{1,\cdots,K\}$, and such that (20) holds for all $l\geq 1+6N$. (Recall that θ^{K+1} has been specified so that player i's continuation payoff after $h_{-i}^{[K]}\in H_{-i}^{[K]}$ is equal to $V_i(h_{-i}^{[K]})=0$ and such that (20) holds for all $l\geq 1+6N+K(1+2N)$.)

We consider θ^k which has the following form:

$$\theta^k(h_{i-1}^{[k]},I_{-i}^{k-1})=\tilde{\theta}^k(h_{i-1}^{[k,m]},I_{-i}^{k-1})+\sum_{t=1}^{2NT}\frac{u_i(a_{i-1}^t,\omega_{i-1}^t)}{\delta^{2NT+1-t}}. \tag{33}$$

Here, $(a_{i-1}^t,\omega_{i-1}^t)$ is player $(i-1)$'s action and signal in the tth period of the kth supplemental round, and $\tilde{\theta}^k$ is a real-valued function of $h_{i-1}^{[k,m]}$ and I_{-i}^{k-1}. Although $\tilde{\theta}^k$ has not been specified yet, the following lemma is established.

Lemma 8. *Player i is indifferent over all actions in every period of the kth supplemental round regardless of the past history, and hence (20) holds for all* $l\geq 2+6N+(k-1)(1+2N)$.

Proof. As in the report round, the second term in the right-hand side of (33) offsets the stage game payoffs in the kth supplemental round. Note also that the term $\tilde{\theta}^k(h_{i-1}^{[k,m]},I_{-i}^{k-1})$ does not depend on the outcome in the kth supplemental round. Thus, it suffices to show that player i's action in the kth supplemental round does not affect the continuation payoff after the kth supplemental round with $(\theta^{k+1},\cdots,\theta^{K+1})$.

Recall that this continuation payoff is assumed to be equal to $V_i(h_{-i}^{[k]})$. By definition, the value $V_i(h_{-i}^{[k]})$ is independent of players $-i$'s inferences on player i's message in the kth supplemental round. Therefore, player i's actions in the $(2i-1)$st and $2i$th review phases cannot affect the continuation payoff. Also, player i cannot manipulate player j's inference on messages from the other players, since player i's action cannot affect the realization of the corresponding random events. Hence, player i's actions in other periods cannot affect the continuation payoff as well. *Q.E.D.*

To specify the real-valued function $\tilde{\theta}^k$, the following notation is useful. For each $h_{-i}^{[k-1]}\in H_{-i}^{[k-1]}$ and $a_i\in A_i$, let $\tilde{W}_i(h_{-i}^{[k-1]},a_i)$ denote player i's continuation payoff from the kth main round, augmented by $(\theta^{k+1},\cdots,\theta^{K+1})$ and by the second term of the right-hand side of (33), when player i plays a_i constantly in the kth main round and plays a best reply thereafter. That is, $\tilde{W}_i(h_{-i}^{[k-1]},a_i)$ denotes the value

$$\sum_{t=1}^{KT}\delta^{t-1}\pi_i(a_i,s_{-i}^{x_{-i}}(h_{-i}^{[k-1]}))+\delta^{KT+2NT}\sum_{h_{-i}^{[k]}\in H_{-i}^{[k]}}\Pr(h_{-i}^{[k]}|h_{-i}^{[k-1]},a_i)V_i(h_{-i}^{[k]})$$

where $\Pr(h_{-i}^{[k]}|h_{-i}^{[k-1]},a_i)$ denotes the probability that $h_{-i}^{[k]}$ realizes when player i plays a_i constantly in the kth main round and sends $m_i^k=0$ in the kth supplemental round while the opponents' play $s^x|_{h_{-i}^{[k-1]}}$. Note that the first term is the stage game payoff in the kth main round, and the second is the continuation payoff after the kth supplemental round augmented by $(\theta^{k+1},\cdots,\theta^{K+1})$. The stage game payoffs in the kth supplemental round does not appear here, as the second term of the right-hand side of (33) offset them.

For each $j\neq i$ and $h_j^{[k-1]}\in H_j^{[k-1]}$, let J_j^{k-1} denote the history in the confirmation round and the past supplemental rounds, i.e., $J_j^{k-1}=(m^0(h_j^{[k-1]}),\cdots,m^k(h_j^{[k-1]}))$. Let $J_{-i}^{k-1}=(J_j^{k-1})_{j\neq i}\in(M^0\times\cdots\times M^{k-1})^{N-1}$. Note that $\tilde{W}_i(h_{-i}^{[k-1]},a_i)$ does not depend on the entire information of $h_{-i}^{[k-1]}$ but on J_{-i}^{k-1}. Hence, one can write $\tilde{W}_i(J_{-i}^{k-1},a_i)$ instead of $\tilde{W}_i(h_{-i}^{[k-1]},a_i)$.

For each $J_{-i}^{k-1}\in(M^0\times\cdots\times M^{k-1})^{N-1}$, let $(a_i^1(J_{-i}^{k-1}),\cdots,a_i^{|A_i|}(J_{-i}^{k-1}))$ be an ordering of all actions $a_i\in A_i$ such that

$$\lim_{T\to\infty}\lim_{\delta\to 1}\frac{\tilde{W}_i(J_{-i}^{k-1},a_i^1(J_{-i}^{k-1}))}{T}\geq\cdots\geq\lim_{T\to\infty}\lim_{\delta\to 1}\frac{\tilde{W}_i(J_{-i}^{k-1},a_i^{|A_i|}(J_{-i}^{k-1}))}{T}.$$

For each J_{-i}^{k-1} and $l\in\{1,\cdots,|A_i|\}$, let $1_{[J_{-i}^{k-1},l]}:H_{i-1}^{[k,m]}\to\{0,1\}$ be the indicator function such that $1_{[J_{-i}^{k-1},l]}(h_{i-1}^{[k,m]})=1$ if the random event $\psi_{i-1}(\{a_i^l(J_{-i}^{k-1}),\cdots,a_i^{|A_i|}(J_{-i}^{k-1})\})$ is counted more than Z_{KT} times in the kth main round (according to the history $h_{i-1}^{[k,m]}$), and $1_{[J_{-i}^{k-1},l]}(h_{i-1}^{[k,m]})=0$ otherwise. Likewise, for each $a_i\in A_i$, let $1_{a_i}:H_{i-1}^{[k,m]}\to\{0,1\}$ be the indicator function such that $1_{a_i}(h_{i-1}^{[k,m]})=1$ if and only if the random event $\psi_{i-1}(\{a_i\})$ is counted more than Z_{KT} times in the kth main round.

From (13), there is a positive number $\eta>0$ satisfying

$$\left(1-\frac{K^2T}{T_b}\right)3\bar{u}_i<3\eta<\underline{w}_i-\lim_{\delta\to 1}\max_{s_i^{T_b}\in S_i^{T_b}}w_i^P(s_i^{T_b},s_{-i}^{x_{-i}})\tag{34}$$

for all $x_{-i}\in X_{-i}^B$. Then, let $\tilde{\theta}^k$ be such that

$$\tilde{\theta}^k(h_{i-1}^{[k,m]},I_{-i}^{k-1})=\sum_{a_i\in A_i}1_{a_i}(h_{i-1}^{[k,m]})KT\eta+\sum_{l=1}^{|A_i|}1_{[I_{-i}^{k-1},l]}(h_{i-1}^{[k,m]})\lambda^k(I_{-i}^{k-1},l),\tag{35}$$

where the values $(\lambda^k(I_{-i}^{k-1}, l))_{I_{-i}^{k-1}, l}$ solve

$$V_i(h_{-i}^{[k-1]}) = \tilde{W}_i(h_{-i}^{[k-1]}, a_i) + \delta^{KT+2NT} \sum_{\tilde{a}_i \in A_i} \Pr(1_{\tilde{a}_i} | h_{-i}^{[k-1]}, a_i) KT\eta$$
$$+ \delta^{KT+2NT} \sum_{I_{-i}^{k-1}} \Pr(I_{-i}^{k-1} | h_{-i}^{[k-1]}) \sum_{l=1}^{|A_i|} \Pr(1_{[I_{-i}^{k-1}, l]} | h_{-i}^{[k-1]}, a_i) \lambda^k(I_{-i}^{k-1}, l) \quad (36)$$

for all $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}$ and $a_i \in A_i$. Here, $\Pr(1_{[\cdot]} | h_{-i}^{[k-1]}, a_i)$ denotes the probability that the indicator function $1_{[\cdot]}(h_{i-1}^{[k,m]})$ takes one given that player i chooses the constant action a_i while players $-i$ play the action $s_{-i}^{x_{-i}}(h_{-i}^{[k-1]})$ constantly in the kth main round; and $\Pr(I_{-i}^{k-1} | h_{-i}^{[k-1]})$ denotes the probability that I_{-i}^{k-1} realizes given that the history at the beginning of the kth main round is $h_{-i}^{[k-1]}$. In words, the values $(\lambda^k(I_{-i}^{k-1}, l))_{I_{-i}^{k-1}, l}$ are determined so that player i's unnormalized continuation payoff after $h_{-i}^{[k-1]}$ augmented by $(\theta^k, \cdots, \theta^{K+1})$ equals $V_i(h_{-i}^{[k-1]})$, no matter what constant action player i chooses in the kth main round. Indeed, the right-hand side of (36) denotes player i's continuation payoff after $h_{-i}^{[k-1]}$ when player i chooses a_i constantly in the kth main round and plays a best reply thereafter.

Lemma 9. *There is $\overline{T}$ such that for all $T > \overline{T}$, there is $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$, system (36) has a unique solution, and it satisfies*

$$-2NT\overline{u}_i - 2KT\eta < \theta^k(h_{i-1}^{[k]}, I_{-i}^{k-1}) < \frac{\overline{w}_i - \underline{w}_i}{(K+3)(1-\delta)} \quad (37)$$

for all $h_{i-1}^{[k]}$ and I_{-i}^{k-1}. Also, using this transfer scheme, (20) holds for all $l \geq 1 + 6N + (k-1)(1+2N)$, $h^{t_l} \in H^{t_l}$, and $x \in X^B$, and player i's continuation payoff after history $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}$ equals $V_i(h_{-i}^{[k-1]})$.

Proof. See Appendix C.5.1. *Q.E.D.*

C.3 Constructing θ^0

Let

$$\theta^0(h_{i-1}^{[0]}) = \sum_{t=1}^{6NT} \frac{u_i(a_{i-1}^t, \omega_{i-1}^t)}{\delta^{6NT+1-t}}$$

where $(a_{i-1}^t, \omega_{i-1}^t)$ is player $(i-1)$'s action and signal in the tth period of the confirmation round. As in the report round, this θ^0 offsets the stage game payoffs in the tth period of the confirmation round. In addition, as in the proof of Lemma 8, one can show that player i's actions in the confirmation round cannot affect the expected continuation payoffs from the first main round augmented by $(\theta^1, \cdots, \theta^{K+1})$. Therefore, player i is indifferent among all actions in every period of the confirmation round regardless of the past history, and hence (20) holds for all $l \geq 1$, $h^{t_l} \in H^{t_l}$, and $x \in X^B$.

Also, as in step 1,

$$-6NT\overline{u}_i < \theta^0(h_{i-1}^{[0]}) < \frac{\overline{w}_i - \underline{w}_i}{(K+3)(1-\delta)} \tag{38}$$

for all $h_{i-1}^{[0]} \in H_{i-1}^{[0]}$, provided that δ is close to one.

C.4 Constructing θ^{-1}

For each $x \in X$, let $H_{i-1}^{[-1]}(x)$ denote the set of all $h_{i-1}^{[-1]} \in H_{i-1}^{[-1]}$ such that $\hat{x}_{-i}(h_{i-1}^{[-1]}) = x_{-i}$ and $\psi_{i-1}(\{a_i^{x_i}\})$ is counted more than Z_T times during the T-period interval from period $(i-1)T+1$ to period iT. Then, for each $x \in X$, let $1_x : H_{i-1}^{[-1]} \to \{0,1\}$ denote the indicator function of $H_{i-1}^{[-1]}(x)$. That is, $1_x(h_{i-1}^{[-1]}) = 1$ if and only if $h_{i-1}^{[-1]} \in H_{i-1}^{[-1]}(x)$.

