A reference case for mean field games models

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Abstract

In this article, we present a reference case of mean field games. This case can be seen as a reference for two main reasons. First, the case is simple enough to allow for explicit resolution: Bellman functions are quadratic, stationary measures are normal and stability can be dealt with explicitly using Hermite polynomials. Second, in spite of its simplicity, the case is rich enough in terms of mathematics to be generalized and to inspire the study of more complex models that may not be as tractable as this one.

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1 A short introduction to mean field games

Mean field games have been introduced by J.-M. Lasry and P.-L. Lions (2006) in two seminal papers. They have been used in economic models and noticeably to model endogeneous growth. Here, we want to detail a reference case that can be used to build and study a lot of mean field games models.

1.1 An idea from physics...

To well understand the nature of mean field games, the best thing to do is certainly to focus on the notion of "mean field". This notion is in fact inspired from particle physics. Typically, particle physicists are interested in interactions between so many particles that they cannot use traditional physics and study each interaction among couples or triples of particles.

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Instead of that they rely on a statistical idea: to study interactions between particles they use a media and the mean field is this media. To clarify, this field is created, in a certain sense, by the particles and impact the behavior of the particles "who" created it. Hence, the interactions between particles is summed up by the interaction between every single particle and the mean field, which is, in some sense, representative of the particles as a whole.
The simplest example is air pressure: pressure is created by the microscopic movements of the particles and impacts particles in a macroscopic way, creating winds for instance. Clearly, this approach is more meaningful than a complete description of the interactions between air particles. Although this example is simple, the type of reasoning is important and used in quantum mechanics.

1.2 ... that can be used in game theory and economics

The same reasoning can be used when it comes to model strategic interactions between many agents in economics. A "mean field" could be used to have a relevant representation of reality. This remark is the starting point of mean field game theory.
As a first example, the heart of modern economics that is the general equilibrium theory, can be considered a mean field game where the mean field is obviously the vector of prices. Prices are indeed a relevant summary of interactions between agents and, in turn, they influence each agent behavior. This approach certainly clarifies what a market is: the market exists because of agents interactions and, in turn, the market induces individual behaviors.
The market is an example of mean field games but mean field games are in fact a general tool to embed externality in models since mean fields are not constrained to be prices. Penetration rates for technologies such as wind turbines or solar panels are instances of mean field. Other examples can be found like page ranking on the internet or ranking of fund managers.

The new theory developed to study mean field games brings a comprehensive mathematical framework, some new concepts and a new way to build models.

1.3 The definition of mean field games

The general framework of mean field game theory is given by four hypotheses:

- Rational expectations
- Continuum of agents
- Agents anonymity

- Social interactions of the mean field type

The first three hypotheses are common in game theory. The first one - the rational expectation hypothesis - has been introduced in the 60's and is now widely accepted among game theorists. The second hypothesis is often used to model games with a large number of players. It's a rather well accepted approximation that has been used for tractability purposes and here, for mean field games, the limit of a game with N players as N goes to infinity has been studied in Lasry et al. (2007) to support this hypothesis. The third hypothesis has always been implicit in game theory but is worth recalling. Basically, it says that agents are anonymous in the sense that any permutation of the agents does not change the outcome of the game.

The fourth hypothesis is specific to mean field games and is an hypothesis on interactions between players[1]. The main idea is that a given agent cannot take into account every single agent she is going to interact with. Therefore, every agent is going to make a decision according to some statistics regarding the overall community of agents. Moreover, this fourth assumption means that an agent is really atomized in the continuum and has no power but a marginal one. Since she cannot influence (but marginally) the behavior of other agents, she has no other choices than considering a strategy that depends only on herself and on information about the overall community, this information being enclosed in the mean field.

In other words, the couple (x, m) is sufficient and exhaustive to explain interactions, where x is a personal characteristics and m the distribution of those characteristics in the population.

1.4 A first class of mean field games

From a mathematical point of view, a first class of mean field games, and this class is the purpose of this paper, appeared in a stochastic control form. With this representation, each agent can control - with a cost - the drift and/or the volatility of a diffusion process and maximizes (in expectancy) a utility criterium that depends on this dynamical process and on the mean field of the problem. This type of framework is really common in finance, in economics or in engineering and corresponds, in the deterministic case to variations calculus. Noticeably, even in the stochastic case, the problem, as far as the players are concerned as a whole, stays deterministic because of the continuum of players and the law of large numbers. In what follows, we are going to see that the equations of this first class of mean field games have a forward/backward structure: a backward PDE (Hamilton-Jacobi-Bellman) to model the individual backward induction process that explains each agent's choices ; a forward PDE (Kolmogorov) to model the evolution of the players as a whole, the evolution of the community.

[1]When we say interactions here, we mean it in the micro sense since we want a micro-foundation of the behaviors.

We are going to present this first class of mean field games in an abstract way, in the sense that we want the reader to understand the tools and hence we focus on a problem that has explicit solutions. Typically, one may understand the problem as a stochastic control problem in which each individual, in the continuum, chooses a characteristic in a state space to resemble other people (in addition to the wish to be at a given place in some parts - see below). Problems of that kind are quite common (even though specification may be different) if one thinks of technology choices for instance since agents may want to have a good technology but a technology that is widespread among others to avoid paying too much.

2 The general framework

In what follows, we consider a continuum of individuals (hereafter a population) that have preferences about resembling each other. This type of problem is typically of the mean field game sort where individuals pay a price to move from one point to another in the state space and have a utility flow that is a function of the overall distribution of individuals in the population. We are going to model it as follows:

- The state space is an n-dimensional space.
- Each agent has a "utility" function v that can be decomposed in two parts: a pure preference part $g : (t, x) \mapsto g(m(t, x))$ (where g is increasing to model the willingness to be like others) that represents what she gets from being having the characteristics x at time t (m is the distribution function of the population) and a pure cost part $h : \alpha \mapsto h(\alpha)$ that corresponds to the price to pay to make a move of size α in the state space (h is typically supposed to be increasing, strictly convex and such that $h(0) = 0$).
- Each agent discounts the time at rate ρ.
- Each agent's characteristics is moved by a brownian motion in dimension n (specific to herself).

The problem we are dealing with can therefore be written as a control problem:

$$u(t,x) = Max_{(\alpha_s)_{s>t}, X_t=x}\mathbb{E}\left[\int_t^T \left(g(m(s,X_s)) - h(|\alpha(s,X_s)|)\right) e^{-\rho(s-t)} ds\right]$$

with $dX_t = \alpha(t, X_t)dt + \sigma dW_t$.

