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Kohta Takehara
Graduate School of Economics, University of Tokyo

Masashi Toda
Graduate School of Economics, University of Tokyo

Akihiko Takahashi
University of Tokyo

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1

APPLICATION OF A HIGH-ORDER ASYMPTOTIC EXPANSION SCHEME TO LONG-TERM CURRENCY OPTIONS [1]

Kohta Takehara[2], Masashi Toda and Akihiko Takahashi
Graduate School of Economics, the University of Tokyo
7-3-1, Hongo, Bunkyo-ku, Tokyo, 113-0033, Japan, +81-3-3812-2111

ABSTRACT

Recently, not only academic researchers but also many practitioners have used the methodology so-called ``an asymptotic expansion method" in their proposed techniques for a variety of financial issues. e.g. pricing or hedging complex derivatives under high-dimensional stochastic environments. This methodology is mathematically justified by Watanabe theory(Watanabe [1987], Yoshida [1992a,b]) in Malliavin calculus and essentially based on the framework initiated by Kunitomo and Takahashi [2003], Takahashi [1995,1999] in a financial context.
In practical applications, it is desirable to investigate the accuracy and stability of the method especially with expansion up to high orders in situations where the underlying processes are highly volatile as seen in recent financial markets.
After Takahashi [1995,1999] and Takahashi and Takehara [2007] had provided explicit formulas for the expansion up to the third order, Takahashi, Takehara and Toda [2009] develops general computation schemes and formulas for an arbitrary-order expansion under general diffusion-type stochastic environments.
In this paper, we describe them in a simple setting to illustrate thier key idea, and to demonstrate their effectiveness apply them to pricing long-term currency options under a cross-currency Libor market model and a general stochastic volatility of a spot exchange rate with maturities up to twenty years.
Keywords: Asymptotic Expansion, Malliavin Calculus, Stochastic Volatility, Libor Market Model, Currency Options
JEL Classification: C63, G13

1. INTRODUCTION

This paper explains two alternative schemes for computation proposed by Takahashi, Takehara and Toda [2009] in the method so-called “an asymptotic expansion approach" based on Watanabe theory(Watanabe [1987]) in Malliavin calculus in a simple setting and apply them to pricing long-term currency options under a cross-currency Libor market model and a general stochastic volatility of a spot exchange rate.

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1. Aim of This Paper

To our best knowledge, the asymptotic expansion is first applied to finance for evaluation of an average option that is a popular derivative in commodity markets.
Recently, not only academic researchers but also many practitioners have used the asymptotic expansion method based on Watanabe theory in their proposed techniques for a variety of financial issues. e.g. pricing or hedging complex derivatives under high-dimensional underlying stochastic environments. These methods are fully or partially based on the framework developed by Kunitomo and Takahashi [1992], Takahashi [1995,1999] in a financial literature.
In theory, this method provides us the expansion of underlying stochastic processes which has a proper meaning in the limit of some ideal situations such as cases where they come deterministic ones (for details see Watanabe [1987], Yoshida [1992a] or Kunitomo and Takahashi [2003]).
In practice, however, we are often interested in cases far from that situation, where the underlying processes are highly volatile as seen in the recent financial markets especially after the crisis on 2008. Then from the view point of the accuracy or stability of the techniques in practical uses, it is desirable to investigate behaviors of its estimators in such situations especially with expansion up to high orders.
In application of the asymptotic expansion, the crucial step is computation of certain conditional expectations appearing in the expansions, especially in the expansion up to high orders which is important in the cases with long maturities or/and with highly volatile underlying variables. Takahashi, Takehara and Toda [2009] developed two alternative schemes for these computations in a general diffusion-type stochastic environment.
This paper describes the essence of their method in a much simpler setting and to demonstrate their effectiveness, apply them to evaluation of long-term currency options with maturities up to twenty years under a cross-currency Libor market model and a general stochastic volatility of a spot exchange rate, which is very complex to obtain closed-form formulas.

2. Literature Reviewing

In this subsection we briefly review related literatures of the asymptotic expansion.
As we mentioned earlier, to our best knowledge, the first application of the asymptotic expasion based on Watanabe theory in finance was Kunitomo and Takahashi [1992] which evaluated average options. Kunitomo and Takahashi [1992] and Takahashi [1995] derive the approximation formulas for an average option by an asymptotic method based on log-normal approximations of an average price distribution when the underlying asset price follows a geometric Brownian motion. Yoshida [1992b] applies a formula derived by the asymptotic expansion of certain statistical estimators for small diffusion processes.
Thereafter, the asymptotic expansion have been applied to a broad class of problems in finance: The basic framework of the method in a general setting was described in Kunitomo and Takahashi [2003], Takahashi [1999,2009]. Kunitomo and Takahashi [2001] generalized and applied the method to the interest derivatives where the underlying model was not necessarily Markovian. Matsuoka, Takahshi and Uchida [2004] computed Greeks, the sensitivities of derivatives with respect to parameters. In Takahashi and Yoshida [2004,2005] the method was used for the optimal portfolio problem and a new variance reduction technique for Monte Carlo simulations with the asymptotiv expansion eas developed. Muroi [2005] considered credit derivatives. Similar issues to that in this paper, pricing currency options under the cross-currency

Libor market model and the exchanfge rate with a sotchastic volatility and/or jumps, were examined in Takahashi and Takehara [2007, 2008a,b]. Takahashi, Takehara and Toda [2009] introduced the genral procedures for actual computation in the method which are applied in this paper.

Organization of this paper is as follows: After Section 2 will develop our methods in the simple setting, Section 3 applies our algorithms described in the previous section to the concrete financial models, and confirms the effectiveness of the higher order expansions by numerical examples in a cross-currency Libor market model with a general stochastic volatility model of the spot foreign exchange rate. Detailed proofs, formulas, or argument of the applied technique in a general setting including our complex example are found in Takahashi, Takehara and Toda [2009].

2. An Asymptotic Expansion Approach in a Black-Scholes Economy

In this section, our essential idea is explained in a simple Black-Scholes-type economy. For discussions in more general settings, refer to Takahashi, Takehara and Toda [2009].