Let

$$\theta^{-1}(h_{i-1}^{[-1]}) = 3T_b\eta + \sum_{t=1}^{NT} \frac{u_i(a_{i-1}^t, \omega_{i-1}^t)}{\delta^{NT-t}} + \sum_{x \in X^B} 1_x(h_{i-1}^{[-1]})\lambda^{-1}(x)$$

where $(a_{i-1}^t, \omega_{i-1}^t)$ is player $(i-1)$'s private history in the tth period of the block game, and the values $(\lambda^{-1}(x))_{x \in X^B}$ solve

$$\sum_{t=1}^{T_b} \delta^{t-1}\underline{w}_i = \delta^{NT}\left[3T_b\eta + \sum_{\tilde{x} \in X^B} \sum_{h_{i-1}^{[-1]} \in H_{i-1}^{[-1]}(\tilde{x})} \Pr(h_{i-1}^{[-1]}|s^x)\lambda^{-1}(\tilde{x})\right] + \delta^{7NT} \sum_{h_{-i}^{[0]} \in H_{-i}^{[0]}} \Pr(h_{-i}^{[0]}|s^x)V_i(h_{-i}^{[0]}) \tag{39}$$

for all $x_{-i} \in X_{-i}^B$. Here, $\Pr(h_{i-1}^{[-1]}|s^x)$ denotes the probability that $h_{i-1}^{[-1]}$ realizes when players perform the block strategy profile s^x, and $\Pr(h_{-i}^{[0]}|s^x)$ denotes the probability of $h_{-i}^{[0]}$. Intuitively, the values $(\lambda^{-1}(x))_{x \in X^B}$ are determined so that (21) holds. Indeed, the right-hand side of (39) denotes player i's auxiliary scenario payoff from the block strategy profile s^x. (Precisely, the first term denotes the expectation of the payment θ^{-1} other than the term u_i, and the second term denotes the expectation of the continuation payoff after the confirmation round. The stage game payoffs in the signaling round and the confirmation round do not appear here, since these payoffs and the term u_i in θ^{-1} and θ^0 cancel out.)

Lemma 10. *There is* $\overline{T}$ *such that for all* $T > \overline{T}$, *there is* $\overline{\delta} \in (0,1)$ *such that for all* $\delta \in (\overline{\delta}, 1)$, *system (39) has a unique solution, and it satisfies*

$$3T_b\eta - NT\overline{u}_i < \theta^{-1}(h_{i-1}^{[-1]}) < \frac{\overline{w}_i - \underline{w}_i}{(K+3)(1-\delta)}. \tag{40}$$

for all $h_{i-1}^{[-1]} \in H_{i-1}^{[-1]}$. *Also, under this transfer scheme, (20) and (21) hold for all* $l \geq 0$, $h^{t_l} \in H^{t_l}$, *and* $x \in X^B$.

Proof. See Appendix C.5.3. *Q.E.D.*

This lemma asserts that the specified U_i^B satisfies (20) and (21). Finally, (22) follows from (13), (14), (32), (37), (38), and (40).

C.5 Remaining Proofs

C.5.1 Proof of Lemma 9

Part 1. Proof of Uniqueness. Observe that the value $V_i(h_{-i}^{[k-1]})$ does not depend on the entire information of $h_{-i}^{[k-1]}$ but on J_{-i}^{k-1}, since the continuation strategy of players $-i$ from the kth main round depends only on J_{-i}^{k-1}. Thus write $V_i(J_{-i}^{k-1})$ instead of $V_i(h_{-i}^{[k-1]})$. Likewise, one can replace $h_{-i}^{[k-1]}$ with J_{-i}^{k-1} in each term of the right-hand side of (36). Therefore, solving (36) is equivalent to considering the following system:

$$\begin{aligned} V_i(J_{-i}^{k-1}) = \tilde{W}_i(J_{-i}^{k-1}, a_i) &+ \delta^{KT+2NT} \sum_{\tilde{a}_i \in A_i} \Pr(1_{\tilde{a}_i} | J_{-i}^{k-1}, a_i) KT\eta \\ &+ \delta^{KT+2NT} \sum_{I_{-i}^{k-1}} \Pr(I_{-i}^{k-1} | J_{-i}^{k-1}) \sum_{l=1}^{|A_i|} \Pr(1_{[I_{-i}^{k-1}, l]} | J_{-i}^{k-1}, a_i) \lambda^k(I_{-i}^{k-1}, l) \end{aligned} \tag{41}$$

for all J_{-i}^{k-1} and $a_i \in A_i$.

Note that (41) is represented by the matrix form

$$Q\lambda^k = b. \tag{42}$$

Here, λ^k is a column vector with elements $\lambda^k(I_{-i}^{k-1}, l)$ for all I_{-i}^{k-1} and $l \in \{1, \cdots, |A_i|\}$; Q is a coefficient matrix, each entry of which denotes the product of $\Pr(I_{-i}^{k-1} | J_{-i}^{k-1})$ and $\Pr(1_{[I_{-i}^{k-1}, l]} | J_{-i}^{k-1}, a_i)$; and b is a column vector denoting the remaining terms, i.e., each entry of b is denoted by

$$b(J_{-i}^{k-1}, a_i) = \frac{1}{\delta^{KT+2NT}} \left[V_i(J_{-i}^{k-1}) - \tilde{W}_i(J_{-i}^{k-1}, a_i) \right] - \sum_{\tilde{a}_i \in A_i} \Pr(1_{\tilde{a}_i} | J_{-i}^{k-1}, a_i) KT\eta.$$

Without loss of generality, assume that for each n, there is I_{-i}^{k-1} such that for each $l \in \{1, \cdots, |A_i|\}$, the $(n|A_i| + l)$th coordinate of λ^k is $\lambda^k(I_{-i}^{k-1}, l)$ and the $(n|A_i| + l)$th coordinate of b is $b(J_{-i}^{k-1}, a_i^l(J_{-i}^{k-1}))$ where J_{-i}^{k-1} satisfies $I_{-i}^{k-1} = J_{-i}^{k-1}$. (This assumption is indeed satisfied by exchanging rows of λ^k and b in an appropriate way.)

It follows from (17) and (18) that, as $T \to \infty$, $\Pr(I_{-i}^{k-1}|J_{-i}^{k-1})$ converges to one if $I_{-i}^{k-1} = J_{-i}^{k-1}$, and to zero otherwise. Likewise, $\Pr(1_{[I_{-i}^{k-1},l]}|J_{-i}^{k-1}, a_i)$ converges to one if a_i is in the set $\{a_i^l(I_{-i}^{k-1}), \cdots, a_i^{|A_i|}(I_{-i}^{k-1})\}$, and to zero otherwise. Hence, the matrix Q converges to

$$\begin{pmatrix} D & & 0 \\ & \ddots & \\ 0 & & D \end{pmatrix}$$

where D is the $|A_i| \times |A_i|$ matrix such that its ij-element equals one if $i \geq j$ and zero if $i < j$. Since the above matrix is invertible, there is an inverse of Q for sufficiently large T, and (42) has a unique solution $Q^{-1}b$.

Part 2. Proof of (37). From (17) and (18), $\Pr(1_{\tilde{a}_i}|J_{-i}^{k-1}, a_i)$ converges as $T \to \infty$ to one if $\tilde{a}_i = a_i$, and to zero otherwise. Therefore,

$$\lim_{T\to\infty}\lim_{\delta\to 1} \frac{b(J_{-i}^{k-1}, a_i)}{T} = \lim_{T\to\infty}\lim_{\delta\to 1} \frac{V_i(J_{-i}^{k-1}) - \tilde{W}_i(J_{-i}^{k-1}, a_i)}{T} - K\eta$$

for all J_{-i}^{k-1} and $a_i \in A_i$. Plugging this into $\lambda^k = Q^{-1}b$,

$$\lim_{T\to\infty}\lim_{\delta\to 1} \frac{\lambda^k(J_{-i}^{k-1}, l)}{T} = \begin{cases} \lim_{T\to\infty}\lim_{\delta\to 1} \dfrac{V_i(J_{-i}^{k-1}) - \tilde{W}_i(J_{-i}^{k-1}, l)}{T} - K\eta & \text{if} \quad l = 1 \\ \\ \lim_{T\to\infty}\lim_{\delta\to 1} \dfrac{\tilde{W}_i(J_{-i}^{k-1}, l-1) - \tilde{W}_i(J_{-i}^{k-1}, l)}{T} & \text{if} \quad l \geq 2 \end{cases}$$

By construction, $\lim_{T\to\infty}\lim_{\delta\to 1} \frac{V_i(J_{-i}^{k-1}) - \tilde{W}_i(J_{-i}^{k-1},1)}{T} \geq 0$ and $\lim_{T\to\infty}\lim_{\delta\to 1} \frac{\tilde{W}_i(J_{-i}^{k-1},l-1) - \tilde{W}_i(J_{-i}^{k-1},l)}{T} \geq 0$ for each $l \geq 2$. Thus, from (35), $\lim_{T\to\infty}\lim_{\delta\to 1} \frac{\tilde{\theta}^k}{T}$ is at least $-K\eta$. Substituting this into (33) and using continuity, it follows that there is $\overline{T}$ such that for all $T > \overline{T}$, there is $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$, (37) still holds.

Part 3. Proof of (20). Since λ^k solves (36), player i's continuation payoff after $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}$ equals $V_i(h_{-i}^{[k-1]})$ if she chooses a constant action in the kth main round and plays a best reply thereafter. This implies that player i is indifferent over all constant actions. Hence, it suffices to show that player i is worse off if she does not take a constant action in the kth main round.

For each $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}$ and $(a_i^t)_{t=1}^{KT} \in (A_i)^{KT}$, let $W_i(h_{-i}^{[k-1]}, (a_i^t)_{t=1}^{KT})$ denote player i's continuation payoff after $h_{-i}^{[k-1]}$, augmented by $(\theta^k, \cdots, \theta^{K+1})$, when player i performs a KT-length sequence of actions $(a_i^t)_{t=1}^{KT}$ in the kth main round and plays a best reply thereafter. That is, $W_i(h_{-i}^{[k-1]}, (a_i^t)_{t=1}^{KT})$ denotes the value

$$\begin{aligned}
&\sum_{t=1}^{KT} \delta^{t-1} \pi_i(a_i^t, s_{-i}^{x_{-i}}(h_{-i}^{[k-1]})) \\
&\qquad + \delta^{KT+2NT} \sum_{h_{-i}^{[k]} \in H_{-i}^{[k]}} \Pr(h_{-i}^{[k]} | h_{-i}^{[k-1]}, (a_i^t)_{t=1}^{KT}) V_i(h_{-i}^{[k]}) \\
&\qquad + \delta^{KT+2NT} \sum_{\tilde{a}_i \in A_i} \Pr(1_{\tilde{a}_i} | h_{-i}^{[k-1]}, (a_i^t)_{t=1}^{KT}) KT\eta \\
&\qquad + \delta^{KT+2NT} \sum_{I_{-i}^{k-1}} \Pr(I_{-i}^{k-1} | h_{-i}^{[k-1]}) \sum_{l=1}^{|A_i|} \Pr(1_{[I_{-i}^{k-1}, l]} | h_{-i}^{[k-1]}, (a_i^t)_{t=1}^{KT}) \lambda^k (I_{-i}^{k-1}, l) \qquad (43)
\end{aligned}$$

where $\Pr(\cdot | h_{-i}^{[k-1]}, (a_i^t)_{t=1}^{KT})$ is defined as $\Pr(\cdot | h_{-i}^{[k-1]}, a_i)$ but player i plays the sequence $(a_i^t)_{t=1}^{KT}$, rather than the constant action a_i, in the kth main round.

Lemma 11. *Suppose that $\delta = 1$. Then, there is $\overline{T}$ such that for all $T > \overline{T}$, $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}$, and $(a_i^t)_{t=1}^{KT} \in (A_i)^{KT}$ satisfying $a_i^t \neq a_i^{\tilde{t}}$ for some t and $\tilde{t}$,*

$$V_i(h_{-i}^{[k-1]}) > W_i(h_{-i}^{[k-1]}, (a_i^t)_{t=1}^{KT}). \qquad (44)$$

Proof. See Appendix C.5.2. *Q.E.D.*

This lemma asserts that in the case of $\delta = 1$, player i is worse off by playing a sequence $(a_i^t)_{t=1}^{KT}$ in the kth main round, provided that $a_i^t \neq a_i^{\tilde{t}}$ for some t and $\tilde{t}$. Also, it follows from Lemma 3 that player i cannot earn profit even if she conditions the play on private signals. Hence, if $\delta = 1$, (20) holds for all $l \geq 1 + 6N + (k-1)(1+2N)$, $h^{t_l} \in H^{t_l}$, and $x \in X^B$.

Since V_i and W_i are continuous with respect to δ, (44) is still satisfied after perturbing δ. Hence, (20) holds, provided that δ is large enough.

C.5.2 Proof of Lemma 11

Without loss of generality, consider a particular history $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}$. Pick arbitrary actions $a_i^* \in A_i$ and $a_i^{**} \neq a_i^*$. For each $\tau \in \{0, \cdots, KT\}$, let $W_i(\tau)$ denote the value $W_i(h_{-i}^{[k-1]}, (a_i^t)_{t=1}^{KT})$ when

$$(a_i^t)_{t=1}^{KT} = (\underbrace{a_i^*, \cdots, a_i^*}_{\tau}, \underbrace{a_i^{**}, \cdots, a_i^{**}}_{KT-\tau}).$$

Also, let $\Pr(\cdot | h_{-i}^{[k-1]}, \tau)$ denote $\Pr(\cdot | h_{-i}^{[k-1]}, (a_i^t)_{t=1}^{KT})$ for such a $(a_i^t)_{t=1}^{KT}$. In what follows, it is shown that $W_i(\tau) < V_i(h_{-i}^{[k-1]})$ for each $\tau \in \{1, \cdots, KT-1\}$, that is, playing a constant action is better than mixing two actions, a_i^* and a_i^{**}.