As for any mean field game we use Lasry et al. (2006) to write the associated system of partial differential equations:

Proposition 1 (Mean Field Games PDEs). *The control problem is equivalent to the following system of PDEs*

$$(Hamilton - Jacobi) \qquad \partial_t u + \frac{\sigma^2}{2}\Delta u + H(\nabla u) - \rho u = -g(m)$$

$$(Kolmogorov) \qquad \partial_t m + \nabla \cdot (mH'(\nabla u)) = \frac{\sigma^2}{2}\Delta m$$

where $H(p) = Max_a\,(ap - h(a))$
Additional conditions are: $m(0,\cdot)$ *given,* $u(T,\cdot) = 0$ *and* $\forall t, m(t,\cdot)$ *is a probability distribution function.*

Our goal here is to find stationary solutions of this problem in several special cases where we always suppose that T is replaced by $+\infty$ [2]:

$$(Hamilton - Jacobi) \qquad \frac{\sigma^2}{2}\Delta u + H(\nabla u) - \rho u = -g(m)$$

$$(Kolmogorov) \qquad \nabla \cdot (mH'(\nabla u)) = \frac{\sigma^2}{2}\Delta m$$

with m a probability distribution function.

3 The quadratic cost framework

3.1 Presentation

One of the simplest framework to deal with mean field games is to consider the special case of quadratic cost: $h(a) = \frac{1}{2}a^2$. This case is indeed simpler since it allows to replace the system of coupled PDEs by a single PDE, either on u or on m (the good variable is actually $\psi = \sqrt{m}$ as we will see later on). Consequently, we focus extensively on quadratic costs even though more complex models can be used to deal with problems involving congestion for instance.

The quadratic cost framework is characterized by a simple Hamiltonian ($H(p) = \frac{1}{2}p^2$) and therefore the system to solve is simplified:

Proposition 2 (Mean Field Games PDEs with quadratic costs). *With quadratic costs, the system can be written as:*

$$(Hamilton - Jacobi) \qquad \partial_t u + \frac{\sigma^2}{2}\Delta u + \frac{1}{2}|\nabla u|^2 - \rho u = -g(m)$$

$$(Kolmogorov) \qquad \partial_t m + \nabla \cdot (m\nabla u) = \frac{\sigma^2}{2}\Delta m$$

In its stationary form, the system is simply:

$$(Hamilton - Jacobi) \qquad \frac{\sigma^2}{2}\Delta u + \frac{1}{2}|\nabla u|^2 - \rho u = -g(m)$$

$$(Kolmogorov) \qquad \nabla \cdot (m\nabla u) = \frac{\sigma^2}{2}\Delta m$$

[2] The transversality condition that appears in the case of infinite horizon is not relevant in the stationary case.

3.2 From two coupled PDEs to one

We are going to enounce two propositions that show the interest of the quadratic costs.

Proposition 3 (One PDE in u). *Let's consider a couple (K, u) where K is a scalar. If (K, u) is a solution of the equations (1) and (1′) then $(u, K\exp(\frac{2u}{\sigma^2}))$ is a solution of our initial stationary problem.*

$$\frac{\sigma^2}{2}\Delta u(x) + \frac{1}{2}|\nabla u(x)|^2 - \rho u(x) = -g\left(x, K\exp(\frac{2u(x)}{\sigma^2})\right) \qquad (1)$$

$$\int K\exp(\frac{2u(x)}{\sigma^2}) = 1 \qquad (1')$$

Another way to look at the problem is to consider an equation in m or more exactly an equation in ψ where ψ is defined as the square root of m.

Proposition 4 (One PDE in $\psi = \sqrt{m}$). *Let's consider a couple (K, ψ) where K is a scalar. If (K, ψ) is a solution of the equations (2) and (2′) then, $m = \psi^2$ and $u = \sigma^2 \ln\left(\frac{\psi}{K}\right)$ are solutions of our initial stationary problem.*

$$\frac{\sigma^4}{2}\frac{\Delta\psi(x)}{\psi(x)} = \rho\sigma^2\ln(\frac{\psi(x)}{K}) - g(\psi^2(x)) \qquad (2)$$

$$\int_x \psi(x)^2 dx = 1 \qquad (2')$$

The partial differential equation in ψ invites us to consider the case where $(t, x) \mapsto g(m(t, x))$ is the logarithm function $\ln(m(t, x))$ as an example of our population problem that may be solved easily and explicitly. This is our next application.

4 Application to the logarithmic utility function

4.1 The basic framework

4.1.1 Presentation

We are going to build a very precise and explicit example that goes into the quadratic cost framework. We consider one population and we suppose that all people in the population have the same preference function which is simply $g(m(t, x)) = \ln(m(t, x))$.

These preferences mean that inside the population, people want to resemble one another. However, they are prevented to do so by the noise and our problem is to find the optimal behavior of individuals in such a context.

To sum up, we want to find stationary solutions to the problem:

$$u(t,x) = Max_{(\alpha_s)_{s>t},X_t=x}\mathbb{E}\left[\int_t^\infty \left(\ln(m(s,X_s)) - \frac{|\alpha(s,X_s)|^2}{2}\right)e^{-\rho(s-t)}ds\right]$$

with $dX_t = \alpha(t,X_t)dt + \sigma dW_t$

In other words, we want to find a solution of the following system of PDEs:

$$(Hamilton - Jacobi) \qquad \frac{\sigma^2}{2}\Delta u + \frac{1}{2}|\nabla u|^2 - \rho u = -\ln(m)$$

$$(Kolmogorov) \qquad \nabla\cdot(m\nabla u) = \frac{\sigma^2}{2}\Delta m$$

4.1.2 Resolution

Proposition 5 (Gaussian solutions). *Suppose that $\rho < \frac{2}{\sigma^2}$.*
There exist three constants, $s^2 > 0$, $\eta > 0$ and ω such that $\forall\mu \in \mathbb{R}^n$, if m is the probability distribution function associated to a gaussian variable $\mathcal{N}(\mu, s^2 I_n)$ and $u(x) = -\eta|x-\mu|^2 + \omega$, then (u,m) is a solution of our problem.

