ISSN 1440-771X

MONASH UNIVERSITY AUSTRALIA

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Averaging Lorenz Curves

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Working Paper 22/2003

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Abstract
A large number of functional forms have been suggested in the literature for estimating Lorenz curves that describe the relationship between income and population shares. One way of choosing a particular functional form is to pick the one that best fits the data in some sense. Another approach, and the one followed here, is to use Bayesian model averaging to average the alternative functional forms. In this averaging process, the different Lorenz curves are weighted by their posterior probabilities of being correct. Unlike a strategy of picking the best-fitting function, Bayesian model averaging gives posterior standard deviations that reflect the functional form uncertainty. Building on our earlier work (Chotikapanich and Griffiths 2002), we construct likelihood functions using the Dirichlet distribution and estimate a number of Lorenz functions for Australian income units. Prior information is formulated in terms of the Gini coefficient and the income shares of the poorest 10% and poorest 90% of the population. Posterior density functions for these quantities are derived for each Lorenz function and are averaged over all the Lorenz functions.

Keywords: Gini coefficient; Bayesian inference; Dirichlet distribution.

JEL CLASSIFICATION: C11, D31

1. Introduction

The Lorenz curve is an important tool for the measurement of income inequality. For a given economy or region, it relates the cumulative proportion of income to the cumulative proportion of population, after ordering the population according to increasing level of income. The most common measure of income inequality that is based on the Lorenz curve is the Gini coefficient. It is equal to twice the area between the Lorenz curve and a 45 degree line in a graph that has the cumulative proportions of income and population as its axes. When income distribution data are available in grouped form, comprising the proportion of population in a number of income categories, the Gini coefficient can be estimated by approximating the Lorenz curve by a series of linear segments. However, because such an approach ignores inequality within each income class, it understates the extent of the inequality. An alternative way to proceed is to assume a particular functional form for the Lorenz curve, to estimate it using the grouped data, and to estimate the Gini coefficient as twice the area between the estimated Lorenz curve and the 45 degree line. With this strategy in mind, several authors have suggested possible functional forms for Lorenz curves. See, for example, Kakwani and Podder (1973, 1976), Kakwani (1980), Rasche et al (1980), Basmann et al (1990), Ortega et al (1991), Chotikapanich (1993) and Sarabia et al (1999). Having available a large number of possible functional forms raises questions about how to choose between them when carrying out estimation. One possibility is to estimate a number of functions and to choose the one that best fits the data in some sense. The best-fitting model could be one that maximizes the likelihood function, or minimizes an information criterion, or, it could be chosen via a sequence of formal hypothesis tests. In any event, one problem with this practice is that, once a particular model has been chosen, the fact that a number of other models have been discarded is usually ignored. No allowance is made for the possibility of sample statistics yielding an incorrect choice. Also, standard errors used to assess the precision of estimation of parameters, and of functions of interest such as the Gini coefficient and the income shares of the poorest and wealthiest segments of the population, make no provision for the preliminary-test nature of the inference.

As a strategy for overcoming these difficulties, in this paper we describe and illustrate how Bayesian model-averaging can be used to average the results from a number of alternative Lorenz functional forms. Suppose that the main reason for estimating a Lorenz curve is to obtain estimates of (1) the Gini coefficient, (2) the income share of the poorest 10% of the population, and (3) the income share of the poorest 90% of the population. We call these three quantities our ‘economic quantities of interest’. Each estimated Lorenz functional form will yield different estimates, and different estimates of the precision of estimation, of the economic quantities of interest (EQI). Instead of choosing one set of estimates of the EQI, as one might do when proceeding with sampling theory inference, we find a weighted average of the estimates from the different functional forms, using posterior model probabilities as the weights. Following this procedure recognizes that the best fitting model is not necessarily the correct one, and the posterior standard deviations of the averaged results provide measures of precision that reflect the uncertainty of model choice.

Our framework for estimation is that suggested in Chotikapanich and Griffiths (2002) where the parameters of a Dirichlet distribution are related to Lorenz curve differences to allow for the cumulative proportional nature of the Lorenz curve data. Most earlier studies used linear or nonlinear least squares, ignoring the proportional nature of the data. We extend the maximum likelihood approach adopted by Chotikapanich and Griffiths (2002) to Bayesian estimation, and then show how to average the results from the different Lorenz functional forms.

The methodology for Bayesian estimation and model averaging for the Lorenz curves is described in Section 2. In Section 3 we describe the application and the results. Some concluding remarks are made in Section 4.

2. Bayesian Estimation of Lorenz Curves

Suppose we have available observations on cumulative proportions of population ($\pi_1, \pi_2, \ldots, \pi_M$ with $\pi_M = 1$) and corresponding cumulative proportions of income ($\eta_1, \eta_2, \ldots, \eta_M$ with $\eta_M = 1$) obtained after ordering population units according to increasing income. We wish to use these observations to estimate a parametric version of a Lorenz curve that we write as $\eta = L(\pi;\beta)$ where β is an $(n \times 1)$ vector of unknown parameters. Following Chotikapanich and Griffiths (2002), we assume that, conditional on the population proportions π_i, the income shares $q_i = \eta_i - \eta_{i-1}$ follow a Dirichlet distribution with probability density function (pdf)

$$f(q \mid \theta) = \Gamma(\lambda)\prod_{i=1}^{M} \frac{q_i^{\lambda[L(\pi_i;\beta)-L(\pi_{i-1};\beta)]-1}}{\Gamma\left(\lambda[L(\pi_i;\beta)-L(\pi_{i-1};\beta)]\right)} \tag{1}$$

where $\theta = (\beta', \lambda)'$ and $\Gamma(\cdot)$ is the gamma function. This pdf is such that

$$E(q_i) \;=\; E(\eta_i) - E(\eta_{i-1}) \;=\; L(\pi_i;\beta) - L(\pi_{i-1};\beta) \tag{2}$$