For each I_{-i}^{k-1}, let $l^*(I_{-i}^{k-1})$ be the integer l such that $a_i^l(I_{-i}^{k-1}) = a_i^*$, and $l^{**}(I_{-i}^{k-1})$ be the integer l such that $a_i^l(I_{-i}^{k-1}) = a_i^{**}$. Let $\mathscr{I}_{-i}^{k-1}$ denote the set of all I_{-i}^{k-1} satisfying $l^*(I_{-i}^{k-1}) > l^{**}(I_{-i}^{k-1})$. It follows from Lemma 3 that

$$\Pr(1_{\tilde{a}_i}|h_{-i}^{[k-1]},\tau) - \Pr(1_{\tilde{a}_i}|h_{-i}^{[k-1]},\tau-1) = \begin{cases} (q_3-q_2)F(KT-\tau) & \text{if } \tilde{a}_i = a_i^* \\ (q_2-q_3)F(\tau-1) & \text{if } \tilde{a}_i = a_i^{**} \\ 0 & \text{otherwise} \end{cases}, \quad (45)$$

where $F_1(\tau)$ denotes $F_1(\tau, KT-1, Z_{KT})$. Likewise,

$$\begin{aligned}&\Pr(1_{[I_{-i}^{k-1},l]}|h_{-i}^{[k-1]},\tau) - \Pr(1_{[I_{-i}^{k-1},l]}|h_{-i}^{[k-1]},\tau-1) \\ &\qquad = \begin{cases} (q_2-q_3)F(\tau-1) & \text{if } I_{-i}^{k-1} \in \mathscr{I}_{-i}^{k-1} \\ (q_3-q_2)F(KT-\tau) & \text{if } I_{-i}^{k-1} \notin \mathscr{I}_{-i}^{k-1} \end{cases},\end{aligned}$$

Substituting these into (43),

$$\begin{aligned} &W_i(\tau) - W_i(\tau-1) \\ &= \pi_i(a_i^*, s_{-i}^{x_{-i}}(h_{-i}^{[k-1]})) - \pi_i(a_i^{**}, s_{-i}^{x_{-i}}(h_{-i}^{[k-1]})) \\ &\quad + \sum_{h_{-i}^{[k]} \in H_{-i}^{[k]}} \Pr(h_{-i}^{[k]}|h_{-i}^{[k-1]},\tau) V_i(h_{-i}^{[k]}) - \sum_{h_{-i}^{[k]} \in H_{-i}^{[k]}} \Pr(h_{-i}^{[k]}|h_{-i}^{[k-1]},\tau-1) V_i(h_{-i}^{[k]}) \\ &\quad - (q_3-q_2)F_1(\tau-1)KT\eta + (q_3-q_2)F_1(KT-\tau)KT\eta \\ &\quad - \sum_{I_{-i}^{k-1} \in \mathscr{I}_{-i}^{k-1}} \Pr(I_{-i}^{k-1}|h_{-i}^{[k-1]}) \sum_{l=1+l^{**}(I_{-i}^{k-1})}^{l^*(I_{-i}^{k-1})} (q_3-q_2)F_1(\tau-1)\lambda^k(I_{-i}^{k-1},l) \\ &\quad + \sum_{I_{-i}^{k-1} \notin \mathscr{I}_{-i}^{k-1}} \Pr(I_{-i}^{k-1}|h_{-i}^{[k-1]}) \sum_{l=1+l^{*}(I_{-i}^{k-1})}^{l^{**}(I_{-i}^{k-1})} (q_3-q_2)F_1(KT-\tau)\lambda^k(I_{-i}^{k-1},l). \end{aligned} \quad (46)$$

Let $\Delta_1(\tau)$ be the terms in the second line of the right-hand side, and $\Delta_2(\tau)$ be the remaining terms. The following lemma is useful to evaluate $\Delta_2(\tau)$.

Lemma 12. *There is $\overline{T} > 0$ such that for all $T > \overline{T}$, there is $\overline{\tau}$ such that $\Delta_2(\tau)$ is negative for $\tau = 1$; non-positive for all $\tau \leq \overline{\tau}$; non-negative for all $\tau > \overline{\tau}$; and positive for $\tau = KT$.*

Proof. Observe that $-\Delta_2(\tau)$ is identified as $T\tilde{W}_i(\tau)$ in (48) of Yamamoto (2007). Indeed, the terms in the first line of the right-hand side of (46) corresponds to the term $\overline{\pi}_i$, the first term in the third line and the term in the fifth line correspond to the term $\sum_{j \in I_C} K_j^T TF(\tau-1)$, and the second term in the third line and the term in the fifth line correspond to the term $\sum_{j \in I_D} K_j^T TF(T-\tau)$. Thus, there exists the desired $\overline{\tau}$ as shown by (54) of Yamamoto (2007). *Q.E.D.*

The next two lemmas show that $\Delta_1(\tau) = 0$ for some cases.

Lemma 13. *Suppose that $h_{-i}^{[k-1]} \notin \overline{H}_{-i}^{[k-1]}$ or $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}(x,i)$ for some $x \in X^B$. Then $\Delta_1(\tau) = 0$ for all τ.*

Proof. Suppose that $h_{-i}^{[k-1]} \notin \overline{H}_{-i}^{[k-1]}$. Then, by definition, $\sum_{h_{-i}^{[k]} \notin \overline{H}_{-i}^{[k]}} \Pr(h_{-i}^{[k]} | h_{-i}^{[k-1]}, \tau) = 1$. This shows that $\Delta_1(\tau) = 0$, since $V_i(h_{-i}^{[k]})$ is constant over all $h_{-i}^{[k]} \notin \overline{H}_{-i}^{[k]}$. Likewise, if $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}(x,i)$ for some $x \in X^B$, then by definition $\sum_{h_{-i}^{[k]} \in H_{-i}^{[k]}(x,i)} \Pr(h_{-i}^{[k]} | h_{-i}^{[k-1]}, \tau) = 1$. Hence, $\Delta_1(\tau) = 0$ as for $h_{-i}^{[k-1]} \notin \overline{H}_{-i}^{[k-1]}$. *Q.E.D.*

Lemma 14. *Suppose that $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}(x)$ for some $x \in X^B$, $a_i^* \neq a_i^{x,k}$, and $a_i^{**} \neq a_i^{x,k}$. Then $\Delta_1(\tau) = 0$ for all τ.*

Proof. Observe that player i's action in the kth main round affects the realization of $h_{-i}^{[k]}$ only through the random events $\psi_{i-1}(x,k)$ and $\psi_{i+1}(x,k)$. The probability that $\psi_{i-1}(x,k)$ is counted against the action profile $(a_i^*, a_{-i}^{x,k})$ is the same as against $(a_i^{**}, a_{-i}^{x,k})$. Likewise, the probability that $\psi_{i+1}(x,k)$ occurs against $(a_i^*, a_{-i}^{x,k})$ is the same as against $(a_i^{**}, a_{-i}^{x,k})$. Therefore, $\Pr(h_{-i}^{[k]} | h_{-i}^{[k-1]}, \tau) = \Pr(h_{-i}^{[k]} | h_{-i}^{[k-1]}, \tau - 1)$ for all τ, and hence $\Delta_1(\tau) = 0$. *Q.E.D.*

Lemma 13 shows that if $h_{-i}^{[k-1]} \notin \overline{H}_{-i}^{[k-1]}$ or $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}(x,i)$ for some $x \in X^B$, then $W_i(\tau) - W_i(\tau - 1) = \Delta_1(\tau) + \Delta_2(\tau) = \Delta_2(\tau)$. Thus, from Lemma 12 and $V_i(h_{-i}^{[k-1]}) = W_i(0) = W_i(KT)$, $V_i(h_{-i}^{[k-1]}) > W_i(\tau)$ for all $\tau \in \{1, \cdots, KT-1\}$, as desired.

Likewise, using Lemmas 12 and 14, one can show that $V_i(h_{-i}^{[k-1]}) > W_i(\tau)$ for all $\tau \in \{1, \cdots, KT-1\}$ if $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}(x)$ for some $x \in X^B$, $a_i^* \neq a_i^{x,k}$, and $a_i^{**} \neq a_i^{x,k}$.

Therefore, it remains to consider the case of $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}(x)$ for some $x \in X^B$ and $a_i^* = a_i^{x,k}$ (the case of $a_i^{**} = a_i^{x,k}$ is analogous).

Lemma 15. *Suppose that $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}(x)$ for some $x \in X^B$ and $a_i^* = a_i^{x,k}$. If T is large enough, then $\Delta_1(\tau)$ is non-negative for all τ.*

Proof. It follows from Lemma 3 that playing the action a_i^* exactly τ times instead of $\tau - 1$ times in the kth main round decreases the probability that player $i-1$ sends the message $m_{i-1}^k = i$ and the probability that player $i+1$ sends the message $m_{i+1}^k = i$, while it increases the probability that player $i-1$ sends the message $m_{i-1}^k = i-2$ and the probability that player $i+1$ sends the message $m_{i+1}^k = i+2$. Then,

$$\sum_{h_{-i}^{[k]} \in H_{-i}^{[k]}(x,i)} \Pr(h_{-i}^{[k]} | h_{-i}^{[k-1]}, \tau) < \sum_{h_{-i}^{[k]} \in H_{-i}^{[k]}(x,i)} \Pr(h_{-i}^{[k]} | h_{-i}^{[k-1]}, \tau - 1)$$

and

$$\sum_{h_{-i}^{[k]} \notin \overline{H}_{-i}^{[k]}} \Pr(h_{-i}^{[k]} | h_{-i}^{[k-1]}, \tau) > \sum_{h_{-i}^{[k]} \notin \overline{H}_{-i}^{[k]}} \Pr(h_{-i}^{[k]} | h_{-i}^{[k-1]}, \tau - 1).$$

These inequalities show that $\Delta_1(\tau)$ is non-negative, since $V_i(h_{-i}^{[k]})$ takes the highest value when $h_{-i}^{[k]} \notin \overline{H}_{-i}^{[k]}$, the second highest value when $h_{-i}^{[k]} \in H_{-i}^{[k]}(x)$, and the lowest value when $h_{-i}^{[k]} \in H_{-i}^{[k]}(x,i)$. *Q.E.D.*

Lemma 16. *Suppose that $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}(x)$ for some $x \in X^B$ and $a_i^* = a_i^{x,k}$. Then, for any $\rho \in (0,1)$ and $n \geq 1$, $\sum_{\tau=1}^{[\rho KT]} \Delta_1(\tau) = o(T^{-n})$. Here, $[\rho KT]$ denotes the integer part of ρKT.*

Proof. Let $\Delta \Pr(m_{i-1}^k = i|\tau)$ denote the decrease in the probability of $m_{i-1}^k = i$ when player i chooses the action a_i^* τ times rather than $\tau - 1$ times in the kth main round. Likewise, let $\Delta \Pr(m_{i+1}^k = i|\tau)$ be the decrease in the probability of $m_{i+1}^k = i$, $\Delta \Pr(m_{i-1}^k = i-2|\tau)$ be the increase in the probability of $m_{i-1}^k = i-2$, and $\Delta \Pr(m_{i+1}^k = i+2|\tau)$ be the increase in the probability of $m_{i+1}^k = i+2$.

Following the proof of Lemma 15, $\Delta_1(\tau)$ is represented by

$$\begin{aligned}\Delta_1(\tau) = C_1(\tau)\Delta \Pr(m_{i-1}^k = i|\tau) + C_2(\tau)\Delta \Pr(m_{i+1}^k = i|\tau) \\ + C_3(\tau)\Delta \Pr(m_{i-1}^k = i-2|\tau) + C_4(\tau)\Delta \Pr(m_{i+1}^k = i+2|\tau). \quad (47)\end{aligned}$$

Here, $C_1(\tau)$ measures how much the expected value of $V_i(h_{-i}^{[k]})$ increases when the probability of $m_{i-1}^k = i$ decreases, while the probability of $m_{i+1}^k = i$, the probability of $m_{i-1}^k = i-2$, and the probability of $m_{i+1}^k = i+2$ are fixed (these probabilities are calculated as if a_i^* is chosen $\tau - 1$ times in the kth main round); $C_2(\tau)$ denotes how much the expected value of $V_i(h_{-i}^{[k]})$ increases when the probability of $m_{i+1}^k = i$ decreases, while the probabilities of $m_{i-1}^k = i$, the probability of $m_{i-1}^k = i-2$, and the probability of $m_{i+1}^k = i+2$ are fixed (the probability of $m_{i-1}^k = i$ is calculated as if a_i^* is chosen τ times, and the others are calculated as if a_i^* is chosen $\tau - 1$ times); and so on. By definition,

$$0 \leq C_n(\tau) \leq \Delta V_i \quad (48)$$

where $\Delta V_i = \max_{h_{-i}^{[k]} \in H_{-i}^{[k]}} V_i(h_{-i}^{[k]}) - \min_{h_{-i}^{[k]} \in H_{-i}^{[k]}} V_i(h_{-i}^{[k]})$.

