CIRJE-F-758

On Properties of Separating Information Maximum Likelihood Estimation of Realized Volatility and Covariance with Micro-Market Noise

Naoto Kunitomo
University of Tokyo

Seisho Sato
Institute of Statistical Mathematics

August 2010

On Properties of Separating Information Maximum Likelihood Estimation of Realized Volatility and Covariance with Micro-Market Noise *

Naoto Kunitomo †

and

Seisho Sato ‡

August 19, 2010

Abstract

For estimating the realized volatility and covariance by using high frequency data, we have introduced the Separating Information Maximum Likelihood (SIML) method when there are possibly micro-market noises by Kunitomo and Sato (2008a, 2008b, 2010a, 2010b). The resulting estimator is simple and it has the representation as a specific quadratic form of returns. We show that the SIML estimator has reasonable asymptotic properties; it is consistent and it has the asymptotic normality (or the stable convergence in the general case) when the sample size is large under general conditions including *some non-Gaussian processes* and *some volatility models*. Based on simulations, we find that the SIML estimator has reasonable finite sample properties and thus it would be useful for practice. The SIML estimator has the asymptotic robustness properties in the sense it is consistent when the noise terms are weakly dependent and they are endogenously correlated with the efficient market price process. We also apply our method to an analysis of Nikkei-225 Futures, which has been the major stock index in the Japanese financial sector.

Key Words

Realized Volatility, Realized Covariance, Micro-Market Noise, High-Frequency Data, Separating Information Maximum Likelihood (SIML), Asymptotic Distributions, Nikkei-225 Futures.

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*KSI10-8-19. This is an extensive revision of Discussion Paper CIRJE-F-581, Graduate School of Economics, University of Tokyo (Kunitomo and Sato (2008a)) and its previous version was presented at the International Conference organized by the Osaka Securities Exchange (OSE) on September 2, 2008. The research was initiated while the first author was visiting the Center for the Study of Finance and Insurance (CSFI) of Osaka University as the OSE Professor. We thank OSE for providing the original data and Kosuke Oya for his helpful comments to an earlier version of the paper.

†Graduate School of Economics, University of Tokyo, Bunkyo-ku, Hongo 7-3-1 Tokyo, 113-0033, JAPAN, kunitomo@e.u-tokyo.ac.jp, Phone: +81-3-5841-5614.

‡Institute of Statistical Mathematics, Midori-cho 10-3, Tachikawa-shi, Tokyo 190-8562, JAPAN

1. Introduction

Recently a considerable interest has been paid on the estimation problem of the realized volatility by using high-frequency data in financial econometrics. Now it is possible to use a large number of high-frequency data in financial markets including the foreign exchange rates markets and stock markets. Although there were some discussion on the estimation of continuous stochastic processes in the statistical literature, the earlier studies often had ignored the presence of micro-market noises in financial markets when they tried to estimate the underlying stochastic process. Because there are several reasons why the micro-market noises are important in high-frequency financial data both in economic theory and in statistical measurement, several new statistical estimation methods have been developed. See Zhou, B. (1998), Anderson, T.G., Bollerslev, T. Diebold, F.K. and Labys, P. (2000), Gloter and Jacod (2001), Ait-Sahalia, Y., P. Mykland and L. Zhang (2005), Hayashi and Yoshida (2005), Zhang, L., P. Mykland and Ait-Sahalia (2005), Hansen P. and A. Lunde (2006), Ubukata and Oya (2009), Barndorff-Nielsen, O., P. Hansen, A. Lunde and N. Shepard (2008), Hansen, P., J. Large and A. Lunde (2008), Xiu (2008), Zhang (2008), and Christensen, Kinnebrock and Podolskij (2009) for further discussions on the related topics.

The main purpose of this paper is to develop a new statistical method for estimating the realized volatility and the realized covariance by using high frequency data in the presence of possible micro-market noise. The estimation method we have proposed in Kunitomo and Sato (2008a, 2008b, 2010a, 2010b) is called the Separating Information Maximum Likelihood (SIML) estimator, which is regarded as a modification of the standard Maximum Likelihood (ML) method under the Gaussian process. The SIML estimator of the realized volatility and covariance for the underlying continuous (diffusion type) process has the representation as a specific quadratic form of returns. As we shall show in this paper, the SIML estimator has reasonable asymptotic properties; it is consistent and it has the asymptotic normality (or the stable convergence in the general case) when the sample size is large under general situations including *some non-Gaussian processes* and *some volatility models*. There has been a theoretical development of the ML estimation of one dimension diffusion process with measurement errors by Gloter and Jacod (2001). Our method could be interpreted as a modification of their procedure and also it is related to the method proposed by Zhou (1998) and other estimation methods. However, the SIML approach has some different features from their methods and it is a new estimation method. (Recently Cai, Munk and Schmidt (2010) have developed a similar approach and examined the estimation problem of the realized volatility in a constant volatility model. As we shall see later, there are several different aspects of their approach from the SIML estimation method in this paper.)

The main motivation of our study is the fact that it is usually difficult to handle the exact likelihood function and calculate the exact ML estimator of unknown parameters from a large number of data for the underlying continuous stochastic process with micro-market noise in the multivariate non-Gaussian cases. This aspect is quite important for the analysis of multivariate high frequency data in stock markets and their futures markets. Instead of calculating the exact likelihood function, we try to separate the information of the signal and noise from the likelihood function and then use each information separately. This procedure simplifies the maximization of the likelihood function and make the estimation procedure applicable to multivariate high frequency data in a straightforward manner. We denote our estimation method as the Separating Information Maximum Likelihood (SIML) estimator because it gives an interesting extension of the standard ML estimation method. The main merit of the SIML estimation is its simplicity and then it can be practically used for the multivariate (high frequency) financial time series. The SIML estimator does have not only desirable asymptotic properties in the situations including some non-Gaussian processes and volatility models, but also it has reasonable finite sample properties. Also Kunitomo and Sato (2010a,b) have shown that the SIML estimator has the asymptotic robustness properties in the sense it is consistent when the noise terms are weakly dependent and they are endogenously correlated with the efficient market price process. Although the real motivation of this study is an application of a multivariate high frequency data, we shall introduce and discuss only the basic properties of the SIML estimation method with equidistance observations in this paper because of the resulting simplicity. For applications we need to discuss additional features such as the micro-market structure with the multivariate high frequency data and some details of the application to the analysis of Nikkei-225 futures market have been reported in Kunitomo and Sato (2008b, 2010b).

In Section 2 we introduce the standard model and the SIML estimation of the realized volatility and the realized covariance with micro-market noise. We give the asymptotic properties of the SIML estimator in the situation first when the instantaneous covariance function is constant, that is, the standard (or simple) case, and then in the time-varying deterministic case and the time varying stochastic case. Then in Section 3 we shall report some finite sample properties of the SIML estimator and a comparison with the realized kernel estimation by Banforff-Nielsen et al. (2008) based on a set of simulations. In Section 4 we shall discuss an application of the SIML method to the Nikkei-225 futures data at Osaka Securities Exchange (OSE) and then in Section 5 some brief remarks will be given. The mathematical derivations of our theoretical results will be given in Section 6. Tables and Figures based on the simulations will be presented in Appendix.

2. The SIML Estimation of Realized Volatility and Covariance with Micro-Market Noise

2.1 The Statistical Models in Continuous Time and Discrete Time

Let y_{ij} be the $i-$th observation of the $j-$th (log-) price at t_i^n for $i = 1, \cdots, n; j = 1, \cdots, p; 0 = t_0^n \leq t_1^n \leq \cdots \leq t_n^n = 1$. We set $\mathbf{y}_i = (y_{i1}, \cdots, y_{ip})'$ be a $p \times 1$ vector and $\mathbf{Y}_n = (\mathbf{y}_i')$ be an $n \times p$ matrix of observations. The underlying continuous process $\mathbf{x}_i$ at t_i^n $(i = 1, \cdots, n)$ is not necessarily the same as the observed prices and let

$\mathbf{v}_i' = (v_{i1}, \cdots, v_{ip})$ be the vector of the additive micro-market noise at t_i^n, which is independent of $\mathbf{x}_i$. Then we have

$$\mathbf{y}_i = \mathbf{x}_i + \mathbf{v}_i \tag{2.1}$$

where $\mathbf{v}_i$ are a sequence of independent random variables with $\mathcal{E}(\mathbf{v}_i) = \mathbf{0}$ and $\mathcal{E}(\mathbf{v}_i\mathbf{v}_i') = \mathbf{\Sigma}_v$. In this paper we focus on the equi-distance case with $h_n = t_i^n - t_{i-1}^n = 1/n$ $(i = 1, \cdots, n)$.

We assume that

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{C}_s^{(x)} d\mathbf{B}_s \ \ (0 \le t \le 1), \tag{2.2}$$

where $\mathbf{B}_s$ is a $q \times 1$ $(q \ge 1)$ vector of the standard Brownian motions, and $\mathbf{C}_s^{(x)} = (c_{gh}^{(x)}(s))$ is a $p \times q$ matrix which is progressively measurable in $[0, s] \times \mathcal{F}_s$ and predictable. We write the instantaneous diffusion functions $\mathbf{\Sigma}_s^{(x)}$ $(= (\sigma_{gh}^{(x)(s)}))$ $= \mathbf{C}_s^{(x)}\mathbf{C}_s^{(x)\prime}$ ($\mathcal{F}_s$ is the $\sigma-$field generated by $\{\mathbf{B}_r, r \le s\}$). For the construction of stochastic integration and the Ito's stochastic calculus, see Chapters II and III of Ikeda and Watanabe [1989], for instance.

Then the main statistical problem in this paper is to estimate the quadratic variations and co-variations

$$\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)}) = \int_0^1 \mathbf{\Sigma}_s^{(x)} ds \tag{2.3}$$

of the underlying continuous process $\{\mathbf{x}_t\}$ and also the variance-covariance $\mathbf{\Sigma}_v = (\sigma_{gh}^{(v)})$ of the noise process from the observed discrete time process $\mathbf{y}_i$ $(i = 1, \cdots, n)$. We use the notation that $\sigma_{gh}^{(x)}(s)$ and $\sigma_{gh}^{(v)}$ are the (gh)-th element of $\mathbf{\Sigma}_s^{(x)}$ and $\mathbf{\Sigma}_v$, respectively.

In this paper three different situations on the instantaneous covariance function shall be considered and the related problems shall be discussed mainly for the expository purpose. (i) When the coefficient matrix is constant, (i.e. $\mathbf{C}_s^{(x)} = \mathbf{C}^{(x)}$), we call the standard case or the simple case. Since the instantaneous variance and covariance are constant over time, the realized variance and covariance are constant. (ii) When the coefficient matrix is time-varying, but it is a deterministic function of time $(\mathbf{C}_s^{(x)})$, we call the deterministic time-varying case. We shall give the asymptotic properties of the SIML estimator when the instantaneous variance and covariance are time-varying, but the realized variance and covariance are constant (or deterministic). (iii) When the coefficient matrix is time-varying and it is a stochastic function of time ($\mathbf{C}_s^{(x)}$), we call the stochastic case. Then we shall investigate the asymptotic properties of the SIML estimator under some additional conditions when the realized variance and covariance are stochastic. In the last case it may be convenient to consider the situation when $\mathbf{C}_s^{(x)} = \mathbf{C}_{t_{i-1}^n}^{(x)}$ in (2.2) ($t_{i-1}^n \leq s < t_i^n; i = 1, \cdots, n$) (we may call the *locally constant* case), which is measurable with respect to the $\sigma-$field $\mathcal{F}_{t_{i-1}^n}$ (we denote $\mathcal{F}_{n,i-1}$ in the following) and thus they can be stochastic. This formulation may be the standard situation when we have the underlying diffusion process as the signal term and the discrete (observed) time series models by using the framework of the Ito's stochastic calculus. (See Chapters II and III of Ikeda and Watanabe [1989], for instance.) Then we write the conditional covariance function of the (underlying) price returns without micro-market noise as

$$\mathcal{E}\left[(\mathbf{x}_i - \mathbf{x}_{i-1})(\mathbf{x}_i - \mathbf{x}_{i-1})'|\mathcal{F}_{n,i-1}\right] = \int_{t_{i-1}}^{t_i} \mathbf{\Sigma}_s^{(x)} ds \ ,$$

which corresponds to $\frac{1}{n}\mathbf{\Sigma}_{t_{i-1}^n}^{(x)} = \frac{1}{n}\mathbf{C}_{t_{i-1}^n}^{(x)}\mathbf{C}_{t_{i-1}^n}^{(x)'}$, where $\mathbf{x}_i - \mathbf{x}_{i-1}$ is a sequence of martingale differences, $\mathbf{\Sigma}_s^{(x)}$ are the time-dependent (instantaneous) conditional variance and $\mathcal{F}_{n,i-1}$ is the $\sigma-$field generated by $\mathbf{x}_j$ ($j \leq i-1$) with (2.2) and $\mathbf{v}_j$ ($j \leq i-1$). More generally, as $n \to \infty$ we can consider the situation that the (true) realized covariance of the returns

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{\Sigma}_{t_{i-1}^n}^{(x)} \longrightarrow \mathbf{\Sigma}_x = \int_0^1 \mathbf{\Sigma}_s^{(x)} ds \ , \tag{2.4}$$

which is a deterministic and constant matrix, and $\mathbf{\Sigma}_0^{(x)}$ is the (fixed) initial condition. It is the case when the (instantaneous) covariance function $\mathbf{\Sigma}_s^{(x)}$ ($0 \leq s \leq 1$) is time-varying and also it can be stochastic. In this paper we assume that

$$\sup_{0\leq s\leq 1} \|\mathbf{\Sigma}_s^{(x)}\| \ < \infty \ (a.s.) \tag{2.5}$$

for the instantaneous covariance function. We shall discuss some examples of the deterministic time varying (instantaneous) volatility function and the stochastic time varying (instantaneous) volatility function in Section 3.