Thus, the income shares have means that are consistent with the Lorenz curve specification $\eta = L(\pi;\beta)$. Also, the Dirichlet distribution assumption is consistent with the proportional share nature of the data, unlike the normal distributional assumption that was implicit in earlier work that used nonlinear least squares to estimate Lorenz curves. The additional parameter λ that appears in equation (1) can be viewed as a measure of the precision of the fitted relationship; the variances and covariances of the q_i are given by

$$\text{var}(q_i) = \frac{E(q_i)[1-E(q_i)]}{\lambda+1}$$

$$\text{cov}(q_i, q_j) = -\frac{E(q_i)E(q_j)}{\lambda+1} \tag{3}$$

After specifying the likelihood function in equation (1), Chotikapanich and Griffiths (2002) use it to obtain maximum likelihood estimates of several Lorenz curve specifications that have been popular in the literature. In this paper, instead of proceeding with maximum likelihood estimation, we use Bayesian inference to obtain posterior pdfs for the parameters of a number of curves as well as for the three economic quantities of interest (EQI) described in the introduction. Also, instead of starting with prior pdfs on the parameters of each of the Lorenz curves, we begin with prior pdfs on the three EQI and transform them to obtain prior pdfs on the parameters of the Lorenz curves. Given our objective is to obtain an ‘average’ Lorenz curve and averaged EQI, it is important that similar prior information is used for each Lorenz curve at the starting point of our investigation.

As mentioned in the introduction, the three EQI that we consider are the Gini coefficient (G), and the income shares for the poorest 10% and poorest 90% of the population $(\eta_{0.1} \text{ and } \eta_{0.9})$. Beta prior pdfs were specified for each of these three quantities. Specifically,

$$f(G) \propto G^{v-1}(1-G)^{w-1} \qquad 0 < G < 1 \tag{4}$$

$$f(\eta_{0.1}) \propto \left(\frac{\eta_{0.1}}{0.1}\right)^{v_1-1}\left(1-\frac{\eta_{0.1}}{0.1}\right)^{w_1-1} \qquad 0 < \eta_{0.1} < 0.1 \tag{5}$$

$$f(\eta_{0.9}) \propto \left(\frac{\eta_{0.9}}{0.9}\right)^{v_9-1}\left(1-\frac{\eta_{0.9}}{0.9}\right)^{w_9-1} \qquad \eta_{0.1} < \eta_{0.9} < 0.9 \tag{6}$$

The inequalities in equations (5) and (6) must hold given the ordering of the population from poorest to richest. The restriction $\eta_{0.1} < \eta_{0.9}$ means that $\eta_{0.1}$ and $\eta_{0.9}$ are not a prior independent even though we specify the joint prior pdf for the parameters as

$$f(G, \eta_{0.1}, \eta_{0.9}) = f(G) f(\eta_{0.1}) f(\eta_{0.9}) \tag{7}$$

It also implies that the conventional normalising constant from the product of the two beta densities $f(\eta_{0.1}) f(\eta_{0.9})$ must be adjusted to allow for the area where they would overlap if no restriction was imposed. This adjustment was made by generating 20,000 observations from independent beta pdfs specified by equations (5) and (6) and counting the proportion where $\eta_{0.1} < \eta_{0.9}$. For our prior parameter settings this proportion was 0.99395, and so only a minor adjustment was necessary. The prior parameter settings chosen were $v = 1.1$, $w = 2$, $v_1 = 1.4$, $w_1 = 1.6$, $v_9 = 1.8611$ and $w_9 = 1.13889$. These settings yield priors that are proper, but with relatively large spreads, motivated by a desire to let the data dominate the prior, and to avoid setting a prior that tends to favour one Lorenz curve over another. A trial-and-error procedure was used to find the prior parameter settings. Cumulative distribution functions were computed for several values of the prior parameters; the chosen values were those that led to suitable prior distribution functions.

The prior pdfs defined by equations (4) to (7) are used to derive a prior pdf for the Lorenz curve parameters β. When a Lorenz curve has a single parameter, only the prior pdf on the Gini coefficient $f(G)$ is used and we have

$$f(\beta) = f(G)\left|\frac{dG}{d\beta}\right| \tag{8}$$

For Lorenz curves where β is of dimension 2, the prior pdf for β is derived from those for the Gini coefficient G and the 10% share $\eta_{0.1}$

$$f(\beta) = f(G,\eta_{0.1})\left|\frac{dG}{d\beta} \quad \frac{d\eta_{0.1}}{d\beta}\right| \tag{9}$$

Finally, when a Lorenz curve has 3 unknown parameters, the prior pdfs on all three EQI are used to obtain the pdf

$$f(\beta) = f(G,\eta_{0.1},\eta_{0.9})\left|\frac{dG}{d\beta} \quad \frac{d\eta_{0.1}}{d\beta} \quad \frac{d\eta_{0.9}}{d\beta}\right| \tag{10}$$

The remaining parameter for which a prior pdf is required is λ. We chose the gamma pdf

$$f(\lambda) \propto \lambda^{g-1}\exp\{-\lambda / p\} \tag{11}$$

with $g = 0.4$ and $p = 7{,}000$. Also, β and λ were taken to be priori independent. The pdf in (11) has a relatively large spread with $\Pr(\lambda \le 200) = 0.27$ and $\Pr(\lambda > 3500) = 0.25$.