For notational convenience, let $F_n'(\tau) = F_n(\tau, KT-1, Z_{KT}')$ and $F_n''(\tau) = F_n(\tau, KT-1, Z_{KT}'')$. Then, from Lemma 3

$$\begin{aligned}\Delta \Pr(m_{i-1}^k = i|\tau) = (q_2 - q_1)F_2''(KT - \tau), \\ \Delta \Pr(m_{i+1}^k = i|\tau) = (q_3 - q_2)F_1'(\tau - 1), \\ \Delta \Pr(m_{i-1}^k = i-2|\tau) = (q_2 - q_1)F_2'(KT - \tau),\end{aligned}$$

and

$$\Delta \Pr(m_{i+1}^k = i+2|\tau) = (q_3 - q_2)F_1''(\tau-1).$$

Substituting these and (48) into (47),

$$\Delta_1(\tau) \leq \Delta V_i \Delta q \left(F_2''(KT-\tau) + F_1'(\tau-1) + F_2'(KT-\tau) + F_1''(\tau-1)\right) \tag{49}$$

where $\Delta q = \max\{q_2 - q_1,\ q_3 - q_2\}$. To complete the proof, one needs to find a bound on the right-hand side. The following claims are useful to obtain such a bound. See Appendix C.5.5 for the proofs.

Claim 1. *For any $\rho \in [0,1)$ and $n \geq 1$, $F_1([\rho T]-1, T-1, Z_T') = o(T^{-n})$ as $T \to \infty$. Also, for any $\rho \in (0,1]$ and $n \geq 1$, $F_2(T-[\rho T], T-1, Z_T'') = o(T^{-n})$ as $T \to \infty$.*

Claim 2. *For any $\rho \in (0,1)$, there exists $\overline{T}$ such that for any $T > \overline{T}$ and any $\tau < [\rho KT]$, $F_2''(KT-\tau) \leq F_2''(KT-[\rho KT])$, $F_1'(\tau-1) \leq F_1'([\rho KT]-1)$, $F_2'(KT-\tau) \leq F_2'(KT-[\rho KT])$, and $F_1''(\tau-1) \leq F_1''([\rho KT]-1)$.*

Claim 3. *For any $\rho \in [0,1)$, $F_1([\rho T]-1, T-1, Z_T'') \leq F_1([\rho T]-1, T-1, Z_T')$ if T is large enough. Also, for any $\rho \in (0,1]$, $F_2(T-[\rho T], T-1, Z_T') \leq F_2(T-[\rho T], T-1, Z_T'')$ if T is large enough.*

Applying Claims 2 and 3 to (49),

$$\sum_{\tau=1}^{[\rho KT]} \Delta_1(\tau) \leq \Delta V_i \Delta q 2[\rho KT]\left(F_2''(KT-[\rho KT]) + F_1'([\rho KT]-1)\right).$$

Notice that $\Delta V_i = O(T)$, since $V_i(h_{-i}^{[k]}) = O(T)$. On the other hand, Claim 1 implies that $F_2''(KT-[\rho KT]) = o(T^{-n})$ and $F_1'([\rho KT]-1) = o(T^{-n})$. Therefore, $\sum_{\tau=1}^{[\rho KT]} \Delta_1(\tau) = o(T^{-n})$. *Q.E.D.*

Lemma 17. *There are $\overline{\rho} \in (0,1)$ and $\overline{T}$ such that for any $T > \overline{T}$ and $\tau > \overline{\rho} KT$, the value $\sum_{\tilde{\tau}=\tau}^{KT} \Delta_2(\tilde{\tau})$ is positive.*

Proof. Let $\Delta\pi_i$ denote the first line of the right-hand side of (46). Let $\overline{\rho} \in (1 + \frac{\eta}{\Delta\pi_i}, 1)$

if $\Delta\pi_i < -2\eta$, and let $\overline{\rho} = \frac{1}{2}$ otherwise. Note that

$$\begin{aligned}
&\frac{\sum_{\tilde{\tau}=\tau}^{KT} \Delta_2(\tilde{\tau})}{T} \\
=&\frac{KT - [\overline{\rho}KT] + 1}{T} \Delta\pi_i \\
&+ \sum_{\tau=[\overline{\rho}KT]}^{KT} (q_3 - q_2) F_1(KT-\tau) K\eta - \sum_{\tau=[\overline{\rho}KT]}^{KT} (q_3 - q_2) F_1(\tau - 1) K\eta \\
&- \sum_{I_{-i}^{k-1} \in \mathscr{I}_{-i}^{k-1}} \Pr(I_{-i}^{k-1} | h_{-i}^{[k-1]}) \sum_{l=1+l^{**}(I_{-i}^{k-1})}^{l^{*}(I_{-i}^{k-1})} \sum_{\tau=[\overline{\rho}KT]}^{KT} (q_3 - q_2) F_1(\tau - 1) \frac{\lambda^k(I_{-i}^{k-1}, l)}{T} \\
&+ \sum_{I_{-i}^{k-1} \notin \mathscr{I}_{-i}^{k-1}} \Pr(I_{-i}^{k-1} | h_{-i}^{[k-1]}) \sum_{l=1+l^{*}(I_{-i}^{k-1})}^{l^{**}(I_{-i}^{k-1})} \sum_{\tau=[\overline{\rho}KT]}^{KT} (q_3 - q_2) F_1(KT-\tau) \frac{\lambda^k(I_{-i}^{k-1}, l)}{T}.
\end{aligned}$$

From (45) and the law of large numbers,

$$\begin{aligned}
&\lim_{T\to\infty} \sum_{\tau=[\overline{\rho}KT]}^{KT} (q_3 - q_2) F(\tau - 1) \\
=&\lim_{T\to\infty} \Pr(1_{a_i^{**}} | h_{-i}^{[k-1]}, [\overline{\rho}KT]) - \lim_{T\to\infty} \Pr(1_{a_i^{**}} | h_{-i}^{[k-1]}, KT \\
=&0 - 0 \\
=&0.
\end{aligned}$$

Likewise, from (18), (45), and the law of large numbers,

$$\begin{aligned}
&\lim_{T\to\infty} \sum_{\tau=[\overline{\rho}KT]}^{KT} (q_3 - q_2) F(KT - \tau) \\
=&\lim_{T\to\infty} \Pr(1_{a_i^{*}} | h_{-i}^{[k-1]}, KT) - \lim_{T\to\infty} \Pr(1_{a_i^{*}} | h_{-i}^{[k-1]}, [\overline{\rho}KT]) \\
=&1 - 0 \\
=&1.
\end{aligned}$$

Substituting these and using $\lambda^k(I_{-i}^{k-1}, l) = O(T)$,

$$\begin{aligned}
\lim_{T\to\infty} \frac{\sum_{\tau=[\overline{\rho}KT]}^{KT} \Delta_2(\tau)}{T} =&(1-\overline{\rho}) K\Delta\pi_i + K\eta \\
&+ \sum_{I_{-i}^{k-1} \notin \mathscr{I}_{-i}^{k-1}} \lim_{T\to\infty} \Pr(I_{-i}^{k-1} | h_{-i}^{[k-1]}) \sum_{l=1+l^{*}(I_{-i}^{k-1})}^{l^{**}(I_{-i}^{k-1})} \lim_{T\to\infty} \frac{\lambda^k(I_{-i}^{k-1}, l)}{T}
\end{aligned}$$

Since $(1-\overline{\rho})K\Delta\pi_i + K\eta > 0$ and $\lim_{T\to\infty} \frac{\lambda^k(I_{-i}^{k-1}, l)}{T} \geq 0$, the right-hand side is positive. Therefore, $\sum_{\tau=[\overline{\rho}KT]}^{KT} \Delta_2(\tau) > 0$ for sufficiently large T. This, together with Lemma 12, completes the proof. *Q.E.D.*

Lemma 18. $\lim_{T\to\infty}\Delta_2(1) = -\infty$.

Proof. Lemma 7 of Yamamoto (2007) asserts that $F_1(KT-1) = o(T^{-1})$ as $T\to\infty$. On the other hand, $\lambda^k(I_{-i}^{k-1}, l) = O(T)$ as $T\to\infty$. Hence, the term in the last line of the right-hand side of (46) and the second term in the third line converge to zero. Meanwhile, the first term in the third line goes to infinity and the limit of the term in the fourth line is non-negative, since (19) holds and $\lim_{T\to\infty}\frac{\lambda^k(I_{-i}^{k-1},l)}{T} \geq 0$. Hence, $\lim_{T\to\infty}\Delta_2(1) = -\infty$. *Q.E.D.*

Let $\overline{\tau}$ be as in Lemma 12, and $\overline{\rho}$ be as in Lemma 17. From Lemmas 12, 15, and 17, if T is sufficiently large, then the value $\sum_{t=\tau}^{KT}(\Delta_1(t)+\Delta_2(t))$ is positive for all $\tau > \min\{\overline{\tau}, \overline{\rho}KT\}$, and hence $W_i(KT) > W_i(\tau)$ for all $\tau > \min\{\overline{\tau}, \overline{\rho}KT\}-1$. Also, using Lemmas 12, 16, and 18, one can show that if T is sufficiently large, then the value $\sum_{t=1}^{\tau}(\Delta_1(t)+\Delta_2(t))$ is negative for all $\tau \leq \min\{\overline{\tau}, \overline{\rho}KT\}$, implying $W_i(0) > W_i(\tau)$ for all $\tau < \min\{\overline{\tau}, \overline{\rho}KT\}$. Using $V_i(h_{-i}^{[k-1]}) = W_i(0) = W_i(KT)$, it follows that $V_i(h_{-i}^{[k-1]}) > W_i(\tau)$ for all $\tau \in \{1, \cdots, KT-1\}$, as desired.

So far it has been shown that player i prefers playing a constant action to mixing two actions. Since a similar argument shows that mixing n actions is better than mixing $n+1$ actions, player i is worse off by deviating from a constant action. This completes the proof.

C.5.3 Proof of Lemma 10

Part 1. Proof of Uniqueness. Notice that (39) is represented by the matrix form

$$Q\lambda^{-1} = b \tag{50}$$

where λ^{-1} is a column vector with elements $\lambda^{-1}(x)$ for all $x \in X^B$, Q is the coefficient matrix, the entry of which is $\sum_{h_{i-1}^{[-1]}\in H_{i-1}^{[-1]}(\tilde{x})}\Pr(h_{i-1}^{[-1]}|s^x)$ for $x\in X^B$ and $\tilde{x}\in X^B$, and b is a column vector representing the remaining terms, i.e., each element of b is

$$b(x) = \sum_{t=1}^{T_b}\delta^{t-1-NT}\underline{w}_i - 3T_b\eta - \delta^{6NT}\sum_{h_{-i}^{[0]}\in H_{-i}^{[0]}}\Pr(h_{-i}^{[0]}|s^x)V_i(h_{-i}^{[0]}). \tag{51}$$

From (17), (18), and the law of large numbers, the term $\sum_{h_{i-1}^{[-1]}\in H_{i-1}^{[-1]}(\tilde{x})}\Pr(h_{i-1}^{[-1]}|s^x)$ converges to one if $\tilde{x}=x$ and to zero otherwise, as $T\to\infty$. This implies that the matrix Q converges to the identity matrix as $T\to\infty$. Hence, for sufficiently large T, there is the inverse of Q, and (50) has a unique solution $\lambda^{-1} = Q^{-1}b$.

Part 2. Proof of (40). From (17) and (18), $\sum_{h_{-i}^{[0]} \in \overline{H}_{-i}^{[0]}} \Pr(h_{-i}^{[0]}|s^x)$ converges to one as $T \to \infty$ for each $x \in X$. By construction, this implies that

$$\lim_{T\to\infty}\lim_{\delta\to 1}\frac{1}{T_b}\sum_{h_{-i}^{[0]}\in H_{-i}^{[0]}}\Pr(h_{-i}^{[0]}|s^x)V_i(h_{-i}^{[0]}) = \lim_{\delta\to 1}\left[\max_{\tilde{x}_{-i}\in X_{-i}^B}\max_{s_i^{T_b}\in S_i^{T_b}} w_i^P(s_i^{T_b}, s_{-i}^{\tilde{x}_{-i}})\right]$$

for all $x \in X^B$. Then, from (51),

$$\lim_{T\to\infty}\lim_{\delta\to 1}\frac{b(x)}{T_b} = \underline{w}_i - 3\eta - \lim_{\delta\to 1}\left[\max_{\tilde{x}_{-i}\in X_{-i}^B}\max_{s_i^{T_b}\in S_i^{T_b}} w_i^P(s_i^{T_b}, s_{-i}^{\tilde{x}_{-i}})\right].$$

Plugging this into $\lambda^{-1} = Q^{-1}b$ and using (34),

$$\lim_{T\to\infty}\lim_{\delta\to 1}\frac{\lambda^{-1}(x)}{T_b} = \lim_{T\to\infty}\lim_{\delta\to 1}\frac{b(x)}{T_b} = \underline{w}_i - 3\eta - \lim_{\delta\to 1}\max_{s_i^{T_b}\in S_i^{T_b}} w_i^P(s_i^{T_b}, s_{-i}^{x_{-i}}) > 0$$

for each $x \in X^B$. Hence, (40) holds.