These three constants are given by:

- $s^2 = \frac{\sigma^4}{4-2\rho\sigma^2}$
- $\eta = \frac{1}{\sigma^2} - \frac{\rho}{2} = \frac{\sigma^2}{4s^2}$
- $\omega = \frac{1}{\rho}\left[\eta n\sigma^2 - \frac{n}{2}\ln\left(\frac{2\eta}{\pi\sigma^2}\right)\right]$

Interestingly, we can come back to the control parameter α. This control parameter describes the move each agent wants to make given her characteristics. We have the following result:

Proposition 6 (Optimal control). *In the framework of the preceding proposition, the optimal control parameter α is given by $\alpha(x) = -2\eta(x-\mu)$. This means that for any agent, her characteristics X_t follows an Ornstein-Uhlenbeck process that mean-reverts around μ:*

$$dX_t = -2\eta(X_t - \mu)dt + \sigma dW_t$$

Proof:

Because $H(p) = \frac{1}{2}p^2$, the optimal control is $\alpha(x) = H'(\nabla u(x)) = \nabla u(x)$.

The preceding proposition gives the result. □

4.2 Comments on the preceding example

The preceding example is interesting in the fact that we have been able to exhibit explicit solutions (this can be generalized to a more complex brownian motion). One caveat, though, is that these solutions are specific

to the logarithmic case.
Another problem with our setting is that we only describe possible stationary solutions and the possible path from an initial distribution to a stationary solution is not dealt with.
This comment leads to the third issue in this example which is the infinite number of solutions. This problem can be dealt with in a very simple way. It's indeed possible to say that, in addition to their willingness to be like each others, agents in the population love a certain characteristics μ^*. In that case, the stochastic control problem can be replaced by:

$$Max_{(\alpha_s)_{s>t}, X_t=x}\mathbb{E}\left[\int_t^{\infty}\left(\ln(m(s,X_s)) - \delta|X_s-\mu^*|^2 - \frac{|\alpha(s,X_s)|^2}{2}\right)e^{-\rho(s-t)}ds\right]$$

with

$$dX_t = \alpha(t, X_t)dt + \sigma dW_t$$

With this quadratic form, one can generalize the preceding computations and we get the following localization result.

Proposition 7 (Localization). *In this new problem any gaussian solution has to be centered in μ^*. The variance coefficient is $s^2 = \frac{\sigma^2}{4\eta}$ where η is now the unique positive solution of:*

$$2\eta^2 - \eta\left(\frac{2}{\sigma^2} - \rho\right) = \delta$$

5 Stability in the logarithmic case

Let's consider the logarithmic case of the last section and let's work for simplicity in dimension 1[3]. We have found stationary solutions of the problem and up to a translation we can consider that $\mu = 0$ so that the stationary solution we consider is:

$$u^*(x) = -\eta x^2 + \omega$$

$$m^*(x) = \frac{1}{\sqrt{2\pi s^2}}\exp\left(-\frac{x^2}{2s^2}\right)$$

An interesting question is the stability of this stationary solution.
We are going to consider two notions of stability. The first notion of stability is the classical physical notion of local stability. If an equilibrium is given, it will be said locally stable in the classical sense if, after a *small* perturbation, the system goes back (perhaps asymptotically) to the initial equilibrium. A second notion of stability is inspired from the eductive viewpoint in economic theory (see Guesnerie (1992)). Typically, the equilibrium will be said to be locally stable in the eductive sense if, the common knowledge that the equilibrium is in a given neighborhood allows agents to find, by a mental process[4] (*i.e.* without any time-dependent learning) the actual equilibrium.

[3]The results we will obtain can be generalized really easily in higher dimension using Hermite Polynomials in higher dimension.

[4]In the seminal articles on eductive stability, the mental process was linked to the notion of rationalizable solutions, see Guesnerie (1992) for more details.

5.1 Local *physical* stability

To work on the local stability in the classical sense, we consider the PDEs of Proposition 3 and we introduce perturbations on the solutions (for $\mu = 0$). These perturbations can be written as:

$$m(0,x) = m^*(x)(1 + \varepsilon\psi(0,x))$$
$$u(T,x) = u^*(x) + \varepsilon\phi(T,x)$$

$\phi(T,\cdot) = \overline{\phi}(\cdot)$ and $\psi(0,\cdot) = \underline{\psi}(\cdot)$ are given and represent respectively the relative perturbation on m^* and the absolution perturbation on u^*[5].

We are going to study the dynamics of the functions ϕ and ψ where we consider the linearized PDEs.

Proposition 8 (Linearized PDEs). *The linearized PDEs around* (u^*, m^*) *are:*

$$(Hamilton - Jacobi) \qquad \dot{\phi} + \frac{\sigma^2}{2}\phi'' - 2\eta x\phi' - \rho\phi = -\psi$$
$$(Kolmogorov) \qquad \dot{\psi} - \frac{\sigma^2}{2}\psi'' + 2\eta x\psi' = -\phi'' + \frac{x}{s^2}\phi'$$

A more convenient way to see these linearized PDEs is to introduce the $\mathcal{L}$ operator: $f \mapsto \mathcal{L}f = -\frac{\sigma^2}{2}f'' + 2\eta x f'$

Proposition 9 (Linearized PDEs). *The above equations can be written as:*

$$(Hamilton - Jacobi) \qquad \dot{\phi} = \mathcal{L}\phi + \rho\phi - \psi$$
$$(Kolmogorov) \qquad \dot{\psi} = -\mathcal{L}\psi + \frac{2}{\sigma^2}\mathcal{L}\phi$$

Proof: The only thing to recall is that $s^2 = \frac{\sigma^2}{4\eta}$. □

Now, we are going to use the properties of the operator $\mathcal{L}$ we have just introduced. To do that we need to use some properties of the Hermite polynomials associated to the space $L^2(m^*(x)dx)$.

Definition 1 (Hermite polynomials). *We defined the n^{th} Hermite polynomial of $L^2(m^*(x)dx)$ by:*

$$H_n(x) = s^n \frac{1}{\sqrt{n!}}(-1)^n \exp(\frac{x^2}{2s^2})\frac{d^n}{dx^n}\exp(-\frac{x^2}{2s^2})$$

Proposition 10 (Hermite polynomials as a basis). *The polynomials $(H_n)_n$ form an orthonormal basis of the Hilbert space $L^2(m^*(x)dx)$.*

Proposition 11 (Hermite polynomials as eigenvectors of $\mathcal{L}$). *The Hermite polynomials H_n are eigenvectors of $\mathcal{L}$ and:*

$$\mathcal{L}H_n = 2\eta n H_n$$

[5]It's in fact really important to consider the relative variation in the case of the probability distribution function m.