The posterior pdf for $\theta = (\beta',\lambda)$ is given by

$$f(\theta \mid q) \propto f(q \mid \theta) f(\beta) f(\lambda) \tag{12}$$

with the components on the right side of this equation given by equation (1) and equations (4) through (11). To complete the specification we need to define the Lorenz curves that are being considered, and the functions of the parameters of these Lorenz curves that define the economic quantities of interest G, $\eta_{0.1}$ and $\eta_{0.9}$. The derivatives in the Jacobian terms in equations (8), (9) and (10) are also required. Five Lorenz curves are estimated. Their equations are:

Exponential $\quad L_1(\pi;a) = \dfrac{e^{a\pi}-1}{e^{a}-1} \quad a > 0 \quad$ (13)

Ortega $\quad L_2(\pi;\alpha,\beta) = \pi^{\alpha}[1-(1-\pi)^{\beta}] \quad \alpha \geq 0,\ 0 < \beta \leq 1 \quad$ (14)

RGKO $\quad L_3(\pi;\alpha,\beta) = [1-(1-\pi)^{\alpha}]^{\beta} \quad \beta \geq 1,\ 0 < \alpha \leq 1 \quad$ (15)

General Pareto $\quad L_4(\pi;\alpha,\delta,\gamma) = \pi^{\alpha}[1-(1-\pi)^{\delta}]^{\gamma} \quad \alpha \geq 0, \gamma \geq 1, 0 < \delta \leq 1 \quad$ (16)

Beta $\quad L_5(\pi;a,b,d) = \pi - a\pi^{d}(1-\pi)^{b} \quad a > 0, 0 < d \leq 1, 0 < b \leq 1$

(17)

The function L_1 is the relatively simple one-parameter function suggested by Chotikapanich (1993); L_2 coincides with the proposal of Ortega et al (1991). L_3 is a well-known form of Lorenz curve suggested by Rasche et al (1980) and L_4 is an extension of L_3 and L_2 introduced by Sarabia et al (1999). Note that L_4 nests both L_2 and L_3, with L_2 being L_4 with $\gamma = 1$ and L_3 being L_4 with $\alpha = 0$. Setting both $\gamma = 1$ and $\alpha = 0$ yields the Lorenz curve $L = 1-(1-\pi)^{\delta}$ which originates from the classical Pareto distribution. The function L_5 is the "beta function" proposed by Kakwani (1980). It is considered one of the best performers among a number of different functional forms for Lorenz curves. See, for example, Datt (1998).

For each of the Lorenz functions the Gini coefficient is defined as

$$G = 1 - 2\int_0^1 L(\pi;\beta)\ d\pi \tag{18}$$

Alternative expressions for G can be found for some of the Lorenz curves. However, with the exception of L_1, they still generally involve a numerical integral. When evaluation of (18) was necessary, it was computed via numerical integration. The other EQI are defined as

$$\eta_{0.1} = L(0.1;\beta) \qquad\qquad \eta_{0.9} = L(0.9;\beta) \tag{19}$$

The partial derivatives of G, $\eta_{0.1}$ and $\eta_{0.9}$ with respect to β, for each of the Lorenz curves, are given in an appendix.

For model averaging we need to recognize that the definition of θ, and the posterior and prior pdfs for θ, depend on the Lorenz curve being considered. To do so, we condition on L_i and rewrite the pdfs that appear in Bayes theorem in equation (12) as

$$f(\theta_i \mid q, L_i) \propto f(q \mid \theta_i, L_i) f(\beta_i \mid L_i) f(\lambda) \tag{20}$$

Given the analytical intractability of the posterior pdfs for the θ_i, either numerical integration or a Markov chain Monte Carlo (MCMC) algorithm is needed to obtain posterior pdfs and their means and standard deviations for the individual Lorenz curve parameters and the corresponding EQI. In the application that follows we used a random walk Metropolis-Hastings algorithm. See, for example, Koop (2003, Ch.5). Once Bayesian estimation of each Lorenz curve is complete, posterior model probabilities are needed to obtain the averaged results. These probabilities are given by

$$P(L_i \mid q) = \frac{f(q \mid L_i) P(L_i)}{\sum_{j=1}^{5} f(q \mid L_j) P(L_j)} \tag{21}$$

where the $P(L_i)$ are the prior model probabilities and the $f(q \mid L_i)$ are the marginal likelihoods defined by

$$f(q \mid L_i) = \int f(q \mid \theta_i, L_i) f(\theta_i \mid L_i) \, d\theta_i \tag{22}$$

where $f(\theta_i \mid L_i) = f(\beta_i \mid L_i) f(\lambda)$. In the application, we made all Lorenz curves equally likely a priori. That is $P(L_i) = 0.2$ for $i = 1,2,\ldots,5$. The marginal likelihoods were estimated using a version of the Gelfand and Dey (1994) procedure recommended by Geweke (1999). Specifically, an estimate of the inverse of the marginal likelihood is given by

$$\left[\hat{f}\left(q \mid L_i\right)\right]^{-1} = \frac{1}{N} \sum_{n=1}^{N} \frac{h\left(\theta_i^{(n)}\right)}{f\left(q \mid \theta_i^{(n)}, L_i\right) f\left(\theta_i^{(n)} \mid L_i\right)} \tag{23}$$

where $\theta_i^{(1)}, \theta_i^{(2)}, \ldots, \theta_i^{(N)}$ are MCMC-generated draws from the posterior pdf $f(\theta_i \mid q, L_i)$, and the pdf $h\left(\theta_i\right)$ is a truncated normal distribution

$$h(\theta_i) = p^{-1}(2\pi)^{-k/2}\left|\hat{\Sigma}_i\right|^{-1/2}\exp\left\{-\frac{1}{2}\left(\theta_i-\overline{\theta}_i\right)'\hat{\Sigma}_i^{-1}\left(\theta_i-\overline{\theta}_i\right)\right\} \quad (24)$$

truncated such that $\left(\theta_i-\overline{\theta}_i\right)'\hat{\Sigma}_i^{-1}\left(\theta_i-\overline{\theta}_i\right)\le K_p$. The value K_p is a critical value from a $\chi^2(k)$ distribution such that $\Pr\left(\chi^2(k)\le K_p\right)=p$, where k is the degrees of freedom (the dimension of θ_i). The quantities $\overline{\theta}_i$ and $\hat{\Sigma}_i$ are the sample mean and covariance matrix from the MCMC-generated draws.