1. An Asymptotic Expansion Approach in a Black-Scholes Economy

Let (W,P) be a one-dimensional Wiener space. Hereafter P is considered as a risk-neutral equivalent martingale measure and a risk-free interest rate is set to be zero for simplicity. Then, the underlying economy is specified with a ($\mathbf{R}_+$ -valued)single risky asset $S^{(\varepsilon)}=\{S_t^{(\varepsilon)}\}$ satisfying

$$S_t^{(\varepsilon)}=S_0+\varepsilon\int_0^t \sigma(S_s^{(\varepsilon)},s)dW_s \tag{1}$$

where $\varepsilon\in(0,1]$ is a constant parameter; $\sigma:\ \mathbf{R}_+^2\mapsto\mathbf{R}$ satisfies some regularity conditions. We will consider the following pricing problem;

$$V(0,T)=\mathbf{E}[\Phi(S_T^{(\varepsilon)})] \tag{2}$$

where Φ is a payoff function written on $S_T^{(\varepsilon)}$ (for example, $\Phi(x)=\max(x-K,0)$ for call options or $\Phi(x)=\delta_x(x)$, a delta function with mass at x for the density function) and $\mathbf{E}[\cdot]$ is an expectation operator under the probability measure P. Rigorously speaking, they are a generalized function on the Wiener functional $S^{(\varepsilon)}$ and a generalized expectation defined for generalized functions respectively, whose mathematically proper definitions will be given in Section 2 of Takahashi, Takehara and Toda [2009].

Let $A_{kt}=\frac{\partial^k S_t^{(\varepsilon)}}{\partial\varepsilon^k}|_{\varepsilon=0}$. Here we represent A_{1t}, A_{2t} and A_{3t} explicitly by

$$A_{1t}=\int_0^t \sigma(S_s^{(0)},s)dW_s, \tag{3}$$

$$A_{2t}=2\int_0^t \partial\sigma(S_s^{(0)},s)A_{1s}dW_s, \tag{4}$$

$$A_{3t}=3\int_0^t\left(\partial^2\sigma(S_s^{(0)},s)(A_{1s})^2+\partial\sigma(S_s^{(0)},s)(A_{2s})\right)dW_s \tag{5}$$

recursively and then $S_T^{(\varepsilon)}$ has its asymptotic expansion

$$S_T^{(\varepsilon)} = S_0 + \varepsilon A_{1T} + \frac{\varepsilon^2}{2!} A_{2T} + \frac{\varepsilon^3}{3!} A_{3T} + o(\varepsilon^3). \tag{6}$$

Note that $S_t^{(0)} = \lim_{\varepsilon \downarrow 0} S_t^{(\varepsilon)} = S_0$ for all t.

Next, normalize $S_T^{(\varepsilon)}$ with respect to ε as $G^{(\varepsilon)} = \frac{S_T^{(\varepsilon)} - S_T^{(0)}}{\varepsilon}$ for $\varepsilon \in (0,1]$. Then,

$$G^{(\varepsilon)} = A_{1T} + \frac{\varepsilon}{2!} A_{2T} + \frac{\varepsilon^2}{3!} A_{3T} + o(\varepsilon^2) \tag{7}$$

in L^P for every $p > 1$. Here the following assumption is made: $\Sigma_T = \int_0^T \sigma^2(S_t^{(0)}, t)dt > 0.$ Note that A_{1T} follows a normal distribution with mean 0 and variance Σ_T, and hence this assumption means that the distribution of A_{1T} does not degenerate. It is clear that this assumption is satisfied when $\sigma(S_t^{(0)}, t) > 0$ for some $t > 0$.

Then, the expectation of $\Phi(G^{(\varepsilon)})$ is expanded around $\varepsilon = 0$ up to ε^2-order in the sense of Watanabe([1987], Yoshida[1992a]) as follows (hereafter the asymptotic expansion of $\mathbf{E}[\Phi(G^{(\varepsilon)})]$ up to the second order will be considered):

$$\begin{aligned}
\mathbf{E}[\Phi(G^{(\varepsilon)})] &= \mathbf{E}\left[\Phi(A_{1T})\right] + \varepsilon \mathbf{E}\left[\Phi^{(1)}(A_{1T})A_{2T}\right] \\
&\quad + \varepsilon^2 \left\{ \mathbf{E}\left[\Phi^{(1)}(A_{1T})A_{3T}\right] + \frac{1}{2}\mathbf{E}\left[\Phi^{(2)}(A_{1T})(A_{2T})^2\right] \right\} + o(\varepsilon^2) \\
&= \mathbf{E}\left[\Phi(A_{1T})\right] + \varepsilon \mathbf{E}\left[\Phi^{(1)}(A_{1T})\mathbf{E}\left[A_{2T} \mid A_{1T}\right]\right] \\
&\quad + \varepsilon^2 \left\{ \mathbf{E}\left[\Phi^{(1)}(A_{1T})\mathbf{E}\left[A_{3T} \mid A_{1T}\right]\right] + \frac{1}{2}\mathbf{E}\left[\Phi^{(2)}(A_{1T})\mathbf{E}\left[(A_{2T})^2 \mid A_{1T}\right]\right] \right\} + o(\varepsilon^2) \\
&= \int_{\mathbf{R}} \Phi(x) f_{A_{1T}}(x)dx + \varepsilon \int_{\mathbf{R}} \Phi^{(1)}(x)\mathbf{E}\left[A_{2T} \mid A_{1T} = x\right] f_{A_{1T}}(x)dx \\
&\quad + \varepsilon^2 \left\{ \int_{\mathbf{R}} \Phi^{(1)}(x)\mathbf{E}\left[A_{3T} \mid A_{1T} = x\right] f_{A_{1T}}(x)dx + \frac{1}{2}\int_{\mathbf{R}} \Phi^{(2)}(x)\mathbf{E}\left[(A_{2T})^2 \mid A_{1T} = x\right] f_{A_{1T}}(x)dx \right\} + o(\varepsilon^2) \\
&= \int_{\mathbf{R}} \Phi(x) f_{A_{1T}}(x)dx + \varepsilon \int_{\mathbf{R}} \Phi(x)(-1)\frac{\partial}{\partial x}\{ \mathbf{E}\left[A_{2T} \middle| A_{1T} = x\right] f_{A_{1T}}(x) \}dx \\
&\quad + \varepsilon^2 \Bigg(\int_{\mathbf{R}} \Phi(x)(-1)\frac{\partial}{\partial x}\{ \mathbf{E}\left[A_{3T} \middle| A_{1T} = x\right] f_{A_{1T}}(x) \}dx \\
&\quad + \frac{1}{2}\int_{\mathbf{R}} \Phi(x)(-1)^2 \frac{\partial^2}{\partial x^2}\{ \mathbf{E}\left[(A_{2T})^2 \middle| A_{1T} = x\right] f_{A_{1T}}(x) \}dx \Bigg) + o(\varepsilon^2).
\end{aligned} \tag{8}$$