Now that we have recalled some basics about the Hermite polynomials we can use them to solve the linearized PDEs of Proposition 12. Let's start first with the matrices $(A_n)_n$ that are going to be involved to solve the problem:

$$A_n = \begin{pmatrix} \rho + 2\eta n & -1 \\ \frac{n}{s^2} & -2\eta n \end{pmatrix}$$

Lemma 1 (Eigenvalues of A_n). *Let's consider $n \geq 2$.*

The eigenvalues of A_n are of opposite signs, $\lambda_n^1 < 0 < \lambda_n^2$ with:

$$\lambda_n^{1,2} = \frac{1}{2}\left[\rho \pm \sqrt{\rho^2 + 16\eta^2 n(n-1)}\right]$$

It's interesting to notice that for a system of two linear PDEs like this we are working on, one equation being forward and the other being backward, the stability result will arise from the opposite signs of the eigenvalues.

Proposition 12. *Let's suppose that the perturbations $\underline{\psi}$ and $\overline{\phi}$ are in the Hilbert space $H = L^2(m^*(x)dx)$.*

Let's consider for $n \geq 2$ the functions $\begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix}$ that verify:

$$\begin{pmatrix} \dot{\phi}_n \\ \dot{\psi}_n \end{pmatrix} = A_n \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix}$$

with $\phi_n(T)$ equal to $\overline{\phi}_n$ and $\psi_n(0)$ equal to $\underline{\psi}_n$.
We have:

$$\phi_n(t) = \mathcal{O}_n\left(\frac{\underline{\psi}_n}{4\eta n} e^{\lambda_n^1 t}\right) + \mathcal{O}_n\left(\overline{\phi}_n e^{-\lambda_n^2 (T-t)}\right)$$

$$\psi_n(t) = \mathcal{O}_n\left(\underline{\psi}_n e^{\lambda_n^1 t}\right) + \mathcal{O}_n\left(\overline{\phi}_n e^{-\lambda_n^2 (T-t)}\right)$$

In particular,

$$\forall t \in (0,T), \forall k \in \mathbb{N}, (n^k \phi_n(t))_n \in l^1 (\subset l^2), (n^k \psi_n(t))_n \in l^1 (\subset l^2)$$

The estimates we established in the preceding proposition are the basis of the regularization property we will obtain in the following proposition. What we will show is indeed that whatever the regularity of the perturbations in the Hilbert space $H = L^2(m^*(x)dx)$, the solutions are going to be in C^∞ on $(0,T) \times \mathbb{R}$.

Proposition 13 (Resolution of the PDEs). *Suppose that:*

- *The perturbations $\underline{\psi}$ and $\overline{\phi}$ are in the Hilbert space $H = L^2(m^*(x)dx)$.*
- *$\int \underline{\psi}(x) m^*(x) dx = 0$ (mass preservation condition)*
- *$\int x \underline{\psi}(x) m^*(x) dx = 0$ (mean preservation condition)*
- *$\int x \overline{\phi}(x) m^*(x) dx = 0$ (this is guaranteed if the perturbation is even)*

Let's define $(\phi_n)_n$ and $(\psi_n)_n$ by:

- *$\phi_0(t) = \overline{\phi} e^{-\rho(T-t)}$ and $\psi_0(t) = 0$.*

- $\phi_1(t) = \psi_1(t) = 0$.
- $\forall n \geq 2$, ϕ_n *and* ψ_n *defined as in the preceding proposition.*

Then $\phi(t,x) = \sum_{n=0}^{\infty} \phi_n(t) H_n(x)$ *and* $\psi(t,x) = \sum_{n=0}^{\infty} \psi_n(t) H_n(x)$ *are well defined in* H, *are in* C^∞ *and are solutions of the PDEs with the boundary conditions associated to* $\overline{\phi}$ *and* $\underline{\psi}$.

Now what we want to demonstrate is a stability result. We want to show that, as T goes to infinity (the initial and final perturbations remaining unchanged), the influence of the perturbation vanish. This is the purpose of the following proposition:

Proposition 14 (Stability I). *Suppose that:*

- *The perturbations* $\underline{\psi}$ *and* $\overline{\phi}$ *are in the Hilbert space* $H = L^2(m^*(x)dx)$.
- $\int \underline{\psi}(x) m^*(x) dx = 0$ *(mass preservation condition)*
- $\int x\underline{\psi}(x) m^*(x) dx = 0$ *(mean preservation condition)*
- $\int x\overline{\phi}(x) m^*(x) dx = 0$ *(this is guaranteed if the perturbation is even)*

Then, $\forall n, \forall \alpha \in (0, \frac{1}{2})$*:*

$$\lim_{T\to\infty} ||\phi_n||_{L^\infty([\alpha T,(1-\alpha)T])} = 0, \qquad \lim_{T\to\infty} ||\psi_n||_{L^\infty([\alpha T,(1-\alpha)T])} = 0$$

It's noticeable that the three conditions on the perturbations are natural to obtain a stability result. First of all, the mass preservation is natural since the total measure must remain the same. Then, the two other conditions are necessary because of the invariance by translation of the problem.

The result we have just obtained is a weak form of stability but stronger stability results can be obtained by using more precise estimations. An example of such an improvement is:

Proposition 15 (Stability II). *Suppose that:*

- *The perturbations* $\underline{\psi}$ *and* $\overline{\phi}$ *are in the Hilbert space* $H = L^2(m^*(x)dx)$.
- $\int \underline{\psi}(x) m^*(x) dx$ *(mass preservation condition)*
- $\int x\underline{\psi}(x) m^*(x) dx$ *(mean preservation condition)*
- $\int x\overline{\phi}(x) m^*(x) dx$ *(this is guaranteed if the perturbation is even)*

Then:

$$\lim_{T\to\infty} sup_{t\in[\alpha T,(1-\alpha)T]} ||\phi(t,\cdot)||_{L^2(m^*(x)dx)} = 0$$
$$\lim_{T\to\infty} sup_{t\in[\alpha T,(1-\alpha)T]} ||\psi(t,\cdot)||_{L^2(m^*(x)dx)} = 0$$

Proof: It's a simple application of the Lebesgue's dominated convergence theorem.