Having obtained:

1) MCMC generated observations for each Lorenz function, $\theta_i^{(1)},\theta_i^{(2)},...,\theta_i^{(N)}$;
2) The corresponding draws from the posterior pdfs of the EQI, $(G_i^{(n)},\eta_{0.1i}^{(n)},\eta_{0.9i}^{(n)})$, $n=1,2,...N$;
3) Estimates of the posterior pdfs, means and standard deviations for θ_i and for $(G_i,\eta_{0.1i},\eta_{0.9i})$; and
4) The posterior model probabilities for each L_i,

we are in a position to proceed with model averaging. We use the result

$$E\left[g(\theta)\,|\,q\right]=\sum_{i=1}^{5}E\left[g(\theta)\,|\,L_i,q\right]P\left(L_i\,|\,q\right) \quad (25)$$

where $g(.)$ is a function of interest. Using suitable choices for $g(.)$, we can average the posterior means for each θ_i and each to get Bayesian point estimates from the averaged pdfs for these quantities. Similarly, posterior variances and standard deviations from the averaged posterior pdfs can be obtained by defining $g(.)$ to give the second moment and then computing the variance in the usual way. To estimate the averaged pdfs we take $g(.)$ as a series of indicator functions, equal to unity when an observation falls into a histogram class, and zero otherwise. In this case equation (25) can be viewed as an averaging of the numbers in each histogram class over the five Lorenz functions. With suitable scaling, the average histogram is an estimate of the average posterior pdf.

3. The Application

The methods are applied to a 1997-98 sample of gross weekly income for one-parent income units in Australia (Australian Bureau of Statistics 1999). The data are grouped into 14 classes. They take the form of the number of sampled income units in each of 14 income classes, as depicted in Table 1. The income classes refer to weekly gross income, measured in dollars, of one-parent income units. The techniques described in Section 2 were applied to these data, with 85,000 observations being drawn using a random-walk Metropolis-Hastings algorithm, and 10,000 of these being discarded as a burn-in. Plots of the observations were taken to confirm the convergence of the Markov chain. Posterior means and standard deviations of the θ_i are presented in Table 2, along with the corresponding maximum likelihood estimates and their standard errors. Also given in this table are the maximum values of the log-likelihood functions and the logs of the marginal likelihoods defined by equation (23). Table 3 contains the posterior means and standard deviations for the $(G_i, \eta_{0.1i}, \eta_{0.9i})$, the posterior probabilities for each of the models and the means and standard deviations for $(G, \eta_{0.1}, \eta_{0.9})$ from their averaged posterior pdf. Plots of the posterior pdfs for the $(G_i, \eta_{0.1i}, \eta_{0.9i})$ appear in Figures 1 through 6.

From Table 2, we see that the Bayesian point estimates are similar to those from maximum likelihood with the exception of the estimates for λ. This outcome suggests the prior information has been relatively mild. For λ the Bayesian point estimates are always lower, possibly reflecting stronger prior information in this case, or a skewed marginal posterior pdf for λ. Bayesian posterior standard deviations are always larger than the maximum likelihood standard errors (again λ is an exception); maximum likelihood standard errors may be understating the finite sample uncertainty.

It is interesting that the model that would be selected on the basis of the largest value of the log-likelihood function is not the one with the highest posterior model probability. See Tables 2 and 3. The three-parameter Lorenz curves (beta and generalized Pareto) have the highest log-likelihood values whereas the two-parameter Lorenz curves (Ortega and RGKO) have the highest posterior probabilities. For the Ortega model this probability is 0.546. For the RGKO model it is 0.213, and for the beta and generalized

Pareto the probabilities are 0.208 and 0.033, respectively. It appears that, relative to the two-parameter Lorenz curves, the Bayesian procedure has substantially penalized the three-parameter curves for the additional uncertainty associated with one more unknown parameter. The generalized Pareto has a low posterior probability (0.033) despite the fact that the standard errors of $\hat{\gamma}$ and $\hat{\alpha}$ for the generalized Pareto model, and the differences in the log-likelihood function values, suggest that the hypotheses $\gamma = 1$ and $\alpha = 0$, that yield the Ortega and RGKO functions, respectively, are likely to be rejected. Since $\gamma = 1$ and $\alpha = 0$ are on the boundary of the parameter space, we cannot say definitely that these hypotheses will be rejected; the sampling theory tests require special treatment (see, for example, Andrews 1998) that we do not pursue here. Nevertheless, the sampling evidence in favour of the 3-parameter generalized Pareto is much stronger than that from Bayesian inference.

The one-parameter exponential curve is not favoured by its log-likelihood function value, or its posterior probability, the latter value being 0.000013. Also, the posterior pdfs for G, $\eta_{0.1}$ and $\eta_{0.9}$ from the exponential function are vastly different from those from the other Lorenz curves.

The posterior pdfs for G, $\eta_{0.1}$ and $\eta_{0.9}$ from each Lorenz curve are plotted in Figures 1, 3 and 5, respectively, with their means and standard deviations given in Table 3. Ignoring the exponential curve because of its poor fit and low posterior probability, the results suggests the Gini coefficient lies between 0.29 and 0.35, with its most likely value being about 0.32. The income share of the poorest 10% of the population is likely to lie between 0.025 and 0.045, although this conclusion, and a conclusion about the most likely 10% share, are more sensitive to the choice of Lorenz function. The means for the 10% share from the generalized Pareto and the RGKO functions are 0.032 and 0.036, respectively. The income share of the poorest 90% of the population is likely to lie between 0.70 and 0.76. The posterior means for this quantity are similar across all models other than the exponential, lying between 0.732 and 0.738, although the spreads of the posterior pdfs are noticeably different for each model.