where $\Phi^{(m)}(x)$ is m-th order derivative of $\Phi(x)$ and $f_{A_{1T}}(x)$ is a probability density function of A_{1T} following a normal distribution; $f_{A_{1T}}(x) := \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{x^2}{2\Sigma_T}\right)$. In particular, letting $\Phi = \delta_x$, we have the asymptotic expansion of the density function of $G^{(\varepsilon)}$ as seen later.

Then, all we have to do to evaluate this expansion is a computation of these conditional expectations. In Particular, we present two alternative approaches.

2. An Approach with an Expansion into Iterated Itô Integrals

In this subsection we show an approach with a further expansion of A_{2T}, A_{3T} and $(A_{2T})^2$ into iterated Itô integrals to compute the conditional expectations in (8).
Recall that we have

$$\begin{aligned}\mathbf{E}[\Phi(G^{(\varepsilon)})] &= \int_{\mathbf{R}}\Phi(x)f_{A_{1T}}(x)dx+\varepsilon\int_{\mathbf{R}}\Phi(x)(-1)\frac{\partial}{\partial x}\{\mathbf{E}\left[A_{2T}\left|A_{1T}=x\right.\right]f_{A_{1T}}(x)\}dx\\ &+\varepsilon^2\left(\int_{\mathbf{R}}\Phi(x)(-1)\frac{\partial}{\partial x}\{\mathbf{E}\left[A_{3T}\left|A_{1T}=x\right.\right]f_{A_{1T}}(x)\}dx\right.\\ &\left.+\frac{1}{2}\int_{\mathbf{R}}\Phi(x)(-1)^2\frac{\partial^2}{\partial x^2}\{\mathbf{E}\left[(A_{2T})^2\left|A_{1T}=x\right.\right]f_{A_{1T}}(x)\}dx\right)+o(\varepsilon^2).\end{aligned} \tag{9}$$

Next, it is shown that A_{2T}, A_{3T}, $(A_{2T})^2$ can be expressed as summations of iterated Itô integrals. First, note that A_{2T} is

$$A_{2T}=2\int_0^T\int_0^{t_1}\partial\sigma(S_{t_1}^{(0)},t_1)\sigma(S_{t_2}^{(0)},t_2)dW_{t_2}dW_{t_1} \tag{10}$$

Next, by application of Itô's formula to (5) we obtain

$$\begin{aligned}A_{3T}&=6\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma(S_{t_1}^{(0)},t_1)\partial\sigma(S_{t_2}^{(0)},t_2)\sigma(S_{t_3}^{(0)},t_3)dW_{t_3}dW_{t_2}dW_{t_1}\\ &+6\int_0^T\int_0^{t_1}\int_0^{t_2}\partial^2\sigma(S_{t_1}^{(0)},t_1)\sigma(S_{t_2}^{(0)},t_2)\sigma(S_{t_3}^{(0)},t_3)dW_{t_3}dW_{t_2}dW_{t_1}\\ &+3\int_0^T\int_0^{t_1}\partial^2\sigma(S_{t_1}^{(0)},t_1)\sigma^2(S_{t_2}^{(0)},t_2)dt_2dW_{t_1}.\end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned}(A_{2T})^2&=16\int_0^T\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\partial\sigma(S_{t_1}^{(0)},t_1)\partial\sigma(S_{t_2}^{(0)},t_2)\sigma(S_{t_3}^{(0)},t_3)\sigma(S_{t_4}^{(0)},t_4)dW_{t_4}dW_{t_3}dW_{t_2}dW_{t_1}\\ &+8\int_0^T\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\partial\sigma(S_{t_1}^{(0)},t_1)\sigma(S_{t_2}^{(0)},t_2)\partial\sigma(S_{t_3}^{(0)},t_3)\sigma(S_{t_4}^{(0)},t_4)dW_{t_4}dW_{t_3}dW_{t_2}dW_{t_1}\\ &+8\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma(S_{t_1}^{(0)},t_1)\partial\sigma(S_{t_2}^{(0)},t_2)\sigma^2(S_{t_3}^{(0)},t_3)dt_3dW_{t_2}dW_{t_1}\\ &+8\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma(S_{t_1}^{(0)},t_1)\partial\sigma(S_{t_2}^{(0)},t_2)\sigma(S_{t_2}^{(0)},t_2)\sigma(S_{t_3}^{(0)},t_3)dW_{t_3}dt_2dW_{t_1}\\ &+8\int_0^T\int_0^{t_1}\int_0^{t_2}\left(\partial\sigma(S_{t_1}^{(0)},t_1)\right)^2\sigma(S_{t_2}^{(0)},t_2)\sigma(S_{t_3}^{(0)},t_3)dW_{t_3}dW_{t_2}dt_1\\ &+4\int_0^T\int_0^{t_1}\left(\partial\sigma(S_{t_1}^{(0)},t_1)\right)^2\sigma^2(S_{t_2}^{(0)},t_2)dt_2dt_1.\end{aligned} \tag{12}$$