5.2 Local eductive stability

Now, we are going to consider another notion of stability that has more to do with the justification of rational expectation hypothesis or with the process through which agents will mentally understand what will be the stationary equilibrium.
The goal in the next paragraphs is in fact to consider an initial guess for the stationary equilibrium (in the neighborhood of the actual equilibrium) and to exhibit a "mental process" (this process is actually a continuous process based on two PDEs involving what we call virtual time) that goes from the initial guess to the true equilibrium.

Let's consider the two equations of Proposition 3:

$$\frac{\sigma^2}{2}u'' + \frac{1}{2}u'^2 - \rho u + \ln(m) = 0$$

$$\frac{\sigma^2}{2}m'' - (mu')' = 0$$

We are going to introduce a variable θ called virtual time and consider, given an initial guess $(u(\theta = 0, x), m(\theta = 0, x))$ for the equilibrium, the mental process associated with the following system of PDEs:

$$\partial_\theta u = \frac{\sigma^2}{2}u'' + \frac{1}{2}u'^2 - \rho u + \ln(m)$$

$$\partial_\theta m = \frac{\sigma^2}{2}m'' - (mu')'$$

Since we only want to consider a local eductive stability, we are going to work with the linearized version of these equations that is given by the following proposition:

Proposition 16 (Linearized mental process). *The linearized mental process around (u^*, m^*) is given by:*

$$\partial_\theta \phi = \frac{\sigma^2}{2}\phi'' - 2\eta x\phi' - \rho\phi + \psi$$

$$\partial_\theta \psi = \frac{\sigma^2}{2}\psi'' + 2\eta x\psi' - \phi'' + \frac{x}{s^2}\phi'$$

where ϕ and ψ are defined as before and where $\phi(0, \cdot)$ and $\psi(0, \cdot)$ are given.

Proof: The proof is identical to the proof of Proposition 8. □

We can write these equations using the $\mathcal{L}$ operator introduced earlier:

Proposition 17. *The above equations can be written as:*

$$\partial_\theta \phi = -\mathcal{L}\phi - \rho\phi + \psi$$

$$\partial_\theta \psi = -\mathcal{L}\psi + \frac{2}{\sigma^2}\mathcal{L}\phi$$

To solve these equations, we need to introduce the matrices $(B_n)_n$:

$$B_n = \begin{pmatrix} -(\rho + 2\eta n) & 1 \\ \frac{n}{s^2} & -2\eta n \end{pmatrix}$$

Lemma 2 (Eigenvalues of B_n). *Let's consider $n \geq 2$.*

The eigenvalues $\xi_n^1 < \xi_n^2$ of B_n are both negative with:

$$\xi_n^{1,2} = \frac{1}{2}\left[-\rho - 4\eta n \pm \sqrt{\rho^2 + \frac{4n}{s^2}}\right]$$

Proposition 18. *Let's suppose that the initial conditions $\phi(0,\cdot)$ and $\psi(0,\cdot)$ are in the Hilbert space $H = L^2(m^*(x)dx)$.*

Let's consider for $n \geq 2$ the functions $\begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix}$ that verify:

$$\begin{pmatrix} \partial_\theta \phi_n \\ \partial_\theta \psi_n \end{pmatrix} = B_n \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix}$$

with $\phi_n(0)$ equal to $\phi(0,\cdot)_n$ and $\psi_n(0)$ equal to $\psi(0,\cdot)_n$.
We have:

$$\phi_n(\theta) = \mathcal{O}_n(|\phi_n(0)|e^{\xi_n^2\theta})$$

$$\psi_n(\theta) = \mathcal{O}_n(\sqrt{n}|\phi_n(0)|e^{\xi_n^2\theta})$$

In particular,

$$\forall\theta > 0, \forall k \in \mathbb{N}, (n^k\phi_n(\theta))_n \in l^1(\subset l^2), (n^k\psi_n(\theta))_n \in l^1(\subset l^2)$$

As before these estimations show that the solutions will be far more regular than the initial conditions.

Proposition 19 (Resolution of the PDEs associated to the mental process). *Suppose that:*

- *The initial conditions $\phi(0,\cdot)$ and $\psi(0,\cdot)$ are in the Hilbert space $H = L^2(m^*(x)dx)$.*
- *$\int \psi(0,x)m^*(x)dx = 0$ (this is guaranteed if the initial guess for m is a probability distribution function)*
- *$\int x\phi(0,x)m^*(x)dx = 0$ (this is guaranteed if the initial guess is even)*
- *$\int x\psi(0,x)m^*(x)dx = 0$ (this is guaranteed if the initial guess is even)*

Let's define $(\phi_n)_n$ and $(\psi_n)_n$ by:

- *$\phi_0(\theta) = \phi_0(0)e^{-\rho\theta}$ and $\psi_0(\theta) = 0$.*
- *$\phi_1(\theta) = \psi_1(\theta) = 0$.*
- *$\forall n \geq 2$, ϕ_n and ψ_n defined as in the preceding proposition.*

Then $\phi(\theta,x) = \sum_{n=0}^{\infty} \phi_n(\theta)H_n(x)$ and $\psi(\theta,x) = \sum_{n=0}^{\infty} \psi_n(\theta)H_n(x)$ are well defined in H, are in C^∞, are solutions of the PDEs and verify the initial conditions.

Proposition 20 (Local eductive stability). *Suppose that:*

- *The initial guesses $\phi(0,\cdot)$ and $\psi(0,\cdot)$ are in the Hilbert space $H = L^2(m^*(x)dx)$.*
- $\int \psi(0,x)m^*(x)dx = 0$ *(this is guaranteed if the initial guess for m is a probability distribution function)*
- $\int x\phi(0,x)m^*(x)dx = 0$ *(this is guaranteed if the initial guess is even)*
- $\int x\psi(0,x)m^*(x)dx = 0$ *(this is guaranteed if the initial guess is even)*

Then the solution (ϕ,ψ) of the mental process converges in the sense that:

$$\lim_{\theta\to\infty} ||\phi(\theta,\cdot)||_{L^2(m^*(x)dx)} = 0 \qquad \lim_{\theta\to\infty} ||\psi(\theta,\cdot)||_{L^2(m^*(x)dx)} = 0$$

This proposition proves that given an initial guess in the neighborhood of a stationary solution, if the initial guess is symmetric around the stationary solution, then, the mental process associated to the PDEs allows agents to find the solution. This is what we called local eductive stability.