Part 3. Proof of (20). From (39), player i's average payoff in the auxiliary scenario from s^x is $\underline{w}_i$ for all $x \in X^B$. Therefore, it suffices to show that playing $s_i^{x_i}$ is a best reply against $s_{-i}^{x_{-i}}$. The next lemma shows that deviating to other actions in period $t \in \{(j-1)T+1, \cdots, jT\}$ for $j \neq i$ is not profitable.

Lemma 19. *For each $j \neq i$ and $t \in \{(j-1)T+1, \cdots, jT\}$, player i is indifferent among all actions in period t of the block game regardless of the past history.*

Proof. In period $t \in \{(j-1)T+1, \cdots, jT\}$, players attempt to receive the message from player j through the random events $(\psi_l(\{a_j^G\}))_{l\neq j}$. Since player i's action cannot affect whether these random events are counted, she is indifferent among all actions. *Q.E.D.*

It remains to consider deviations in period $t \in \{(i-1)T+1, \cdots, iT\}$. As shown in the following lemma, player i is indifferent between a_i^B and $a_i \neq a_i^G, a_i^B$ in these periods.

Lemma 20. *For each $t \in \{(i-1)T+1, \cdots, iT\}$ and $a_i \in A_i \setminus \{a_i^G, a_i^B\}$, player i is indifferent between a_i and a_i^B in period t of the block game independently of the past history.*

Proof. In period $t \in \{(i-1)T+1, \cdots, iT\}$, player j attempts to receive a message from player i through $\psi_j(\{a_i^G\})$, and both $a_i \in A_i \setminus \{a_i^G, a_i^B\}$ and a_i^B induce the same distribution of $\psi_j(\{a_i^G\})$. *Q.E.D.*

Thus it suffices to show that mixing a_i^G and a_i^B in the T-period interval from period $(i-1)T+1$ to period iT is not profitable. For each $x_{-i} \in X_{-i}^B$ and $\tau \in \{0, \cdots, T\}$, let

$W_i(s_{-i}^{x_{-i}},\tau)$ denote player i's (unnormalized) payoff in the auxiliary scenario against $s_{-i}^{x_{-i}}$ when player i follow a sequence

$$(\underbrace{a_i^B,\cdots,a_i^B}_{\tau},\underbrace{a_i^G,\cdots,a_i^G}_{T-\tau})$$

from period $(i-1)T+1$ to period iT, and chooses a best reply in other periods. That is, $W_i(s_{-i}^{x_{-i}},\tau)$ is defined as

$$W_i(s_{-i}^{x_{-i}},\tau) = \delta^{7NT} \sum_{h_{-i}^{[0]}\in H_{-i}^{[0]}} \Pr(h_{-i}^{[0]}|s_{-i}^{x_{-i}},\tau)V_i(h_{-i}^{[0]})$$
$$+\delta^{NT}\left[3T_b\eta + \sum_{\tilde{x}\in X^B}\sum_{h_{i-1}^{[-1]}\in H_{i-1}^{[-1]}(\tilde{x})} \Pr(h_{i-1}^{[-1]}|s_{-i}^{x_{-i}},\tau)\lambda^{-1}(\tilde{x})\right] \quad (52)$$

where $\Pr(h_{i-1}^{[-1]}|s_{-i}^{x_{-i}},\tau)$ and $\Pr(h_{-i}^{[0]}|s_{-i}^{x_{-i}},\tau)$ denote the probability of $h_{i-1}^{[-1]}$ and $h_{-i}^{[0]}$.

Lemma 21. *When $\delta=1$, there is $\overline{T}>0$ such that for all $T>\overline{T}$, $x_{-i}\in X_{-i}^B$, and $\tau\in\{1,\cdots,T-1\}$,*

$$\sum_{t=1}^{T_b}\delta^{t-1}\underline{w}_i > W_i(s_{-i}^{x_{-i}},\tau). \quad (53)$$

Proof. See Appendix C.5.4. *Q.E.D.*

This lemma asserts that player i is worse off by mixing a_i^G and a_i^B in the ith T-period interval of the signaling round (and by taking a best reply in other periods), when δ equals one. By continuity, the result remains true as long as δ is close to one.

C.5.4 Proof of Lemma 21

Use (52) to get

$$W_i(s_{-i}^{x_{-i}},\tau) - W_i(s_{-i}^{x_{-i}},\tau-1)$$
$$= \sum_{h_{-i}^{[0]}\in H_{-i}^{[0]}} V_i(h_{-i}^{[0]})\left[\Pr(h_{-i}^{[0]}|s_{-i}^{x_{-i}},\tau) - \Pr(h_{-i}^{[0]}|s_{-i}^{x_{-i}},\tau-1)\right]$$
$$+\sum_{\tilde{x}\in X^B}\sum_{h_{i-1}^{[-1]}\in H_{i-1}^{[-1]}(\tilde{x})}\lambda^{-1}(\tilde{x})\left[\Pr(h_{i-1}^{[-1]}|s_{-i}^{x_{-i}},\tau) - \Pr(h_{i-1}^{[-1]}|s_{-i}^{x_{-i}},\tau-1)\right]$$

Without loss of generality, consider a particular $x_{-i}\in X_{-i}^B$, and let $\Delta_3(\tau)$ denote the first term of the right-hand side and $\Delta_4(\tau)$ the second term.

Lemma 22. *For any $n\geq 1$, $\max_{\tau\in\{1,\cdots,T\}}|\Delta_3(\tau)| = o(T^{-n})$ as $T\to\infty$.*

Proof. Observe that $V_i(h_{-i}^{[0]})$ is constant for all $h_{-i}^{[0]} \in \overline{H}_{-i}^{[0]}$, and takes a higher value for $h_{-i}^{[0]} \notin \overline{H}_{-i}^{[0]}$. Also, $V_i(h_{-i}^{[0]}) = O(T)$ as $T \to \infty$. Hence, it suffices to show that

$$\max_{\tau \in \{1,\cdots,T\}} \sum_{h_{-i}^{[0]} \notin \overline{H}_{-i}^{[0]}} \left| \Pr(h_{-i}^{[0]}|\tau) - \Pr(h_{-i}^{[0]}|\tau-1) \right| = o(T^{-n}) \tag{54}$$

The following claims are useful. The proofs are found in Appendix C.5.5.

Claim 4. *For any $n \geq 1$, $j \in I$, and $x_j \in X_j$, if player $j \in I$ sends x_j in the signaling round, then the probability of $\hat{x}_j(h_l^t) \neq x_j$ for some $l \neq j$ is $o(T^{-n})$ as $T \to \infty$.*

Claim 5. *For any $n \geq 1$, $j \in I$, and $m_j^0 \in M_j^0$, if player $j \in I$ sends m_j^0 in the confirmation round, then the probability of $\hat{m}_j^0(h_l^t) \neq m_j^0$ for some $l \neq j$ is $o(T^{-n})$ as $T \to \infty$.*

Claim 6. *Let $P(\tau)$ denote the probability that $\hat{x}_i(h_j^t) = G$ and $\hat{x}_i(h_l^t) = B$ for some $j \neq i$ and $l \neq i, j$ when player i chooses a_i^B τ times and a_i^G $T-\tau$ times during the ith T-period interval. Then, for any $n \geq 1$, $\max_{\tau \in \{1,\cdots,T\}} P(\tau) = o(T^{-n})$ as $T \to \infty$.*

By definition, if players $-i$ make correct inferences on each other's message in the signaling round and the confirmation round, and if there is no pair (j,l) such that $\hat{x}_i(h_j^t) = G$ and $\hat{x}_i(h_l^t) = B$, then $h_{-i}^{[0]} \in \overline{H}_{-i}^{[0]}$. Thus it follows from Claims 4 through 6 that the probability of $h_{-i}^{[0]} \in \overline{H}_{-i}^{[0]}$ is $1 - o(T^{-n})$ irrespective of player i's play in the ith T-period interval. This proves (54). *Q.E.D.*

Lemma 23. *There is $\overline{T}$ such that for any $T > \overline{T}$, there is $\overline{\tau}$ such that $\Delta_4(\tau)$ is negative for all $\tau \leq \overline{\tau}$ and is positive for all $\tau > \overline{\tau}$.*

Proof. Analogous to the proof of Lemma 12. *Q.E.D.*

Lemma 24. $\lim_{T\to\infty} \Delta_4(1) = -\infty$, *and* $\lim_{T\to\infty} \Delta_4(T) = \infty$.

Proof. Analogous to the proof of Lemma 18. *Q.E.D.*

Lemmas 22 through 24, together with $W_i(s_{-i}^{x_{-i}}, 0) = W_i(s_{-i}^{x_{-i}}, T) = \sum_{t=1}^{T_b} \delta^{t-1} \underline{w}_i$, establish (53).

C.5.5 Proofs of Claims

The following two claims are useful to prove other claims.

Claim 7. *$F_1(\tau, T, r)$ is single-peaked with respect to τ and r. Also, $F_2(\tau, T, r)$ is single-peaked with respect to τ and r. Here, a function $h(\tau)$ is single-peaked with respect to τ if $h(\tau) \geq h(\tau+1)$ implies $h(\tau+1) \geq h(\tau+2)$.*

Proof. Follows from Lemma 5 of Yamamoto (2007). *Q.E.D.*

Claim 8. *For any* $\rho \in (0,1)$ *satisfying* $\rho \neq q_3$ *and* $n \geq 1$, $F_1(0, T-1, [\rho(T-1)]) = o(T^{-n})$ *as* $T \to \infty$.

Proof. This is a trivial extension of Lemma 6 of Yamamoto (2007). To get the result, just replace T^2 with T^n. *Q.E.D.*

Proof of Claim 1. Using Claim 8, one can show that $F_1([\rho T]-1, T-1, Z'_T) = o(T^{-n})$ for any $\rho \in [q_2, 1)$, as in the proof of Lemma 7 of Yamamoto (2007). However, the proof of Yamamoto (2007) does not work for $\rho \in [0, q_2)$, since $TF_1(T-1, T-1, Z'_T)$ may not go to infinity. An alternative proof is as follows.

Let $f_T(\tau) = [q_2(\tau+1)] + [q_3(T-\tau)]$. Since $f_T(T-1) = 0$, $f_T(0) = q_3 T$, and $f_T(\tau) + 1 \geq f_T(\tau - 1)$ for all τ, and (18) holds, there is a sequence of integers $(\tau_T)_{T=1}^{\infty}$ such that $f_T(\tau_T) = Z'_T$ when T is large enough. By definition,

$$[q_2(\tau_T+1)] + [q_3(T-\tau_T)] = Z'_T \tag{55}$$

for sufficiently large T. Dividing both sides by T and applying (18),

$$\lim_{T\to\infty} \frac{\tau_T}{T} = 1. \tag{56}$$

It follows from (18) and (56) that for sufficiently large T, $[q_3(T-\tau_T)] < Z'_T$. Then.

$$\begin{aligned}
T^2 F_1(\tau_T, T-1, Z'_T) &= T^2 \sum_{r=0}^{Z'_T} F_1(0, T-1-\tau_T, r) F_1(\tau_T, \tau_T, Z'_T - r) \\
&\geq T^2 F_1(0, T-1-\tau_T, [q_3(T-\tau_T)]) F_1(\tau_T, \tau_T, Z'_T - [q_3(T-\tau_T)]) \\
&= T^2 F_1(0, T-1-\tau_T, [q_3(T-\tau_T)]) F_1(\tau_T, \tau_T, [q_2(\tau_T+1)])
\end{aligned}$$

for sufficiently large T. Here, the last equality comes from (55). Observe that $F_1(0, T, r)$ is maximized by $r = [q_3(T+1)]$, since $F_1(0, T, r) \geq F_1(0, T, r-1)$ if and only if $r \leq q_3(T+1)$. Likewise, $F_1(T, T, r)$ is maximized by $r = [q_2(T+1)]$. Therefore, $[q_3(T-\tau_T)]$ and $[q_2(\tau_T+1)]$ are the maximizers of $F_1(0, T-1-\tau_T, r)$ and $F_1(\tau_T, \tau_T, r)$, respectively. Thus $F_1(0, T-1-\tau_T, [q_3(T-\tau_T)]) \geq \frac{1}{T}$ and $F_1(\tau_T, \tau_T, [q_2(\tau_T+1)]) \geq \frac{1}{T}$. Plugging these into the above inequality,

$$T^2 F_1(\tau_T, T-1, Z'_T) \geq 1. \tag{57}$$

Recall that $F_1([\rho T]-1, T-1, Z'_T) = o(T^{-n})$ for any $\rho \in [q_2, 1)$, and hence $F_1([q_2 T]-1, T-1, Z'_T) = o(T^{-n})$. Noting that (56) and (57) hold, it follows that for sufficiently large T, $F_1([q_2 T]-1, T-1, Z'_T) < F_1(\tau_T, T-1, Z'_T)$ and $[q_2 T] - 1 < \tau_T$. Then, from Claim 7, $F_1([\rho T]-1, T-1, Z'_T) \leq F_1([q_2 T]-1, T-1, Z'_T)$ for each $\rho \in [0, q_2)$. This establishes that $F_1([\rho T]-1, T-1, Z'_T) = o(T^{-n})$, since $F_1([q_2(T-1)], T-1, Z'_T) = o(T^{-n})$. The proof of $F_2(T-[\rho T], T-1, Z''_T) = o(T^{-n})$ is analogous. *Q.E.D.*