Then, by Proposition 1 in Takahashi, Takehara and Toda [2009], the conditional expectations in (9) can be computed as

$$\mathbf{E}[A_{2T}\mid A_{1T}=x]=2F_2\left(\partial\sigma^{(0)}\sigma^{(0)},(\sigma^{(0)})^2\right)\frac{H_2(x;\Sigma_T)}{\Sigma_T^2}=:c_2^{2,1}H_2(x;\Sigma_T) \tag{13}$$

$$\mathbf{E}[A_{3T} \mid A_{1T} = x] = \left(6F_3\left(\partial\sigma^{(0)}\sigma^{(0)}, \partial\sigma^{(0)}\sigma^{(0)}, (\sigma^{(0)})^2\right) + 6F_3\left(\partial^2\sigma^{(0)}\sigma^{(0)}, (\sigma^{(0)})^2, (\sigma^{(0)})^2\right)\right)\frac{H_3(x;\Sigma_T)}{\Sigma_T^3}$$
$$+3F_2\left(\partial\sigma^{(0)}\sigma^{(0)}, (\sigma^{(0)})^2\right)\frac{H_1(x;\Sigma_T)}{\Sigma_T} \tag{14}$$
$$=: c_3^{3,1}H_3(x;\Sigma_T) + c_1^{3,1}H_1(x;\Sigma_T)$$

and

$$\mathbf{E}[(A_{2T})^2 \mid A_{1T} = x]$$
$$= \left(16F_4\left(\partial\sigma^{(0)}\sigma^{(0)}, \partial\sigma^{(0)}\sigma^{(0)}, (\sigma^{(0)})^2, (\sigma^{(0)})^2\right) + 8F_4\left(\partial\sigma^{(0)}\sigma^{(0)}, (\sigma^{(0)})^2, \partial\sigma^{(0)}\sigma^{(0)}, (\sigma^{(0)})^2\right)\right)\frac{H_4(x;\Sigma_T)}{\Sigma_T^4}$$
$$+\left(16F_3\left(\partial\sigma^{(0)}\sigma^{(0)}, \partial\sigma^{(0)}\sigma^{(0)}, (\sigma^{(0)})^2\right) + 8F_3\left((\partial\sigma^{(0)})^2, (\sigma^{(0)})^2, (\sigma^{(0)})^2\right)\right)\frac{H_2(x;\Sigma_T)}{\Sigma_T^2}$$
$$+4F_3\left((\partial\sigma^{(0)})^2, (\sigma^{(0)})^2\right)H_0(x;\Sigma_T) \tag{15}$$
$$=: c_4^{2,2}H_4(x;\Sigma_T) + c_2^{2,2}H_2(x;\Sigma_T) + c_0^{2,2}H_0(x;\Sigma_T)$$

where $H_n(x;\Sigma)$ is a n-th order Hermite polynomial defined by

$$H_n(x;\Sigma) := (-\Sigma)^n e^{x^2/2\Sigma}\frac{d^n}{dx^n}e^{-x^2/2\Sigma},$$

with notations $F_n(f_1,\cdots,f_n) := \int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}} f_1(t_1)\cdots f_n(t_n)dt_n\cdots dt_1, n \geq 1,$

$\sigma^{(0)} = \sigma(S_t^{(0)}, t)$ and $\partial^i\sigma^{(0)} = \partial^i\sigma(S_t^{(0)}, t)$.

Substituting these into (9), we have the asymptotic expansion of $\mathbf{E}\left[\Phi(G^{(\varepsilon)})\right]$ up to ε^2-order. Further, letting $\Phi = \delta_x$, we have the expansion of $f_{G^{(\varepsilon)}}$, the density function of $G^{(\varepsilon)}$:

$$f_{G^{(\varepsilon)}} = f_{A_{1T}}(x) + \varepsilon(-1)\frac{\partial}{\partial x}\{\mathbf{E}\left[A_{2T} \middle| A_{1T} = x\right] f_{A_{1T}}(x)\}$$
$$+\varepsilon^2\left((-1)\frac{\partial}{\partial x}\{\mathbf{E}\left[A_{3T} \middle| A_{1T} = x\right] f_{A_{1T}}(x)\} + \frac{1}{2}(-1)^2\frac{\partial^2}{\partial x^2}\{\mathbf{E}\left[(A_{2T})^2 \middle| A_{1T} = x\right] f_{A_{1T}}(x)\}\right) + o(\varepsilon^2)$$
$$= f_{A_{1T}}(x) + \varepsilon(-1)\frac{\partial}{\partial x}\{c_2^{2,1}H_2(x;\Sigma_T) f_{A_{1T}}(x)\} \tag{16}$$
$$+\varepsilon^2\left((-1)\frac{\partial}{\partial x}\{\sum_{i=1,3} c_i^{3,1}H_i(x;\Sigma_T) f_{A_{1T}}(x)\} + \frac{1}{2}(-1)^2\frac{\partial^2}{\partial x^2}\{\sum_{i=0,2,4} c_i^{2,2}H_i(x;\Sigma_T) f_{A_{1T}}(x)\}\right) + o(\varepsilon^2).$$

3. An Alternative Approach with a System of Ordinary Differential Equations

In this subsection, we present an alternative approachin which the conditional expectations are computed through some system of ordinary differential equations.

Again the asymptotic expansion of $\mathbf{E}\left[\Phi(G^{(\varepsilon)})\right]$ up to ε^2-order is considered in this subsection.

Note that the expectations of A_{2T}, A_{3T} and $(A_{2T})^2$ conditional on A_{1T} are expressed by linear combinations of a finite number of Hermite polynomials as in (13), (14) and (15). Thus, by Lemma 4 in Takahashi, Takehara and Toda [2009], we have we have

$$\mathbf{E}[A_{2T} \mid A_{1T} = x] = \sum_{n=0}^{2} a_n^{2,1} H_n(x;\Sigma_T), \tag{17}$$

$$\mathbf{E}[A_{3T} \mid A_{1T} = x] = \sum_{n=0}^{3} a_n^{3,1} H_n(x;\Sigma_T), \tag{18}$$

$$\text{and}\mathbf{E}[(A_{2T})^2 \mid A_{1T} = x] = \sum_{n=0}^{4} a_n^{2,2} H_n(x;\Sigma_T), \tag{19}$$

where the coefficients are given by

$$a_n^{2,1} = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\frac{\partial^n}{\partial\xi^n}\bigg|_{\xi=0}\left\{\mathbf{E}[Z_T^{<\xi>}A_{2T}]\right\}, a_n^{3,1} = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\frac{\partial^n}{\partial\xi^n}\bigg|_{\xi=0}\left\{\mathbf{E}[Z_T^{<\xi>}A_{3T}]\right\},$$

$$a_n^{2,2} = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\frac{\partial^n}{\partial\xi^n}\bigg|_{\xi=0}\left\{\mathbf{E}[Z_T^{<\xi>}(A_{2T})^2]\right\}, \text{and} Z_t^{<\xi>} := \exp\left(i\xi A_{1t} + \frac{\xi^2}{2}\Sigma_t\right).$$