5.3 Remarks on the conditions to have stability results

In both the proof of the physical stability and the proof of the eductive stability, there was a need to impose symmetry conditions on the perturbations or on the initial guesses. These conditions were necessary to ensure stability because both A_1 and B_1 were singular. If one wants to have stability results for more general initial perturbations or initial guesses, the intuitive idea is to break the translation invariance of the problem.
Interestingly, we have done that before in the paragraphs dedicated to localization. This localization idea can be used once again, to have more general stability results. If we center the problem around 0 as before, we know that the only relevant difference between the original problem and the problem with an additional term $-\delta x^2$, that localizes the problem around 0, is the positive constant η that depends on δ according to the equation:

$$2\eta^2 - \eta\left(\frac{2}{\sigma^2} - \rho\right) = \delta$$

Now, in this context we can prove that the eigenvalues of A_n are of opposite signs for $n \geq 1$ and that the eigenvalues of B_n are both negative for $n \geq 1$ (remember that we needed n to be larger than 2 to have these properties in the case where $\delta = 0$).

Lemma 3 (Eigenvalues of A_n and B_n for $\delta > 0$). *Suppose that $\delta > 0$ and $n \geq 1$.*
Then, the eigenvalues $\lambda_n^{1,2}$ of $A_n = \begin{pmatrix} \rho + 2\eta n & -1 \\ \frac{n}{s^2} & -2\eta n \end{pmatrix}$ are of opposite signs.
Similarly, the eigenvalues $\xi_n^{1,2}$ of $B_n = \begin{pmatrix} -(\rho + 2\eta n) & 1 \\ \frac{n}{s^2} & -2\eta n \end{pmatrix}$ are both negative.

This lemma can be used to prove general stability results when $\delta > 0$. It is indeed straightforward that all our stability results can be rewritten exactly the same if one replaces the conditions

$$\left\{ \begin{array}{l} \int x\underline{\psi}(x)m^*(x)dx = 0 \\ \int x\overline{\phi}(x)m^*(x)dx = 0 \end{array} \right. \quad by \quad \delta > 0 \quad (physical\ stability)$$

or

$$\left\{ \begin{array}{l} \int x\psi(0,x)m^*(x)dx = 0 \\ \int x\phi(0,x)m^*(x)dx = 0 \end{array} \right. \quad by \quad \delta > 0 \quad (eductive\ stability)$$

These ideas will be used extensively to study stability in multi-population frameworks.

5.4 Concluding remarks on the two stability notions

Even though the two kinds of stability look like each other, the two notions of stability we used are completely orthogonal. The physical stability is indeed linked to a perturbation of the system. The system is physically stable because, after an *initial* perturbation of m^* and a *final* perturbation of u^*, under some conditions, the solution of the game is stable in the sense that agents go back to the equilibrium. Hence, the physical stability involves forward/backward reasoning. This is not the case of the eductive stability because the mental process is purely forward (in virtual time). We start from a guess not too far from an equilibrium (the equilibrium being *a priori* unknown) and the mental process converges toward this equilibrium.
The fact that our solutions are stable for both the physical stability and the eductive stability backs up the mean field game approach to find relevant solutions.

Conclusion

The model we presented in this paper is the archetype of a dynamical mean field game model in continuous time with a continuous state space. Even though the specification is simple, examples can be built with different specification and several populations that interact with one another. Numerical methods will be the topic of a forthcoming article by Jean-Michel Lasry and Pierre-Louis Lions.

Appendix: Proofs

Proof of proposition 3:

The only thing to prove is that $m(x) = K\exp(\frac{2u(x)}{\sigma^2})$ is a solution of the Kolmogorov equation. Taking logs and deriving we have $\nabla m = \frac{2\nabla u}{\sigma^2}m$. Hence, if we apply the divergence operator to each side we obtain the

Kolmogorov equation. $\square$

Proof of proposition 4:

Let's consider (K, ψ) solution of the preceding equations and let's introduce $m = \psi^2$ and $u = \sigma^2 \ln\left(\frac{\psi}{K}\right)$.

We have the following derivatives:

$$\frac{\nabla m}{m} = 2\frac{\nabla \psi}{\psi}$$

$$\nabla u = \sigma^2 \frac{\nabla \psi}{\psi} = \frac{\sigma^2}{2}\frac{\nabla m}{m}$$

Hence, (u, m) verifies the Kolmogorov equation.

Now,

$$\Delta u = \sigma^2 \left[\frac{\Delta \psi}{\psi} - \frac{|\nabla \psi|^2}{\psi^2}\right] = \sigma^2 \frac{\Delta \psi}{\psi} - \frac{1}{\sigma^2}|\nabla u|^2$$

$$\Rightarrow \frac{\sigma^2}{2}\Delta u(x) + \frac{1}{2}|\nabla u(x)|^2 = \frac{\sigma^4}{2}\frac{\Delta \psi(x)}{\psi(x)} = \rho\sigma^2 \ln(\frac{\psi(x)}{K}) - g(\psi^2(x))$$

$$\Rightarrow \frac{\sigma^2}{2}\Delta u(x) + \frac{1}{2}|\nabla u(x)|^2 - \rho u(x) = -g(m(x))$$

Hence, (u, m) verifies the Hamilton-Jacobi equation. $\square$

Proof of proposition 5:

We are going to use Proposition 3 and the PDE in u.

We are looking for a solution for u of the form:

$$u(x) = -\eta|x - \mu|^2 + \omega$$

If we put this form in the Hamilton-Jacobi equation of Proposition 3 we get:

$$2\eta^2|x - \mu|^2 + \rho\eta|x - \mu|^2 - \rho\omega - \eta n\sigma^2 = -\ln(K) + \frac{2\eta|x - \mu|^2}{\sigma^2} - \frac{2\omega}{\sigma^2}$$

A first condition for this to be true is:

$$2\eta^2 + \rho\eta = \frac{2\eta}{\sigma^2}$$

$$\iff \eta = \frac{1}{\sigma^2} - \frac{\rho}{2}$$

A second condition, to find ω, is related to the fact that m is a probability distribution function (equation $(1')$). This clearly requires η to be positive but this is guaranteed by the hypothesis $\rho\sigma^2 < 2$. This also implies:

$$K \exp\left(\frac{2\omega}{\sigma^2}\right) \int_{\mathbb{R}^n} \exp\left(\frac{-2\eta}{\sigma^2}|x - \mu|^2\right) = K \exp\left(\frac{2\omega}{\sigma^2}\right) \left(\frac{\pi\sigma^2}{2\eta}\right)^{\frac{n}{2}} = 1$$

$$\Rightarrow \rho\omega + \eta n \sigma^2 = \frac{n}{2} \ln\left(\frac{2\eta}{\pi\sigma^2}\right)$$

and this last equation gives ω.

