

Auctions and Shareholdings*

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Abstract

We study how shareholdings affect auctions' revenue and efficiency with independent private values. Two types of shareholdings are analyzed: Vertical (resp: horizontal) toeholds cover situations in which bidders own a fraction of the seller's profit (resp: a share of their competitor's profit). The expected revenue is an increasing (resp : decreasing) function of vertical (resp : horizontal)toeholds. With both types of toeholds, auction formats are not revenue equivalent. The second price auction is more sensitive than the first-price auction to the revenue variations upwards or downwards due to these toeholds.

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1 Introduction

We study how shareholdings and cross-shareholdings affect firms behaviors in an auction context. But first, let us illustrate why shareholdings can matter in an auction framework through two cases.

The Global One case. Global One was a joint-venture created in 1996 by Deutsche Telekom (40%), France Télécom (40%) and Sprint (20%). In 1999, in order to satisfy competition rules, Sprint was forced to sell its shares of Global One. Because of a former agreement, the two remaining ex-partners were the only possible buyers for Sprint's shares. Neither of the two European companies accepted to stay with 40% of Global One while the other owned 60%. The former partners agreed that they had to choose a selling process such that, at the end, only one of the European operators would hold 100% of Global One. The three firms considered using an auction process.¹ The situation was slightly different from a standard auction setting. In the Global One case, the bidders, France Télécom and Deutsche Telekom, were wearing two hats. They were, in the same time, potential buyers of the remaining shares and potential sellers of their own shares. The specificity of this setting is the existence of this seller hat. We intend to study the impact of this extra seller hat.

In the Global One case, bidders owned, ex ante, a fraction of the good for sale. This is not the only situation in which shareholdings affect strategic incentives in an auction framework. We also observe situations in which bidders own a fraction of the capital of other bidders. This can also be a major issue. We illustrate that point through the presentation of a second case.

RVI and Volvo Trucks were two of the main European truck manufacturers. Between 1991 and 1994, they had symmetric crossholdings of 45%. Meanwhile, in some European countries, their joint market share exceeded 50%. Nevertheless, during that period, they still regularly competed with each other in tenders organized by the major haulage companies of these countries. Once again, one can wonder how these toeholds influenced the strategic behaviors of RVI and Volvo Trucks in the tender stages. As a matter of fact, each of the truck manufacturer got back a fraction of the profit of the other. Thus, if RVI lost a market, he preferred Volvo Trucks to obtain it and to make the highest possible profit with this tender.

In both cases, Global One and RVI/Volvo Trucks, because of these toeholds, bidders, conditional on losing the auction, care about the price paid by the winner. We intend to understand how this affects bidders' strategies. Do shareholdings have consequences on the efficiency or the expected revenue of the auction? Do they have the same impact on the different auction formats? Which of the standard auction formats is preferable in these cases? What is the optimal auction format? This paper addresses these issues.

¹The author spoke with the Finance director of France Télécom before the choice of the auction process. Results of the current paper were evoked. The auction format that was effectively used remained secret.

We consider two types of toeholds. When a bidder owns shares of the seller or a fraction of the object for sale² as in the Global One case, we speak of a *vertical toehold*. When a bidder owns shares of another bidder, we speak of an *horizontal toehold*.³ We study how both types of toeholds affect the revenue and the efficiency of auction formats in an independent private value setting in the first price auction and the second price auction.⁴ We also propose an optimal auction format with horizontal toeholds.

The issue of toeholds in an auction context has been the object of several studies. More specifically, Engelbrecht-Wiggans (1994) can be reinterpreted as a study of an auction with vertical toeholds.⁵ In this context, he showed that the expected price is higher with a second price auction than with a first price auction.⁶ He considered that this result unbalances Myerson's Revenue Equivalence Theorem (see Myerson (1981)). In fact, proposition 3 is a reformulation of this result⁷. However, we show, through our proof of this proposition, that, on the contrary, this result can be interpreted as a consequence of the Revenue Equivalence Theorem. The use of the Revenue Equivalence Theorem even allows to better understand the origins and consequences of the difference between the first price auction and the second price auction. Regarding the horizontal toeholds case, Dasgupta and Tsui (2004) contains results very close to ours⁸.

The specificity of our contribution consists in extending the horizontal toehold case (and to a lesser extent the vertical toehold case) mainly by the identification of the optimal auction format and the extension to the n bidders case and in the joint analysis of the two types of toeholds which allows to identify the qualitative differences between them. We show that horizontal toeholds cannot be interpreted as *negative* vertical toeholds. As a matter of fact, a losing bidder owning a vertical toehold only cares about the price paid by the winner. A losing bidder owning an horizontal toehold cares about the price paid by the winner but he also cares about the winner's valuation for the good. Besides, the joint analysis of both types of toeholds allows to have a general interpretation of the results in terms of the sensitivity of auction formats. Let us be more precise and introduce the results.

Firstly, neither vertical toeholds, nor horizontal toeholds, as long as they are symmetric,

²Both events are conceptually identical.

³A toehold is not consubstantially horizontal or vertical. The context determines whether the toehold is horizontal or vertical.

⁴Here, the two other standard auction formats, the descending and the ascending auction are equivalent to the first price auction and second price auction, respectively.

⁵His original motivation comes from Amish estate sales. After the death of a member of the community, the farm is auctioned among the heirs and the resulting revenue is split among them.

⁶This result is also used in Maasland and Onderstal (2006) and Engers and McManus (2004). Independently from this work, they both developed a related framework which can also be interpreted in terms of vertical toeholds.

⁷Let us also mention that propositions 1 and 2 are corollaries of this work which considers a more general setting.

⁸They obtain in a framework with reserve price and independent private signals results equivalent to propositions 5, 6 and 7.

affect efficiency. Secondly, toeholds do have an impact on the expected revenue of auctions. In both formats, vertical toeholds raise the expected revenue and horizontal toeholds decrease it. Thirdly, these variations are exacerbated under the second price auction relative to the first price auction. When bidders have vertical toeholds, the expected revenue is larger under a second price auction than under a first price auction. When toeholds are horizontal, the expected revenue is smaller under the second price auction than under the first price auction. Therefore, which auction format among the first price and the second price auction generates more revenue depends on the type of toehold considered.

The intuition is as follows. When bidders own shares of the seller of the good, they get back, through their shares, a fraction of the amount given to the seller. For both bidders, this is true whether they win or lose the auction. Then, in both auction formats, bidders tend to bid more aggressively and the expected price is higher than without toeholds. However, a losing bidder can have a direct impact on the price only in a second price auction. Then, bidders have an extra incentive to be aggressive with the second price auction. That is why the expected price is higher with a second price auction than with a first price auction.

When bidders have crossholdings, we observe the opposite phenomenon. If a bidder does not obtain the good, he prefers the price to be low. Suppose that bidder i owns a fraction of bidder j . If bidder i loses the auction, he prefers that bidder j pays the lowest possible price. This gives incentives to bid less aggressively than without toeholds. Consequently, with both auction formats, the expected price is lower than in the standard case without toeholds. Once again, a losing bidder can have a direct impact on the price only in a second price auction. Because of this more direct leverage, bidders decrease their bids more strongly⁹ in a second price auction than in a first price auction. Thus, the expected revenue is lower with the second price auction than with the first price auction.

As we already mentioned, the impact of shareholdings has already been discussed in the auction theory literature. Cramton, Gibbons and Klemperer (1987) and de Frutos and Kittsteiner (2006) study a case of vertical toeholds in which the whole target firm is owned, ex ante, by asymmetric competitors. They consider the efficiency issue. In their framework, the question of the auction revenue does not make much sense. Once the allocation of the good is determined, there are only transfers between the bidders. There is no actual sale price. In another vein, Singh (1998) and Burkart (1995) analyze a contested takeover in which one bidder owns a vertical toehold in the target firm. He faces a bidder without toehold. They observe overbiddings and inefficiencies deriving from the asymmetry among bidders. They do not study the equilibrium of the first price auction, extremely difficult to compute in this case. Thus, we do not know if our ranking can be applied to their asymmetric setting. Goeree et al (2005) studies charity sales in which the taste for the financed public good is equivalent to a vertical toehold. They mainly focus on all-pay auctions and introduce

⁹In comparison with the bids they would have submitted if there were no horizontal toeholds.

an optimal auction format. Our approach of the optimal auction format with horizontal toeholds can be seen as a complement to their approach. Bulow, Huang and Klemperer (1999) also considers vertical toeholds but in a common value framework. They argue that the common value paradigm is more appropriate to represent *financial* bidders. In that sense, their approach is complementary to ours in which we consider *strategic*¹⁰ bidders. In their common value framework, they also observe that the revenue of the second price auction is higher than the revenue of the first price auction when toeholds are symmetric.