Note that $Z^{<\xi>}$ is a martingale with $Z_0^{<\xi>} = 1$. Since these conditional expectations can be represented by linear combinations of Hermite polynomials as seen in the previous subsection, the following should hold, which can be confirmed easily with results of this subsection:

$$\begin{cases} a_2^{2,1} = c_2^{2,1}; a_1^{2,1} = a_0^{2,1} = 0; a_3^{3,1} = c_3^{3,1}; a_1^{3,1} = c_1^{3,1}; a_2^{3,1} = a_0^{2,1} = 0; (20) \\ a_4^{2,2} = c_4^{2,2}; a_2^{2,2} = c_2^{2,2}; a_0^{2,2} = c_0^{2,2}; a_3^{2,2} = a_1^{2,2} = 0. \end{cases} \tag{21}$$

Then, computation of these conditional expectations is equivalent to that of the unconditional expectations $\mathbf{E}[Z_T^{<\xi>}A_{2T}]$, $\mathbf{E}[Z_T^{<\xi>}A_{3T}]$ and $\mathbf{E}[Z_T^{<\xi>}(A_{2T})^2]$.

First, applying Itô's formula to $\left(Z_t^{<\xi>}A_{2t}\right)$ we have

$$\begin{aligned}\mathbf{E}\left[Z_t^{<\xi>}A_{2t}\right] &= \mathbf{E}\left[\int_0^t Z_s^{<\xi>}dA_{2s} + \int_0^t A_{2s}dZ_s^{<\xi>} + \left\langle A_2, Z^{<\xi>}\right\rangle_t\right] \\ &= 2(i\xi)\int_0^t \partial\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}A_{1s}\right]ds \end{aligned} \tag{22}$$

Then, applying Itô's formula to $\left(Z_t^{<\xi>}A_{1t}\right)$ again, we also have

$$\begin{aligned}\mathbf{E}\left[Z_t^{<\xi>}A_{1t}\right] &= \mathbf{E}\left[\int_0^t Z_s^{<\xi>}dA_{1s} + \int_0^t A_{1s}dZ_s^{<\xi>} + \left\langle A_1, Z^{<\xi>}\right\rangle_t\right] \\ &= (i\xi)\int_0^t \sigma^2(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}\right]ds = (i\xi)\int_0^t \sigma^2(S_s^{(0)},s)ds \end{aligned} \tag{23}$$

since $\mathbf{E}\left[Z_t^{<\xi>}\right] = 1$ for all t.

Similarly, the followings are obtained;

$$\mathbf{E}\left[Z_t^{<\xi>}A_{3t}\right] = 3(i\xi)\left(\int_0^t \partial^2\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}(A_{1s})^2\right]ds\right.$$

$$+\int_0^t \partial\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}A_{2s}\right]ds\Big) \tag{24}$$

$$\mathbf{E}\left[Z_t^{<\xi>}(A_{1t})^2\right]=\int_0^t \sigma^2(S_s^{(0)},s)ds+2(i\xi)\int_0^t \sigma^2(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}A_{1s}\right]ds \tag{25}$$

$$\mathbf{E}\left[Z_t^{<\xi>}(A_{2t})^2\right]=4\int_0^t \left(\partial\sigma(S_s^{(0)},s)\right)^2\mathbf{E}\left[Z_s^{<\xi>}(A_{1s})^2\right]ds$$

$$+4(i\xi)\int_0^t \partial\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}A_{2s}A_{1s}\right]ds \tag{26}$$

$$\mathbf{E}\left[Z_t^{<\xi>}A_{2t}A_{1t}\right]=2\int_0^t \partial\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}A_{1s}\right]ds \tag{27}$$

$$+(i\xi)\int_0^t (\sigma(S_s^{(0)},s))^2\mathbf{E}\left[Z_s^{<\xi>}A_{2s}\right]ds+2(i\xi)\int_0^t \partial\sigma(S_s^{(0)},s)\sigma(S_s^{(0)},s)\mathbf{E}\left[Z_s^{<\xi>}(A_{1s})^2\right]ds.$$

Then, $\mathbf{E}[Z_T^{<\xi>}A_{2T}]$, $\mathbf{E}[Z_T^{<\xi>}A_{3T}]$ and $\mathbf{E}[Z_T^{<\xi>}(A_{2T})^2]$ can be obtained as solutions of the system of ordinary differential equations (22), (23), (24), (25), (26) and (27). In fact, since they have a grading structure that the higher-order equations depend only on the lower ones, they can be easily solved with substituting each solution into the next ordinary differential equation recursively. Moreover, since these solutions are clearly the polynomial of $(i\xi)$, we can easily implement differentiations with respect to ξ in (17), (18) and (19). It is obvious that the resulting coefficients given by these solutions are equivalent to the results in the previous subsection.

Here, at the end of this section, we state a brief summary. In the Black-Scholes-type economy, we consider the risky asset $S^{(\varepsilon)}$ and evaluate some quantities, expressed as an expectation of the function of the terminal price, such as prices or risk sensitivities of the securities on this asset. First we expand them around the limit to $\varepsilon=0$ so that we obtain the expansion (8) which contains some conditional expectations. Then, by approaches described in Section 2 or 3, we compute these conditional expectation. Finally, substituting computation results into (8), we obtain the asymptotic expansion of those quantities.