The results after model averaging appear in the last row in Table 3 and in Figures 2, 4 and 6 for G, $\eta_{0.1}$ and $\eta_{0.9}$, respectively. In these figures the averaged pdf is included with the pdfs from each of the models, with the exception of those from the exponential model. Because of its low posterior probability, the exponential curve did not contribute to the averaging process. What we observe is that the results from model averaging are very similar to the results from the Ortega curve. This outcome is perhaps surprising. Although the Ortega curve has the highest posterior probability, one would not expect a probability of 0.546 to be sufficiently large to dominate in the averaging process. A closer inspection shows that the pdfs from the Ortega curve tend to lie between the pdfs from the beta curve and the RGKO curve. Consequently, averaging the RGKO and beta pdfs, and then placing a weight of 0.546 on the Ortega pdf, yields results similar to those from the Ortega curve.

4. Concluding Remarks

Many functional forms have been suggested in the literature for estimating Lorenz curves. Choosing a particular functional form, either prior to estimation or on the basis of goodness-of-fit or the outcome of hypothesis tests, means that inequality measures of interest such as the Gini coefficient, or the income shares of certain proportions of the population, will be conditional on the chosen curve. We demonstrate how Bayesian model averaging can be used to obtain estimates of such quantities of interest without conditioning on a particular Lorenz curve. Presenting the results in this way allows for, and expresses, uncertainty resulting from an unknown model and unknown parameters in each model. Also, as part of our description of Bayesian model averaging procedures, we have shown how Bayesian estimation of the parameters of a Lorenz curve can proceed within the framework of a Dirichlet distribution and how prior pdfs on inequality measures of interest can be used to find prior pdfs of the parameters in a Lorenz curve.

Appendix: Derivatives for Jacobian terms in prior pdfs

Exponential with 1 parameter

LC Equation: $\eta = \frac{e^{a\pi}-1}{e^{a}-1}$ $a \geq 0$

$$\frac{\partial \eta}{\partial a} = \frac{\pi e^{a\pi}}{e^{a}-1} - \frac{e^{a}\left(e^{a\pi}-1\right)}{\left(e^{a}-1\right)^{2}}$$

$$\frac{\partial G}{\partial a} = -2\int_{0}^{1}\left[\frac{\pi e^{a\pi}}{e^{a}-1} - \frac{e^{a}\left(e^{a\pi}-1\right)}{\left(e^{a}-1\right)^{2}}\right] d\pi$$

Ortega

LC function: $\eta = \pi^{\alpha}\left[1-\left(1-\pi\right)^{\beta}\right]$ $\alpha \geq 0,\ 0 < \beta \leq 1$

$$\frac{\partial \eta}{\partial \alpha} = \left[1-\left(1-\pi\right)^{\beta}\right]\pi^{\alpha}\log\pi$$

$$\frac{\partial \eta}{\partial \beta} = -\pi^{\alpha}\left[\left(1-\pi\right)^{\beta}\log\left(1-\pi\right)\right]$$

$$\frac{\partial G}{\partial \alpha} = -2\int_{0}^{1}\left[1-\left(1-\pi\right)^{\beta}\right]\pi^{\alpha}\log\pi\ d\pi$$

$$\frac{\partial G}{\partial \beta} = 2\int_{0}^{1}\pi^{\alpha}\left[\left(1-\pi\right)^{\beta}\log\left(1-\pi\right)\right] d\pi$$

RGKO

LC function:

$$\eta = \left[1-(1-\pi)^{\alpha}\right]^{\beta} \qquad \beta \geq 1,\ 0 < \alpha \leq 1$$

$$\log \eta = \beta \log\left[1-(1-\pi)^{\alpha}\right]$$

$$\frac{\partial \log \eta}{\partial \beta} = \log\left[1-(1-\pi)^{\alpha}\right]$$

$$\frac{\partial \eta}{\partial \beta} = \left[1-(1-\pi)^{\alpha}\right]^{\beta} \log\left[1-(1-\pi)^{\alpha}\right]$$

$$\frac{\partial \log \eta}{\partial \alpha} = \frac{-\beta(1-\pi)^{\alpha} \log(1-\pi)}{\left[1-(1-\pi)^{\alpha}\right]}$$

$$\frac{\partial \eta}{\partial \alpha} = -\beta(1-\pi)^{\alpha} \log(1-\pi)\left[1-(1-\pi)^{\alpha}\right]^{\beta-1}$$

$$\frac{\partial G}{\partial \beta} = -2\int_0^1 \left[1-(1-\pi)^{\alpha}\right]^{\beta} \log\left[1-(1-\pi)^{\alpha}\right] d\pi$$

$$\frac{\partial G}{\partial \alpha} = 2\int_0^1 \beta(1-\pi)^{\alpha} \log(1-\pi)\left[1-(1-\pi)^{\alpha}\right]^{\beta-1} d\pi$$