2.2 The Standard Case

We first consider the situation when $\mathbf{x}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are independent with $\mathbf{\Sigma}_s^{(x)} = \mathbf{\Sigma}_x$ $(0 \leq s \leq 1)$, and $\mathbf{v}_i$ are independently, identically and normally distributed as $N_p(\mathbf{0}, \mathbf{\Sigma}_v)$. Then given the initial condition $\mathbf{y}_0$, we have

$$\mathbf{Y}_n \sim N_{n\times p}\left(\mathbf{1}_n \cdot \mathbf{y}_0^{'}, \mathbf{I}_n \otimes \mathbf{\Sigma}_v + \mathbf{C}_n \mathbf{C}_n^{'} \otimes h_n \mathbf{\Sigma}_x\right) , \tag{2.6}$$

where $\mathbf{1}_n^{'} = (1, \cdots, 1)$, $h_n = 1/n$ $(= t_i^n - t_{i-1}^n)$ and

$$\mathbf{C}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix} . \tag{2.7}$$

In order to investigate the likelihood function in the standard case, we prepare the next lemma, which may be of independent interest. (See Appendix A of Kunitomo and Sato (2008a) for the proof.)

Lemma 1 : (i) Define an $n \times n$ matrix $\mathbf{A}_n$ by

$$\mathbf{A}_n = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} . \tag{2.8}$$

Then $\cos \pi(\frac{2k-1}{2n+1})$ $(k = 1, \cdots, n)$ are the eigenvalues of $\mathbf{A}_n$ and the eigen vectors are

$$\begin{bmatrix} \cos[\pi(\frac{2k-1}{2n+1})\frac{1}{2}] \\ \cos[\pi(\frac{2k-1}{2n+1})\frac{3}{2}] \\ \vdots \\ \cos[\pi(\frac{2k-1}{2n+1})(n-\frac{1}{2})] \end{bmatrix} (k = 1, \cdots, n). \tag{2.9}$$

(ii) Then we have the spectral decomposition

$$\mathbf{C}_n^{-1} \mathbf{C}_n^{'-1} = \mathbf{P}_n \mathbf{D}_n \mathbf{P}_n^{'} = 2\mathbf{I}_n - 2\mathbf{A}_n , \tag{2.10}$$

where $\mathbf{D}_n$ is a diagonal matrix with the k-th elements

$$(2.11) \qquad d_k = 2\left[1 - \cos(\pi(\frac{2k-1}{2n+1}))\right] \ (k = 1, \cdots, n) \ ,$$

$$(2.12) \qquad \mathbf{C}_n^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

and

$$(2.13) \qquad \mathbf{P}_n = (p_{jk}) \ , \ p_{jk} = \sqrt{\frac{2}{n+\frac{1}{2}}} \cos\left[\pi(\frac{2k-1}{2n+1})(j-\frac{1}{2})\right] \ .$$

We transform $\mathbf{Y}_n$ to $\mathbf{Z}_n$ $(= (\mathbf{z}_k^{'}))$ by

$$(2.14) \qquad \mathbf{Z}_n = h_n^{-1/2}\mathbf{P}_n^{'}\mathbf{C}_n^{-1}\left(\mathbf{Y}_n - \bar{\mathbf{Y}}_0\right)$$

where

$$(2.15) \qquad \bar{\mathbf{Y}}_0 = \mathbf{1}_n \cdot \mathbf{y}_0^{'} \ .$$

We note that given the initial condition $\mathbf{y}_0$ the transformation is one-to-one and each components of $\mathbf{Z}_n$ are independent in the present situation. Then the likelihood function under the Gaussian noise is given by

$$(2.16) \quad L_n^*(\boldsymbol{\theta}) = \left(\frac{1}{\sqrt{2\pi}}\right)^{np} \prod_{k=1}^{n} |a_{kn}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x|^{-1/2} e^{\left\{-\frac{1}{2}\mathbf{z}_k^{'}\left(a_{kn}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x\right)^{-1}\mathbf{z}_k\right\}} \ ,$$

where

$$(2.17) \qquad a_{kn} = 4n\sin^2\left[\frac{\pi}{2}\left(\frac{2k-1}{2n+1}\right)\right] \ (k = 1, \cdots, n) \ .$$

Hence the maximum likelihood (ML) estimator can be defined as the solution of maximizing

$$(2.18) \qquad L_n(\boldsymbol{\theta}) = \sum_{k=1}^{n} \log|a_{kn}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x|^{-1/2} - \frac{1}{2}\sum_{k=1}^{n} \mathbf{z}_k^{'}[a_{kn}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x]^{-1}\mathbf{z}_k \ .$$

From this representation we find that the ML estimator of unknown parameters is a rather complicated function of all observations in general because each a_{kn} terms depend on k as well as n. Let denote $a_{k_n,n}$ and then we can evaluate that $a_{k_n,n} \to 0$ as $n \to \infty$ when $k_n = O(n^{\alpha})$ $(0 < \alpha < \frac{1}{2})$ since $\sin x \sim x$ as $x \to 0$. Also $a_{n+1-l_n,n} = O(n)$ when $l_n = O(n^{\beta})$ $(0 < \beta < 1)$.

When k_n is small, we expect that $a_{k_n,n}$ is small. Then we may approximate $2 \times L_n(\boldsymbol{\theta})$ by

$$L_n^{(1)}(\boldsymbol{\theta}) = -m \log |\boldsymbol{\Sigma}_x| - \sum_{k=1}^{m} \mathbf{z}_k^{'} \boldsymbol{\Sigma}_x^{-1} \mathbf{z}_k \ . \tag{2.19}$$

It is the standard likelihood function except the fact that we only use the first m terms. (See Lemma 3.2.2 of Anderson (2003).) Then the SIML estimator of $\hat{\boldsymbol{\Sigma}}_x$ is defined by

$$\hat{\boldsymbol{\Sigma}}_x = \frac{1}{m_n} \sum_{k=1}^{m_n} \mathbf{z}_k \mathbf{z}_k^{'} \ . \tag{2.20}$$

On the other hand, when l_n is small and $k_n = n + 1 - l_n$, we expect that $a_{n+1-l_n,n}$ is large. Thus we may approximate $2 \times L_n(\boldsymbol{\theta})$ by

$$L_n^{(2)}(\boldsymbol{\theta}) = - \sum_{k=n+1-l}^{n} \log |a_{kn} \boldsymbol{\Sigma}_v| - \sum_{k=n+1-l}^{n} \mathbf{z}_k^{'} [a_{kn} \boldsymbol{\Sigma}_v]^{-1} \mathbf{z}_k \ . \tag{2.21}$$

It is also the standard likelihood function approach except the fact that we only use the last l terms. Then the SIML estimator of $\hat{\boldsymbol{\Sigma}}^{(v)}$ is defined by

$$\hat{\boldsymbol{\Sigma}}_v = \frac{1}{l_n} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1} \mathbf{z}_k \mathbf{z}_k^{'} \ . \tag{2.22}$$

For both $\hat{\boldsymbol{\Sigma}}_v$ and $\hat{\boldsymbol{\Sigma}}_x$, the number of terms m_n and l_n should be dependent on n. Then we only need the order requirements that $m_n = O(n^{\alpha})$ $(0 < \alpha < \frac{1}{2})$ and $l_n = O(n^{\beta})$ $(0 < \beta < 1)$ for $\boldsymbol{\Sigma}_x$ and $\boldsymbol{\Sigma}_v$, respectively.

In the above construction we define the SIML estimator by approximating the exact likelihood function under the Gaussian micro-market noise and the continuous diffusion process with the deterministic covariance. As we shall discuss later, the convergence rate of estimator of the realized volatility and covariance is not optimal in the light of Gloter and Jacod (2001). However, the SIML estimator has some asymptotic robustness as we have discussed in Kunitomo and Sato (2010a, b). The most important characteristic of the SIML estimator is its simplicity and it has some important aspects for dealing with high-frequency data. It is because the number of observations of tick data becomes enormous from the standard statistical sense. It is quite easy to deal with the multivariate high-frequency data in our approach as we demonstrated in Kunitomo and Sato (2008b, 2010b).

Since we use a linear transformation in (2.14) and Lemma 2 of Section 6, we can alternatively write

$$(2.23)\ \hat{\mathbf{\Sigma}}_x$$
$$= \frac{1}{m}(\frac{2n}{n+\frac{1}{2}})\sum_{k=1}^{m}\left[\sum_{i=1}^{n}\mathbf{R}_i\cos\left[\pi(\frac{2k-1}{2n+1})(i-\frac{1}{2})\right]\right]\left[\sum_{j=1}^{n}\mathbf{R}_j^{'}\cos\left[\pi(\frac{2k-1}{2n+1})(j-\frac{1}{2})\right]\right]^{'}$$
$$= \sum_{i=j=1}^{n} c_{ii}\mathbf{R}_i\mathbf{R}_i^{'} + \sum_{i\neq j=1}^{n} c_{ij}\mathbf{R}_i\mathbf{R}_j^{'}\ ,$$

where $\mathbf{R}_i = \mathbf{y}_i - \mathbf{y}_{i-1}$ and

$$c_{ii} = (\frac{2n}{2n+1})\left[1+\frac{1}{2m}\frac{\sin 4\pi m(\frac{i-1/2}{2n+1})}{\sin(\pi\frac{i-1/2}{2n+1})}\right]\ ,$$
$$c_{ij} = \frac{1}{2m}(\frac{2n}{2n+1})\left[\frac{\sin 2\pi m(\frac{i+j-1}{2n+1})}{\sin(\pi\frac{i+j-1}{2n+1})}+\frac{\sin 2\pi m(\frac{i-j}{2n+1})}{\sin(\pi\frac{i-j}{2n+1})}\right]\ \ (i\neq j)\ .$$

Hence we have an alternative representation of the SIML estimator in terms of asset returns (i.e. $\mathbf{y}_t - \mathbf{y}_{t-1} = (y_{t,j} - y_{t-1,j})$ with the observation interval h_n). Then we may find the relation between the SIML estimator and other estimation methods.

2.3 Asymptotic Properties of the SIML estimator in the Simple Case

Since the SIML estimator has a simple representation, it is not difficult to derive the asymptotic properties of the SIML estimator. In order to make our arguments clear, we first consider the asymptotic normality of the SIML estimator of the realized volatility and the realized covariance in the simple case, i.e., the instantaneous covariance function is constant over time in Section 2.3. Then in Section 2.5 we shall consider the same problem in a more general setting on the time-varying (conditional) covariance function. It may be appropriate here to stress the fact that we do not assume the Gaussianity on the noise process to develop the analysis of the asymptotic properties of the SIML estimator.