3. Numerical Examples: application to long-term currency options

In this section we apply our methods to pricing options on currencies under Libor Market Models(LMMs) of interest rates and a stochastic volatility of the spot foreign exchange rate(Forex), which is much more complex then Black-Scholes-type case in the previous section. Due to limitation of space, only the structure of the stochastic differential equations of our model is described here. For details of the underlying model, see Takahashi and Takehara [2007]. Detailed discussions in a general setting including following examples are found in Section 3 and 4 of Takahashi, Takehara and Toda [2009].

1. Cross-Currency Libor Market Models

Let $(\Omega, F, \tilde{P}, \{F_t\}_{0\le t\le T^*<\infty})$ be a complete probability space with a filtration satisfying the usual conditions. We consider the following pricing problem for the call option with maturity $T\in(0,T^*]$ and strike rate $K>0$;

$$V^C(0;T,K) = P_d(0,T)\times \mathbf{E}^P\left[(S(T)-K)^+\right] = P_d(0,T)\times \mathbf{E}^P\left[(F_T(T)-K)^+\right] \tag{28}$$

where $V^C(0;T,K)$ denotes the value of an European call option at time 0 with maturity T and strike rate K, $S(T)$ denotes the spot exchange rate at time $t \geq 0$ and $F_T(t)$ denotes the time t value of the forex forward rate with maturity T. Similarly, for the put option we consider

$$V^P(0;T,K) = P_d(0,T)\times \mathbf{E}^P\left[(K-S(T))^+\right] = P_d(0,T)\times \mathbf{E}^P\left[(K-F_T(T))^+\right]. \tag{29}$$

It is well known that the arbitrage-free relation between the forex spot rate and the forex forward rate are given by $F_T(t) = S(t)\frac{P_f(t,T)}{P_d(t,T)}$ where $P_d(t,T)$ and $P_f(t,T)$ denote the time t values of domestic and foreign zero coupon bonds with maturity T respectively. $\mathbf{E}^P[\cdot]$ denotes an expectation operator under EMM(Equivalent Martingale Measure) P whose associated numeraire is the domestic zero coupon bond maturing at T.

For these pricing problems, a market model and a stochastic volatility model are applied to modeling interest rates' and the spot exchange rate's dynamics respectively.

We first define domestic and foreign forward interest rates as $f_{dj}(t) = \left(\frac{P_d(t,T_j)}{P_d(t,T_{j+1})}-1\right)\frac{1}{\tau_j}$ and $f_{fj}(t) = \left(\frac{P_f(t,T_j)}{P_f(t,T_{j+1})}-1\right)\frac{1}{\tau_j}$ respectively, where $j = n(t), n(t)+1, \cdots, N$, $\tau_j = T_{j+1} - T_j$, and $P_d(t,T_j)$ and $P_f(t,T_j)$ denote the prices of domestic/foreign zero coupon bonds with maturity T_j at time $t\,(\leq T_j)$ respectively; $n(t) = \min\{i : t \leq T_i\}$. We also define spot interest rates to the nearest fixing date denoted by $f_{d,n(t)-1}(t)$ and $f_{f,n(t)-1}(t)$ as $f_{d,n(t)-1}(t) = \left(\frac{1}{P_d(t,T_{n(t)})}-1\right)\frac{1}{(T_{n(t)}-t)}$ and $f_{f,n(t)-1}(t) = \left(\frac{1}{P_f(t,T_{n(t)})}-1\right)\frac{1}{(T_{n(t)}-t)}$. Finally, we set $T = T_{N+1}$ and will abbreviate $F_{T_{N+1}}(t)$ to $F_{N+1}(t)$ in what follows.

Under the framework of the asymptotic expansion in the standard cross-currency libor market model, we have to consider the following system of stochastic differential equations(henceforth called S.D.E.s) under the domestic terminal measure P to price options. For detailed arguments on the framework of these S.D.E.s see Takahashi and Takehara [2007].

As for the domestic and foreign interest rates we assume forward market models; for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$f_{dj}^{(\varepsilon)}(t) = f_{dj}(0) + \varepsilon^2 \sum_{i=j+1}^{N} \int_0^t g_{di}^{0,(\varepsilon)}(u)'\gamma_{dj}(u) f_{dj}^{(\varepsilon)}(u)du + \varepsilon \int_0^t f_{dj}^{(\varepsilon)}(u)\gamma_{dj}'(u)dW_u, \tag{30}$$

$$\begin{aligned} f_{fj}^{(\varepsilon)}(t) = f_{fj}(0) &- \varepsilon^2 \sum_{i=0}^{j} \int_0^t g_{fi}^{0,(\varepsilon)}(u)'\gamma_{fj}(u) f_{fj}^{(\varepsilon)}(u)du + \varepsilon^2 \sum_{i=0}^{N} \int_0^t g_{di}^{0,(\varepsilon)}(u)'\gamma_{fj}(u) f_{fj}^{(\varepsilon)}(u)du \\ &- \varepsilon^2 \int_0^t \sigma^{(\varepsilon)}(u)\bar{\sigma}'\gamma_{fj}(u) f_{fj}^{(\varepsilon)}(u)du + \varepsilon \int_0^t f_{fj}^{(\varepsilon)}(u)\gamma_{fj}'(u)dW_u, \end{aligned} \tag{31}$$

where $g_{dj}^{0,(\varepsilon)}(t) := \frac{-\tau_j f_{dj}^{(\varepsilon)}(t)}{1+\tau_j f_{dj}^{(\varepsilon)}(t)}\gamma_{dj}(t), g_{fj}^{0,(\varepsilon)}(t) := \frac{-\tau_j f_{fj}^{(\varepsilon)}(t)}{1+\tau_j f_{fj}^{(\varepsilon)}(t)}\gamma_{fj}(t);$ x' denotes the transpose of x and W is a r-dimensional standard Wiener process under the domestic terminal measure P; $\gamma_{dj}(s)$, $\gamma_{fj}(s)$ are r-dimensional vector-valued functions of time-parameter s; $\bar{\sigma}$ denotes a r-dimensional constant vector satisfying $\|\bar{\sigma}\| = 1$ and $\sigma^{(\varepsilon)}(t)$, the volatility of the spot

$$\sigma^{(\varepsilon)}(t) = \sigma(0) + \int_0^t \mu(u,\sigma^{(\varepsilon)}(u))du + \varepsilon^2 \sum_{j=1}^{N} \int_0^t g_{dj}^{0,(\varepsilon)}(u)'\omega(u,\sigma^{(\varepsilon)}(u))du + \varepsilon \int_0^t \omega'(u,\sigma^{(\varepsilon)}(u))dW_u,$$

exchange rate, is specified to follow a $\mathbf{R}_{++}$-valued general time-inhomogeneous Markovian process as follows:

(32)

where $\mu(s,x)$ and $\omega(s,x)$ are functions of s and x.