Proof of Claim 2. As shown in the proof of Claim 1, there exists $(\tau_T)_{T=1}^{\infty}$ such that $F_1([\rho T]-1, T-1, Z'_T) < F_1(\tau_T, T-1, Z'_T)$ and $\tau_T > [\rho T]-1$, provided that T is large enough. Then, it follows from Claim 7 that when T is large enough, $F'_1(\tau-1) \leq F'_1([\rho KT]-1)$ for all $\tau < [\rho KT]$. The remaining inequalities follow from a similar argument. *Q.E.D.*

Proof of Claim 3. Let $r_T = [q_3(T-1-[\rho T])] + [q_2[\rho T]]$. Then, from (18), $r_T > Z'_T$ for sufficiently large T. Also, one can show that $T^2 F_1([\rho T]-1, T-1, r_T) \geq 1$ for sufficiently large T. (The proof is omitted, since this follows from a similar reason for $T^2 F_1(\tau_T, T-1, Z'_T) \geq 1$ in the proof of Claim 1.) Then, (15) and Claims 7 and 1 give the desired inequality, $F_1([\rho T]-1, T-1, Z''_T) \leq F_1([\rho T]-1, T-1, Z'_T)$, for sufficiently large T. A similar argument applies to F_2. *Q.E.D.*

Proof of Claim 4. For each $j \neq i$ and $l \neq i, j$, let $\Pr(\hat{x}_j(h_l^t) \neq B | x_j = B)$ be the probability of $\hat{x}_j(h_l^t) \neq B$ when player j sends $x_j = B$. In what follows, it is shown that $\Pr(\hat{x}_j(h_l^t) \neq B | x_j = B) = o(T^{-n})$.

From Claim 8,

$$F_1(0, T, [\frac{2q_2+q_3}{3}T]) = o(T^{-n}) \tag{58}$$

for $n \geq 1$. Since Claim 7 asserts that $F_1(0,T,r)$ is single-peaked with respect to r, either $F_1(0,T,r) \leq F_1(0,T,[\frac{2q_2+q_3}{3}T])$ for all $r > [\frac{2q_2+q_3}{3}T]$, or $F_1(0,T,r) \leq F_1(0,T,[\frac{2q_2+q_3}{3}T])$ for all $r < [\frac{2q_2+q_3}{3}T]$. If $F_1(0,T,r) \leq F_1(0,T,[\frac{2q_2+q_3}{3}T])$ for all $r < [\frac{2q_2+q_3}{3}T]$, then from (58), $\sum_{r<[\frac{2q_2+q_3}{3}T]} F_1(0,T,r) \leq T F_1(0,T,[\frac{2q_2+q_3}{3}T]) = o(T^{-n})$, but the law of large numbers and Lemma 3 assures that $\lim_{T\to\infty} \sum_{r<[\frac{2q_2+q_3}{3}T]} F_1(0,T,r) = 1$; a contradiction. Therefore, $F_1(0,T,r) \leq F_1(0,T,[\frac{2q_2+q_3}{3}T])$ for all $r > [\frac{2q_2+q_3}{3}T]$. This, together with (58), shows that

$$\begin{aligned}\Pr(\hat{x}_j(h_l^t) \neq B | x_j = B) &= \sum_{r>[\frac{2q_2+q_3}{3}T]} F_1(0,T,r) \\ &< (T - [\tfrac{2q_2+q_3}{3}T]) F_1(0,T,[\tfrac{2q_2+q_3}{3}T]) = o(T^{-n}).\end{aligned}$$

A similar argument applies to $x_j = G$. *Q.E.D.*

Proof of Claim 5. Analogous to the proof of Claim 4. *Q.E.D.*

Proof of Claim 6. Let $\Pr(\hat{x}_i(h_j^t) = G | \tau)$ be the probability of $\hat{x}_i(h_j^t) = G$ when player i chooses a_i^B τ times and a_i^G $T-\tau$ times. Also, let $\Pr(\hat{x}_i(h_l^t) = B | \tau)$ denote the probability of $\hat{x}_i(h_l^t) = B$. It suffices to show that $\max_{\tau \geq [\frac{1}{2}T]} \Pr(\hat{x}_i(h_j^t) = G | \tau) = o(T^{-n})$ and $\max_{\tau \leq [\frac{1}{2}T]} \Pr(\hat{x}_i(h_l^t) = B | \tau) = o(T^{-n})$.

As in the proof of Claim 1, one can show that $F_1([\frac{1}{2}T], T, [\frac{q_2+2q_3}{3}T]) = o(T^{-n})$, and that there exists a sequence of integers $(\tau_T)_{T=1}^{\infty}$ such that when T is large enough,

$F_1([\frac{1}{2}T],T,[\frac{q_2+2q_3}{3}T]) < F_1(\tau_T,T,[\frac{q_2+2q_3}{3}T])$ and $\tau_T < [\frac{1}{2}T]$. Then, it follows from Claim 7 that $F_1([\frac{1}{2}T],T,[\frac{q_2+2q_3}{3}T]) \geq F_1(\tau,T,[\frac{q_2+2q_3}{3}T])$ for all $\tau \geq [\frac{1}{2}T]$, provided that T is large enough. Meanwhile, as in the proof of Claim 1, one can show that $T^2F_1(\tau,T,[q_2(\tau+1)]+[q_3(T-\tau+1)]) \geq 1$ for each $\tau \geq [\frac{1}{2}T]$. Then, from Claim 7,

$$F_1([\tfrac{1}{2}T],T,[\tfrac{q_2+2q_3}{3}T]) \geq F_1(\tau,T,[\tfrac{q_2+2q_3}{3}T]) \geq F_1(\tau,T,r)$$

for all $\tau \geq [\frac{1}{2}T]$ and $r > [\frac{q_2+2q_3}{3}T]$, provided that T is large enough. Therefore,

$$\begin{aligned}\max_{\tau\geq[\frac{1}{2}T]} \Pr(\hat{x}_i(h_j^t)=G|\tau) &= \max_{\tau\geq[\frac{1}{2}T]} \sum_{r>[\frac{q_2+2q_3}{3}T]} F_1(\tau,T,r) \\ &< (T-[\tfrac{q_2+2q_3}{3}T])F_1([\tfrac{1}{2}T],T,[\tfrac{q_2+2q_3}{3}T]) = o(T^{-n})\end{aligned}$$

for all $n \geq 1$. A similar argument shows that

$$\max_{\tau\leq[\frac{1}{2}T]} \Pr(\hat{x}_i(h_l^t)=B|\tau) = o(T^{-n})$$

for all $n \geq 1$. *Q.E.D.*

Appendix D: Proof of Lemma 5

Let $u_i : A_{i-1}\times\Omega_{i-1} \to \boldsymbol{R}$ and $\overline{u}_i$ be as in the proof of Lemma 4. Without loss of generality, consider a particular $i \in I$. To simplify the notation, let X^G be the set of all $x \in X$ satisfying $x_{i-1} = G$, and X_{-i}^G be the set of all $x_{-i} \in X_{-i}$ satisfying $x_{i-1} = G$. Let $\overline{H}_{-i}^{[0]}$ be a union of $H_{-i}^{[0]}(x)$ over all $x \in X^G$ (see the proof of Lemma 4 for the specification of $H_{-i}^{[0]}(x)$). Also, for each $k \in \{1,\cdots,K-1\}$, let $\overline{H}_{-i}^{[k]}$ be the union of $(H_{-i}^{[k]}(x) \cup H_{-i}^{[k]}(x,i))$ over all $x \in X^G$ (again, see the proof of Lemma 4 for the specification of $H_{-i}^{[k]}(x)$ and $H_{-i}^{[k]}(x,i)$).

Suppose that U_i^G is decomposable into real-valued functions $(\theta^{-1},\cdots,\theta^{K+1})$ as in the proof of Lemma 4. In what follows, the transfers $(\theta^{-1},\cdots,\theta^{K+1})$ are specified so that (23) through (25) hold.

Let θ^0 and θ^{K+1} be as in the proof of Lemma 4, i.e., these transfers are the discounted sums of u_i. Then, player i is indifferent among all actions in periods of the confirmation round and the report round.

The transfers $(\theta^1,\cdots,\theta^K)$ are specified by backward induction. To define θ^k, assume that the transfers $(\theta^{k+1},\cdots,\theta^{K+1})$ are determined so that player i's continuation payoff after history $h_{-i}^{[k]} \in H_{-i}^{[k]}$, augmented by $(\theta^{k+1},\cdots,\theta^{K+1})$, is equal to $V_i(h_{-i}^{[k]})$, and that (23) holds for all $l \geq 1+6N+k(1+2N)$, $h^{t_l} \in H^{t_l}$, and $x \in X^G$. Here, for each $k \in \{1,\cdots,K\}$ and $h_{-i}^{[k]} \in \overline{H}_{-i}^{[k]}$, the value $V_i(h_{-i}^{[k]})$ denotes the minimum of player i's continuation payoff after history $h_{-i}^{[k]}$ over all her continuation strategies consistent with

$\mathscr{S}_i^{T_b}$ (i.e., continuation strategies that play some $a_i \in \mathscr{A}^k$ constantly in the kth main round for each k) subject to the constraints that monitoring is perfect and that payoffs in the communication stages are replaced with zero. For each $h_{-i}^{[0]} \in \overline{H}_{-i}^{[0]}$, the value $V_i(h_{-i}^{[0]})$ denotes the minimum of player i's continuation payoff after history $\tilde{h}_{-i}^{[0]}$ over all $\tilde{h}_{-i}^{[0]} \in \overline{H}_{-i}^{[0]}$ and over all her continuation strategies consistent with $\mathscr{S}_i^{T_b}$, subject to the constraints that monitoring is perfect and that payoffs in the communication stages are replaced with zero. For each $k \in \{0, \cdots, K\}$ and $h_{-i}^{[k]} \notin \overline{H}_{-i}^{[k]}$, the value $V_i(h_{-i}^{[k]})$ denotes player i's continuation payoff when she earns $\min_{a \in A} \pi_i(a)$ in periods of the main rounds and zero in the other periods.

In what follows, it is shown that there is θ^k such that player i's continuation payoff after $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}$ is equal to $V_i(h_{-i}^{[k-1]})$, and such that (23) holds for all $l \geq 1 + 6N + (k-1)(1+2N)$, $h^{t_l} \in H^{t_l}$, and $x \in X^G$. Notice that the transfers $(\theta^1, \cdots, \theta^K)$ can be specified by iterating this argument, as in the proof of Lemma 4.

Suppose that θ^k is decomposed as in (33). Then, on the analogy of Lemma 8, (23) holds for all $l \geq 2 + 6N + (k-1)(1+2N)$.

To specify the real-valued function $\tilde{\theta}^k$, the following notation is useful. For each $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}$ and $a_i \in A_i$, let $\tilde{W}_i(h_{-i}^{[k-1]}, a_i)$ denote player i's continuation payoff from the kth main round, augmented by $(\theta^{k+1}, \cdots, \theta^{K+1})$ and by the second term of (33), when player i plays a_i constantly in the kth main round and plays a best reply thereafter. As in the proof of Lemma 4, one can write $\tilde{W}_i(J_{-i}^{k-1}, a_i)$ instead of $\tilde{W}_i(h_{-i}^{[k-1]}, a_i)$.