Our approach is also reminiscent of some aspects of the study of auctions with externalities initiated by Jehiel and Moldovanu. In Jehiel and Moldovanu (2000), for instance, they consider an asymmetric information setting in which a losing bidder derives a positive or negative fixed externality from the allocation of the good to another bidder. A priori, our vertical (resp: horizontal) toeholds could be assimilated to a negative (resp: positive) externality. However, what really matters here is that the externality term depends on the price. This specific issue does not appear in the auction with externalities literature while our comparison of the first price auction and the second price auction crucially relies on it. That is why, they observe a revenue equivalence between the main standard auction formats while we do not obtain this equivalence.

The remainder of the paper is organized as follows. In section 2, we present the model. In section 3, we analyze the impact of vertical toeholds. In section 4, we study the horizontal toehold case. Section 5 concludes.

2 The model

A good is sold through an auction process with two risk-neutral bidders, 1 and 2. Bidders 1 and 2 perfectly represent the interests of firms 1 and 2, respectively.¹¹ Firm i 's valuation¹² for the good is v_i which is bidder i 's private information¹³. It is common knowledge that

¹⁰For a discussion on the difference between a strategic and a financial bidder, see the introduction of Bulow, Huang and Klemperer (1999). In short, A *financial* bidder buys shares of companies in order to sell them later so that he cares about the common value component of the target firm. A *strategic* bidder buys firms in order to merge with this firm and to realize synergies. A *strategic* bidder cares more about the private value component of the target firm.

¹¹In the rest of the paper, we will identify bidders with the firms they represent.

¹²Throughout paper, by default $i = 1, 2$ and $j \neq i$.

¹³We consider the independent private value paradigm in order to identify effects that are specifically due to the presence of toeholds. Our results are independent from the effect of affiliation and the explosive impact of asymmetry in a common value framework as identified in Bulow et al (1999). Besides, in the two introductory examples, the private value hypothesis seems justified. As a matter of fact, in the RVI/Volvo Trucks case, players' private information concerns their production costs whom they almost perfectly know and which does not give them more information about the specific production costs of their opponents. In the Global One case, of course, an important fraction of the value of the good is common among the two buyers. However, we claim that the residual private information of the bidders concerns the independent and private part of their valuations : private synergies, long-run industrial strategies... The common value part of Global One is known by both bidders who have access to all the information concerning the firm and

v_1 and v_2 are independently distributed according to an identical cumulative distribution F with density f on the interval $[0, 1]$. We further assume that F is continuous, differentiable, strictly increasing on $[0, 1]$ and satisfies the monotone hazard rate condition ($\frac{d}{dt} \frac{1-F(t)}{f(t)} < 0$ on all the interval).

We define two categories of toeholds. A bidder has an *horizontal toehold* when he owns a fraction of the capital of the other bidder. A bidder has a *vertical toehold* when he owns a fraction of the capital of the seller. For the sake of simplicity, we assume that the only asset of the seller is the good for sale. Thus, we can identify the seller with his good. For a bidder, owning a vertical toehold is equivalent to possessing a fraction of the good for sale.

We focus on two polar cases regarding distribution of toeholds.¹⁴

- Vertical toeholds: Both firms own an identical fraction $\alpha \in (0, \frac{1}{2})$ of the capital of the seller and no horizontal toehold.
- Horizontal toeholds: Each firm owns an identical fraction $\theta \in (0, \frac{1}{2})$ of the capital of the other firm and no vertical toehold.

In both cases, toeholds are common knowledge.

We consider two different auction formats: The second price auction and the first price auction defined as follows. In both auctions, each bidder simultaneously submits a bid $b \geq 0$ and the bidder who submits the highest bid obtains the good. In the first price auction, the winner pays the amount of his bid. In the second price auction, he pays the second highest bid which coincides here with the bid of his opponent. In both auctions, if bidders submit exactly the same bid, the seller flips a fair coin. The winner obtains the good and pays the common bid.

In the vertical toehold case, we suppose that the loser always accepts to sell his fraction of the good at the price defined by the auction.¹⁵ Bidder i owns a fraction α of the good. Then, if he wins the auction, he buys the remaining $(1 - \alpha)$ shares of the good. On the other hand, if he loses the auction, bidder i sells his toehold. Utilities are then defined as follows.

If i obtains the good and pays a price p :

$$U_i = v_i - p + \alpha p$$

all the studies on the perspectives of the telecom sector by the major consulting firms

¹⁴In both cases, we assume that bidders are symmetric. It is a necessary condition to solve equilibria.

¹⁵Consider, for instance, a contested takeover. In that case, a losing competitor usually prefers not to keep his toehold. If the other bidder takes the control of the target firm, he will probably divert the extra profits he can create. The loser is better off selling his shares before this dilution. On the other side, the winning bidder, in most legislations, cannot refuse to buy his adversary's toehold at the price of the winning tender. Besides, if we interpret toeholds as a fraction of the capital of the seller owned by bidders, through his shares, a bidder always gets a fraction α of the extra profit of the selling company : αp . The losing bidder gets αp and the winning bidder gets $v - p + \alpha p = v - (1 - \alpha)p$.

If j obtains the good and pays a price p :

$$U_i = \alpha p$$

Denoting by q_k the probability that bidder k obtains the good and p_k the expected payment of bidder k , the expected utility of bidder i can be written:

$$U_i = q_i v_i - p_i + \alpha(p_i + p_j).$$

The horizontal toehold case is slightly more complex. We assume that through dividends or the rise of shares' value, any additional profit of a firm goes to its shareholders in proportion to their stakes. If bidder i wins the auction and pays a price p , firm i derives a direct profit from this purchase: $v_i - p$. Consequently, the value of a fraction θ of firm i increases by $\theta(v_i - p)$. Since firm j owns a fraction θ of firm i , whenever firm i wins at the price p , firm j 's value increases by $\theta(v_i - p)$. But, firm i also owns a fraction θ of firm j . Then, because of the increase of firm j 's value, firm i 's value now increases by $\theta^2(v_i - p)$. This mechanism continues *ad infinitum* so that the total increase of firm i 's value is $\sum_{k=0}^{\infty} (\theta^2)^k (v_i - p) = \frac{1}{1-\theta^2} (v_i - p)$. We also obtain that the total increase of firm j 's value is $\sum_{k=0}^{\infty} \theta (\theta^2)^k (v_i - p) = \frac{\theta}{1-\theta^2} (v_i - p)$. Consequently, the total increase of firm i 's value if bidder j buys the good for a price p is $\sum_{k=0}^{\infty} \theta (\theta^2)^k (v_j - p) = \frac{\theta}{1-\theta^2} (v_j - p)$.

Up to a rescaling of payoffs,¹⁶ utility functions can be defined as follows.

If bidder i obtains the good for a price p :

$$U_i = \frac{1}{1+\theta} (v_i - p)$$

If bidder j obtains the good for a price p :

$$U_i = \frac{\theta}{1+\theta} (v_j - p)$$

Using the same notations as in the vertical toehold case, the expected utility of bidder i is:

$$U_i = \frac{1}{1+\theta} (q_i v_i - p_i + \theta(q_j v_j - p_j))$$

Notice that an horizontal toehold cannot be modelled as a negative vertical toehold. There is a specific element that does not appear in the vertical toehold case. Here, a losing bidder not only cares about the price paid by the loser, as in the vertical toehold case. He also cares about the valuation for the good of the winning bidder. That is why these two

¹⁶We multiply the former expressions by $(1-\theta)$. With such a normalization, $\forall i = 1, 2$, if bidder i wins the auction, whatever the price paid, p , is, $U_i + U_j + p = v_i$.

types of toeholds cannot be represented by a unique coefficient whose sign would be positive for a vertical toehold and negative for an horizontal toehold.

In both the first price and the second price auction and for vertical and horizontal toeholds, we restrict attention to symmetric equilibria.

3 Auctions with vertical toeholds

In this section, we consider the setting with vertical toeholds. Bidders own a fraction of the seller or a fraction of the good for sale. We present the equilibria of the two auction formats, compare expected revenues and study the properties of this framework.

Most of the proofs are in the appendix.

3.1 Equilibria Characterization

Proposition 1 *There is a unique symmetric equilibrium of the second price auction. For $i = 1, 2$, bidder i with valuation v_i bids $b_V^{II}(v_i)$ where:*

$$b_V^{II}(v_i) = v_i + \int_{v_i}^1 \left(\frac{1 - F(t)}{1 - F(v_i)} \right)^{\frac{1}{\alpha}} dt$$

Proposition 2 *There is a unique symmetric equilibrium of the first price auction. For $i = 1, 2$, bidder i with valuation v_i bids $b_V^I(v_i)$ where:*

$$b_V^I(v_i) = v_i - \int_0^{v_i} \left(\frac{F(t)}{F(v_i)} \right)^{\frac{1}{1-\alpha}} dt$$

Corollary 1 *Both auction formats are efficient. Except if $v_i = 1$ or 0 , bidding functions are strictly increasing in α and so are the actual and expected revenues.*

Proof: Efficiency derives from the symmetry of the equilibria and the fact that b_V^{II} and b_V^I are strictly increasing functions of v_i . The rest is a direct consequence of proposition 1 and proposition 2. Q.E.D.