Finally, we consider the process of the forex forward $F_{N+1}(t)$. Since $F_{N+1}(t) \equiv F_{T_{N+1}}(t)$ can be expressed as $F_{N+1}(t) = S(t)\frac{P_f(t,T_{N+1})}{P_d(t,T_{N+1})}$, we easily notice that it is a martingale under the domestic terminal measure. In particular, it satisfies the following stochastic differential equation

$$F_{N+1}^{(\varepsilon)}(t) = F_{N+1}(0) + \varepsilon\int_0^t \sigma_F^{(\varepsilon)}(u)' F_{N+1}^{(\varepsilon)}(u)dW_u \tag{33}$$

where $\sigma_F^{(\varepsilon)}(t) := \sum_{j=0}^N \left(g_{fj}^{0,(\varepsilon)}(t) - g_{dj}^{0,(\varepsilon)}(t)\right) + \sigma^{(\varepsilon)}(t).$

2. Numerical Examples

We here specify our model and parameters, and confirm the effectiveness of our method in this cross-currency framework. First of all, the processes of domestic and foreign forward interest rates and of the volatility of the spot exchange rate are specified. We suppose $r = 4$, that is the dimension of a Brownian motion is set to be four; it represents the uncertainty of domestic and foreign interest rates, the spot exchange rate, and its volatility. Note that in this framework correlations among all factors are allowed. We also suppose $S(0) = 100$.

Next, we specify a volatility process of the spot exchange rate in (32) with

$$\begin{cases} \mu(s,x) = \kappa(\theta - x), (34) \\ \omega(s,x) = \omega\sqrt{x}, \end{cases} \tag{35}$$

where θ and κ represent the level and speed of its mean-reversion respectively, and ω denotes a volatility vector on the volatility. In this section the parameters are set as follows; $\varepsilon = 1$, $\sigma(0) = \theta = 0.1$, and $\kappa = 0.1$; $\omega = \omega^* \bar{v}$ where $\omega^* = 0.3$ and $\bar{v}$ denotes a four dimensional constant vector given below.

We further suppose that initial term structures of domestic and foreign forward interest rates are flat, and their volatilities also have flat structures and are constant over time: that is, for all $j, f_{dj}(0) = f_d$, $f_{fj}(0) = f_f$, $\gamma_{dj}(t) = \gamma_d^* \bar{\gamma}_d 1_{\{t<T_j\}}(t)$ and $\gamma_{fj}(t) = \gamma_f^* \bar{\gamma}_f 1_{\{t<T_j\}}(t)$. Here, γ_d^* and γ_f^* are constant scalars, and $\bar{\gamma}_d$ and $\bar{\gamma}_f$ denote four dimensional constant vectors. Moreover, given a correlation matrix $\underline{C}$ among all four factors, the constant vectors $\bar{\gamma}_d$, $\bar{\gamma}_f$, $\bar{\sigma}$ and $\bar{v}$ can be determined to satisfy $\|\bar{\gamma}_d\| = \|\bar{\gamma}_f\| = \|\bar{\sigma}\| = \|\bar{v}\| = 1$ and $V'V = \underline{C}$ where $V := (\bar{\gamma}_d, \bar{\gamma}_f, \bar{\sigma}, \bar{v})$.

In this subsection, we consider four different cases for f_d, γ_d^*, f_f and γ_f^* as in Table 1. For correlations, the parameters are set as follows: the correlation between interest rates and the spot exchange rate are allowed while there are no correlations among the others; the correlation between domestic ones and the spot forex is 0.5($\bar{\gamma}_d'\bar{\sigma} = 0.5$) and the correlation between foreign ones and the spot forex is -0.5($\bar{\gamma}_f'\bar{\sigma} = -0.5$): It is well known that (both of exact and approximate)evaluation of the long-term options is a hard task in the case with such complex structures of correlations.

Table 1. Initial domestic/foreign forward interest rates and their volatilities

	f_d	γ_d^*	f_f	γ_f^*
case (i)	0.05	0.12	0.05	0.12
case (ii)	0.02	0.3	0.05	0.12
case (iii)	0.05	0.12	0.02	0.3
case (iv)	0.02	0.3	0.02	0.3

This table shows the initial term structures of domestic and foreign forward interest rates and those of their volatilities, which are assumed to be flat. The figures in the first and second column are the initial value of domestic interest rates and their volatility. The figures in the third and fourth columun are thoes of foregin interest rates.

Lastly, we make an assumption that $\gamma_{dn(t)-1}(t)$ and $\gamma_{fn(t)-1}(t)$, volatilities of the domestic and foreign interest rates applied to the period from t to the next fixing date $T_{n(t)}$, are equal to be zero for arbitrary $t \in [t, T_{n(t)}]$.

In Figure 1, we compare our estimations of the values of call and put options whose maturities are from ten to twenty years by an asymptotic expansion up to the fourth order to the benchmarks estimated by 10^6 trials of Monte Carlo simulation. The simulation we discretized the underlying processes by Euler-Maruyama scheme with time step 0.05 and applied the Antithetic Variable Method. For the moneynesses(defined by $K / F_{N+1}(0)$) less than one, the prices of put options are shown; otherwise, the prices of call options are displayed.

As seen in this figure, in general the estimators seems more accurate as the order of the expansion increases. Especially, for the deep OTM-put options the fourth order approximation performs much better and is stabler than the approximation with the lower order.