Let $\mathbf{r}_i = \mathbf{x}_i - \mathbf{x}_{i-1}$ and the (constant) covariance matrix is given by

$$\mathcal{E}\left[n\ \mathbf{r}_i\mathbf{r}_i^{'}|\mathcal{F}_{n,i-1}\right] = \mathbf{\Sigma}_x \tag{2.24}$$

for all i $(i = 1, \cdots, n)$. When $\mathbf{C}_s^{(x)}$ $(0 \leq s \leq 1)$ does not depend on s, we write $\mathbf{C}_s^{(x)} = \mathbf{C}_x$) and the realized covariance matrix $\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)})$ is a constant (non-negative definite) matrix. Then we have the next result and the proof will be given in Section 6.

Theorem 1 : We assume that $\mathbf{x}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are independent and they follow (2.1) and (2.2) with $\mathbf{C}_s^{(x)} = \mathbf{C}_x$, $\mathbf{\Sigma}_s^{(x)} = \mathbf{C}_s^{(x)}\mathbf{C}_x^{(x)'} = \mathbf{\Sigma}_x \geq 0$(non-negative definite) for $s \in [0, 1]$ and $\mathbf{\Sigma}_v \geq 0$. Define the SIML estimator $\hat{\mathbf{\Sigma}}_x = (\hat{\sigma}_{gh}^{(x)})$ of $\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)})$ and $\hat{\mathbf{\Sigma}}_v = (\hat{\sigma}_{gh}^{(v)})$ of $\mathbf{\Sigma}_v = (\sigma_{gh}^{(v)})$ by (2.20) and (2.22), respectively.
(i) For $m_n = n^\alpha$ and $0 < \alpha < 1/2$, as $n \longrightarrow \infty$

$$\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x \xrightarrow{p} \mathbf{O}\ . \tag{2.25}$$

(ii) For $m_n = n^\alpha$ and $0 < \alpha < 0.4$, as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}\right] \xrightarrow{w} N\left(0, \sigma_{gg}^{(x)}\sigma_{hh}^{(x)} + \left[\sigma_{gh}^{(x)}\right]^2\right)\ . \tag{2.26}$$

The covariance of the limiting distributions of $\sqrt{m_n}[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}]$ and $\sqrt{m_n}[\hat{\sigma}_{kl}^{(x)} - \sigma_{kl}^{(x)}]$ is given by $\sigma_{gk}^{(x)}\sigma_{hl}^{(x)} + \sigma_{gl}^{(x)}\sigma_{hk}^{(x)}$ $(g, h, k, l = 1, \cdots, p)$.
(iii) For $l_n = n^\beta$ and $0 < \beta < 1$, as $n \longrightarrow \infty$

$$\hat{\mathbf{\Sigma}}_v - \mathbf{\Sigma}_v \xrightarrow{p} \mathbf{O}\ . \tag{2.27}$$

(iv) Furthermore, assume the moment condition $\mathcal{E}[v_{ig}^2 v_{jh}^2] < \infty$ for all $i, j = 1, \cdots, n; g, h = 1, \cdots, p$. Then

$$\sqrt{l_n}\left[\hat{\sigma}_{gh}^{(v)} - \sigma_{gh}^{(v)}\right] \xrightarrow{w} N\left(0, \sigma_{gg}^{(v)}\sigma_{hh}^{(v)} + \left[\sigma_{gh}^{(v)}\right]^2\right)\ . \tag{2.28}$$

The covariance of the limiting distributions of $\sqrt{l_n}[\hat{\sigma}_{gh}^{(v)} - \sigma_{gh}^{(v)}]$ and $\sqrt{l_n}[\hat{\sigma}_{kl}^{(v)} - \sigma_{kl}^{(v)}]$ is given by $\sigma_{gk}^{(v)}\sigma_{hl}^{(v)} + \sigma_{gl}^{(v)}\sigma_{hk}^{(v)}$ $(g,h,k,l = 1,\cdots,p)$.

It may be obvious that we have the joint normality of $\hat{\mathbf{\Sigma}}_x$ and $\hat{\mathbf{\Sigma}}_v$ as the limiting distributions of the SIML estimator if we take a look at the proofs of Section 6. One interesting observation is the result that the asymptotic covariance in (2.26) and (2.28) do not depend on the fourth order moments of the noise term. This feature has an important implication for the testing problems on the realized volatility in the presence of micro-market noise. In the SIML approach the testing procedures and confidence regions can be constructed rather directly by using (2.26) and (2.28) for the covariance of the underlying continuous stochastic process and the covariance of the noises. We can utilize the decomposition

$$
\begin{aligned}
(2.29)\quad & \frac{1}{n}\sum_{k=1}^{n}\mathbf{z}_k\mathbf{z}_k^{'} \\
= \ & \left(\frac{m}{n}\right)\frac{1}{m}\sum_{k=1}^{m}\mathbf{z}_k\mathbf{z}_k^{'} + \left(\frac{n-l-m}{n}\right)\frac{1}{n-l-m}\sum_{k=m+1}^{n+1-l}\mathbf{z}_k\mathbf{z}_k^{'} + \left(\frac{l}{n}\right)\frac{1}{l}\sum_{k=n+1-l}^{n}\mathbf{z}_k\mathbf{z}_k^{'} \\
= \ & \frac{m}{n}\hat{\mathbf{\Sigma}}_x^{(1)} + \frac{n-l-m}{n}\hat{\mathbf{\Sigma}}_x^{(2)} + \frac{l}{n}\hat{\mathbf{\Sigma}}_x^{(3)} \quad (m+l<n),
\end{aligned}
$$

where $\hat{\mathbf{\Sigma}}_x^{(i)}$ $(i = 1,2,3)$ are defined accordingly. Since they are asymptotically independent, we can construct the testing procedure and confidence region on any elements of $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_v$ based on them.

One simple testing example is to test the null-hypothesis $H_0 : \sigma_{gg}^{(v)} = 0$ vs. $H_1 : \sigma_{gg}^{(v)} > 0$ for some j, where $\sigma_{gg}^{(v)}$ is the (g,g)-th element of $\mathbf{\Sigma}_v$ $(g = 1,\cdots,p)$. For this problem we consider the test statistic

$$
(2.30)\qquad T_1 = \sqrt{m_n}\left[\frac{\frac{1}{l_n}\sum_{k=n+1-l_n}^{n} z_{kg}^2}{\frac{1}{m_n}\sum_{k=1}^{m_n} z_{kg}^2} - 1\right],
$$

where $\mathbf{z}_k = (z_{kg})$ $(k = 1,\cdots,n; g = 1,\cdots,p)$.
When H_0 is true, $(1/l_n)\sum_{k=n+1-l_n}^{n} z_{kg}^2$ diverges in probability because of (2.17) while

$(1/m_n)\sum_{k=1}^{m_n} z_{kg}^2$ converges to $\sigma_{gg}^{(x)}$ in probability. Hence it may be reasonable to use this statistic for testing the null hypothesis H_0. Under the null hypothesis H_0, we have the next result and the proof is given at the end of Section 6.

Corollary 1 : Assume $0 < \alpha < \beta < 1$ and the conditions of Theorem 1. Under $H_0 \ : \ \sigma_{gg}^{(v)} = 0$ for some g $(1 \leq g \leq p)$,

$$T_1 \xrightarrow{d} N(0,2) \tag{2.31}$$

as $n \to \infty$.

It is straightforward to construct test statistics and testing procedures based on the SIML estimator, which are valid asymptotically as the standard statistical procedure, which is one of nice properties of the SIML approach.

2.4 An Optimal Choice of m_n

Because the properties of the SIML estimation method crucially depend on the choice of m_n, which are dependent on n, we have investigated the asymptotic effects as well as the small sample effects of several possibilities.

By using Lemma 4 and Lemma 5 with (6.9) and (6.12), the main order of the bias of the SIML estimator is $n^{-1}\sum_{i=1}^{m_n} a_{kn} = O(n^{2\alpha-1})$. Since the normalization of the SIML estimator is in the form of $\sqrt{m_n}[\hat{\sigma}_{gg}^{(x)} - \sigma_{gg}^{(x)}] = O_p(1)$, its variance is of the order $O(n^{-\alpha})$. Hence when n is large we can approximate the mean squared error of $\hat{\sigma}_{gg}^{(x)}$ $(g = 1, \cdots, p)$ as

$$\mathrm{MSE}_n(\alpha) = c_{1g}\frac{1}{n^{\alpha}} + c_{2g}n^{4\alpha-2} \ , \tag{2.32}$$

where c_{1g} and c_{2g} are some constants.
The first term and the second term of (2.32) correspond to the order of the variance and the squared bias, respectively. By minimizing $\mathrm{MSE}_n(\alpha)$ with respect to α, we can obtain an optimal choice of m_n.

Theorem 2 : An optimal choice of $m_n = n^{\alpha}$ $(0 < \alpha < 0.5)$ to minimize (2.32) with respect to α, when n is large, is given by $\alpha^* = 0.4$.

By using Theorem 1, Lemma 3 and Lemma 4 of Section 6, we find that when $\alpha = 0.4$

$$\sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)} - \frac{a}{\sqrt{m_n}}\right] \xrightarrow{w} N\left(0, \sigma_{gg}^{(x)}\sigma_{hh}^{(x)} + \left[\sigma_{gh}^{(x)}\right]^2\right) , \tag{2.33}$$

where

$$a = \sigma_{gh}^{(v)} \lim_{n\to\infty} \frac{1}{\sqrt{m_n}} \sum_{k=1}^{n} a_{kn} = \sigma_{gh}^{(v)} \frac{\pi^2}{3} . \tag{2.34}$$

In this case we have some asymptotic bias, which is dependent upon the covariance $\sigma_{gh}^{(v)}$. By using a set of simulations, we have investigated the finite sample properties of the SIML estimator by choosing different m_n.

It is possible to generalize the rule $m_n = [d\ n^\alpha]$ and d is a constant. When $p = q = 1$, for instance, we use the notation $\mathbf{\Sigma}_x = \sigma_x^2$ and $\mathbf{\Sigma}_v = \sigma_v^2$. Then it is straightforward to show that an asymptotically optimal choice is $\alpha^* = 0.4$ and

$$d^* = \left[\frac{9}{2\pi^4}\frac{\sigma_x^4}{\sigma_v^4}\right]^{1/5} \sim 0.541\left(\frac{\sigma_x^2}{\sigma_v^2}\right)^{0.4} . \tag{2.35}$$

In most cases of our simulations we have reasonable estimates when we set $\alpha = 0.4$ and $d = 1$. We may have a problem to use an estimate of the unknown signal-noise ratio for d except $d = 1$ in practical applications. For l_n we only have the condition $0 < \beta < 1$ and we have reasonable estimate when we set $\beta = 0.8$ by using our results in simulations. There could be some improvements on the finite sample properties if we use different criteria for choosing m_n.

2.5 Asymptotic Properties of the SIML estimator when the Instantaneous Covariance function is Time-varying

It is important to investigate the asymptotic properties of the SIML estimator when the instantaneous volatility function $\mathbf{\Sigma}_s^{(x)}$ of the underlying asset price is not constant over time. When the realized volatility is a positive (deterministic) constant a.s. (i.e.

$\sigma_{gh}^{(x)} = \int_0^1 \sigma_{gh}^{(x)}(s)ds$ is not stochastic) while the instantaneous covariance function is time varying, we have the consistency and the asymptotic normality of the SIML estimator as $n \to \infty$. For the deterministic time varying case we summarize the asymptotic properties of the SIML estimator and the proof is given in Section 6.

Theorem 3 : We assume that $\mathbf{x}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ in (2.1) and (2.2) are independent and $\boldsymbol{\Sigma}_s^{(x)} = \mathbf{C}_s^{(x)} \mathbf{C}_s^{(x)\prime} \geq 0$. Assume (2.4), (2.5) and $\boldsymbol{\Sigma}_x$ is a constant (or deterministic) matrix (a.s.). Define the SIML estimator $\hat{\boldsymbol{\Sigma}}_x = (\hat{\sigma}_{gh}^{(x)})$ of $\boldsymbol{\Sigma}_x = (\sigma_{gh}^{(x)})$ by (2.20) and (2.22), respectively.
(i) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$

$$\hat{\boldsymbol{\Sigma}}_x - \boldsymbol{\Sigma}_x \xrightarrow{p} \mathbf{O} \ . \tag{2.36}$$

(ii) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.4$, as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}\right] \xrightarrow{d} N\left[0, V_{gh}\right] \ , \tag{2.37}$$

provided that we assume the convergence of the asymptotic variance which is given by

$$V_{gh} = \left[\int_0^1 \sigma_{gg}^{(x)}(s)ds\right]\left[\int_0^1 \sigma_{hh}^{(x)}(s)ds\right] + \left[\int_0^1 \sigma_{gh}^{(x)}(s)ds\right]^2$$
$$+ \ \text{plim}_{n\to\infty} \sum_{i,j=1}^{n} (m_n c_{ij}^2 - 1)\left[\int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds \int_{t_{j-1}}^{t_j} \sigma_{hh}^{(x)}(s)ds + \int_{t_{i-1}}^{t_i} \sigma_{gh}^{(x)}(s)ds \int_{t_{j-1}}^{t_j} \sigma_{gh}^{(x)}(s)ds\right]$$

and it is a positive constant.