For each J_{-i}^{k-1}, let $(a_i^1(J_{-i}^{k-1}), \cdots, a_i^{|\mathscr{A}_i^k|}(J_{-i}^{k-1}))$ be a ordering of all elements of $\mathscr{A}_i^k$ such that

$$\lim_{T\to\infty} \lim_{\delta\to 1} \frac{\tilde{W}_i(J_{-i}^{k-1}, a_i^1(J_{-i}^{k-1}))}{T} \geq \cdots \geq \lim_{T\to\infty} \lim_{\delta\to 1} \frac{\tilde{W}_i(J_{-i}^{k-1}, a_i^{|\mathscr{A}_i^k|}(J_{-i}^{k-1}))}{T}.$$

For each J_{-i}^{k-1} and $l \in \{1, \cdots, |\mathscr{A}_i^k|\}$, let $1_{[J_{-i}^{k-1}, l]} : H_{i-1}^{[k,m]} \to \{0,1\}$ be the indicator function such that $1_{[J_{-i}^{k-1}, l]}(h_{i-1}^{[k,m]}) = 1$ if the random event $\psi_{i-1}(\{a_i^l(J_{-i}^{k-1}), \cdots, a_i^{|\mathscr{A}_i^k|}(J_{-i}^{k-1})\})$ is counted more than Z_{KT} times in the kth main round (according to history $h_{i-1}^{[k,m]}$) and $1_{[J_{-i}^{k-1}, l]}(h_{i-1}^{[k,m]}) = 0$ otherwise. For each $a_i \in A_i$, let $1_{a_i} : H_{i-1}^{[k,m]} \to \{0,1\}$ be the indicator function such that $1_{a_i}(h_{i-1}^{[k,m]}) = 1$ if and only if the random event $\psi_{i-1}(\{a_i\})$ is counted more than Z_{KT} times in the kth main round (according to history $h_{i-1}^{[k,m]}$).

Since (14) holds, there is a positive number $\eta > 0$ satisfying

$$\left(1 - \frac{K^2 T}{T_b}\right)(|A_i| + 2)\overline{u}_i < (|A_i| + 2)\eta < \lim_{\delta\to 1} \min_{s_i^{T_b} \in \mathscr{S}_i^{T_b}} w_i^P(s_i^{T_b}, s_{-i}^{x_{-i}}) - \overline{w}_i$$

for all $x_{-i} \in X_{-i}^G$. Let C be a real number satisfying

$$C > \max_{a \in A} \pi_i(a) - \min_{a \in A} \pi_i(a),$$

and let $\tilde{\theta}^k$ be such that

$$\tilde{\theta}^k(h_{i-1}^{[k,m]}, I_{-i}^{k-1}) = -T_b C + \sum_{a_i \in \mathscr{A}_i^k} 1_{a_i}(h_{i-1}^{[k,m]}) KT\eta + \sum_{l=1}^{|\mathscr{A}_i^k|} 1_{[I_{-i}^{k-1},l]}(h_{i-1}^{[k,m]}) \lambda^k(I_{-i}^{k-1}, l),$$

where the values $(\lambda^k(I_{-i}^{k-1}, l))_{I_{-i}^{k-1},l}$ solve

$$V_i(h_{-i}^{[k-1]}) = \tilde{W}_i(h_{-i}^{[k-1]}, a_i) - \delta^{KT+2NT} T_b C + \delta^{KT+2NT} \sum_{\tilde{a}_i \in \mathscr{A}_i^k} \Pr(1_{\tilde{a}_i} | h_{-i}^{[k-1]}, a_i) KT\eta$$

$$+ \delta^{KT+2NT} \sum_{I_{-i}^{k-1}} \Pr(I_{-i}^{k-1} | h_{-i}^{[k-1]}) \sum_{l=1}^{|\mathscr{A}_i^k|} \Pr(1_{[I_{-i}^{k-1},l]} | h_{-i}^{[k-1]}, a_i) \lambda^k(I_{-i}^{k-1}, l) \quad (59)$$

for all $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}$ and $a_i \in \mathscr{A}_i^k$. Note that the right-hand side of (59) denotes player i's continuation payoff after $h_{-i}^{[k-1]}$, augmented by $(\theta^k, \cdots, \theta^{K+1})$, when player i chooses $a_i \in \mathscr{A}_i^k$ constantly in the kth main round and plays a best reply thereafter. The next lemma assures that the above θ^k satisfies the desired property.

Lemma 25. *There is $\overline{T}$ such that for all $T > \overline{T}$, there is $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$, system (59) has a unique solution, and it satisfies*

$$-\frac{\overline{w}_i - \underline{w}_i}{(K+3)(1-\delta)} < \theta^k(h_{i-1}^{[k]}, I_{-i}^{k-1}) < 2NT\overline{u}_i + |\mathscr{A}_i^k| KT\eta$$

for all $h_{i-1}^{[k]}$ and I_{-i}^{k-1}. Also, under this transfer scheme, (23) holds for all $l \geq 1 + 6N + (k-1)(1+2N)$, $h^{t_l} \in H^{t_l}$, and $x \in X^G$, and and player i's continuation payoff after history $h_{-i}^{[k-1]} \in H_{-i}^{[k-1]}$ equals $V_i(h_{-i}^{[k-1]})$.

Proof. Analogous to the proof of Lemma 9. *Q.E.D.*

Let θ^{-1} be such that

$$\theta^{-1}(h_{i-1}^{[-1]}) = -T_b \tilde{C} + \sum_{t=1}^{NT} \frac{u_i(a_{i-1}^t, \omega_{i-1}^t)}{\delta^{NT-t}} + \sum_{x \in X^G} 1_x(h_{i-1}^{[-1]}) \lambda^{-1}(x)$$

where

$$\tilde{C} = \eta + \lim_{\delta \to 1} \left(\min_{x_{-i} \in X_{-i}^G} \min_{s_i^{T_b} \in \mathscr{S}_i^{T_b}} w_i^P(s_i^{T_b}, s_{-i}^{x_{-i}}) - \overline{w}_i \right),$$

$(a_{i-1}^t, \omega_{i-1}^t)$ is player $(i-1)$'s private history in the tth period of the block game, and the values $(\lambda^{-1}(x))_{x \in X^G}$ solve

$$\sum_{t=1}^{T_b} \delta^{t-1} \underline{w}_i = \delta^{NT} \left[-T_b \tilde{C} + \sum_{\tilde{x} \in X^G} \sum_{h_{i-1}^{[-1]} \in H_{i-1}^{[-1]}(\tilde{x})} \Pr(h_{i-1}^{[-1]} | s^x) \lambda^{-1}(\tilde{x}) \right]$$

$$+ \delta^{7NT} \sum_{h_{-i}^{[0]} \in H_{-i}^{[0]}} \Pr(h_{-i}^{[0]} | s^x) V_i(h_{-i}^{[0]}) \quad (60)$$

for all $x_{-i} \in X^G_{-i}$. Note that the right-hand side of (60) denotes player i's auxiliary payoff from the block strategy profile s^x.

Lemma 26. *There is $\overline{T}$ such that for all $T > \overline{T}$, there is $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$, system (60) has a unique solution, and it satisfies*

$$\frac{\overline{w}_i - \underline{w}_i}{(K+3)(1-\delta)} < \theta^{-1}(h^{[-1]}_{i-1}) < NT\overline{u}_i - (|A_i|+1)T_b\eta.$$

for all $h^{[-1]}_{i-1} \in H^{[-1]}_{i-1}$. Also, under this transfer scheme, (23) and (24) hold for all $l \geq 0$, $h^{t_l} \in H^{t_l}$, and $x \in X^G$.

Proof. Analogous to the proof of Lemma 10. *Q.E.D.*

This lemma asserts that the specified U^G_i satisfies (23) and (24). Finally, (25) follows as in the proof of Lemma 4.

Appendix E: Proof of Proposition 3

In this appendix, we prove Proposition 3. Let $\mathscr{A} \in \mathscr{J}$ be such that $\mathscr{A}_i = \{a^*_i, a^{**}_i\}$ for each $i \in I$. Then by assumption, one can check that $\underline{v}_i(\mathscr{A}) \leq \max_{a_i \in A_i} \pi_i(a_i, a^{**}_{-i}) < \pi_i(a^*) \leq \overline{v}_i(\mathscr{A})$. Let $p \in \triangle\mathscr{J}$ be such that $p(\mathscr{A}) = 1$ and $p(\tilde{\mathscr{A}}) = 0$ for $\tilde{\mathscr{A}} \neq \mathscr{A}$. Then by definition, $\pi(a^*) \in V(p)$. Also, $\pi(a^*) \in \times_{i \in I}[p \cdot \underline{v}_i, p \cdot \overline{v}_i]$, since $\underline{v}_i(\mathscr{A}) < \pi_i(a^*) \leq \overline{v}_i(\mathscr{A})$. Therefore, $\pi(a^*)$ is an element of the right-hand side of (2).

It remains to check that the stage game corresponds to the positive case. Let $\tilde{p} \in \triangle\mathscr{J}$ be such that $\tilde{p}(\mathscr{A}) = 1 - \varepsilon$, $\tilde{p}(A) = \varepsilon$, and $\tilde{p}(\tilde{\mathscr{A}}) = 0$ for other $\tilde{\mathscr{A}}$, where $\varepsilon > 0$. It suffices to show that for a sufficiently small $\varepsilon > 0$, the intersection of $V(\tilde{p})$ and $\times_{i \in I}[\tilde{p} \cdot \underline{v}_i, \tilde{p} \cdot \overline{v}_i]$ is N-dimensional.

Since $\underline{v}_i(\mathscr{A}) < \pi_i(a^*) \leq \overline{v}_i(\mathscr{A})$ and $\pi_i(a^{**}) < \pi_i(a^*)$ for all $i \in I$, there is a payoff vector v such that v is a convex combination of $\pi(a^{**})$ and $\pi(a^*)$ and $p \cdot \underline{v}_i < v_i < p \cdot \overline{v}_i$ for all $i \in I$. By definition, this payoff vector v is in the feasible payoff set, and is an element of $V(p)$. Therefore, v is an element of $V(\tilde{p})$. Also, since $\tilde{p} \cdot \underline{v}_i$ and $\tilde{p} \cdot \overline{v}_i$ converge to $p \cdot \underline{v}_i$ and $p \cdot \overline{v}_i$ as $\varepsilon \to 0$, v is an interior point of $\times_{i \in I}[\tilde{p} \cdot \underline{v}_i, \tilde{p} \cdot \overline{v}_i]$ for a sufficiently small ε. Fix such a ε.

Recall that the feasible payoff set is full dimensional, and so is the set $V(\tilde{p})$. Let $\tilde{v}$ be an interior point of $V(\tilde{p})$, and let $\hat{v} = \kappa v + (1-\kappa)\tilde{v}$ for $\kappa \in (0,1)$. Since v is an element of $V(\tilde{p})$ and $\tilde{v}$ is an interior point of $V(\tilde{p})$, $\hat{v}$ is an interior point of $V(\tilde{p})$. In addition, $\hat{v}$ is an interior point $\times_{i \in I}[\tilde{p} \cdot \underline{v}_i, \tilde{p} \cdot \overline{v}_i]$ for κ sufficiently close to one, since v is an interior point of $\times_{i \in I}[\tilde{p} \cdot \underline{v}_i, \tilde{p} \cdot \overline{v}_i]$. These facts show that $\hat{v}$ is an interior point of the intersection of $V(\tilde{p})$ and $\times_{i \in I}[\tilde{p} \cdot \underline{v}_i, \tilde{p} \cdot \overline{v}_i]$. Hence, this intersection is N-dimensional.

Appendix F: Characterizing $E(\delta)$ for the Abnormal Case

Theorem 1 characterizes the limit set of belief-free review-strategy equilibrium payoffs for the positive, negative, and empty cases, but it does not apply to the abnormal case. In this appendix, we show that in the abnormal case, the equilibrium payoff set is either empty or the convex hull of the set of pure-strategy Nash equilibrium payoffs of the stage game, for generic payoff functions.

Given a stage game, let $\mathscr{J}^*$ be the maximal set $\mathscr{J}' \subseteq \mathscr{J}$ such that there is $p \in \triangle \mathscr{J}$ such that

(i) $p(\mathscr{A}) = 0$ for all $\mathscr{A} \notin \mathscr{J}'$;

(ii) $p\bar{v}_i \geq p\underline{v}_i$ for all $i \in I$;

(iii) $\pi_i(a) \geq \pi_i(\tilde{a}_i, a_{-i})$ for all $i \in I$ satisfying $p\bar{v}_i = p\underline{v}_i$, for all $\mathscr{A} \in \mathscr{J}'$, for all $a \in \mathscr{A}$, and for all $\tilde{a}_i \in A_i$ with equality if $\tilde{a}_i \in \mathscr{A}_i$; and

(iv) $\pi_i(a) = \pi_i(\tilde{a})$ for all $i \in I$ satisfying $p\bar{v}_i = p\underline{v}_i$, for all $\mathscr{A} \in \mathscr{J}'$, for all $a \in \mathscr{A}$, and for all $\tilde{a} \in \mathscr{A}$.

Notice that, if $\mathscr{J}' \subseteq \mathscr{J}$ and $\mathscr{J}'' \subseteq \mathscr{J}$ satisfy the above conditions, then does the union of $\mathscr{J}'$ and $\mathscr{J}''$. Therefore, the maximal set indeed exists. With an abuse of notation, for each $p \in \triangle \mathscr{J}^*$, let $p\bar{v}_i$ denote $\sum_{\mathscr{A} \in \mathscr{J}^*} p(\mathscr{A})\bar{v}_i(\mathscr{A})$ and $p\underline{v}_i$ denote $\sum_{\mathscr{A} \in \mathscr{J}^*} p(\mathscr{A})\underline{v}_i(\mathscr{A})$. Likewise, let $V(p)$ denote the set of feasible payoffs when the public randomization $p \in \triangle \mathscr{J}^*$ determines the recommended action set $\mathscr{A}$.