4. Conclusions

In this paper, we reviewed the general procedures for the explicit computation necessary for practical computations of the asymptotic expansion method: One is that of conditional expectations based on the approach for iterated Ito integrals, and the other is the alternative but equivalent calculation algorithm which computes the unconditional expectations directly instead of the conditional ones. For simplicity and limitation of space, we focused on the simple case of Black-Scholes-type economy as in Section 2, which illustrated our key ideas(for further explanations in more general environment, see Takahashi, Takehara and Toda [2009]).

Moreover, we applied the methods to option pricing in the cross currency Libor market model with a stochastic volatility of the spot exchange rate to illustrate the usefulness and accuracy of our approximation with high order expansions. In this practically important example, satisfactory results were confirmed even for options with a twenty-year maturity.

In this paper only path-independet European derivatives under the situation where there is no jumps are considered. Thus, at the end of this section, we state our future plans: We will develop a similar result in the case with a jump component; we will also pursue an efficient method for the evaluation of multi-factor path-dependent or/and American derivatives. In fact, our proposed scheme can be applied to average options under a general setting of the underlying factors.

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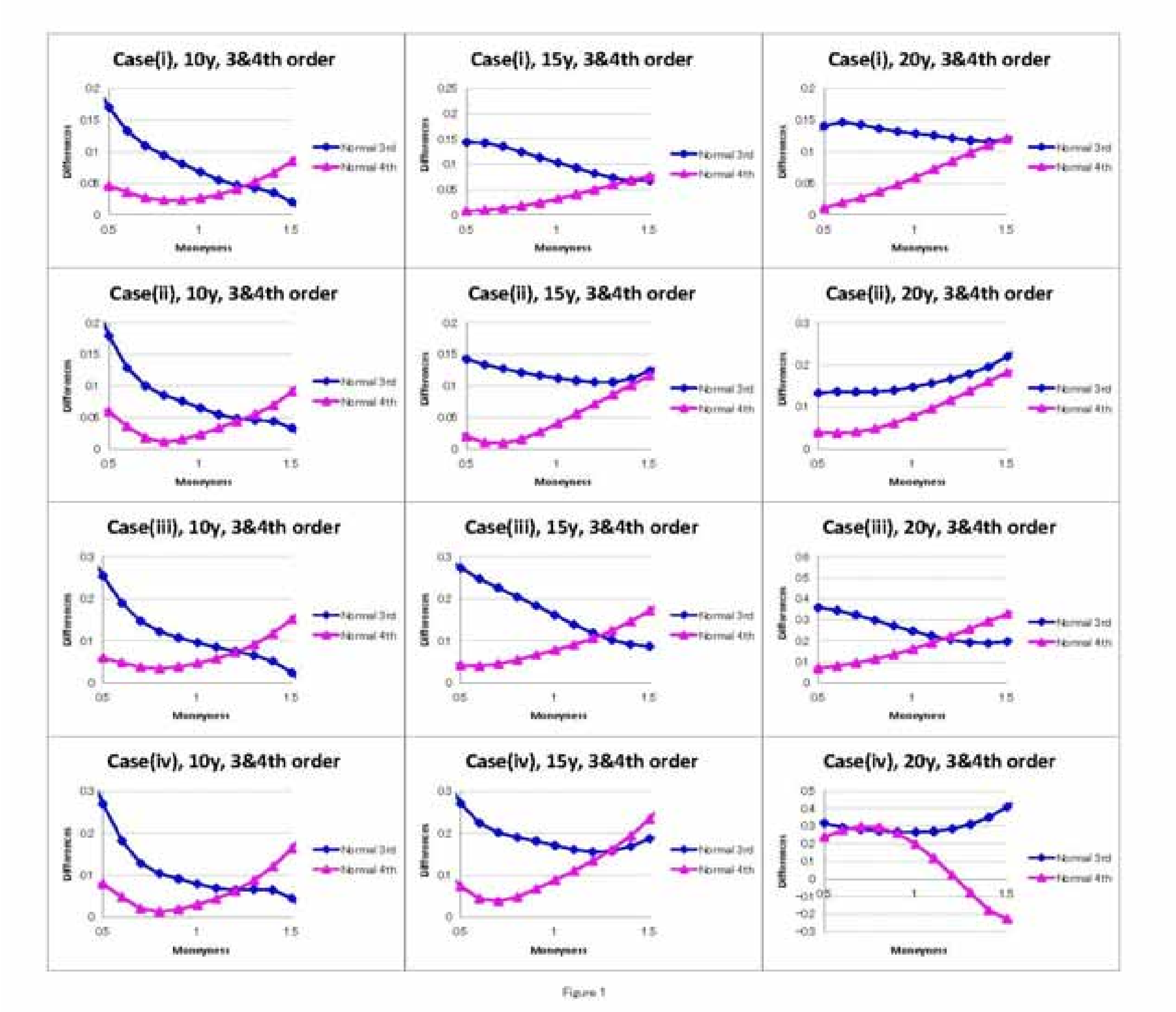
BIOGRAPHY

Dr. Akihiko Takahashi is a Professor of Graduate School of Economics, the University of Tokyo. He can be contacted at: Graduate School of Economics, the University of Tokyo, 7-3-1, Hongo, Bunkyo-ku, Tokyo, 113-0033, Japan, +81-3-3812-2111.

Kohta Takehara is a Ph.D student of Graduate School of Economics, the University of Tokyo. He can be contacted at: Graduate School of Economics, the University of Tokyo, 7-3-1, Hongo, Bunkyo-ku, Tokyo, 113-0033, Japan, +81-3-3812-2111. E-mail: fin.tk.house@gmail.com

Masashi Toda is a Ph.D student of Graduate School of Economics, the University of Tokyo. He can be contacted at: Graduate School of Economics, the University of Tokyo, 7-3-1, Hongo, Bunkyo-ku, Tokyo, 113-0033, Japan, +81-3-3812-2111. E-mail: masashi.toda@gmail.com

Figure 1: Comparsions of the estimators by the asymptotic expansion and simulations.



This figure shows the differences between our estimators of option prices by the asymptotic expansion up to the third(blue lines) and fourth order(pink lines) and those by Monte Carlo simulations. The defferences are defined by (the estimate by the asymptotic expansion – that by simulation). "Moneyness" is defined by (Strike Rate / Spot Rate).