There are some remarks on the limiting distribution of the SIML estimator and its asymptotic variance formula in Theorem 3. The quantity $V_{gh.n}^{(2)}$ defined by

$$V_{gh.n}^{(2)} = \sum_{i,j=1}^{n} (m_n c_{ij}^2 - 1)\left[\int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds \int_{t_{j-1}}^{t_j} \sigma_{hh}^{(x)}(s)ds + \int_{t_{i-1}}^{t_i} \sigma_{gh}^{(x)}(s)ds \int_{t_{j-1}}^{t_j} \sigma_{gh}^{(x)}(s)ds\right] \tag{2.38}$$

is a bounded function by using (2.5) and Lemma 3 of Section 6. Hence it may be reasonable to assume the convergence of $V_{gh.n}^{(2)}$ to the second part of V_{gh} ($V_{gh}^{(2)}$, say).

When the instantaneous covariance $\sigma_{gh}^{(x)}(s) = \sigma_{gh}^{(x)}$ is constant, then

$$V_{gh} = \left[\int_0^1 \sigma_{gg}^{(x)}(s)ds\right]\left[\int_0^1 \sigma_{hh}^{(x)}(s)ds\right] + \left[\int_0^1 \sigma_{gh}^{(x)}(s)ds\right]^2 ,$$

which is equivalent to (2.26). (We shall give the detailed derivations of these formulas in Section 6.)

Also Kunitomo and Sato (2010a,b) have reported an alternative expression of the limiting distribution of $\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}$ in terms of

(2.39) $$\sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)} - \sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i} \sigma_{gh}^{(x)}(s)ds\right] \xrightarrow{d} N\left[0, V_{gh}\right] ,$$

However, Lemma 5 of Section 6 shows that

(2.40) $$\sqrt{m_n}\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i} \sigma_{gh}^{(x)}(s)ds = o_p(1) .$$

Hence we have found that this term is actually the higher order bias of the SIML estimator.

When $\mathbf{\Sigma}_x$ is a random matrix, we need the concept of stable convergence, which has been explained by Chapter 3 of Hall and Heyde (1980) or Gloter and Jacod (2001). The results of Theorem 3 can be held in the stochastic case with an additional assumption. By applying Theorem 3.5 and Corollary 3.3 of Hall and Heyde (1980) to our present formulation, we obtain the next result and the proof is given in Section 6.

Theorem 4 : We assume that $\mathbf{x}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ in (2.1) and (2.2) are independent and $\mathbf{\Sigma}_s^{(x)} = \mathbf{C}_s^{(x)}\mathbf{C}_s^{(x)'} \geq 0$. Additionally we assume that $\mathbf{\Sigma}_x$ can be a (non-negative) random variable (a.s.) in (2.4) and (2.5) while both each elements of $\mathbf{C}_s^{(x)}$ $(0 \leq s \leq 1)$ and $\mathbf{\Sigma}_x$ are *bounded*. Define the SIML estimator $\hat{\mathbf{\Sigma}}_x = (\hat{\sigma}_{gh}^{(x)})$ of $\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)})$ by (2.20).

(i) For $m_n = n^\alpha$ and $0 < \alpha < 1/2$, as $n \longrightarrow \infty$

(2.41) $$\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x \xrightarrow{p} \mathbf{O} .$$

(ii) For $m_n = n^\alpha$ and $0 < \alpha < 0.4$, as $n \longrightarrow \infty$

$$Z_{gh.n} = \sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}\right] \xrightarrow{d} Z_{gh}^{*} \tag{2.42}$$

and the characteristic function $g_n(t) = \mathcal{E}[\exp(itZ_{gh.n})]$ converges to the characteristic function of Z_{gh}^{*}, which is written as

$$g(t) = \mathcal{E}[e^{-\frac{V_{gh}t^2}{2}}]\ , \tag{2.43}$$

provided that we assume the probability convergence of the random variable, which is given by

$$\begin{aligned} V_{gh} &= \left[\int_0^1 \sigma_{gg}^{(x)}(s)ds\right]\left[\int_0^1 \sigma_{hh}^{(x)}(s)ds\right] + \left[\int_0^1 \sigma_{gh}^{(x)}(s)ds\right]^2 \\ &+ \ \mathrm{plim}_{n\to\infty}\sum_{i,j=1}^{n}(m_n c_{ij}^2 - 1)\left[\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\int_{t_{j-1}}^{t_j}\sigma_{hh}^{(x)}(s)ds + \int_{t_{i-1}}^{t_i}\sigma_{gh}^{(x)}(s)ds\int_{t_{j-1}}^{t_j}\sigma_{gh}^{(x)}(s)ds\right]. \end{aligned}$$

We have additional remarks on the stochastic case when the instantaneous covariance function is time-varying. The boundedness condition of $\mathbf{C}_s^{(x)}$ $(0 \le s \le 1)$ and $\mathbf{\Sigma}_x$ in Theorem 4 has been used to simplify the proof given in Section 6, but it may not be essential for the result. Also Lemma 3 of Section 6 shows the condition that

$$(\frac{1}{n})^2\sum_{i,j=1}^{n}(m_n c_{ij}^2 - 1) \to 0 \tag{2.44}$$

as $n \to \infty$. Hence when the underlying instantaneous volatility function is locally smooth in some sense, we have the next result under an additional condition (2.46).

Corollary 2 : Let

$$V_{gh}(t_{i-1}, t_i) = \frac{1}{h_n}\int_{t_{i-1}}^{t_i}\sigma_{gh}^{(x)}(s)ds \quad (i = 1, \cdots, n) \tag{2.45}$$

be the local covariance functions, where $h_n = t_i - t_{i-1} = 1/n$. We assume that

$$V_{gh}(t_{i-1}, t_i) \xrightarrow{p} \sigma_{gh}^{(x)*} \tag{2.46}$$

uniformly as $n \to \infty$. Then the asymptotic (stochastic) variance of the SIML estimator V_{gh} in Theorem 4 is given by

$$V_{gh} = \sigma_{gg}^{(x)*}\sigma_{hh}^{(x)*} + [\sigma_{gh}^{(x)*}]^2 \ . \tag{2.47}$$

When $\mathbf{\Sigma}_s^{(x)} = \mathbf{\Sigma}_{t_{i-1}^n}^{(x)}$ for $t_{i-1}^n \leq s < t_i^n$, we immediately find that the condition (2.46) implies that $\mathbf{\Sigma}_s^{(x)} \xrightarrow{p} \mathbf{\Sigma}_x$. In general V_{gh} can be a random variable (i.e. in the stochastic case) and some stochastic volatility models may satisfy the present condition.

2.6 Discussions

Although we have introduced the SIML estimator as a modification of the ML estimator in the standard situation, Theorem 1, Theorem 3 and Theorem 4 show that it has the consistency and the asymptotic normality under more general conditions. Also Kunitomo and Sato (2010a, b) have shown that the asymptotic properties of the SIML estimator essentially remains the same even when the noise terms are weakly dependent and they can be correlated with the signal terms, which has been sometimes called *the efficient price process* in financial economics. In the SIML approach we can separate the information on the covariance matrix of the underlying price volatilities and the covariance matrix of the micro-market noise in an asymptotic sense. Then the resulting estimators of the realized volatility and the covariance do not depend on the independence assumption among $\mathbf{x}_t$ (the state vector) and $\mathbf{v}_i$ (the noise vector). Also we have already conducted a large number of simulations on these issues and have found that the finite sample properties explained in Section 3 are not changed essentially. (See Kunitomo and Sato (2010a, b).)

Although there are merits in the SIML estimation, there could be naturally some cost. The convergence rate of the SIML estimator of $\mathbf{\Sigma}_x$ in Theorem 1 implies that it is slightly less than 1/4 if we take $\alpha = 0.4$. It has been known that the asymptotic bound is 1/4 in the standard case. (See Gloter and Jacod (2001) for instance.) Thus the SIML estimation sacrifices some efficiency loss against the ML estimator based on the MA(1) process when the standard assumptions hold without any misspecification. It is because we have pursued the simplicity of the estimation method and an *asymptotic robustness* of the estimation procedure for multivariate high frequency data with possible misspecification. Kunitomo and Sato (2010a,b) have investigated the related problems and found that the SIML estimator has the asymptotic robustness.

3. Simulations

We have investigated the finite sample distributions of the SIML estimators for the realized variance and the realized covariance based on a set of simulations. The number of replications is 1000. As a reasonable setting we have taken $n = 5000$ and $n = 20000$, and we have chosen $\alpha = 0.4$ and $\beta = 0.8$. In our experiments we have considered the situation that the variance of noise $10^{-2} \sim 10^{-6}$ of the realized variances of the underlying signals. We have reported additional simulation results in Kunitomo and Sato (2008b, 2010a) with some multivariate settings.

3.1 Basic Simulations

In our basic simulations we consider two cases when the observations are the sum of signal and micro-market noise. with $p = q = 1$. Thus we use the notation $\mathbf{\Sigma}_s^{(s)} = \sigma_x^2(s)$, $\mathbf{\Sigma}_x = \sigma_x^2$ and $\mathbf{\Sigma}_v = \sigma_v^2$. In the first example the signal is the Brownian motion with the instantaneous volatility function

$$\sigma_x^2(s) = \sigma(0)^2 \left[a_0 + a_1 s + a_2 s^2 \right], \tag{3.1}$$

where a_i $(i = 0, 1, 2)$ are constants and we have some restrictions such that $\sigma_x(s)^2 > 0$ for $s \in [0, 1]$. In this case the realized variance $\mathbf{\Sigma}_x = \sigma_x^2$ is given by

$$\sigma_x^2 = \int_0^1 \sigma_x(s)^2 ds = \sigma_x(0)^2 \left[a_0 + \frac{a_1}{2} + \frac{a_2}{3} \right] . \tag{3.2}$$

In this example we have taken several intra-day instantaneous volatility patterns including the flat (or constant) volatility, the monotone (decreasing or increasing) movements and the U-shaped movements.
In the second example the volatility function follows the stochastic volatility model such that $\mathbf{\Sigma}_x(s) = \sigma_x^2(s)$ and

$$\sigma_x^2 = \frac{1}{n}\sum_{i=1}^{n}\sigma_x(t_i^n)^2, \tag{3.3}$$

where $\sigma_x(t_i^n)^2 = e^{h(t_i)}$ $(s = t_i, 0 < t_1^n < \cdots < t_n^n \leq 1)$ and

$$h(t_i^n) = \gamma\, h(t_{i-1}^n) + c\, u(t_i^n)\ . \tag{3.4}$$

We have set that $u(t_i^n)$ is independent of $v(t_i^n)$. Then we have the condition given by (2.4) when we have $|\gamma| < 1$. In our experiments we have set $\gamma = 0.9, c = 0.2$ and $u(t_i^n)$ are the white noise process followed by $N(0,1)$ as a typical situation.

We summarize our estimation results of the first example in Tables 3.1∼3.4 and the second example in Table 3.5, respectively. (See Tables in Appendix.) In each table we have also calculated the value of the historical volatility as HI for comparison. When there are micro-market noise components with the martingale signal part, the value of HI often differs from the true realized volatility of the signal part substantially. However, we have found that it is possible to estimate the realized variance and the noise variance when we have the signal-noise ratio as $10^{-2} \sim 10^{-6}$ at least by the SIML estimation method. Although we have omitted the details of the second example, the estimation results are similar in the stochastic volatility model.

By our basic simulations we may conclude that we can estimate both the realized volatility of the hidden martingale part and the market noise part reasonably in all cases we have examined by the SIML estimation. When the market noises are extremely small, we have some difficulty to estimate the noise variance, which may be a natural phenomenon. In that case, however, we can detect that fact by using the testing procedure and the confidence interval constructed by the SIML estimation method. We also have conducted a number of further simulations and the details of our results have been given in Kunitomo and Sato (2008b, 2010b). They have reported some additional results on the robustness of the SIML estimator and the optimal hedging problem when $p = q = 2$.