In both auction formats, we derive a unique symmetric equilibrium in which bidding functions are increasing in the size of the toehold. The expected price also increases with the size of the toehold and the allocation is efficient. These proprieties are shared by both auction formats. However, vertical toeholds affect bidding behaviors in the two auction formats through two different channels that we will now describe.

Let us consider the second price auction first. Bidders care about the price paid, p , even if they fail to obtain the good since they receive αp in that case. Therefore, contrary to the standard case without toeholds, bidding its own valuation is not a dominant strategy.

Bidders tend to bid more than their valuations to raise the price, conditional on losing the auction. On the other side, bidding too high is dangerous. In doing so, a bidder can win and pay a price above his valuation for the good. The equilibrium bid is the result of this trade-off.

For higher values of α , the motivation for raising bids increases since bidders receive a larger fraction of the price paid conditional on losing the auction. Besides, even if a bidder wins he only buys a fraction $(1 - \alpha)$ of the good. The higher α is, the lower the *risk* conditional on winning the auction. These two elements go in the same direction and equilibrium bidding functions are strictly increasing in α (except for $v = 1$).

In the first price auction, losing bids have no effects on the price paid by the winner. There is no direct strategic way for the loser to raise the price paid by the winner. The choice of a bid only fixes the probability of winning and the price paid conditional on winning the auction. However, toeholds have an impact on bidders' incentives, even in a first price auction.

Each bidder wears two hats: A buyer hat and a seller hat. If he wins, he is the buyer of a fraction $(1 - \alpha)$ of the good. If he loses, he is the seller of a fraction α of the good. As a potential buyer, a bidder, if he bids ε more and wins the auction, does not pay ε more but rather $(1 - \alpha)\varepsilon$ more. Therefore, conditional on winning the auction, bidding more is less costly than without toehold. As a potential seller, the expected utility of bidder i , if he eventually loses the auction while he had submitted a bid d , is:¹⁷ $\alpha \int_{b_j^{-1}(d)}^1 \frac{b_j(t)f(t)}{1-F(b_j^{-1}(d))} dt$. This is strictly increasing in d . By increasing his bid, a bidder reduces his probability to sell his toehold for a low price.

For higher values of α , bidders have two reasons to increase their bids. First, if they lose the auction, they sell a bigger fraction of the good. Thus, it is even more important not to sell for too low a price. Second, if they win, since they buy a smaller fraction $(1 - \alpha)$ of the good, paying a higher price is less costly. As a result, in the first price auction also, equilibrium bids increase in α .

Now, let us reconsider bidders' strategies in the second price auction. Bidders submit bids above their valuations. As a result, bidder i can win the auction and buy the good for a price strictly above his valuation. It happens whenever $v_i < b^{II}(v_j) < b^{II}(v_i)$. Bidder i wins the auction and pays a price $b^{II}(v_j)$ that is strictly higher than his valuation. As this arises because of his toehold, we say that bidder i is victim of the *owner's curse*.¹⁸ Since the bidder only buys a fraction $(1 - \alpha)$ of the good, he does not always derive a negative utility. However, for some realizations, he may actually derive a strictly negative utility ex post.¹⁹

¹⁷At the equilibrium, b_j , the bidding function of j , is a strictly increasing function of v_j .

¹⁸This expression is due to Singh (1998). He used it to describe the same phenomenon in the asymmetric case, with only one bidder owning a toehold.

¹⁹As in the *winner's curse* case, the ex-post negative utility is only a possibility. For any v_i and α , the expected utility of bidder i is always strictly positive.

The following example illustrate this issue.

Example 1 Suppose that F is a uniform distribution, $\alpha = \frac{1}{5}$ and $(v_1, v_2) = (0, \frac{1}{10})$. At the equilibrium, bidder 2 wins the auction, the price is $b^{II}(0) = \frac{1}{6}$ and bidder 2's utility is equal to: $(\frac{1}{10} - \frac{1}{6}) + \frac{1}{5} \times \frac{1}{6} = -\frac{1}{30}$.

This owner's curse only appears in second price auctions since in the first price auction bidders always submit less than their valuations. This seems to indicate that the second price auction strengthens more competition than the first price auction does. This intuition is confirmed by the following proposition which establishes the non equivalence in terms of revenue between the two auction formats.

3.2 Revenue Comparison

Proposition 3 For any $\alpha \in (0, \frac{1}{2})$, the expected price is strictly higher with a second price auction than with a first price auction.

This result only superficially contradicts the Equivalence Revenue Theorem. Myerson's Theorem says that for the revenues of two auction formats to be equivalent: (i) The same allocation rule must arise in the two formats. This is clearly met in the present context. (ii) The reservation utility obtained by the lowest bidder must be the same in both auction formats. As the analysis above shows it, this is not so in the present context. The reservation utility is strictly higher with the first price auction. As a result, the expected revenue is lower with the first price auction than with the second price auction.

Now, let us propose an intuitive explanation for this revenue ranking. In the second price auction, bidders have a more immediate reason to raise their bids. It is a direct way to increase the price conditional on their losing the auction. By the very definition of auction formats, this motivation cannot appear in a first price auction. That is why vertical toeholds affect more directly bidding strategies in the second price auction than in the first price auction. As a result, the expected price is higher with the second price auction. This format is more sensitive to vertical toeholds.

We established what is the preferred format for the seller. We may also consider bidders' preferences regarding auction formats.

Corollary 2 For any $\alpha \in (0, \frac{1}{2})$, $i = 1, 2$ and $v_i \in [0, 1]$, the expected utility of bidder i with a first price auction is higher than his expected utility with a second price auction by an amount D_V that is independent of his valuation v_i :

$$D_V = \int_0^1 \alpha(1 - F(t))^{\frac{1}{\alpha}} - \alpha + F(t) - (1 - \alpha)F(t)^{\frac{1}{1-\alpha}} dt > 0$$

Proof: The expected utility of a bidder, whatever his valuation is, is a function of the allocation plus his reservation utility (Revenue Equivalence Theorem). Here, the allocation rule is the same with both auction formats. Therefore, whatever v_i , for bidder i , the difference between his expected utility in the two auction formats is the same, equal to what we compute for $v_i = 0$. Q.E.D.

This uniformity result, although it relies on standard regularity properties, is not so intuitive. It means that the preference of a bidder for an auction format over another one depends on α , the size of his toehold but not on his valuation. The use of a first price auction rather than a second price auction is worth a fixed amount to a bidder which does not change with his valuation. If a bidder has a high valuation and he is almost sure to win, the extra utility that he derives from the choice of a first price auction rather than a second price auction is the same as when he has a low valuation and he is almost sure to lose the auction. In the first case, he gets an extra utility because he pays a lower price. In the second case, he gets an extra utility because his opponent pays a higher price. Corollary 2 tells us that the two effects perfectly compensate for each other.

Now, for $\alpha = \frac{1}{2}$, bidders' expected utilities are identical with both auction formats since for any $x \in [0, 1]$, $-(1-x)^{\frac{1}{\alpha}} + 1 - \frac{1}{\alpha}x + \frac{1-\alpha}{\alpha}x^{\frac{1}{1-\alpha}} = 0$ whenever $\alpha = \frac{1}{2}$.²⁰ In fact, this case corresponds to the partnership problem studied by Cramton and al (1987) showing that the first price and the second price auction is equivalent. Besides, it is a well-known result that in the standard case, for $\alpha = 0$, bidders' expected utilities are the same with both auction formats. From these results, we derive the following remark.

Remark 1 *For any $i = 1, 2$ and $v_i \in [0, 1]$, the difference between bidder i 's expected utility with a first price auction and his expected utility with a second price auction is a non-monotonic function of α .*

For low values of α , toeholds do not affect much bidding strategies. Thus, the differences between the two auction formats are minor. For high values of α , close to $\frac{1}{2}$, the auction tends to be equivalent to the allocation of a good between his two exclusive owners. In that case, expected utilities uniquely depend on the allocation rule. Since, with both auction formats, the allocation is the same, expected utilities of bidders are also identical with both auction formats. Thus, the choice of an auction format or of the other really matters for bidders only for intermediary values of α .

3.3 Using vertical toeholds to raise the seller's revenue

We observed that the presence of vertical toehold raises the expected price paid by the winning buyer, with both the first price and the second price auction. Therefore, we may

²⁰All the former proofs remain true for $\alpha = \frac{1}{2}$.

wonder whether giving shares away to the bidders could be profitable for the seller. Can the rise in the price due to the existence of vertical toeholds overrules the cost of giving shares to the bidders?

Let us be more specific. Of course, giving shares away cannot be part of an optimal selling mechanism (as defined by Myerson (1981) or Riley and Samuelson (1981)). However, we intend to see if giving shares to the bidders can raise the seller's revenue in a restricted framework in which the seller could only use standard auction formats without entry cost and reserve price (i.e. the good is always sold). We can establish some results in this restricted framework.