Generalized Pareto

LC function:

$$\eta = \pi^{\alpha}\left[1-(1-\pi)^{\delta}\right]^{\gamma} \qquad \alpha \geq 0,\ \gamma \geq 1,\ 0 < \delta \leq 1$$

$$\log \eta = \alpha \log \pi + \gamma \log\left[1-(1-\pi)^{\delta}\right]$$

$$\frac{\partial \eta}{\partial \alpha} = \pi^{\alpha}\left[1-(1-\pi)^{\delta}\right]^{\gamma} \log \pi$$

$$\frac{\partial \eta}{\partial \gamma} = \pi^{\alpha}\left[1-(1-\pi)^{\delta}\right]^{\gamma} \log\left[1-(1-\pi)^{\delta}\right]$$

$$\frac{\partial \eta}{\partial \delta} = -\gamma(1-\pi)^{\delta} \log(1-\pi)\pi^{\alpha}\left[1-(1-\pi)^{\delta}\right]^{\gamma-1}$$

$$\frac{\partial G}{\partial \alpha} = -2\int_0^1 \pi^{\alpha}\left[1-(1-\pi)^{\delta}\right]^{\gamma} \log\pi \, d\pi$$

$$\frac{\partial G}{\partial \delta} = 2\int_0^1 \gamma(1-\pi)^{\delta} \log(1-\pi)\,\pi^{\alpha}\left[1-(1-\pi)^{\delta}\right]^{\gamma-1} d\pi$$

$$\frac{\partial G}{\partial \gamma} = -2\int_0^1 \pi^{\alpha}\left[1-(1-\pi)^{\delta}\right]^{\gamma} \log\left[1-(1-\pi)^{\delta}\right] d\pi$$

Beta

LC function: $\eta = \pi - a\pi^{d}(1-\pi)^{b}$ $\quad a>0,\ 0<d\le 1,\ 0<b\le 1$

$$\frac{\partial \eta}{\partial a} = -\pi^{d}(1-\pi)^{b}$$

$$\frac{\partial \eta}{\partial d} = -a\pi^{d}(1-\pi)^{b}\log\pi$$

$$\frac{\partial \eta}{\partial b} = -a\pi^{d}(1-\pi)^{b}\log(1-\pi)$$

$$\frac{\partial G}{\partial a} = 2\int_0^1 \pi^{d}(1-\pi)^{b}\, d\pi = 2B(d+1,b+1) = 2\,\frac{\Gamma(d+1)\Gamma(b+1)}{\Gamma(d+b+2)}$$

$$\frac{\partial G}{\partial d} = 2\int_0^1 a\pi^{d}(1-\pi)^{b}\log\pi \, d\pi$$

$$\frac{\partial G}{\partial b} = 2\int_0^1 a\pi^{d}(1-\pi)^{b}\log(1-\pi)\, d\pi$$

Table 1: Data for example

Income class	Number of income units	π	η
1 - 119	7	0.0123	0.0015
120 - 159	5	0.0211	0.004
160 - 199	14	0.0456	0.0131
200 - 299	154	0.3158	0.1515
300 – 399	120	0.5263	0.3027
400 – 499	76	0.6596	0.4258
500 – 599	54	0.7544	0.5327
600 – 699	49	0.8404	0.6474
700 - 799	22	0.8789	0.7068
800 - 999	43	0.9544	0.8462
1000 - 1199	12	0.9754	0.8937
1200 - 1499	10	0.993	0.9423
1500 - 1999	1	0.9947	0.9487
$\geq$2000	3	1	1

Table 2: ML and Bayesian estimates of the parameters of the Lorenz functions[a]

	ML	Bayes		ML	Bayes
Beta			**Gen Pareto**		
a	0.5642 (0.0272)	0.5673 (0.0345)	α	0.7948 (0.1839)	0.7110 (0.2319)
d	0.9071 (0.0199)	0.9139 (0.0256)	δ	0.4816 (0.0565)	0.4955 (0.0657)
b	0.4964 (0.0201)	0.4989 (0.0251)	γ	0.5476 (0.1448)	0.6146 (0.1888)
λ	3452.4 (1350.3)	2685.3 (1174.7)	λ	2748.2 (1073.3)	2157.9 (912.21)
log l'hood	57.81	63.52	log l'hood	56.13	61.67
RGKO			**Ortega**		
α	0.6506 (0.0221)	0.6489 (0.0263)	α	0.2661 (0.0439)	0.2590 (0.0503)
β	1.2344 (0.0444)	1.2290 (0.0517)	β	0.6083 (0.0169)	0.6080 (0.0198)
λ	1628.1 (631.41)	1428.6 (581.25)	λ	1958.3 (730.01)	1702.0 (705.50)
log l'hood	53.0562	63.55	log l'hood	54.17	64.49
Exponential					
a	1.9582 (0.2566)	1.9420 (0.2721)			
λ	233.43 (81.245)	231.79 (86.229)			
log l'hood	39.09	52.48			

[a] The log l'hood entries in the ML columns are the maximum values of the log-likelihood functions. Those in the Bayes columns are the logs of the marginal likelihoods.

Table 3: Posterior means and standard deviations for the Gini coefficient and the income shares for 10% and 90% of the population

	10% Share $\eta_{0.1}$	**90% Share** $\eta_{0.9}$	**Gini** γ	**Posterior Prob.**
Beta	0.0344 (0.0028)	0.7368 (0.0077)	0.3212 (0.0122)	0.208
Gen Pareto	0.0319 (0.0034)	0.7381 (0.0090)	0.3189 (0.0128)	0.033
RGKO	0.0357 (0.0039)	0.7317 (0.0109)	0.3229 (0.0154)	0.213
Ortega	0.0343 (0.0037)	0.7328 (0.0099)	0.3230 (0.0144)	0.546
Exponential	0.0363 (0.0059)	0.7938 (0.0167)	0.3041 (0.0379)	0.000
Average	0.0346 (0.0036)	0.7336 (0.0099)	0.3225 (0.0142)	