3.2 A Finite Sample Behavior of the Distribution of SIML Estimator

In order to examine the finite sample and asymptotic behabior of the SIML estimator, we have done a large number of simulations on the shape of the distribution function of the SIML estimator in the form of (2.20) and (2.22). We have found that the (higher order) bias term of (2.40) is numerically small and negligible for practical purposes. For an illustration, we only give two figures (Figures 3.1 and 3.2 in Appendix) of the histograms on the SIML estimator when the instantaneous covariance function has a deterministic time varying U-shape and with $n = 5000$ and $n = 30000$.

When $n = 5000$, the distribution of the SIML estimator is skewed considerably as we expected as a kind of variance estimator, while its distribution becomes symmetric when $n = 30000$. Because $m_n = n^{0.4}$, the convergence rate toward the limiting normal distribution is not so fast although it depends crucially on the sample size, the realized covariance and the noise variance. It may be typical when we estimate the realized variance and covariance in a nonparametric or semi-parametric way.

3.3 A Simple Comparison with the Realized Kernel method

The Realized Kernel (RK) method developed by Bandorff-Nielsen et al. (2008) has been influential on the estimation problem of the realized volatility. Since there is a natural question on the comparison of the RK estimator and the SIML estimator, we give two tables Table 3.6 and Table 3.7. In order to make a fair comparison we have tried to follow the recommendation by Bandorff-Nielsen et al. (2008) on the choice of kernel (Tukey-Hanning) and the band width parameter H. One important issue in the RK method has been to choose H, which depends on the noise variance and the instantaneous variance and so we have chosen some reasonable velues of $H = c\sqrt{\sigma_v^2/[\sigma_x^2/n]}$ when $p = q = 1$ in our experiments. Although we have done a large number of simulations, we only give two tables for the case when the noise variance is comparable to the instantaneous variance.

From Tables 3.6 and 3.7 we have found that the RK estimation gives a reasonable estimate if we had taken the reasonable value of the key parameter H. In some cases of such situations the variance of the RK estimator is smaller than the variance of the SIML estimator. On the other hand, the variance of the RK estimator can be larger than the variance of the SIML estimator while the latter is quite stable because the SIML estimator is quite robust against the possible values of the variance ratio. On the whole, we have confirmed that the SIML estimator gives the robustness property.

In addition to these observations we should note that we need a priori information on the variance ratio for the Realized Kernel method while we do not need such information in advance for the SIML estimation.

4. An Application to Nikkei-225 Futures

One of important futures market in Japan was formally started in September 1987 at the Osaka Securities Exchanges (OSE), which is the second largest securities exchanges after Tokyo Securities Exchange (TSE) and it has been developed in the trading size and scale over the past 20 years. The Nikkei-225 futures, the successful products of OSE, correspond to the Nikkei-225 Spot-Index as its future contracts. The Nikkei-225 spot index has been the most important stock index in the Japanese financial sector. The trading volume of the Nikkei225 futures at OSE has been heavy and there have been usually trades occurred within one second in most days. The Nikkei-225 Futures have been the major financial tool in the financial industry because the Nikkei-225 is the major index in Japan. We have high frequency data less than 1 second of Nikkei-225 Futures in most times and in our analysis we have used data in 1 second, 5 seconds, 10 seconds, 30 seconds and 60 seconds.

We have picked one day in April 2007 and estimated the realized volatility with different time intervals in Table 4.1 by both the traditional historical volatility (HI) estimation and the SIML estimation as a typical example. Then we found that the values of the estimated HI heavily depend on the observation intervals while our estimation does not depend on them very much. The problem of significant biases of the estimated HI has been pointed out recently by several researchers and our analysis has been consistent with them. Also by using the test statistic in (2.30) we find that $T_1 = 103.56(1s), 43.26(5s), 19.15(10s), 11.29(30s), 3.07(60s)$. Thus we have also confirmed that the presence of micro-market noises is an important factor with high frequencies in the Nikkei-225 futures market.

The analysis of Nikkei-225 spot and futures markets with the bivariate high frequency data was the real motivation of our study and we have illustrated the results briefly. Some details of our analysis and results including the realized hedging have been reported in Kunitomo and Sato (2008b, 2010b).

5. Concluding Remarks

In this paper, we have developed a new statistical method for estimating the realized variance and the realized covariance by using high-frequency financial data under the presence of noise. The Separating Information Maximum Likelihood (SIML) estimator proposed by Kunitomo and Sato (2008a,b) can be regarded as a modification of the standard Maximum Likelihood (ML) method and it has the representation as a quadratic form of returns. We have shown that the SIML estimator has reasonable asymptotic properties; it is consistent and it has the asymptotic normality (or the stable convergence in the general case) when the sample size is large and the data frequency interval is small under some conditions including non-Gaussian processes and volatility models. The SIML estimator has reasonable finite sample properties and also it has the asymptotic robustness properties as shown in Kunitomo (2010a,b).

The SIML estimator is so simple that it can be practically used not only for the realized volatility but also the realized covariance of the multivariate high frequency financial series. As an application we have applied the SIML estimation to investigate a set of high frequency data of Nikkei-225 Futures at OSE (Osaka Securities Exchange). We have confirmed that the presence of micro-market noises is an important factor in the Nikkei-225 futures market. Some further empirical analysis have been discussed in Kunitomo and Sato (2008b, 2010b).

6 Mathematical Derivations

We first prepare some useful formulas and evaluations. The derivations are the results of elementary use of trigonometric functions, which are straightforward and thus they are omitted.

Lemma 2 : For any integer l and m $(1 \leq l, m \leq n)$

(6.1)
$$\sum_{k=1}^{m}\left[\cos \pi \frac{2k-1}{2n+1} l\right] = \frac{1}{2}\frac{\sin 2\pi m \frac{l}{2n+1}}{\sin \pi \frac{l}{2n+1}}$$

and

(6.2)
$$\sum_{k=1}^{m}\left[\cos \pi \frac{2k-1}{2n+1} l\right]^2 = \frac{m}{2} + \frac{1}{4}\frac{\sin 4\pi m \frac{l}{2n+1}}{\sin 2\pi \frac{l}{2n+1}} .$$

Lemma 3 : Let $c_{ij} = (2/m)\sum_{k=1}^{m} s_{ik}s_{jk}$ $(i, j = 1, \cdots, n; k = 1, \cdots, m)$ and

(6.3)
$$s_{jk} = \cos\left[\frac{2\pi}{2n+1}(j-\frac{1}{2})(k-\frac{1}{2})\right] .$$

Then we have

(i) for any integers j, k

(6.4)
$$\sum_{i=1}^{n} c_{ij}c_{ik} = \frac{2}{m}(n+\frac{1}{2})c_{jk}$$

and

(6.5)
$$\sum_{i,j=1}^{n} c_{ij}^2 = \frac{4}{m}\left[\frac{n}{2}+\frac{1}{4}\right]^2 .$$

(ii) As $n \to \infty$,

(6.6)
$$\frac{1}{n}\sum_{i=1}^{n}(c_{ii}-1) \to 0$$

and

(6.7)
$$\frac{1}{n}\sum_{i=1}^{n}(c_{ii}-1)^2 \to 0 .$$

We shall give the proofs of Theorem 1, Theorem 3 and Theorem 4 in Section 2. Since Theorem 1 is a special case of Theorem 3, we shall mainly focus on the proofs of Theorem 3 and Theorem 4. For any unit vector $\mathbf{e}_g = (0, \cdots, 0, 1, 0, \cdots, 0)'$ ($g = 1, \cdots, p$), we define $\sigma_{gh}^{(x)} = \mathbf{e}_g^{'} \mathbf{\Sigma}_x \mathbf{e}_h$, $\hat{\sigma}_{gh}^{(x)} = \mathbf{e}_g^{'} \hat{\mathbf{\Sigma}}_x \mathbf{e}_h$, $\sigma_{gh}^{(v)} = \mathbf{e}_g^{'} \mathbf{\Sigma}_v \mathbf{e}_h$ and $\hat{\sigma}_{gh}^{(v)} = \mathbf{e}_g^{'} \hat{\mathbf{\Sigma}}_v \mathbf{e}_h$. From the transformation (2.14) we set $x_{kg} = \mathbf{e}_g^{'} \mathbf{z}_k$ ($k = 1, \cdots, n$) and $x_{kg} = x_{kg}^{(1)} + x_{kg}^{(2)}$, where $x_{kg}^{(1)}$ and $x_{kg}^{(2)}$ correspond to the $(k, g)-$elements of $\mathbf{X}_n^{(1)} = h_n^{-1/2} \mathbf{P}_n^{'} \mathbf{C}_n^{-1} (\mathbf{X}_n - \mathbf{Y}_0)$ and $\mathbf{X}_n^{(2)} = h_n^{-1/2} \mathbf{P}_n^{'} \mathbf{C}_n^{-1} \mathbf{V}_n$, respectively. By using Lemma 1, we have $\mathcal{E}[\mathbf{X}_n^{(1)} \mathbf{e}_g] = \mathbf{0}$, $\mathcal{E}[\mathbf{X}_n^{(2)} \mathbf{e}_g] = \mathbf{0}$ and

$$\mathcal{E}[\mathbf{X}_n^{(2)} \mathbf{e}_g \mathbf{e}_h^{'} \mathbf{X}_n^{(2)'}] = (\mathbf{e}_g^{'} \mathbf{\Sigma}_v \mathbf{e}_h) h_n^{-1} \mathbf{P}_n^{'} \mathbf{C}_n^{-1} \mathbf{C}_n^{'-1} \mathbf{P}_n^{'} = (\mathbf{e}_g^{'} \mathbf{\Sigma}_v \mathbf{e}_h) h_n^{-1} \mathbf{D}_n \ . \tag{6.8}$$

In the following derivations, we mainly discuss the estimation of the realized variance (or the realized volatility). It is because the estimation of the realized covariance is quite similar with additional notations. One important difference is to use the fact that in the limiting distribution $2(\mathcal{E}[X_g^2])^2$ should be replaced by $(\mathcal{E}[X_g^2])(\mathcal{E}[X_h^2]) + (\mathcal{E}[X_g X_h])^2$ when $\mathbf{X} = (X_g)$ follows the multivariate normal distribution for any $g, h = 1, \cdots, p$. It has been a standard practice in the statistical multivariate analysis (see Anderson (2003), for instance). Also without loss of generality we often use the case when $q = 1$ in our derivations because the resulting expressions become simple. In our proofs of theorems we shall extensively use the decomposition

$$\begin{aligned} (6.9)\ & \hat{\sigma}_{gg}^{(x)} - \sigma_{gg}^{(x)} \\ = & \ \frac{1}{m_n} \sum_{k=1}^{m_n} \left[x_{kg}^2 - \sigma_{gg}^{(x)} \right] \\ = & \ \frac{1}{m_n} \sum_{k=1}^{m_n} \left[x_{kg}^{(1)2} - \sigma_{gg}^{(x)} + \sigma_{gg}^{(v)} a_{kn} \right] + \frac{1}{m_n} \sum_{k=1}^{m_n} \left[x_{kg}^{(2)2} - \sigma_{gg}^{(v)} a_{kn} \right] + 2 \frac{1}{m_n} \sum_{k=1}^{m_n} \left[x_{kg}^{(1)} x_{kg}^{(2)} \right] \ . \end{aligned}$$

Lemma 4 : Assume the assumptions of Theorem 3.

(i) For $0 < \alpha < 0.5$,

$$\hat{\sigma}_{gg}^{(x)} - \sigma_{gg}^{(x)} \xrightarrow{p} 0 \tag{6.10}$$

as $n \to \infty$.

(ii) For $0 < \alpha < 0.4$,

$$
\begin{aligned}
(6.11) \qquad & \sqrt{m_n}\left[\hat{\sigma}_{gg}^{(x)} - \sigma_{gg}^{(x)} - \sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right] \\
& -\sqrt{m_n}\left[\frac{1}{m}\sum_{k=1}^{m}\left(x_{kg}^{(1)2}\right) - \sigma_{gg}^{(x)} - \sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right] \\
\xrightarrow{p} \;\; & 0
\end{aligned}
$$

as $n \to \infty$.