First of all, when bidders are ex-ante symmetric (identical distribution functions for their valuations), even though vertical toeholds increase the expected price, the seller cannot increase his expected revenue by giving shares away to the bidders. This result is stated in the following proposition.

Proposition 4 *If both bidders have the same distribution function for their valuations, satisfying the monotone hazard rate condition and no vertical toeholds, the seller cannot increase his expected revenue by giving shares to the bidders. This is true with both the first price and the second price auction.*

Starting from the standard symmetric case, the seller cannot use shares' distribution to raise his revenue. Now, if bidders' valuations distribution functions differ, this result does not stand any more and the seller can increase his revenue by giving shares away. This can be shown with a simple example.

Example 2 *Suppose that F_1 is a uniform distribution function on $[0, 1/2]$ and F_2 is a uniform distribution function on $[0, 1]$. We consider the second price auction.*

Without vertical toeholds, the expected revenue for the seller, equal to the expected price, in the unique equilibrium with undominated strategy (i.e. : bidders submit their valuations for the good) is equal to $\frac{5}{24} \simeq 0.2083$.

Suppose that the seller gives 20% of the good to bidder 1. Then, there is a unique equilibrium with undominated strategies : bidder 1 submit $\frac{1+5v_1}{6}$ and bidder 2 submits v_2 . If this equilibrium is played, the expected price reached by the auction is equal to : $\frac{257}{864}$ and the expected revenue for the seller is : $(\frac{4}{5})(\frac{257}{864}) \simeq 0.238$.

The seller can increase his revenue by giving away a fraction of the sale revenue to one of the bidders. In this example, his revenue rises by approximately 15%.

The best way to get the intuition of these two seemingly opposite results is to think in terms of optimal auction format (with a myersonian perspective). It is a well-known result that, in order to maximize his expected revenue, a seller must choose an auction which

allocates the good to the bidder with the highest marginal revenue²¹ (the marginal revenue of a bidder with valuation v and distribution function F is : $v - \frac{1-F(v)}{f(v)}$).

When bidders are symmetric with a distribution function satisfying the monotone hazard rate condition, the bidder with the highest valuation has the highest marginal revenue. A second price auction already allocates the good to the bidder with the highest valuation. Therefore, there is no need to use vertical toeholds in order to "improve" the allocation.

When bidders' distribution functions are not identical, the highest marginal revenue and the highest valuation may not coincide. For instance, in example 2, when bidders have the same valuation, bidder 1 has a strictly higher marginal revenue than bidder 2. Therefore a revenue maximizing mechanism should favor bidder 1 and give him the good in some configurations in which he has a lower valuation than bidder 2. By giving shares to bidder 1, the seller distorts the way the second price auction allocates the good in this direction. Bidder 1 increases his bids and obtains the good with a higher probability. Giving a vertical toehold makes the allocation closer to the revenue maximizing allocation and raises the expected price reached by the auction. Example 2 shows that this effect can counterbalance the loss of a fraction of the sale's revenue²² so that the expected revenue may rise.

4 Auctions with horizontal toeholds

In this section, we study the horizontal toeholds case. Two bidders with cross-shareholdings compete in an auction process. We first characterize the symmetric equilibria, then we compare the generated revenues, we also introduce an optimal auction format and extend the results to the $n > 2$ bidders case.

With horizontal toeholds, incentives are completely opposite to what we have observed with vertical toeholds. If bidder i loses the auction, he has nothing to sell to the winner. In contrast, as he owns a fraction of firm j , he prefers that firm j makes the highest possible profit. He prefers the price to be low.

Thus, for reasons symmetric to those evoked in the case of vertical toeholds, at the equilibrium, bidders submit lower bids than in the standard case. The following propositions show precisely in which way horizontal toeholds affect the equilibrium bidding functions of the two auction formats.

²¹When the highest marginal revenue is negative, the seller is better of keeping the good. But this is not possible in the restricted framework we consider.

²²By giving shares away to a bidder.

4.1 Equilibria Characterization

Proposition 5 *There is a unique symmetric equilibrium of the second price auction. For $i = 1, 2$, bidder i bids $b_H^{II}(v_i)$ where:*

$$b_H^{II}(v_i) = v_i - \int_0^{v_i} \left(\frac{1 - F(v_i)}{1 - F(t)} \right)^{\frac{1-\theta}{\theta}} dt$$

Proposition 6 *There is a unique symmetric equilibrium of the first price auction. For $i = 1, 2$, bidder i bids $b_H^I(v_i)$ where:*

$$b_H^I(v_i) = v_i - \int_0^{v_i} \left(\frac{F(t)}{F(v_i)} \right)^{1-\theta} dt$$

Corollary 3 *Both auction formats are efficient. Except if $v_i = 0$, bidding functions are strictly decreasing in θ and so are the actual and expected revenues.*

Proof: Efficiency derives from the symmetry of the equilibria and the fact that b_H^{II} and b_H^I are strictly increasing functions of v_i . The rest is a direct consequence of proposition 5 and proposition 6. Q.E.D.

Both auction formats are efficient. Equilibrium bids are decreasing functions of θ . This result is not surprising, but it raises an issue concerning competition regulation. Let us illustrate our point quoting the Commission of the European Communities. In a 1990 report (Case N°IV/M.0004 (1990)), it explained that a shareholdings exchange of 25% between two competitors need not be controlled by the competition authority provided that the exchange “does not in itself either give sole control of one party over the other or create a situation of common control” (in application of Council Regulation N° 4064/89, article 3). Here, even if no common decision is taken, crossholdings distort bidders’ behaviors and affect the price. Therefore, such an exchange, because of its possible consequences, should also be controlled by the authorities in charge of market regulation. This point was already established in the context of a Cournot model (see Reynolds and Snapp (1986)). We showed that it remains true with both the first price auction and the second price auction.

Now, let us be more explicit about how crossholdings affect the two auction formats.

In the second price auction, bidding its own valuation is not a dominant strategy. Bidder i , if he submits a bid b_i and fails to win the auction, derives a utility, $\frac{\theta}{1+\theta}(v_j - b_i)$ which is decreasing in b_i . Thus, bidders tend to bid less than their valuations. Nevertheless, an extremely low bid cannot be part of an equilibrium bidding strategy. As a matter of fact, if bidder i was bidding that way, for some values of v_i , he would leave the good to bidder j for a low price while he would be better off bidding just a little bit more and obtaining the good. Bidding strategies, in the second price auction, are the result of this trade-off. As θ grows, bidder i gets back a bigger fraction of bidder j ’s profit. Thus, it becomes more and

more important for bidder i , conditional on losing, to submit a low bid. That is why the equilibrium bidding function is decreasing in θ .

In the first price auction, losing bids do not fix the price. Nevertheless, toeholds still affect bidders' strategies. As a matter of fact, if a bidder does not obtain the good, his utility is the profit of the other bidder multiplied by $\frac{\theta}{1+\theta}$ and not zero. Losing is not as negative an event as it is in the absence of toehold. The equilibrium bidding function which is the result of a trade-off between the fear to lose and the will to make a higher profit when obtaining the good is consequently affected. *Ceteris paribus*, the expected utility of bidder i , conditional on losing the auction, is increasing in θ . For larger values of θ , bidders are less eager to win the auction with a small difference between their valuations and their bids. As a result, equilibrium bidding functions are decreasing in θ .

Thus, through two different channels, in both auction formats, vertical toeholds have a decreasing effect on the bids. We may also ask whether, as in the vertical toehold case, there is a general ranking in terms of expected revenue between the two auction formats.

4.2 Revenue Comparison

Proposition 7 *For any $\theta \in (0, \frac{1}{2})$, the expected price is strictly higher with a first price auction than with a second price auction.*

As in the former section, we observe that the equivalence in terms of allocation, even in an independent private value framework, is not sufficient to derive the revenue equivalence of the two auction formats. Here also, reservation utilities differ. A bidder with valuation zero strictly prefers the second price auction in which he bids zero and the actual price paid by the other bidder is zero. Therefore, the expected revenue is lower with a second price auction.

Even though the revenue ranking is opposite, the intuition of this result is similar to the intuition in the vertical toehold case. As a matter of fact, as in the vertical toehold case, a losing bidder can more directly influence the price in the second price auction. Thus, the downwards variations of expected revenue due to horizontal toeholds are exacerbated in the second price auction.

Finally, in the case of horizontal toeholds, a seller should be well advised to choose a first price auction in order to generate more revenue.

We established what is the preferred format for the seller. We may also consider bidders' preferences regarding auction formats.