Figure1: Posterior pdf's for the Gini Coefficient

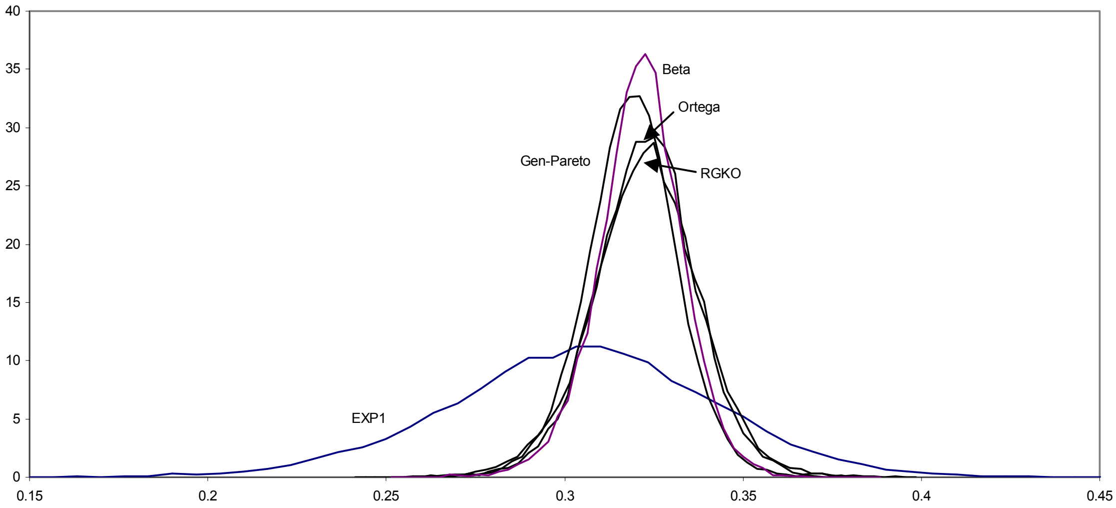


Figure2: Posterior pdf's and Average Posterior pdf for the Gini Coefficient

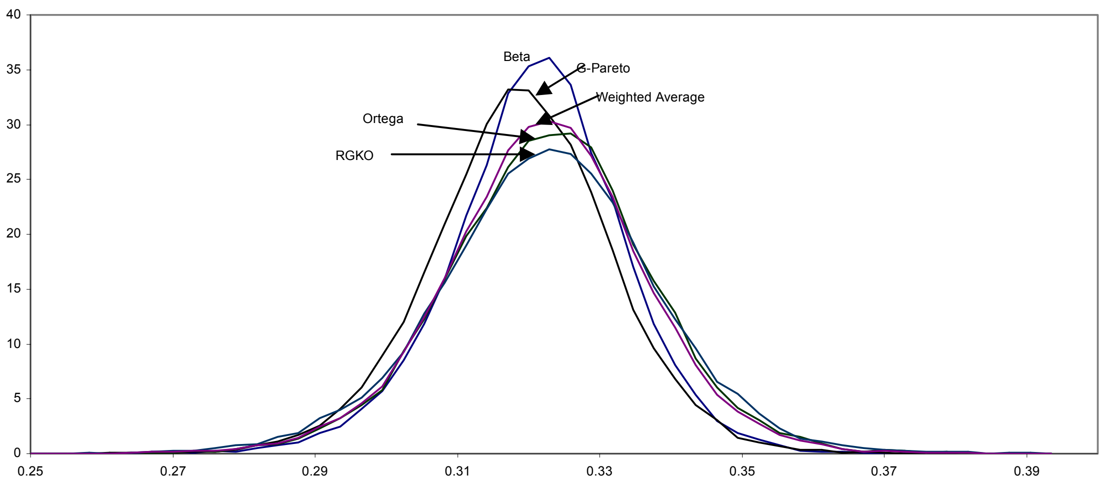


Figure 3: Posterior pdf's for the 10% Share

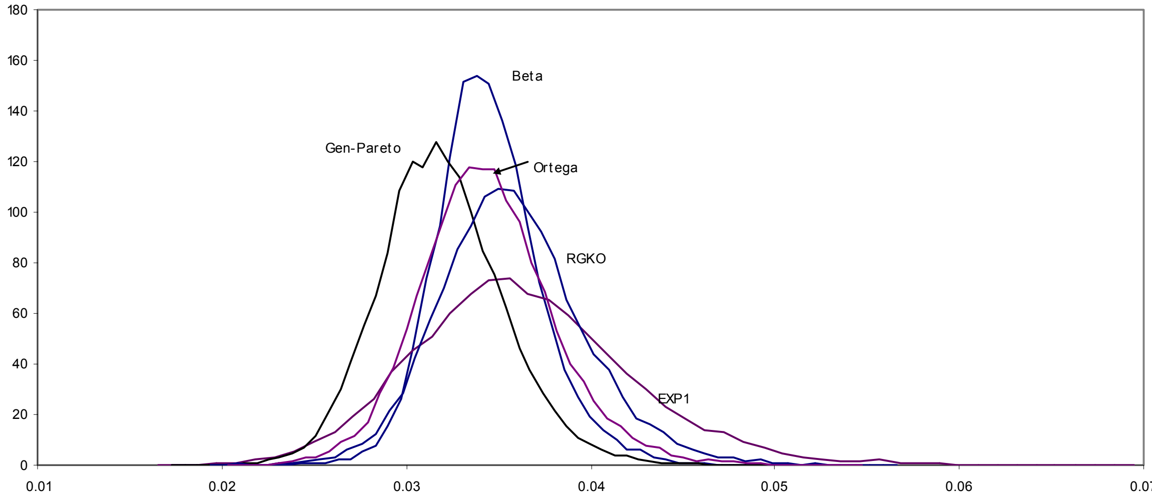