Proof of Lemma 4 : By using (2.17) and the relation $\sin x = x - (1/6)x^3 + (1/120)x^5 + O(x^7)$,

$$
\begin{aligned}
(6.12) \qquad \frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn} &= \frac{1}{m_n}2n\sum_{k=1}^{m_n}\left[1-\cos(\pi\frac{2k-1}{2n+1})\right] \\
&= \frac{n}{m_n}\left[2m_n - \frac{\sin\pi\frac{2m_n}{2n+1}}{\sin\pi\frac{1}{2n+1}}\right] \\
&\sim \frac{n}{m_n}\left[2m_n - \frac{(\pi\frac{2m_n}{2n+1}) - \frac{1}{6}(\pi\frac{2m_n}{2n+1})^3}{(\frac{\pi}{2n+1}) - \frac{1}{6}(\frac{\pi}{2n+1})^3}\right] \\
&= O(\frac{m_n^2}{n})
\end{aligned}
$$

and

$$
\begin{aligned}
(6.13) \qquad \frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn}^2 &= \frac{1}{m_n}4n^2\sum_{k=1}^{m_n}\left[1-2\cos(\pi\frac{2k-1}{2n+1}) + \frac{1}{2}(1+\cos(2\pi\frac{2k-1}{2n+1}))\right] \\
&= \frac{4n^2}{m_n}\left[\frac{3}{2}m_n - \frac{\sin\pi\frac{2m_n}{2n+1}}{\sin\pi\frac{1}{2n+1}} + \frac{1}{4}\frac{\sin\pi\frac{4m_n}{2n+1}}{\sin\pi\frac{2}{2n+1}}\right] \\
&= O(\frac{m_n^4}{n^2})
\end{aligned}
$$

as $n \to \infty$. Then (6.12) and (6.13) are $o(1)$ when we have the condition that $m_n^2/n \to 0$ $(n \to \infty)$. Hence for the first term of (6.9) we need $0 < \alpha < 0.5$ for the consistency and $0 < \alpha < 0.4$ for the asymptotic normality as the mimimum requirements, respectively, because $(1/\sqrt{m_n})\sum_{i=1}^{m_n} a_{kn}$ should be negligible in the latter case. In order to show that these conditions are sufficient, we shall evaluate each terms of $\sqrt{m_n}[\hat{\sigma}_{gg}^{(x)} - \sigma_{gg}^{(x)}]$ based on the decomposition (6.9).

For the third term of (6.9), there exists a positive constant K_1

$$
\begin{aligned}
(6.14)\qquad & \mathcal{E}\left[\frac{1}{\sqrt{m_n}}\sum_{j=1}^{m_n} x_{kg}^{(1)}x_{kg}^{(2)}\right]^2 \\
&= \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathcal{E}\left[x_{kg}^{(1)}x_{k',g}^{(1)}x_{kg}^{(2)}x_{k',g}^{(2)}\right] \\
&= \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathcal{E}\left[2\sum_{j,j'=1}^{n} s_{jk}s_{j'k'}\mathcal{E}(r_{jg}r_{j',g}|\mathcal{F}_{\min(j,j')})x_{kg}^{(2)}x_{k',g}^{(2)}\right] \\
&= \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathcal{E}\left[2\sum_{j=1}^{n} s_{jk}s_{j,k'}\mathcal{E}(r_{jg}^2|\mathcal{F}_{j-1})x_{kg}^{(2)}x_{k',g}^{(2)}\right] \\
&\leq K_1\left[\sup_{0\leq s\leq 1}\mathcal{E}(c_{gg}^{(x)}(s))\right]\frac{2}{n}(\frac{n}{2}+\frac{1}{4})\frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn} \\
&= O(\frac{m_n^2}{n})
\end{aligned}
$$

where we use the notation $\mathbf{C}_s^{(x)} = (c_{gh}(s))$. In the above evaluation we have used the independence of $x_{kg}^{(1)}$ and $x_{kg}^{(2)}$, and the relation

$$
(6.15)\qquad \sum_{j=1}^{n} s_{jk}^2 = n/2 + 1/4 \ \text{ for any } k \geq 1 \ .
$$

For the second term of (6.9), let $\mathbf{b}_k = \mathbf{e}_k^{'}\mathbf{P'}_n\mathbf{C}_n^{-1} = (b_{kj})$ and $\mathbf{e}_k^{'} = (0,\cdots,1,0,\cdots)$ is an $n\times 1$ vector. Then we can write $x_{kg}^{(2)} = \sum_{j=1}^{n} b_{kj}v_{jg}$ and

$$
\begin{aligned}
(6.16)\qquad & \mathcal{E}\left[\frac{1}{\sqrt{m_m}}\sum_{j=1}^{m_n}(x_{kg}^{(2)2}-\sigma_{gg}^{(v)}a_{kn})\right]^2 \\
&= \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathcal{E}\left[(x_{kg}^{(2)2}-\sigma_{gg}^{(v)}a_{kn})(x_{k',g}^{(2)2}-\sigma_{gg}^{(v)}a_{k'n})\right] \\
&= \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathcal{E}\left[(\sum_{j=1}^{n} b_{kj}v_{jg})^2(\sum_{j'=1}^{n} b_{k'j'}v_{j',g})^2-\sigma_{gg}^{(v)}a_{kn}a_{k'n}\right] \\
&\leq K_2\frac{1}{m_n}\sum_{k=1}^{m_n}\sum_{j=1}^{n} b_{kj}^4 \\
&\leq K_2\frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn}^2 = O(\frac{m_n^4}{n^2}) \ ,
\end{aligned}
$$

where K_2 is a positive constant.

Hence we have found that the main effect of the sampling errors associated with the SIML estimator of the realized variance is the first term of (6.9). Then we shall show the consistency and the variance formula in (2.26) and (2.28). We write $\mathbf{r}_i = (r_{ig}) = \mathbf{x}_i - \mathbf{x}_{i-1}$ $(i,j = 1,\cdots,n; g = 1,\cdots,p)$ and by using the fact that $\mathbf{r}_i = (r_{ig})$ $(i = 1,\cdots,n; g = 1,\cdots,p)$ are a sequence of martingale differences,

$$
\begin{aligned}
(6.17)\qquad & \mathcal{E}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}(x_{kg}^{(1)2} - \sigma_{gg}^{(x)})\right]^2 \\
= & \left[\frac{2n}{2n+1}\right]^2 \mathcal{E}\left\{\sum_{i,j=1}^{n}\left[c_{ij}\ r_{ig}r_{jg} - \delta_{ij}\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\right]\right\}^2 \\
= & \left[\frac{2n}{2n+1}\right]^2 \mathcal{E}\left\{\sum_{i=j=1}^{n}\left[c_{ij}r_{ig}^2 - \int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\right]\right\}^2 + \mathcal{E}\left\{\sum_{i\neq j=1}^{n}\left[c_{ij}r_{ig}r_{jg}\right]\right\}^2 ,
\end{aligned}
$$

where $\delta_{ij} = 1$ $(i = j)$; $\delta_{ij} = 0$ $(i \neq j)$. Then we need to evaluate

$$
\begin{aligned}
& \mathcal{E}\left\{\sum_{i=1}^{n}\left[c_{ii}r_{ig}r_{ig} - \int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\right]\right\}^2 \\
= & \ \mathcal{E}\left[\sum_{i,j=1}^{n}c_{ii}c_{jj}(r_{ig}^2 - \int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds)(r_{jg}^2 - \int_{t_{j-1}}^{t_j}\sigma_{gg}^{(x)}(s)ds)\right] \\
= & \ \sum_{i=1}^{n}c_{ii}^2\mathcal{E}\left[r_{ig}^2 - \int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\right]^2
\end{aligned}
$$

and

$$
(6.18)\qquad \mathcal{E}\left\{\left[\sum_{i\neq j=1}^{n}c_{ij}r_{ig}r_{jg}\right]^2\right\} = 2\sum_{i\neq j=1}^{n}c_{ij}^2\mathcal{E}(r_{ig}^2)\mathcal{E}(r_{jg}^2) .
$$

By using (2.5), we have the relation

$$
\mathcal{E}(r_{ig}^2) = \mathcal{E}(\int_{t_{i-1}}^{t_i}[c_{gg}^{(x)}(s)]^2 ds) \leq \frac{K_3}{n} ,
$$

where K_3 is a positive constant. Then by using Lemma 3, we find that the second term of the right-hand side of (6.17) is of the order $O(\frac{1}{m})$. As we shall show immediately, the first term of the right-hand side of (6.17) is of the order $o(\frac{1}{m})$, we have

$$
\begin{aligned}
(6.19)\qquad \mathcal{E}\left[\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}(x_{kg}^{(1)2} - \sigma_{gg}^{(x)})\right]^2 &= \mathcal{E}\left\{\sqrt{m_n}\sum_{i\neq j=1}^{n}c_{ij}r_{ig}r_{jg}\right\}^2 + o(1) \\
&= O(1) .
\end{aligned}
$$

It is because the first term of (6.17) the right-hand side of is approximately equivalent to

$$(6.20)\qquad \sum_{i=1}^{n}\left[r_{ig}^2-\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds+(c_{ii}-1)r_{ig}^2\right]$$

$$= \sqrt{\frac{1}{n}}\sqrt{n}\sum_{i=1}^{n}\left[r_{ig}^2-\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\right]+\left[\sum_{i=1}^{n}(c_{ii}-1)(r_{ig}^2-\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds)\right]$$

$$+\left[\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}ds\right] .$$

Then by using the basic evaluation obtained by Jacod-Protter (1998) as

$$\sqrt{n}\sum_{i=1}^{n}\left[r_{ig}^2-\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right]=O_p(1) .$$

Also we have the desired result by applying the inequalities

$$\begin{aligned}|\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds|^2 &= \left[\sum_{i=1}^{n}(c_{ii}-1)^2\right]\left[\sum_{i=1}^{n}(\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds)^2\right]\\ &\le \left[\frac{1}{n}\sum_{i=1}^{n}(c_{ii}-1)^2\right]\left[\sup_{0\le s\le t}\sigma_{gg}^{(x)}(s)\right]^2\\ &= O(\frac{1}{m_n})\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}\left[|\sum_{i=1}^{n}(c_{ii}-1)(r_{ig}^2-\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds)|^2\right] &= \left[\sum_{i=1}^{n}(c_{ii}-1)^2\mathcal{E}(r_{ig}^2-\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds)^2\right]\\ &= O(\frac{1}{n}) .\end{aligned}$$

Q.E.D.

Lemma 5 : For $0<\alpha\le 0.4$,

$$(6.21)\qquad \sqrt{m_n}\left[\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\right]\xrightarrow{p}0$$

as $n\to\infty$.

Proof of Lemma 5 : We use the relation

$$(6.22)\qquad \sqrt{m_n}\left[\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\right]$$

$$= \sum_{i=1}^{n}\frac{1}{2\sqrt{m_n}}\left[\frac{\sin[\frac{2\pi m_n}{2n+1}(2i-1)]}{\sin[\frac{\pi}{2n+1}(2i-1)]}\right]\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds .$$

We take a positive constant γ $(0 < \gamma < 1)$ and divide the summation of the right-hand side of (6.22) from 1 to n into two parts, that is, (i) $1 \leq i \leq n^\gamma$ and (ii) $n^\gamma + 1 \leq i \leq n$. For (i) there exists a positive K_4 such that the first part of the summation is less that

$$K_4 \frac{1}{m_n} \sum_{i=1}^{n^\gamma} \frac{n}{i} \left[\frac{1}{n} \sup_{0 \leq s \leq 1} \sigma_{gg}^{(x)}(s) \right] = O_p\left(\frac{\log n^\gamma}{\sqrt{m_n}}\right) . \tag{6.23}$$

For (ii) there exists a positive K_5 such that the second part of the summation is less that

$$K_5 \frac{1}{\sqrt{m_n}} \sum_{i=n^\gamma+1}^{n} \frac{n}{n^\gamma} \left[\frac{1}{n} \sup_{0 \leq s \leq 1} \sigma_{gg}^{(x)}(s) \right] = O_p\left(\frac{n}{n^{\gamma+\alpha/2}}\right) . \tag{6.24}$$

Hence if we impose the condition $\gamma + \alpha/2 > 1$, both terms converge to zero as $n \longrightarrow \infty$ by using (2.5). Actually we can take γ satisfying this condition.
Q.E.D.

We note that in Lemma 5 we have intentionally used $O_p(\,\cdot\,)$ instead of $O(\,\cdot\,)$ because of the stochastic case below.