Corollary 4 *For any $\theta \in (0, \frac{1}{2})$, $i = 1, 2$ and $v_i \in [0, 1]$, the expected utility of bidder i with a second price auction is higher than his expected utility with a first price auction by an*

amount D_H that is independent of v_i :

$$D_H = \frac{\theta}{1 + \theta} \int_0^1 (t - \int_0^t (\frac{F(u)}{F(t)})^{1-\theta} du) f(t) dt > 0$$

Proof: From the Equivalence Revenue Theorem we derive that in any auction mechanism, the expected utility of a bidder is a function of the allocation rule and his valuation plus his reservation utility. Here, the allocation is the same with both auction formats. Therefore, whatever v_i , for bidder i , the difference between his expected utility in the two auction formats is the same, equal to what we computed in zero. Q.E.D.

Bidders' preferences for an auction format over another one depends on θ , the size of the toehold but not on their valuations. The use of a second price auction rather than a first price auction is worth a fixed amount to a bidder. This amount does not change with his valuation. For low valuations, bidders prefer the second price auction mainly because it reduces the price paid conditional on losing. For high values, they prefer the second price auction because it reduces the price he pays, if he wins. Corollary 4 tells us that the two effects perfectly compensate for each other.

4.3 The optimal auction format

We compared the expected revenue of the standard auction formats in presence of horizontal toeholds. A complementary approach would consist in defining the expected revenue maximizing (optimal) auction format²³.

It is a well-known result that a revenue maximizing auction must be such that the bidder with the highest marginal revenue²⁴ provided that this marginal revenue is positive and bidders' reservation utility is equal to zero. In the standard case, with symmetric distribution functions and without toeholds, an ascending auction with a reserve price R^* such that $R^* = \frac{1-F(R^*)}{f(R^*)}$ has these two properties.

When bidders have horizontal toeholds, this is no longer the case. A bidder reservation is not zero, it is equal to the share of the winning firm profit that he will obtain through the toeholds. Therefore, to maximize the seller's revenue, we must find a way to reduce this reservation utility. This can be done by setting a pre-auction round in which bidders would be asked to pay their reservation utilities. The seller would commit not to sell the good if one of the bidders refuses to pay this "organization fee". A pre-auction round with organization fee works as follows.

²³The optimal auction format with vertical toeholds has already been defined in Goeree et al (2005). That is why we did not introduce it in the previous section. Our definition of the optimal auction format is inspired by their work.

²⁴We suppose that distribution functions satisfies the monotone hazard rate condition so that the marginal revenue is increasing in v .

Step 1 : The seller asks all the bidders to pay him an organization fee, c . If one of the bidder refuses to pay, the auction process ends up and the good is not sold. Otherwise, the auction process goes on.

Between step 1 and 2, if all the bidders accepted to pay, they pay c to the seller.

Step 2 : If all the bidders paid the organization fee, the seller proceeds with the auction²⁵.

We can build an optimal auction format using this type of pre-auction round²⁶.

Proposition 8 *Consider a second price auction with reserve price R such that $R = \frac{1-F(R)}{f(R)}$ and a pre-auction round with an organization fee equal to $\frac{\theta}{1+\theta} \int_R^1 (t-R)f(t)dt$. This auction has a unique symmetric equilibrium with undominated strategies in which bidders participate with a strictly positive probability. In step 1, both bidders accept to pay the organization fee whatever their valuations are. In step 2, a bidder i does not make a submission if $v_i < R$ and submits $v_i - \int_R^{v_i} \left(\frac{1-F(t)}{1-F(R)}\right)^{\frac{1-\theta}{\theta}} dt$ if $v_i \geq R$.*

Proposition 9 *There is no allocation mechanism in which buyers have a positive reservation utility that gives a strictly higher expected revenue to the seller than the auction format described in proposition 8.*

We saw that a key element that makes the auction with horizontal toeholds different from a standard auction is the strictly positive reservation utility of the bidders. With the organization fee, the seller can fix the reservation utility to any positive level, zero being the best choice for him. Therefore, it is always possible to build an optimal auction with the help of this organization fee. We built an optimal auction based on a second price auction, it could have been done as well with a first price auction and a different (lower) value for the organization fee. As long as the seller can credibly commit not to sell the good if one of the bidders refuses to pay the organization fee, he can build a revenue maximizing auction based on any standard auction format.

4.4 Extension to the n bidders case

We derived all our results in an environment with 2 bidders which seems more plausible in the case of horizontal toeholds. However, we may wonder whether these results can be extended to a wider framework with more than two bidders. In fact, we intend to show that the revenue ranking of the auction formats remains true with $n > 2$ bidders. To do so, we consider the following framework.

There are n bidders, with $n \geq 2$. Each bidder i has a valuation drawn from an independent distribution function F_i . All the F_i are identical, equal to F which satisfies the

²⁵The details of the auction format are given to the sellers before the beginning of step 1.

²⁶We assume that bidders have lexicographic preferences, if it does not affect their expected profit, they strictly prefer the good to be sold.

monotone hazard rate condition. Each bidder i owns a fraction $\theta \in (0, \frac{1}{n-1})$ of the capital of all his opponents. If we define q_k as the probability that bidder k wins the auction and p_k the expected payment from bidder k to the seller, then the preferences of bidder i can be represented by the following utility function :²⁷

$$U_i = \frac{1 + (2 - n)\theta}{1 + (3 - n)\theta}(q_i v_i - p_i) + \frac{\theta}{1 + (3 - n)\theta} \sum_{k \neq i} q_k v_k - p_k$$

As in the case with 2 bidders, we can identify the unique symmetric equilibrium with both auction formats.

Proposition 10 *There is a unique symmetric equilibrium of the second price auction. For any $i = 1, 2, \dots, n$ bidder i bids $b_H^{II}(v_i, n)$ where:*

$$b_H^{II}(v_i, n) = v_i - \int_0^{v_i} \left(\frac{1 - F(v_i)}{1 - F(t)} \right)^{\frac{1+(1-n)\theta}{\theta}} dt$$

Proposition 11 *There is a unique symmetric equilibrium of the first price auction. For $i = 1, 2, \dots, n$ bidder i bids $b_H^I(v_i, n)$ where:*

$$b_H^I(v_i, n) = v_i - \int_0^{v_i} \left(\frac{F(t)}{F(v_i)} \right)^{\frac{(1+(1-n)\theta)(n-1)}{1+(2-n)\theta}} dt$$

Whatever the number of bidder is, there always exists a symmetric and efficient equilibrium with both auction formats. We can also establish that the ranking of the auction formats (in terms of expected price) remains the same independently from the number of bidders

Proposition 12 *For any $n \geq 2$ and for any $\theta \in (0, \frac{1}{n-1})$, the expected price is strictly higher with a first price auction than with a second price auction.*

This proposition shows that the results we obtained do not depend on the number of bidders. They remain true with $n > 2$ bidders.

²⁷Let $h(\theta, n)$, be the fraction of an extra profit of firm $k \neq i$ that firm i gets depending on θ and n . If firm k receives 1 Euro, all the other firms get $h(\theta, n)$ Euro and firm k gets $1 + \theta(n - 1)h(\theta, n)$ Euro. The value of all the firms raises by : $1 + (1 + \theta)(n - 1)h(\theta, n)$. Since we suppose that any extra profit is distributed to the shareholders through dividends, none of the firm keeps any money from this Euro. This means that the amount of the dividends given to the "other shareholders", those who do not have crossholdings is equal to 1. This can be written as follows : $(1 - (n - 1)\theta)(1 + (1 + \theta)(n - 1)h(\theta, n)) = 1$ which gives $h(\theta, n) = \frac{\theta}{(1+\theta)(1-(n-1)\theta)}$ and $1 + \theta(n - 1)h(\theta, n) = \frac{1+(2-n)\theta}{(1+\theta)(1-(n-1)\theta)}$. we can normalize and use $\frac{1+(2-n)\theta}{1+(3-n)\theta}$ and $\frac{\theta}{1+(3-n)\theta}$.

5 Conclusion

In the presence of cross-shareholdings, we showed that an expected revenue maximizer seller is always better off choosing a first price auction. We can apply this result, for instance, to the RVI/Volvo Trucks case that we presented in the introduction. We derive that for the clients of RVI and Volvo Trucks, among the standard auction formats, the first price auction was the best choice in order to strengthen the competition among them and to avoid as much as possible the negative effects of crossholdings.²⁸

When bidders own a share of the seller or a fraction of the good for sale, the expected revenue is higher with the second price auction than with the first price auction. If we apply this result to the Global One case, we derive that a second price auction was more in favor of Sprint, the seller and a first price auction was more in favor of France Télécom and Deutsche Telekom, the buyers.

We may also notice that, in that case, it is not obvious that the seller was the one who chose the auction format. As a matter of fact, the two possible *buyers* owned 80% of Global One while, the *seller* owned only 20% of Global One. Therefore, it is unclear who really controlled the agenda.

This remark suggests a natural extension to this work. It would consist in modelling a pre-auction bargaining about the choice of the auction procedure. This may be treated in future work. For the time being, we can make the following remark. We established in corollary 2 that bidders' preferences for an auction format over another one do not depend on their valuations. Then, without going more into the details of this pre-auction bargaining, we can say that it will not allow to directly extract information about bidders' valuations²⁹.