Figure 4: Posterior pdf's and Average Posterior pdf for the 10% Share

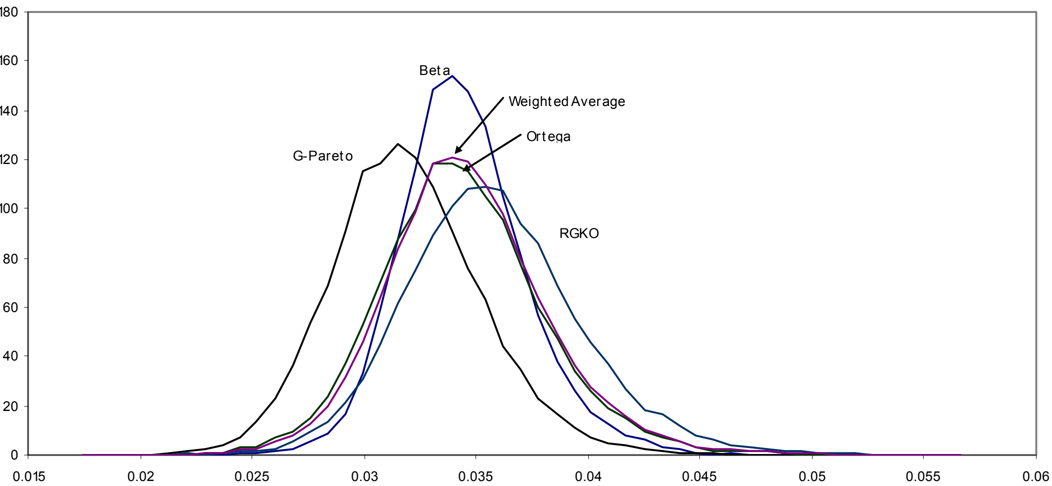


Figure 5: Posterior pdf's for the 90% Share

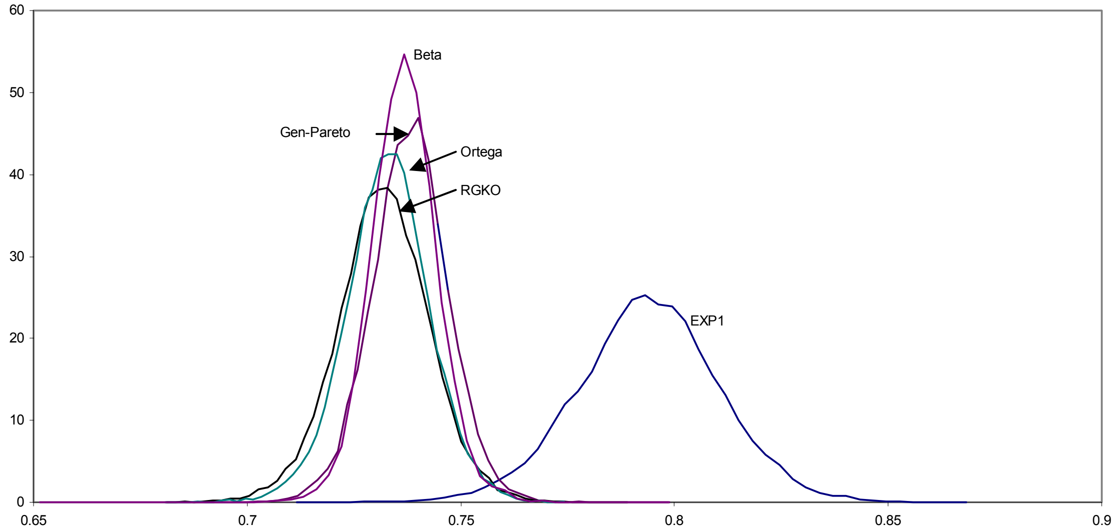
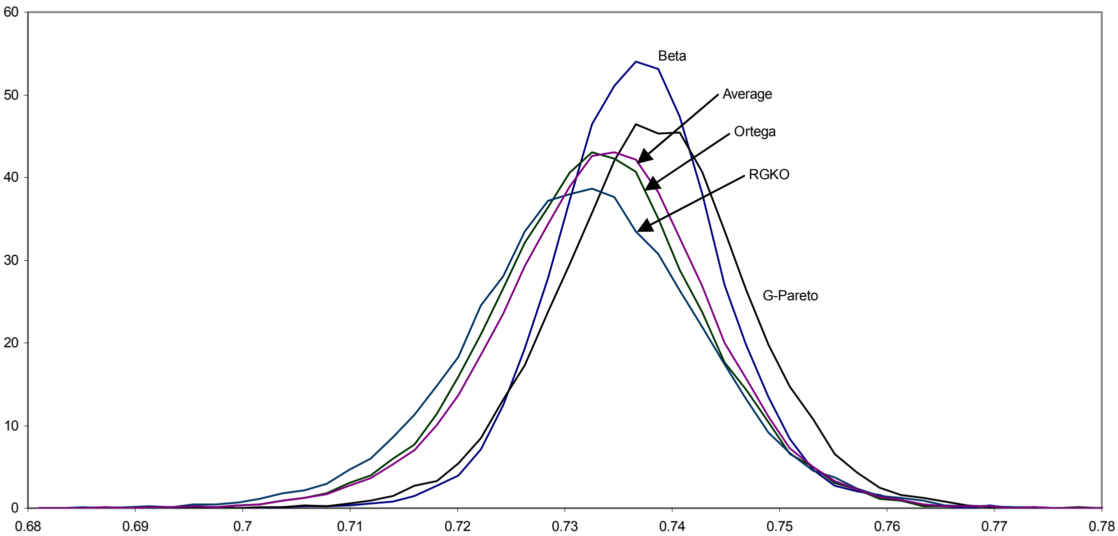


Figure 6: Posterior pdf's and Average Posterior pdf for the 90% Share



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