Lemma 6 : Under the assumptions of Theorem 1 with the condition $0 < \beta < 1$, as $n \to \infty$,

$$\hat{\sigma}_{gh}^{(v)} - \sigma_{gh}^{(v)} \stackrel{p}{\longrightarrow} 0 \tag{6.25}$$

and

$$\sqrt{l_n} \left[\hat{\sigma}_{gh}^{(v)} - \sigma_{gh}^{(v)} \right] = O_p(1) . \tag{6.26}$$

Proof of Lemma 6 : We only give a brief proof for the estimation problem of noise variance $\sigma_{gg}^{(v)}$ because the argument on the estimation of the noise covariance is quite similar. For this purpose we use the decomposition

$$\begin{aligned} & \hat{\sigma}_{gg}^{(v)} - \sigma_{gg}^{(v)} \\ &= \frac{1}{l_n} \sum_{k=n+1-l}^{n} a_{kn}^{-1} \left[x_{kg}^2 - \sigma_{gg}^{(x)} a_{kn} \right] \\ &= \frac{1}{l_n} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1} \left[x_{kg}^{(2)2} - \sigma_{gg}^{(v)} a_{kn} \right] + \frac{1}{l_n} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1} \left[x_{kg}^{(1)2} + 2 x_{kg}^{(1)} x_{kg}^{(2)} \right] . \end{aligned} \tag{6.27}$$

Then the main argument of the proof is similar to that of Lemma 4 except l_n instead of m_n. For the variance of noise term, we use the fact that $l_n/n = o(1)$ and for $n+1-l_n \le k \le n$ and $l_n/n = o(1)$, $a_{kn} = 2n[1+\cos\pi(\frac{2l_n}{2n+1})] \ge n$ for a sufficiently large n. Since $a_{kn}^{-1} = o(n^{-1})$,

$$(6.28)\qquad \mathcal{E}[\sum_{k=n+1-l_n}^{n} a_{kn}^{-1}(x_{kg}^{(1)2})] = \sigma_{gg}^{(x)} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1} = O(\frac{l_n}{n}) \ .$$

Then by using the similar evaluations as (6.14) and (6.16)

$$\mathcal{E}\left[\frac{1}{\sqrt{l_n}} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1} x_{kg}^{(1)2}\right]^2 = o(1)$$

and

$$(6.29)\qquad \mathcal{E}\left[\frac{1}{\sqrt{l_n}} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1} x_{kg}^{(1)} x_{kg}^{(2)}\right]^2 = o(1) \ .$$

Hence we can ignore the last two terms of the right-hand side of (6.27) and we need to evaluate the leading term. Then by using the similar evaluation as (6.16), it is possible to evaluate

$$(6.30)\qquad \mathcal{E}\left[\frac{1}{l_n} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1}\left(x_{kg}^{(2)2} - \sigma_{gg}^{(v)} a_{kn}\right)\right]^2 = o(1)$$

and

$$(6.31)\qquad \mathcal{E}\left[\frac{1}{\sqrt{l_n}} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1}\left(\sum_{i,j=1;i\neq j}^{n} b_{ik} b_{jk} v_{ig} v_{jg}\right)\right]^2 = O(1) \ .$$

Q.E.D.

Proofs of Theorem 1, Theorem 3 and Theorem 4 :

(**Step 1**) : We shall give only the proof of the asymptotic normality (and the stable convergence in Theorem 4) of the realized variance $\sigma_{gg}^{(x)}$ $(g = 1, \cdots, p)$. (The proof of the realized covariance is quite similar with some extra notations. We give some brief comments on the related problem between Lemma 3 and Lemma 4.) We first use the proofs of Lemma 4 and Lemma 5 for the consistency and the asymptotic behavior of the SIML estimator.

We write $\sqrt{m_n}[\hat{\sigma}_{gg}^{(x)} - \sigma_{gg}^{(x)}]$ as

$$\begin{aligned}
(6.32) \qquad & \sqrt{m}\left[\sum_{i,j=1}^{n} c_{ij} r_{ig} r_{jg} - \delta_{ij}\int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds\right] \\
&= 2\sqrt{m}\sum_{i>j} c_{ij} r_{ig} r_{jg} + \sqrt{m}\left[\sum_{i=1}^{n} c_{ii} r_{ig}^2 - \int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds\right] ,
\end{aligned}$$

where $\delta_{ij} = 1\ (i=j); \delta_{ij} = 0\ (i \neq j)$ and the second term is equivalent to

$$\begin{aligned}
(6.33) \quad & \sqrt{m}\sum_{i=1}^{n}\left[r_i^2 - \int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds + (c_{ii}-1)r_{ig}^2\right] \\
= \quad & \sqrt{\frac{m}{n}}\sqrt{n}\sum_{i=1}^{n}\left[r_{ig}^2 - \int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds\right] + \sqrt{m}\left[\sum_{i=1}^{n}(c_{ii}-1)(r_{ig}^2 - \int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds)\right] \\
& +\sqrt{m}\left[\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds\right] ,
\end{aligned}$$

which is $o_p(1)$.
From the proofs of Lemma 4 and Lemma 5, we can ignore each terms of (6.33) for the limiting distribution $\sqrt{m_n}[\hat{\sigma}_{gg}^{(x)} - \sigma_{gg}^{(x)}]$.

The summation of the conditional covariances associated with the first term of the right-hand side of (6.32) is

$$\begin{aligned}
& 4\sum_{i<j} m_n c_{ij}^2 \mathcal{E}_{i-1}[r_{ig}^2]\mathcal{E}_{j-1}[r_{jg}^2] \\
= \quad & 2\sum_{i,j=1}^{n} m_n c_{ij}^2 \mathcal{E}_{i-1}[r_{ig}^2]\mathcal{E}_{j-1}[r_{jg}^2] - 2\sum_{i=1}^{n} m_n c_{ii}^2 (\mathcal{E}_{i-1}[r_{ig}^2])^2 \\
= \quad & 2\sum_{i,j=1}^{n} \mathcal{E}_{i-1}[r_{ig}^2]\mathcal{E}_{j-1}[r_{jg}^2] + 2\sum_{i,j=1}^{n}(m_n c_{ij}^2 - 1)\mathcal{E}_{i-1}[r_{ig}^2]\mathcal{E}_{j-1}[r_{jg}^2] - 2\sum_{i=1}^{n} m_n c_{ii}^2 (\mathcal{E}_{i-1}[r_{ig}^2])^2 ,
\end{aligned}$$

where we use the notation $\mathcal{E}_{i-1}[r_{ig}^2] = \mathcal{E}[r_{ig}^2|\mathcal{F}_{n,i-1}]$.
For the third term, we have

$$(6.34) \qquad \sum_{i=1}^{n} m c_{ii}^2 \left(\mathcal{E}_{i-1}[r_{ig}^2]\right)^2 \leq \left[\sup_{0\leq\leq 1} \sigma_{gg}^{(x)}(s)\right]^2 \frac{m_n}{n^2}\sum_{i=1}^{n} c_{ii}^2 \to 0$$

as $m/n \to 0$. Then the main part of the asymptotic variance of (2.37) becomes

(6.35) $V_{gg.n}$

$$
\begin{aligned}
&= 2\left[\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\right]^2 + 2\sum_{i,j=1}^{n}(m_n c_{ij}^2 - 1)\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\int_{t_{j-1}}^{t_j}\sigma_{gg}^{(x)}(s)ds \\
&\to 2\left[\int_0^1\sigma_{gg}^{(x)}(s)ds\right]^2 + 2\lim_{n\to\infty}\sum_{i,j=1}^{n}(m_n c_{ij}^2 - 1)\left[\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\right]\left[\int_{t_{j-1}}^{t_j}\sigma_{gg}^{(x)}(s)ds\right] \\
&= V_{gg}\ .
\end{aligned}
$$

The second term is bounded because by using Lemma 3 we have

$$
\frac{1}{n^2}\sum_{i,j=1}^{n}|m_n c_{ij}^2 - 1| \leq 1 + \frac{m_n}{n^2}\sum_{i,j=1}^{n}c_{ij}^2\ .
$$

When the volatility function is constant ($\sigma_{gg}^{(x)}(s) = \sigma_{gg}^{(x)}$),

$$
\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds = \sigma_{gg}^{(x)}\frac{1}{n}\ ,
$$

the second term of (6.35) vanishes because Lemma 3 again implies

$$
\frac{1}{n^2}\sum_{i,j=1}^{n}(m_n c_{ij}^2 - 1) = \frac{1}{n^2}\left[n^2 + n + \frac{1}{4} - n^2\right] \to 0
$$

and then

$$
V_{gg} = 2\left[\sigma_{gg}^{(x)}\right]^2\ . \tag{6.36}
$$

(**Step 2**) : Next, we need to show that the SIML estimator has the asymptotic normality. For this purpose, we construct a sequence of $\sigma-$fields such that $\mathcal{F}_{n,i} \subseteq \mathcal{F}_{n+1,i}$ and we apply the Martingale Central Limit Theorem (MCLT) to the first part of (6.32). We shall use Theorem 3.5 and Corollary 3.3 of Hall and Heyde (1980) in particular. In order to do this, we need the condition that $V_{gg.n}^{(2)}$ in (2.38) converges to V_{gg} and V_{gg} is positive a.s. (We need that V_{gg} takes a non-negative value in Theorem 3 while it is non-negative a.s. in the stochastic case for Theorem 4. When the probability limit V_{gg} is a random variable, the MCLT gives the stable convergence. See Chapter 3 of Hall and Heyde (1980) for the detailed discussion we need for the present purpose.)

In our proof we shall use of a sequence of random variables

$$
U_n = \sum_{j=2}^{n}[2\sum_{i=1}^{j-1}\sqrt{m_n}c_{ij}r_{ig}]r_{jg}, \tag{6.37}
$$

which is a martingale. Then we apply Theorem 3.5 of Hall and Heyde (1980) to U_n by setting $X_{nj} = (2\sum_{i=1}^{j-1} \sqrt{m_n} c_{ij} r_{ig}) r_{jg}$ $(j = 2, \cdots, n)$ and $V_{gg.n}^{*} = \sum_{j=2}^{n} \mathcal{E}[X_{nj}^2 | \mathcal{F}_{n,j-1}]$. Under the assumptions of Theorems we have enough moment conditions on r_{ig}. Then in our situation it is sufficient to check Condition (A)

$$\max_{1 \le j \le n} \mathcal{E}[X_{nj}^2 | \mathcal{F}_{n,j-1}] \xrightarrow{p} 0 \ , \tag{6.38}$$

Condition (B)

$$\sum_{j=1}^{n} \mathcal{E}[X_{nj}^4] \longrightarrow 0 \tag{6.39}$$

and Condition (C)

$$\mathcal{E}[(V_{gg.n}^{*} - V_{gg})^2] \longrightarrow 0 \tag{6.40}$$

as $n \longrightarrow \infty$.

First we notice that Condition (B) implies Condition (A) in the present formulation because for $Y_{nj} = \mathcal{E}[X_{nj}^2 | \mathcal{F}_{n,j-1}]$ and any $\epsilon > 0$

$$\mathrm{P}(\max_{1 \le j \le n} Y_{nj} > \epsilon) \le \sum_{j=1}^{n} \mathrm{P}(Y_{nj} > \epsilon) \le (\frac{1}{\epsilon})^2 \sum_{j=1}^{n} \mathcal{E}[\mathrm{Y}_{nj}^2] \ . \tag{6.41}$$

Then Lemma 7 below shows Condition (B).

Second, we have assumed the condition $V_{gg.n} \xrightarrow{p} V_{gg}$ in Theorems and $V_{gg.n}$ and V_{gg} are bounded. Then we can find a positive K_6 such that for any $\epsilon > 0$

$$\begin{aligned} \mathcal{E}[(V_{gg.n} - V_{gg})^2] &= \mathcal{E}[(V_{gg.n} - V_{gg})^2 I(|V_{gg.n} - V_{gg}| \ge \epsilon)] \\ &\quad + \mathcal{E}[(V_{gg.n} - V_{gg})^2 I(|V_{gg.n} - V_{gg}| < \epsilon)] \\ &\le K_6 \mathrm{P}(|V_{gg.n} - V_{gg}| \ge \epsilon) + \epsilon^2 \ . \end{aligned}$$

Hence we only need to show Condition (D)

$$\mathcal{E}[(V_{gg.n}^{*} - V_{gg.n})^2] \longrightarrow 0 \tag{6.42}$$

as $n \longrightarrow \infty$. Then Lemma 8 below shows Condition (D).