We can also give a broader interpretation of these results. As a matter of fact, losing bidders may care about the final price in many other situations. We could analyze such situations with our model: α and θ would be, respectively, the coefficient of mutual *malevolence* and the coefficient of mutual *benevolence* among bidders. With this interpretation, we could extend the application field of our results. We would derive the following. In order to benefit more from the effects of mutual malevolence among bidders, a seller should choose the second price auction. On the other side, to protect himself from mutual benevolence among bidders, he should choose a first price auction.

This interpretation is reminiscent of some insights of the existing auction theory literature, although there exists no general study of this issue. Let us give two examples. First, the collusion situation. In that case, we may consider that bidders are mutually benevolent. It has been shown that collusion is much easier to sustain with a second price auction than

²⁸As far as we know, the format that was effectively used was close to a first price auction. A priori, it was not chosen because of the considerations we studied.

²⁹Contrary to what de Frutos and Kittsteiner (2006) derive in a partnership dissolution environment, here, the choice of the auction format does not affect the allocation, it only affects the surplus distribution

with a first price auction (see Robinson (1985) on this issue). Second, budget-constrained bidders in two sequential auctions. In the first item auction, the losing bidder prefers that the winning bidder pays a high price. If he spends more in the first auction, this winning bidder will be a less dangerous opponent in the following auction. Thus, we can talk of mutual malevolence in the first auction. Pitchik and Schotter (1988) studied that case, focusing on the revenue of the first auction. They showed that the first price auction raises a lower revenue than the second price auction, in the first item auction, both in theory and in practice.

In these two examples, the revenue ranking follows the same general logic as the one we observed in our model. In the benevolent (resp: malevolent) case, a first price auction (resp: second price auction) generates more revenue.³⁰ This tends to show that the results we found should be applicable, more generally, to situations in which a seller has to deal with mutual benevolence or mutual malevolence among bidders. Proving this intuition will be the object of future research.

6 Appendix

6.1 Proof of proposition 1

As a first step, we must show that any symmetric equilibrium bidding function of the second price auction, b must satisfy the following conditions: b is continuous in the interval $[0, 1)$, strictly increasing in the interval $[0, 1]$ and $\lim_{v \rightarrow 1^-} b(v) = 1$.³¹

First, let us prove that b is nondecreasing: If $\underline{v} < \bar{v}$, then $b(\bar{v}) < b(\underline{v})$ is impossible. As a matter of fact, as $b(\underline{v})$ is a best response for a bidder with valuation \underline{v} , a bidder with valuation \bar{v} can profitably deviate by submitting $b(\underline{v})$ rather than $b(\bar{v}) < b(\underline{v})$. Thus, b must be nondecreasing. We can also exclude the possibility that b has an atom (an interval of valuations for which bidder i submits the same bid). As a matter of fact, it is impossible that, an interval of types, bidder 2 prefers to quit simultaneously with bidder 1's atom rather than leave just before or just after.

Now, let us show that b must be continuous on $[0, 1)$. Suppose that b has a gap in $v^* \in (0, 1)$. Since bidders strictly prefer to sell their shares for the highest possible price and b is strictly increasing, there always exist an $\varepsilon > 0$ such that a bidder with valuation $v^* - \varepsilon$ is strictly better off submitting $\frac{\lim_{+} b(v^*) + \lim_{-} b(v^*)}{2}$ rather than $b(v^* - \varepsilon)$ which means that b is not constitutive of a symmetric equilibrium. Thus, b must be continuous on $(0, 1)$. The

³⁰We would not obtain such results if we assumed that losing bidders derive a fixed externality that only depends on the identity of the loser. Jehiel and Moldovanu (2000) showed that, in that case, there is no such revenue ranking. The revenue ranking relies on bidders caring about the price paid by the winner.

³¹At the equilibrium, the exact value of $b(1)$ has no importance provided that $b(1) \geq 1$. For the sake of simplicity, we will consider that $b(1) = 1$.

continuity in 0 can be proved the same way.

Finally, suppose that $\lim_{v \rightarrow 1^-} b(v) \neq 1$. As bidding less than his valuation is a dominated strategy, we must have $\lim_{v \rightarrow 1^-} b(v) > 1$. Since b is increasing, we also have $1 < \lim_{v \rightarrow 1^-} b(v) \leq b(1)$. However, if a bidder has a valuation 1 for the good, he strictly prefers that his opponent wins the auction for a price 1 rather than buying the good for a price exceeding 1. Thus, a bidder whose valuation for the good is 1 would be strictly better off bidding 1 than $b(1) > 1$. We obtain a contradiction, therefore, $\lim_{v \rightarrow 1^-} b(v) = 1$.

Now, consider a bidding function b respecting these conditions. If bidder j bids according to b , it is a dominated strategy for bidder i to bid less than $b(0)$ and he cannot be better off bidding more than $b(1)$ than he would be bidding $b(1)$. Thus, we can restrict bidder i 's strategy to the choice of a $g : [0, 1] \rightarrow [0, 1]$, such that he bids $b(g(v_i))$. Let us define $U_i^{II}(v_i, \hat{v}_i)$ as the expected utility of bidder i with valuation v_i bidding $b(\hat{v}_i)$. As we can limit our study to the case $\hat{v}_i \in [0, 1]$, we obtain the following expression

$$U_i^{II}(v_i, \hat{v}_i) = \int_0^{\hat{v}_i} (v_i - (1 - \alpha)b(t))f(t)dt + \alpha \int_{\hat{v}_i}^1 b(\hat{v}_i)f(t)dt$$

We obtain the following necessary and sufficient condition³² for b to be a symmetric equilibrium strategy:

$$\frac{\partial U_i^{II}(v, \hat{v}_i)}{\partial \hat{v}_i} = 0 \text{ for } \hat{v}_i = v_i, \text{ for } v_i \in [0, 1]$$

This can be written:³³

$$v_i f(v_i) - b(v_i)f(v_i) + \alpha(1 - F(v_i))b'(v_i) = 0 \quad (1)$$

To solve this differential equation, we note that

$$d[b(v_i)(1 - F(v_i))^{1/\alpha}] = [b'(v_i)(1 - F(v_i))^{1/\alpha} - \frac{1}{\alpha}f(v_i)b(v_i)(1 - F(v_i))^{\frac{1-\alpha}{\alpha}}]dv_i \quad (2)$$

Then, from expressions (A.1) and (A.2) we derive

$$d[b(v_i)(1 - F(v_i))^{\frac{1}{\alpha}}] = [-\frac{1}{\alpha}(1 - F(v_i))^{\frac{1-\alpha}{\alpha}}v_i f(v_i)]dv_i$$

$$\int_{v_i}^1 d[b(t)(1 - F(t))^{\frac{1}{\alpha}}]dt = \int_{v_i}^1 -\frac{1}{\alpha}(1 - F(t))^{\frac{1-\alpha}{\alpha}}t f(t)dt$$

³²We can exclude corner solutions.

³³We assume that b' is well defined on the considered interval, a condition that is verified at the equilibrium.

Integrating by parts, we obtain

$$-b(v_i)(1 - F(v_i))^{\frac{1}{\alpha}} = -v_i(1 - F(v_i))^{\frac{1}{\alpha}} - \int_{v_i}^1 (1 - F(t))^{\frac{1}{\alpha}} dt$$

And finally

$$b(v_i) = v_i + \int_{v_i}^1 \left(\frac{1 - F(t)}{1 - F(v_i)} \right)^{\frac{1}{\alpha}} dt$$

Q.E.D.

6.2 Proof of proposition 2

The reasoning is the same as for the proof of proposition 1. For the same reasons, b is continuous and strictly increasing. As it is a dominated strategy for a bidder to bid more than his valuation, we derive $b(0) = 0$ (which replaces the limit condition $\lim_{v \rightarrow 1^-} b(v) = 1$). Without repeating the arguments of the proof of proposition 1, we can directly study the expression

$$U_i^I(v_i, \hat{v}_i) = F(\hat{v}_i)(v_i - (1 - \alpha)b(\hat{v}_i)) + \alpha \int_{\hat{v}_i}^1 b(t)f(t)dt$$

The equivalent of expressions (A.1) and (A.2) are

$$v_i f(v_i) - b(v_i)f(v_i) - (1 - \alpha)F(v_i)b'(v_i) = 0 \quad (3)$$

$$d[b(v_i)F(v_i)^{\frac{1}{1-\alpha}}] = [b'(v_i)F(v_i)^{\frac{1}{1-\alpha}} + \frac{1}{1-\alpha}f(v_i)b(v_i)F(v_i)^{\frac{\alpha}{1-\alpha}}]dv_i \quad (4)$$

Then, from equations (A.3) and (A.4) we derive

$$d[b(v_i)F(v_i)^{\frac{1}{1-\alpha}}] = \frac{1}{1-\alpha}v_i f(v_i)F(v_i)^{\frac{\alpha}{1-\alpha}}$$

And with transformations identical to these of the proof of proposition 1, we obtain

$$b(v_i) = v_i - \int_0^{v_i} \left(\frac{F(t)}{F(v_i)} \right)^{\frac{1}{1-\alpha}} dt$$

Q.E.D.