From this solution for u we can find a solution for m. We indeed know that m is a probability distribution function and that m is given by

$$m(x) = K \exp(\frac{2u(x)}{\sigma^2})$$

As a consequence, m is the probability distribution function of an n-dimensional gaussian random variable with variance equal to $s^2 I_n$ where $s^2 = \frac{\sigma^2}{4\eta}$ i.e. $s^2 = \frac{\sigma^4}{4-2\rho\sigma^2}$. □

Proof of proposition 8:

A Taylor expansion of the ln is the only thing needed to obtain the HJB equation.

For the Kolmogorov equation, the linearized PDE first appears as:

$$\dot{\psi} m^* - \frac{\sigma^2}{2}(\psi m^*)'' + (-2\eta x \psi m^*)' = -(\phi' m^*)'$$

Since $(m^*)' = -\frac{x}{s^2} m^*$ and $(m^*)'' = \left(\frac{x^2}{s^4} - \frac{1}{s^2}\right) m^*$, we obtain:

$$\dot{\psi} - \frac{\sigma^2}{2}\left(\psi'' - 2\frac{x}{s^2}\psi' + \left(\frac{x^2}{s^4} - \frac{1}{s^2}\right)\psi\right) - 2\eta\psi - 2\eta x\psi' + 2\eta\frac{x^2}{s^2}\psi = -\phi'' + \frac{x}{s^2}\phi'$$

Using now the fact that $s^2 = \frac{\sigma^2}{4\eta}$, we obtain the result. □

Proof of Lemma 1:

The eigenvalues are the roots of the polynomials $X^2 - \rho X - 2\eta n(\rho + 2\eta n) + \frac{n}{s^2}$. We can compute Δ:

$$\Delta = \rho^2 + 8\eta n \left(\rho - \frac{2}{\sigma^2} + 2\eta n\right)$$

Hence, using the relations between η and ρ we get:

$$\Delta = \rho^2 + 16\eta^2 n(n-1)$$

Since $n \geq 2$ we have $\Delta > \rho^2$ and therefore the two roots are real, one is positive and the other is negative. □

Proof of proposition 12:

If we use the preceding lemma, we see that we can write:

$$\begin{pmatrix} \phi_n(t) \\ \psi_n(t) \end{pmatrix} = C^1_{n,T} e^{\lambda^1_n t} \begin{pmatrix} 1 \\ v^1_n \end{pmatrix} + C^2_{n,T} e^{\lambda^2_n t} \begin{pmatrix} 1 \\ v^2_n \end{pmatrix}$$

where the v's are found using eigenvectors of the matrix A_n:

$$v_n^1 = \rho + 2\eta n - \lambda_n^1, \qquad v_n^2 = \rho + 2\eta n - \lambda_n^2$$

Now, to find the two constants we need to use the conditions on $\phi_n(T)$ and $\psi_n(0)$:

$$\begin{cases} \phi_n(T) = \overline{\phi}_n = C_{n,T}^1 e^{\lambda_n^1 T} + C_{n,T}^2 e^{\lambda_n^2 T} \\ \psi_n(0) = \underline{\psi}_n = C_{n,T}^1 v_n^1 + C_{n,T}^2 v_n^2 \end{cases}$$

Hence:

$$\begin{cases} C_{n,T}^1 = \frac{v_n^2 \overline{\phi}_n - e^{\lambda_n^2 T} \underline{\psi}_n}{v_n^2 e^{\lambda_n^1 T} - v_n^1 e^{\lambda_n^2 T}} \\ C_{n,T}^2 = \frac{v_n^1 \overline{\phi}_n - e^{\lambda_n^1 T} \underline{\psi}_n}{v_n^1 e^{\lambda_n^2 T} - v_n^2 e^{\lambda_n^1 T}} \end{cases}$$

Using the fact that $v_n^1 \sim 4\eta n$ and $v_n^2 \sim \frac{\rho}{2} + \eta$ we can deduce the asymptotic behavior[6] of $C_{n,T}^{1,2}$ as n goes to infinity (with T fixed).

$$C_{n,T}^1 \sim_{n\to\infty} \frac{\underline{\psi}_n}{4\eta n}, \qquad C_{n,T}^2 \sim_{n\to\infty} \overline{\phi}_n e^{-\lambda_n^2 T}$$

Hence:

$$\phi_n(t) = \mathcal{O}_n\left(\frac{\underline{\psi}_n}{4\eta n} e^{\lambda_n^1 t}\right) + \mathcal{O}_n\left(\overline{\phi}_n e^{-\lambda_n^2 (T-t)}\right)$$

$$\psi_n(t) = \mathcal{O}_n\left(\underline{\psi}_n e^{\lambda_n^1 t}\right) + \mathcal{O}_n\left(\overline{\phi}_n e^{-\lambda_n^2 (T-t)}\right)$$

These two estimations prove the results. □

Proof of proposition 13:

First of all, the preceding proposition ensure that the two functions ϕ and ψ are well defined, in C^∞, and that we can differentiate formally the expressions. Then, the first three conditions can be translated as $\underline{\psi}_0 = 0$, $\underline{\psi}_1 = 0$ and $\overline{\phi}_1 = 0$ and so the conditions at time 0 and time T are verified. The fact that the PDEs are verified is due to the definition of ϕ_n and ψ_n and also to the fact that we can differentiate under the sum sign because of the estimates of the preceding proposition. □

Proof of proposition 14:

The result is obvious for $n = 0$ and $n = 1$. For $n \geq 2$, we need to go back to the expressions of $\phi_n(t)$ and $\psi_n(t)$.

First of all, let's go back to the two constants:

$$\begin{cases} C_{n,T}^1 = \frac{v_n^2 \overline{\phi}_n - e^{\lambda_n^2 T} \underline{\psi}_n}{v_n^2 e^{\lambda_n^1 T} - v_n^1 e^{\lambda_n^2 T}} \\ C_{n,T}^2 = \frac{v_n^1 \overline{\phi}_n - e^{\lambda_n^1 T} \underline{\psi}_n}{v_n^1 e^{\lambda_n^2 T} - v_n^2 e^{\lambda_n^1 T}} \end{cases}$$

[6] Here we assume that $\underline{\psi}_n \neq 0$ and $\overline{\phi}_n \neq 0$. If one of these coefficients is equal to 0, the estimates of the proposition are still true and can even be improved.