The following proposition characterizes the equilibrium payoff set for the abnormal case with generic payoff functions.

Proposition 6. *Suppose that the stage game is abnormal. If $\mathscr{J}^*$ is empty, then $E(\delta) = \emptyset$ for every $\delta \in (0,1)$. If $\mathscr{J}^*$ is not empty and if $\pi_i(a) \neq \pi_i(\tilde{a})$ for all $i \in I$, $a \in A$, and $\tilde{a} \neq a$, then $\lim_{\delta \to 1} E(\delta)$ is equal to the convex hull of the set of pure-strategy Nash equilibrium payoffs of the stage game.*

To prove this proposition, the following lemma is useful.

Lemma 27. *If $\mathscr{J}^*$ is not empty, then*

$$E(\delta) \subseteq \bigcup_{p \in \triangle \mathscr{J}^*} (V(p) \cap \times_{i \in I}[p\underline{v}_i, p\bar{v}_i]). \tag{61}$$

for any $\delta \in (0,1)$. If $\mathscr{J}^*$ is empty, then $E(\delta) = \emptyset$ for any $\delta \in (0,1)$.

Proof. Suppose that there is a belief-free review-strategy equilibrium $s \in S$. As in the proof of Proposition 1, player i's continuation payoff from the lth review phase is in the interval $[p^l \underline{v}_i, p^l \overline{v}_i]$, where p^l is as in (30) and $\mathscr{A}(l)$ is the union of the support of $s(h^{t_{l-1}})$ over all $h^{t_{l-1}} \in H^{t_{l-1}}$. In particular, together with the feasibility constraint, the equilibrium payoff vector is in the set $V(p^1) \bigcap \times_{i\in I} [p^1 \underline{v}_i, p^1 \overline{v}_i]$. Thus, it suffices to show that $\mathscr{J}^{**} \subseteq \mathscr{J}^*$, where $\mathscr{J}^{**}$ denotes the support of p^1.

Let I^* be the set of all $i \in I$ such that $p^l \underline{v}_i = p^l \overline{v}_i$ for all $l \geq 1$. Suppose first that $I = I^*$. Set $p = p^1$. This p satisfies both (i) and (ii) for $\mathscr{J}' = \mathscr{J}^{**}$, as $\mathscr{J}^{**}$ is the support of p and the equilibrium payoff of player i is in the interval $[p\underline{v}_i, p\overline{v}_i]$. Also, since no dynamic incentive is provided in this equilibrium, any $a \in \mathscr{A}(l)$ must be a Nash equilibrium. This shows that (iii) holds for $\mathscr{J}' = \mathscr{J}^{**}$.

Moreover, since $p^l \underline{v}_i = p^l \overline{v}_i$ for all $i \in I$ and $l \geq 1$, $\underline{v}_i(\mathscr{A}(l)) = \overline{v}_i(\mathscr{A}(l))$ for all $i \in I$ and $l \geq 1$. This, together with (iii), proves that p satisfies (iv) for $\mathscr{J}' = \mathscr{J}^{**}$. To see this, suppose not so that there is $\mathscr{A} \in \mathscr{J}^{**}$ such that $\pi_i(a) < \pi_i(\tilde{a})$ for some $a \in \mathscr{A}$ and $\tilde{a} \in \mathscr{A}$. Then from (iii), $\underline{v}_i(\mathscr{A}) \leq \max_{a'_i \in A_i} \pi_i(a'_i, a_{-i}) = \pi_i(a) < \pi_i(\tilde{a}) = \min_{a''_i \in \mathscr{A}_i} \pi_i(a''_i, a_{-i}) \leq \overline{v}_i(\mathscr{A})$. But this implies that there is $l \geq 1$ such that $\underline{v}_i(\mathscr{A}(l)) < \overline{v}_i(\mathscr{A}(l))$, a contradiction.

Overall, this p satisfies (i) through (iv) for $\mathscr{J}' = \mathscr{J}^{**}$. Since $\mathscr{J}^*$ is the maximal set, $\mathscr{J}^{**} \subseteq \mathscr{J}^*$, as desired.

Next, suppose that $I \neq I^*$. By definition, for each $i \notin I^*$, there is a natural number l_i satisfying $p^{l_i} \underline{v}_i < p^{l_i} \overline{v}_i$. Let $p = \frac{1}{|I|-|I^*|} \sum_{i \notin I^*} p^{l_i}$. This p satisfies (i) for $\mathscr{J}' = \mathscr{J}^{**}$, as the support of p^l is a subset of the support of p^1 for all $l \geq 1$. Also, (ii) follows, since player i's continuation payoff from the lth review phase is in the interval $[p^l \underline{v}_i, p^l \overline{v}_i]$ for all $l \geq 1$. Moreover, (iii) and (iv) hold as in the case of $I = I^*$, since $p\underline{v}_i = p\overline{v}_i$ for all $i \in I^*$, and $p^l \underline{v}_i = p^l \overline{v}_i$ for all $i \in I^*$ and $l \geq 1$. This proves that $\mathscr{J}^{**} \subseteq \mathscr{J}^*$. *Q.E.D.*

Proof of Proposition 6. Lemma 27 asserts that if $\mathscr{J}^*$ is empty, then $E(\delta) = \emptyset$. What remains is to show that $\lim_{\delta \to 1} E(\delta)$ is equal to the convex hull of the set of pure-strategy Nash equilibrium payoffs when $\mathscr{J}^*$ is not empty. To do so, it suffices to verify that the right-hand side of (61) is included in the convex hull of the set of pure-strategy Nash equilibrium payoffs, as the reverse side is obvious.

Since this is the abnormal case, for each $p \in \triangle \mathscr{J}$ satisfying $p\overline{v}_i \geq p\underline{v}_i$ for all $i \in I$, there is $i \in I$ such that $p\overline{v}_i = p\underline{v}_i$. Then, by definition of $\mathscr{J}^*$, there is $i \in I$ such that $\pi_i(a) = \pi_i(\tilde{a})$ for all $\mathscr{A} \in \mathscr{J}^*$, $a \in \mathscr{A}$, and $\tilde{a} \in \mathscr{A}$. This implies that each $\mathscr{A} \in \mathscr{J}^*$ is a singleton, since $\pi_i(a) \neq \pi_i(\tilde{a})$ whenever $\tilde{a} \neq a$.

Notice that if $\mathscr{A}$ is a singleton, then $\underline{v}_i(\mathscr{A}) \geq \overline{v}_i(\mathscr{A})$ for all $i \in I$. This shows that $p\underline{v}_i \geq p\overline{v}_i$ for all $p \in \triangle \mathscr{J}^*$ and $i \in I$, and then by definition of $\mathscr{J}^*$, $\pi_i(a) \geq \pi_i(\tilde{a}_i, a_{-i})$ for all $i \in I$, $\mathscr{A} \in \mathscr{J}^*$, $a \in \mathscr{A}$, and $\tilde{a}_i \in A_i$. This establishes that for each $\mathscr{A} \in \mathscr{J}^*$ and $a \in \mathscr{A}$, a is a pure-strategy Nash equilibrium and $\pi_i(a(\mathscr{A})) = \underline{v}_i(\mathscr{A}) = \overline{v}_i(\mathscr{A})$ for all $i \in I$. Therefore, for any $\mathscr{A} \in \mathscr{J}^*$, the set $\times_{i\in I}[\underline{v}_i(\mathscr{A}), \bar{v}_i(\mathscr{A})]$ is equal to the set of pure-strategy Nash equilibrium payoffs, as desired. *Q.E.D.*

Appendix G: Relaxing Conditional Independence

In Remark 1, we argued that (CI) is stronger than necessary for Theorem 1, and can be replaced with a weaker condition. In this appendix, we assume that the monitoring structure is *weakly conditionally independent* in the sense that players observe statistically independent signals conditional on an action profile a and on a hidden common shock ω_0, and show that Theorem 1 is valid under this assumption. Formally, we assume the following:

Condition Weak-CI. There is a finite set Ω_0, $q_0 : A \to \triangle\Omega_0$, and $q_i : A \times \Omega_0 \to \triangle\Omega_i$ for each $i \in I$ satisfying the following properties.

(i′) For each $a \in A$ and $\omega \in \Omega$,

$$q(\omega|a) = \sum_{\omega_0 \in \Omega_0} q_0(\omega_0|a) \prod_{i\in I} q_i(\omega_i|a, \omega_0).$$

(ii′) For each $i \in I$ and $a_i \in A_i$, $\text{rank}Q_i(a_i) = |A_{-i}| \times |\Omega_0|$ where $Q_i(a_i)$ is a matrix with rows $(q_i(\omega_i|a_i, a_{-i}, \omega_0))_{\omega_i \in \Omega_i}$ for all $a_{-i} \in A_{-i}$ and $\omega_0 \in \Omega_0$.

In words, clause (i) says that given an action profile a, a common shock ω_0 is randomly chosen following the distribution $q_0(\cdot|a)$, and then players observe statistically independent signals conditional on (a, ω_0). Clause (ii) is a version of individual full-rank condition.

Under (Weak-CI), players' signals are correlated through a common shock ω_0, so that a player's signal has some information about the opponents' signals. But we can construct random events such that a player's private signal has no information about whether the opponents' random events are counted, and thus no feedback on what the opponents will do in a continuation play. Therefore, a player has no incentive to play a history-dependent strategy within a review phase and hence the equilibrium construction in Sections 3.2 and 3.3 are valid under (Weak-CI).

Specifically, we can show the following lemmas under (Weak-CI). Let $P(\psi_i|a, \omega_{-i})$ denote the probability that the random event ψ_i is counted when the action profile a is played and players $-i$ observe ω_{-i}. The proofs of the lemmas are similar to Lemma 1 of Yamamoto (2007), and hence omitted.

Lemma 28. *Suppose that there are only two players and that (Weak-CI) holds. Then, for some q_1 and q_2 satisfying $0 < q_1 < q_2 < 1$, there is a random event $\psi_i(\mathscr{A}_{-i})$:*

$A_i \times \Omega_i \to [0,1]$ *for all* $i \in I$ *and* $\mathscr{A}_{-i} \in \mathscr{J}_{-i}$ *such that for all* $a \in A$ *and* $\omega_{-i} \in \Omega_{-i}$,

$$P(\psi_i(\mathscr{A}_{-i})|a) = \begin{cases} q_2 & \text{if} \quad a_{-i} \in \mathscr{A}_{-i} \\ q_1 & \text{otherwise} \end{cases}$$

and

$$P(\psi_i(\mathscr{A}_{-i})|a, \omega_{-i}) = P(\psi_i(\mathscr{A}_{-i})|a).$$

Lemma 29. *Suppose that there are three or more players and that (Weak-CI) holds. Then, for some* q_1, q_2, *and* q_3 *satisfying* $0 < q_1 < q_2 < q_3 < 1$, *there are random events* $\psi_i(\mathscr{A}_j) : A_i \times \Omega_i \to [0,1]$ *and* $\psi_i(a_{i+1}, a_{i-1}) : A_i \times \Omega_i \to [0,1]$ *for all* i, j, $a \in A$, *and* $\mathscr{A}_j \in \mathscr{J}_j$ *such that for all* $\tilde{a} \in A$ *and* $\omega_{-i} \in \Omega_{-i}$,

$$P(\psi_i(\mathscr{A}_j)|\tilde{a}) = \begin{cases} q_3 & \text{if} \quad a_j \in \mathscr{A}_j \\ q_2 & \text{otherwise} \end{cases},$$

$$P(\psi_i(a_{i+1}, a_{i-1})|\tilde{a}) = \begin{cases} q_1 & \text{if} \quad a_{i-1} = \tilde{a}_{i-1} \text{ and } a_{i+1} \neq \tilde{a}_{i+1} \\ q_3 & \text{if} \quad a_{i+1} = \tilde{a}_{i+1} \text{ and } a_{i-1} \neq \tilde{a}_{i-1} \\ q_2 & \text{otherwise} \end{cases},$$

$$P(\psi_i(\mathscr{A}_j)|\tilde{a}) = P(\psi_i(\mathscr{A}_j)|\tilde{a}, \omega_{-i}),$$

$$P(\psi_i(a_{i+1}, a_{i-1})|\tilde{a}) = P(\psi_i(a_{i+1}, a_{i-1})|\tilde{a}, \omega_{-i}),$$

and for each i *and* $j \neq i$, *player* i*'s random event and player* j*'s random event are statistically independent conditional on any* $a \in A$.

Note that $P(\psi_i|a, \omega_{-i}) = P(\psi_i|a)$ means that players $-i$'s signal ω_{-i} has no information about whether the random event ψ is counted, and hence players have no incentive to use a history-dependent strategy within a review phase. This shows that Theorem 1 remains valid even if (CI) is replaced with (Weak-CI).

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