6.3 Proof of proposition 3

The Revenue Equivalence Theorem says that the revenue of an auction is a function of the allocation rule minus bidders' reservation utilities of bidders. In the present case, the

allocation is identical with both auction formats. Thus, in order to compare the expected revenues of these two auction formats, we can focus on the comparison of expected utility of lowest type, $v = 0$.

In the first price auction, the reservation utility is equal to

$$\alpha E(b^I(v_j)) = \alpha \int_0^1 [t - \int_0^t (\frac{F(x)}{F(t)})^{\frac{1}{1-\alpha}} dx] f(t) dt$$

In the second price auction, the reservation utility is equal to

$$\alpha b^{II}(0) = \alpha \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} dt$$

As $\alpha > 0$, comparing these two formulae is equivalent to determining the sign of

$$S_V = \int_0^1 [(1 - F(t))^{\frac{1}{\alpha}} - (tf(t) - \int_0^t f(t) (\frac{F(x)}{F(t)})^{\frac{1}{1-\alpha}} dx)] dt$$

Which we can rewrite as:

$$\begin{aligned} S_V &= \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} dt - \int_0^1 \int_0^t [1 - (\frac{F(x)}{F(t)})^{\frac{1}{1-\alpha}}] dx f(t) dt \\ &= \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} dt - \int_0^1 \int_x^1 [1 - (\frac{F(x)}{F(t)})^{\frac{1}{1-\alpha}}] f(t) dt dx \\ &= \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} dt - \int_0^1 [F(t) + \frac{1-\alpha}{\alpha} F(x)^{\frac{1}{1-\alpha}} F(t)^{\frac{-\alpha}{1-\alpha}}]_{t=x}^{t=1} dx \\ &= \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} dt - \int_0^1 [1 + \frac{1-\alpha}{\alpha} F(x)^{\frac{1}{1-\alpha}} - \frac{1}{\alpha} F(x)] dx \\ S_V &= \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} - 1 + \frac{1}{\alpha} F(t) - \frac{1-\alpha}{\alpha} F(t)^{\frac{1}{1-\alpha}} dt \end{aligned}$$

As $h(x) = -(1-x)^{\frac{1}{\alpha}} + 1 - \frac{1}{\alpha}x + \frac{1-\alpha}{\alpha}x^{\frac{1}{1-\alpha}} > 0$,³⁴ for $x \in (0, 1)$, $\alpha \in (0, \frac{1}{2})$, S_V is strictly negative. The reservation utility is higher with a first price auction than with a second price auction. Therefore, the expected revenue of the second price auction is higher than the expected revenue of the first price auction. Q.E.D.

6.4 Proof of proposition 4

It is a well-known result that in the class of mechanism which always allocates the good, an expected revenue maximizing mechanism must satisfy the two following properties : 1)

³⁴This can be demonstrated showing that $\forall \alpha \in (0, \frac{1}{2})$, $h(x)$ on the considered interval is convex then concave and convex. Besides $h(0) = h(1) = 0$ and $h'(0) \geq 0, h'(1) \leq 0, \forall \alpha \in (0, \frac{1}{2})$.

The good must be allocated to the bidder with the highest marginal revenue and 2) Bidders' reservation utility must be equal to zero.

When bidders have the same distribution function for their valuations satisfying the monotone hazard rate condition and no vertical toeholds, both the first price and the second price auction satisfy these properties. Therefore, they are constrained³⁵ revenue maximizing auction format and the seller cannot increase his expected revenue by giving shares away. In fact, he could only reduce its expected revenue by giving shares since he cannot "improve" the allocation rule and he would raise bidders' reservation utility. Q.E.D.

6.5 Proof of proposition 5

Once more, the reasoning of the proof is the same as for the proof of proposition 1. All the first part is equivalent, we can directly study the expected utility

$$U_i^{II}(v_i, \hat{v}_i) = \frac{1}{1+\theta} \left(\int_0^{\hat{v}_i} (v_i - b(t))f(t)dt + \theta \int_{\hat{v}_i}^1 (t - b(\hat{v}_i))f(t)dt \right)$$

A necessary and sufficient condition³⁶ for b to be a symmetric equilibrium strategy is

$$\frac{\partial U_i^{II}(v_i, \hat{v}_i)}{\partial \hat{v}_i} = 0 \text{ for } \hat{v}_i = v_i, \forall v_i \in [0, 1]$$

This can be written:³⁷

$$v_i f(v_i) - b(v_i) f(v_i) - \frac{\theta}{1-\theta} (1 - F(v_i)) b'(v_i) = 0$$

As $b(0) = 0$, the solution of the differential equation is

$$b(v_i) = (1 - F(v_i))^{\frac{1-\theta}{\theta}} \int_0^{v_i} \frac{1-\theta}{\theta} (1 - F(t))^{\frac{-1}{\theta}} f(t) dt$$

Integrating by parts, we obtain

$$b(v_i) = v_i - \int_0^{v_i} \left(\frac{1 - F(v_i)}{1 - F(t)} \right)^{\frac{1-\theta}{\theta}} dt$$

Q.E.D.

³⁵Constrained because we restrict our attention to auction mechanisms which always allocate the good;

³⁶We can exclude corner solutions.

³⁷We assume that b' is well defined on the considered interval, a condition that is verified at the equilibrium.

6.6 Proof of proposition 6

We apply the same arguments as in the proof of proposition 2 and directly study the expression

$$U_i^I(v_i, \widehat{v}_i) = \frac{1}{1+\theta} (F(\widehat{v}_i)(v_i - b(\widehat{v}_i)) + \theta \int_{\widehat{v}_i}^1 (t - b(t))f(t)dt)$$

The first-order condition is

$$v_i f(v_i) - b(v_i) f(v_i) - \frac{1}{1-\theta} F(v_i) b'(v_i) = 0$$

As $b(0) = 0$, the solution of the differential equation is

$$b(v_i) = \lim_{x \rightarrow 0} \exp\left[\int_x^{v_i} -(1-\theta) \frac{f(t)}{F(t)} dt\right] \int_x^{v_i} \exp\left[\int_0^t (1-\theta) \frac{f(s)}{F(s)} ds\right] (1-\theta) \frac{f(t)}{F(t)} t dt$$

Integrating by parts, we obtain

$$b(v_i) = v_i - \int_{v_i}^1 \left(\frac{F(t)}{F(v_i)}\right)^{1-\theta} dt$$

Q.E.D.

6.7 Proof of proposition 7

As in the proof of proposition 3, we can use the Revenue Equivalence Theorem. The allocation is the same with the first price auction and the second price auction. Therefore, we directly focus on the reservation utilities comparison.

In the first price auction the reservation utility of both bidders is:

$$\frac{\theta}{1+\theta} E[v_j - b_H^I(v_j)] = \frac{\theta}{1+\theta} \int_0^1 (t - b_H^I(t)) f(t) dt$$

In the second price auction, the reservation utility is:

$$\frac{\theta}{1+\theta} E[v_j - b_H^{II}(0)] = \frac{\theta}{1+\theta} \int_0^1 t f(t) dt$$

$\theta > 0$ and $\int_0^1 b_H^I(t) f(t) dt > 0$, then the reservation utility is strictly higher with the second price auction than with the first price auction. As a result, the expected revenue is higher with the first price auction than with the second price auction. Q.E.D.

6.8 Proof of proposition 8

First, for any \underline{v} and \bar{v} such that $0 \leq \underline{v} < \bar{v} \leq 1$, if at the equilibrium, a bidder with valuation \underline{v} accepts to pay the organization fee, then he also accepts to pay it when his valuation is \bar{v} (because, by participating he can get at least what he would get if he were to participate with a valuation \underline{v}). Then, the strategy of bidder i in the first step can be represented by a threshold (for a valuation below this threshold, a bidder does not accept to pay the organization fee and, for a valuation above this threshold he accepts to pay the organization fee). Let us call this threshold v^* .

Suppose that $v^* \in [R, 1]$ ³⁸, then we can show that, in the second step, bidders would submit bids increasing with their valuations and higher than R . Now, bidder i with valuation 0 would be better off accepting to pay the organization fee in step 1 than refusing. If he refuses, his utility is zero. If he accepts, either bidder j refuses and bidder i gets zero or bidder j also accepts. If bidder j accepts, in step 2, bidder i can submit 0. Bidder j wins the auction with probability 1 and the expected utility of bidder i is $\frac{1}{1+\theta}(-\frac{\theta}{1+\theta} \int_R^1 (t-R)f(t)dt + \theta(\int_{v^*}^1 (t-R)\frac{f(t)}{1-F(v^*)}dt - \frac{\theta}{1+\theta} \int_R^1 (t-R)f(t)dt)$ which is strictly positive since $v^* \geq R > 0$. Then $v^* \in [R, 1]$ cannot be part of an equilibrium.