Then[7],

$$\lim_{T\to\infty} C^1_{n,T} = \frac{\psi_n}{v^1_n}, \qquad C^2_{n,T} \sim_{T\to\infty} \overline{\phi}_n e^{-\lambda^2_n T}$$

Using now the expressions for the functions,

$$\phi_n(t) = C^1_{n,T} e^{\lambda^1_n t} + C^2_{n,T} e^{\lambda^2_n t}$$

$$\psi_n(t) = C^1_{n,T} v^1_n e^{\lambda^1_n t} + C^2_{n,T} v^2_n e^{\lambda^2_n t}$$

we get:

$$||\phi_n||_{L^\infty([\alpha T,(1-\alpha)T])} \le |C^1_{n,T}| e^{\lambda^1_n \alpha T} + |C^2_{n,T}| e^{\lambda^2_n (1-\alpha)T}$$

$$||\psi_n||_{L^\infty([\alpha T,(1-\alpha)T])} \le |C^1_{n,T} v^1_n| e^{\lambda^1_n \alpha T} + |C^2_{n,T} v^2_n| e^{\lambda^2_n (1-\alpha)T}$$

and this leads to the result really easily. □

Proof of lemma 2:

The eigenvalues are the roots of the polynomials $X^2 + (\rho + 4\eta n)X + 2\eta n(\rho + 2\eta n) - \frac{n}{s^2}$. We can compute Δ:

$$\Delta = \rho^2 + \frac{4n}{s^2} > 0$$

Hence, the eigenvalues are real and are of the form given in the proposition. Since $tr(B_n) < 0$ and $\det(B_n) = 2\eta n(\rho + 2\eta n) - \frac{4\eta n}{\sigma^2} = 4\eta^2 n(n-1) > 0$, the two eigenvalues are negative. □

Proof of proposition 18:

The proof is similar to the proof of Proposition 12.

$$\begin{pmatrix} \phi_n(\theta) \\ \psi_n(\theta) \end{pmatrix} = A_n e^{\xi^1_n \theta} \begin{pmatrix} 1 \\ a_n \end{pmatrix} + B_n e^{\xi^2_n \theta} \begin{pmatrix} 1 \\ b_n \end{pmatrix}$$

where:

$$a_n = \rho + 2\eta n + \xi^1_n, \qquad b_n = \rho + 2\eta n + \xi^2_n$$

Now, to find the two constants we need to use the conditions on $\phi_n(0)$ and $\psi_n(0)$:

$$\begin{cases} \phi_n(0) = A_n + B_n \\ \psi_n(0) = a_n A_n + b_n B_n \end{cases}$$

Hence:

$$\begin{cases} A_n = \frac{b_n \phi_n(0) - \psi_n(0)}{b_n - a_n} \\ B_n = \frac{a_n \phi_n(0) - \psi_n(0)}{a_n - b_n} \end{cases}$$

[7] Here we assume that $\overline{\phi}_n \neq 0$. If this coefficient is equal to 0, the result is still true but the estimate for $C^2_{n,T}$ cannot be written this way and is in fact better than the estimate presented below.

Using the fact that $a_n \sim -\frac{\sqrt{\eta}}{\sigma}\sqrt{n}$ and $b_n \sim \frac{\sqrt{\eta}}{\sigma}\sqrt{n}$ we can deduce the asymptotic behavior of the constants as n goes to infinity.

$$A_n \sim_{n\to\infty} \frac{\phi_n(0)}{2}, \qquad B_n \sim_{n\to\infty} \frac{\phi_n(0)}{2}$$

Hence, since $\xi_n^1 < \xi_n^2$,

$$\phi_n(\theta) = \mathcal{O}_n(|\phi_n(0)|e^{\xi_n^2\theta})$$
$$\psi_n(\theta) = \mathcal{O}_n(\sqrt{n}|\phi_n(0)|e^{\xi_n^2\theta})$$

These two estimations prove the results. □

Proof of proposition 19:

First of all, the preceding proposition ensure that the two functions ϕ and ψ are well defined, in C^∞, and that we can differentiate formally the expressions. Then, the first three conditions can be translated as $\psi_0(0,\cdot) = 0$, $\phi_1(0,\cdot) = 0$ and $\psi_1(0,\cdot) = 0$ and so the conditions at time 0 is verified.
The fact that the PDEs are verified is due to the definition of ϕ_n and ψ_n and also to the fact that we can differentiate under the sum sign because of the estimates of the preceding proposition. □

Proof of proposition 20:

We basically want to show that:

$$\sum_{n=0}^{+\infty} |\phi_n(\theta)|^2 \to_{\theta\to+\infty} 0, \qquad \sum_{n=0}^{+\infty} |\psi_n(\theta)|^2 \to_{\theta\to+\infty} 0$$

This is actually a pure consequence of the estimates proved in Proposition 21 and of the Lebesgue's dominated convergence theorem. □

Proof of lemma 3:

$\lambda_n^{1,2}$ are the two roots of the polynomial $X^2 - \rho X - 2\eta n(\rho + 2\eta n) + \frac{n}{s^2}$. The associated Δ is given by

$$\Delta = \rho^2 + 8\eta n\left(\rho - \frac{2}{\sigma^2} + 2\eta n\right)$$
$$\Delta = \rho^2 + 16\eta^2 n(n-1) + 8n\delta$$

Hence, the eigenvalues $\lambda_n^{1,2} = \frac{1}{2}\left(\rho \pm \sqrt{\Delta}\right)$ are of opposite signs for $n \geq 1$ since $\Delta > \rho^2$.

Now, $\xi_n^{1,2}$ are the two roots of the polynomial $X^2 + (\rho + 4\eta n)X + 2\eta n(\rho + 2\eta n) - \frac{n}{s^2}$. The associated Δ is given by

$$\Delta = \rho^2 + 4\frac{n}{s^2}$$

Hence, $\xi_n^{1,2} = \frac{1}{2}\left[-\rho - 4\eta n \pm \sqrt{\rho^2 + \frac{4n}{s^2}}\right]$. These two eigenvalues are negative if and only if:

$$\rho + 4\eta n > \sqrt{\rho^2 + \frac{4n}{s^2}}$$
$$\iff 8\rho\eta n + 16\eta^2 n^2 > \frac{16\eta n}{\sigma^2}$$
$$\iff 2\eta n > \frac{2}{\sigma^2} - \rho$$

and this is true for $n \geq 1$. □

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