Suppose that $v^* \in (0, R)$. Bidder i with valuation v^* pays the organization fee and he has a dominant strategy in the auction : not to participate. His expected utility is at least equal to zero (otherwise, he would be better off not paying the organization fee). Now if bidder i has a valuation \hat{v} strictly lower than v^* , he can also pay the organization fee, not participate in the auction and obtain the same positive expected utility (which he prefers to a situation in which the good would remain unsold because of his lexicographic preferences). Therefore, $v^* \in (0, R)$ cannot be part of an equilibrium either.

Now, suppose that $v^* = 0$. In the auction itself, a bidder with valuation strictly lower than R has a dominant strategy : not to participate in the auction. If there exists a bidding function, b , such that, at the symmetric equilibrium, a bidder with valuation $\hat{v} \geq R$ submits $b(\hat{v}) \geq R$ with b strictly increasing, it must be such that (following the proof of proposition 5):

$$v_i f(v_i) - b(v_i) f(v_i) - \frac{\theta}{1-\theta} (1 - F(v_i)) b'(v_i) = 0$$

As $b(R) = R$, the solution of the differential equation is

$$b(v_i) = v_i - \int_R^{v_i} \left(\frac{1 - F(v_i)}{1 - F(t)} \right)^{\frac{1-\theta}{\theta}} dt$$

We need to check that with such a bidding strategy in step 2, it is an equilibrium for the bidders always to pay the organization fee in step 1. To do so, we only need to check

³⁸Note that if $v^* = 1$, bidders participate in the auction with a probability zero. We excluded this kind of equilibrium.

that a bidder with a valuation $v = 0$ is better off paying the organization fee. If he pays, his expected utility is equal to :

$$\begin{aligned} & \frac{1}{1-\theta} \left(-\frac{\theta}{1+\theta} \int_R^1 (v-R)dv + \theta \left(-\frac{\theta}{1+\theta} \int_R^1 (v-R)dv + \int_R^1 (v-R)dv \right) \right) \\ &= \frac{1}{1-\theta} \left(-(1+\theta) \frac{\theta}{1+\theta} \int_R^1 (v-R)dv + \theta \int_R^1 (v-R)dv \right) \end{aligned}$$

This is equal to zero and a bidder with valuation zero prefers paying the organization fee because of his lexicographic preferences.

Q.E.D.

6.9 Proof of proposition 9

It is a well known result that an auction is optimal if and only if it allocates the good to the bidder with the highest marginal revenue (provided that this marginal revenue is positive) and bidders' reservation utility is equal to zero.

Since we assumed that the distribution function satisfies the monotone hazard rate condition, the marginal revenue of a bidder is strictly increasing with his valuation and positive when $v \geq v^*$ with v^* such that $v^* = \frac{1-F(v^*)}{f(v^*)}$.

The reserve price of the auction is precisely equal to v^* and the equilibrium of the auction is such that the bidder with the highest valuation above v^* wins the auction. Then, the allocation of the auction coincides with the optimal allocation rule.

Now, we need to check bidders' reservation valuation. A bidder with a valuation equal to zero pays the organization fee equal to $\frac{\theta}{1+\theta} \int_R^1 (t-R)f(t)dt$ and derives an expected profit from the sale of the good equal to $\frac{\theta}{1+\theta} \left(\int_R^1 (t-R)f(t)dt - \frac{\theta}{1+\theta} \int_R^1 (t-R)f(t)dt \right)$. Therefore, his expected utility is equal to

$$\begin{aligned} & -\frac{1}{1+\theta} \frac{\theta}{1+\theta} \int_R^1 (t-R)f(t)dt + \frac{\theta}{1+\theta} \left(\int_R^1 (t-R)f(t)dt - \left(\frac{\theta}{1+\theta} \right)^2 \int_R^1 (t-R)f(t)dt \right) \\ &= \frac{1}{(1+\theta)^2} \left[-\theta \int_R^1 (t-R)f(t)dt + (\theta + \theta^2) \int_R^1 (t-R)f(t)dt - \theta^2 \int_R^1 (t-R)f(t)dt \right] \end{aligned}$$

which is equal to 0.

Q.E.D.

6.10 Proof of proposition 10

Everything is identical to the proof of proposition 5 except that the first equation is replaced by

$$U_i^{II}(v_i, \widehat{v}_i) = (1 + (2 - n)\theta)(n - 1) \int_0^{\widehat{v}_i} (v_i - b(t))f(t)F^{n-2}dt + \theta(n - 1) \int_{\widehat{v}_i}^1 (t - b(\widehat{v}_i))f(t)F^{n-2}dt \\ + \theta(n - 2)(n - 1) \int_{\widehat{v}_i}^1 \left(\int_t^1 (x - b(t))f(x)dx \right) f(t)F^{n-3}(t)dt$$

Which gives the following differential equation :

$$(v_i - b(v_i))f(v_i) - \frac{\theta}{1 + (1 - n)\theta}(1 - F(v_i))b'(v_i) = 0$$

As $b(0) = 0$, the solution of the differential equation is

$$b(v_i) = v_i - \int_0^{v_i} \left(\frac{1 - F(v_i)}{1 - F(t)} \right)^{\frac{1+(1-n)\theta}{\theta}} dt$$

Q.E.D.

6.11 Proof of proposition 11

We apply the same arguments as in the proof of proposition 6 and directly study the expression

$$(1 + (2 - n)\theta)(F^{n-1}(\widehat{v}_i)(v_i - b(\widehat{v}_i)) + \theta(n - 1) \int_{\widehat{v}_i}^1 (t - b(t))F^{n-2}f(t)dt)$$

The first-order condition is

$$v_i f(v_i) - b(v_i) f(v_i) - \frac{1 + (2 - n)\theta}{(n - 1)(1 + (1 - n)\theta)} F(v_i) b'(v_i) = 0$$

As $b(0) = 0$, the solution of the differential equation is

$$b(v_i) = v_i - \int_{v_i}^1 \left(\frac{F(t)}{F(v_i)} \right)^{\frac{(n-1)(1+(1-n)\theta)}{1+(2-n)\theta}} dt$$

Q.E.D.

6.12 Proof of Proposition 12

We prove the proposition by induction.

Note that we already established this proposition for $n = 2$. Now, suppose that it is verified for $n = \widetilde{n} \geq 2$ for any value of $\theta \in (0; \frac{1}{\widetilde{n}-1})$. Let us consider the case with $\widetilde{n} + 1$ bidders and any $\widehat{\theta} \in (0, \frac{1}{\widetilde{n}})$.

First, let us observe that since $\hat{\theta} < \frac{1}{\tilde{n}}$, there always exists a $\theta' < \frac{1}{\tilde{n}-1}$ such that $\frac{\hat{\theta}}{1+(1-\tilde{n})\hat{\theta}} = \frac{\theta'}{1+(2-\tilde{n})\theta'}$. In a second price auction, bidders make the same equilibrium submissions when there are \tilde{n} bidders and a toehold of size θ' or when there are $\tilde{n} + 1$ bidders and a toehold of size $\hat{\theta}$. In a first price auction, bidders make higher equilibrium submissions when there are $\tilde{n} + 1$ bidders and a toehold of size $\hat{\theta}$ or when there are \tilde{n} bidders and a toehold of size θ' .

Since, with both auctions, the allocation is the same (i.e. the bidder with the highest valuation wins the auction), we just need to compare reservation utility. The reservation utility is equal to $\hat{\theta}$ times the expected utility of the bidder with the highest valuation conditional on one bidder having a valuation equal to zero. In a second price auction, this is equal to the expected utility of the bidder with the highest valuation, with \tilde{n} bidders and a toehold of size θ' (we will denote it : $E\bar{U}^{II}(\theta', \tilde{n})$). In a first price auction, this is less than the expected utility of the bidder with the highest valuation, with \tilde{n} bidders and a toehold of size θ' (we will denote it : $E\bar{U}^I(\theta', \tilde{n})$).

With \tilde{n} bidders, whatever the size of the toehold is (strictly positive), the expected price paid is strictly higher with a first price auction than with a second price auction and the allocation is the same. Therefore, since the allocation is the same with both auction formats, whatever his valuation is, the expected utility of a bidder is strictly higher with a second price than with a first price auction. Then $E\bar{U}^{II}(\theta', \tilde{n}) > E\bar{U}^I(\theta', \tilde{n})$. This means that, with $\tilde{n} + 1$ bidders and a toehold of size $\hat{\theta}$, the reservation utility is strictly higher in a second-price than in a first price auction and the expected revenue of the seller is strictly higher in a first price auction.

Q.E.D.

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