(**Step 3**) : The proof of the asymptotic normality of the variance and covariance of the noise terms can be given, which are very similar to the ones in (i) and (ii)

for the integrated variance. Since Lemma 6 gives the proof of the consistency and the order of the SIML estimator, the remaining task is to calculate the asymptotic variance and to apply Theorem 3.5 of Hall and Heyde (1980). Since the arguments are lengthy, but most of them are parallel to the arguments in (i) and (ii) because the method of our proof does not depend on the Gaussianity of the underlying process much and we have omitted its details.
Q.E.D.

Lemma 7 : Under the assumptions of Theorem 4, we have Condition (B).

Proof of Lemma 7 : Without loss of generality we consider the stochastic case when $p = q = 1$ and we denote $\mathbf{C}_s^{(x)} = c_s$. We shall show Condition (B) under the assumptions of Theorem 4. Let

$$Z_{nj}(t) = \int_{t_{j-1}}^{t} c_s dB_s \ (t_{j-1} \le t \le t_j, j = 1, \cdots, n)$$

and

$$W_{nj} = \sum_{i=1}^{j-1} \sqrt{m_n} c_{ij} \int_{t_{i-1}}^{t_i} c_s dB_s \ (j = 2, \cdots, n) \ .$$

Then we need to show that

$$\sum_{j=2}^{n} \mathcal{E}[W_{nj}^4 Z_{nj}(t_j)^4] \longrightarrow 0 \tag{6.43}$$

as $n \to \infty$.
First by using Ito's Lemma, we have

$$Z_{nj}(t)^4 = \int_{t_{j-1}}^{t} 4[Z_{nj}(s)]^3 c_s dB_s + \int_{t_{j-1}}^{t} 6[Z_{nj}(s)]^2 c_s^2 ds \ .$$

Then by taking the conditional expectation of both sides given $\mathcal{F}_{n,j-1}$ (we denote $\mathcal{E}[\ \cdot \ |\mathcal{F}_{n,j-1}] = \mathcal{E}_{j-1}[\ \cdot \ ]$), we have

$$\mathcal{E}_{j-1}[Z_{nj}(t)^4] = \int_{t_{j-1}}^{t} 6\mathcal{E}_{j-1}[(Z_{nj}(t))^2 c_s^2] ds \tag{6.44}$$

and then

$$\mathcal{E}_{j-1}[Z_{nj}(t)^4] \le 3\int_{t_{j-1}}^{t} \mathcal{E}_{j-1}[(Z_{nj}(s))^4] ds + 3\int_{t_{j-1}}^{t} \mathcal{E}_{j-1}[c_s^4] ds \ . \tag{6.45}$$

By using the boundedness condition, we have $\int_{t_{j-1}}^{t} c_s^4 ds = O(\frac{1}{n})$. Then by using the standard argument in stochastic calculus on the evaluation of moments (i.e. Chapter III of Ikeda and Watanabe (1989), for instance), we can find a positive constant K_7 such that

$$\mathcal{E}_{j-1}[Z_{nj}(t)^4] \leq K_7(\frac{1}{n}) \ .$$

By using the Cauchy-Schwartz inequality, we can find a positive constant K_8 such that

$$\mathcal{E}_{j-1}[Z_{nj}(t)^4] \leq 6 \int_{t_{j-1}}^{t} \left(\mathcal{E}_{j-1}[(Z_{nj}(s))^4]\right)^{1/2} \left(\mathcal{E}_{j-1}[c_s^4]\right)^{1/2} ds \leq K_8 \left[\frac{1}{n}\right]^{1+\frac{1}{2}} \ .$$

By repeating the above substitution procedure, we have the bound of the fourth order moment as $K_8^{'}(\frac{1}{n})^{1+1/2+(1/2)^2+\cdots+(1/2)^r}$ for an arbitrary positive integer r $(r \geq 2)$ and a positive constant $K_8^{'}$. Then we can find that for an arbitrary small ϵ (> 0) and $t_{j-1} \leq t \leq t_j$,

$$\mathcal{E}_{j-1}[Z_{nj}(t)^4] = O((\frac{1}{n})^{2(1-\epsilon)}) \ . \tag{6.46}$$

Next, we shall evaluate the expectation $\mathcal{E}[W_{nj}(t)^4]$, that is

$$\begin{aligned} & \mathcal{E}[W_{nj}(t)^4] \\ = \ & \mathcal{E}\left[\sum_{i_1,i_2,i_3,i_4=1}^{j-1} m_n^2 c_{i_1,j} c_{i_2,j} c_{i_3,j} c_{i_4,j} \int_{t_{i_1-1}}^{t_{i_1}} c_{s_1} dB_{s_1} \int_{t_{i_2-1}}^{t_{i_2}} c_{s_2} dB_{s_2} \int_{t_{i_3-1}}^{t_{i_3}} c_{s_3} dB_{s_3} \int_{t_{i_4-1}}^{t_{i_4}} c_{s_4} dB_{s_4}\right] . \end{aligned}$$

In this form we only need to consider the summations of the forms (i) $\sum_{i_1=i_2,i_3,i_4}[\ \cdot\ ]$ and (ii) $\sum_{i_1=i_2,i_3=i_4}[\ \cdot\ ]$ because $\int_{t_{i_1-1}}^{t_{i_1}} c_{s_1} dB_{s_1}$ is a martingale difference. We first consider Case (i) and we set $i_1 = i_2 > i_3 > i_4$. In this case we can utilize the fact that $\mathcal{E}[\int_{t_{i_1-1}}^{t_{i_1}} c_{s_1} dB_{s_1}]^2 = \int_{t_{i_1-1}}^{t_{i_1}} c_{s_1}^2 ds$ and
$|2\int_{t_{i_3-1}}^{t_{i_3}} c_{s_3} dB_{s_3} \int_{t_{i_4-1}}^{t_{i_4}} c_{s_4} dB_{s_4}| \leq [\int_{t_{i_3-1}}^{t_{i_3}} c_{s_3} dB_{s_3}]^2 + [\int_{t_{i_4-1}}^{t_{i_4}} c_{s_4} dB_{s_4}]^2$.
By using the assumption that c_s are bounded in Theorem 4 and $\int_{t_{i_1-1}}^{t_{i_1}} c_{s_1} dB_{s_1}$ is a martingale difference, we can find a positive constant K_9 such that

$$\mathcal{E}\left[|\int_{t_{i_1-1}}^{t_{i_1}} c_{s_1} dB_{s_1} \int_{t_{i_2-1}}^{t_{i_2}} c_{s_2} dB_{s_2} \int_{t_{i_3-1}}^{t_{i_3}} c_{s_3} dB_{s_3} \int_{t_{i_4-1}}^{t_{i_4}} c_{s_4} dB_{s_4}|\right] \leq K_9(\frac{1}{n})^2 \ . \tag{6.47}$$

Hence we have

$$\mathcal{E}[Z_{nj}(t)^4](\frac{1}{n})^{2(1-\epsilon)} = o(\frac{1}{n})$$

if we can show

$$\left[\sum_{i_1,i_3,i_4}^{j-1} m_n^2 c_{i_1,j}^2 c_{i_3,j} c_{i_4,j} (\frac{1}{n})^{1+2(1-\epsilon)}\right] \longrightarrow 0 \tag{6.48}$$

as $n \to \infty$. By using the similar method to the proof of Lemma 5 to

$$\sum_{i=1}^{q} c_{ij} = \frac{2}{m_n} \sum_{k=1}^{m_n} \left[\frac{\sin[\frac{\pi q}{2n+1}(2k-1)]}{2\sin[\frac{\pi}{2n+1}(2k-1)]}\right] s_{jk} \ ,$$

for a sufficiently small ϵ (> 0) we have

$$\frac{\sqrt{m_n}}{n^{1-\epsilon}} \sum_{i=1}^{q} c_{ij} = o(1) \tag{6.49}$$

as $n \to \infty$. Hence we have the result for Case (i). We can use the same method of evaluation for Case (ii) and then we have obtained the order of $\mathcal{E}[Z_{nj}(t)^4]$. Finally, because we set the assumption that c_s and $\sigma_{gg}^{(x)}(s)$ are bounded ($g = 1$) and $W_{nj} Z_{nj}(t)$ ($t_{j-1} \leq t \leq t_j; j = 2, \cdots, n$) is a sequence of martingale differences, we have the desired result. **Q.E.D.**

We shall give the proof of Condition (D) for the time-varying deterministic case because the arguments we use are rather clear and straightforward. However, it is possible to show the result in the stochastic case with additional arguments illustrated in the proof of Lemma 7.

Lemma 8 : Under the assumptions in Theorem 3, we have Condition (D).

Proof of Lemma 8 : Without loss of generality we consider the case when $p = q = 1$. By using (6.34) and (6.35) ($\mathcal{E}_{i-1}[r_{ig}^2] = \mathcal{E}[r_{ig}^2]$ in the present case), it is sufficient to evaluate

$$\begin{aligned} D_n &= \mathcal{E}\left\{\left(\sum_{j=1}^{n}[\mathcal{E}_{j-1}(X_{nj}^2) - \mathcal{E}(X_{nj}^2)]\right)^2\right\} \\ &= \mathcal{E}\left\{\sum_{j=1}^{n}\left(\sum_{i_1,i_2=1}^{n} m_n c_{i_1,j} c_{i_2,j} (\int_{t_{j-1}}^{t_j} c_s^2 ds)\right.\right. \\ &\quad \left.\left.\left[(\int_{t_{i_1-1}}^{t_{i_1}} c_{s_{i_1}} dB_{s_{i_1}})(\int_{t_{i_2-1}}^{t_{i_2}} c_{s_{i_2}} dB_{s_{i_2}}) - \delta(i_1,i_2)(\int_{t_{i_1-1}}^{t_{i_1}} c_{s_{i_1}}^2 ds_{i_1})\right]\right)\right\}^2 \ , \end{aligned}$$

where $\mathcal{E}_{j-1}(X_{nj}^2) = \mathcal{E}(X_{nj}^2|\mathcal{F}_{n,j-1})$, $\delta(i_1,i_2) = 1$ for $i_1 = i_2$ and $\delta(i_1,i_2) = 0$ for $i_1 \neq i_2$.

We use (6.47) for $t_{i_1} = t_{i_2}, t_{i_1} = t_{i_3}$ or $t_{i_1} = t_{i_4}$ in the proof of Lemma 7. Because we have $\mathcal{E}[\int_{t_{i-1}}^{t_i} c_s dB_s]^2 = O_p(\frac{1}{n})$, which is bounded a.s. under the present formulation, and we use Lemma 3 (i.e. (6.5)), there exist positive constants K_{10} and K_{11} such that

$$\begin{aligned} D_n &\leq K_{10}(\frac{1}{n})^2 \sum_{i_1.i_2=1}^{n} \left[\sum_{j,j'=1}^{n} m_n^2 c_{i_1,j} c_{i_2,j} c_{i_1.j'} c_{i_2,j'} (\int_{t_{j-1}}^{t_j} c_s^2 ds)(\int_{t_{j'-1}}^{t_{j'}} c_{s'}^2 ds') \right] \\ &= K_{10}(\frac{1}{n})^2 \left[\sum_{j,j'=1}^{n} (\sum_{i_1=1}^{n} m_n c_{i_1,j} c_{i_1.j'})(\sum_{i_2=1}^{n} m_n c_{i_2,j} c_{i_2.j'})(\int_{t_{j-1}}^{t_j} c_s^2 ds)(\int_{t_{j'-1}}^{t_{j'}} c_{s'}^2 ds') \right] \\ &\leq K_{11}(\frac{1}{n})^2 (n + \frac{1}{2})^2 \left[\sum_{j,j'=1}^{n} c_{j.j'}^2 \right] (\frac{1}{n})^2 . \end{aligned}$$

Then by using Lemma 3 (i.e. (6.5)) and the fact that $\mathcal{E}[\int_{t_{j-1}}^{t_j} c_s dB_s]^2 = O(\frac{1}{n})$, finally we find that

$$D_n = O(\frac{1}{m_n}) . \tag{6.50}$$

Q.E.D.

Proof of Corollary 1 : When $\sigma_{gg}^{(v)} = 0$, we have $\mathbf{X}_n^{(2)} \mathbf{e}_g = \mathbf{0}$ and then $\mathbf{Z}_n \mathbf{e}_g = h_n^{-1/2} \mathbf{P}_n \mathbf{C}_n (\mathbf{X}_n - \bar{\mathbf{Y}}_0 \mathbf{e}_g$. We use the relation

$$\begin{aligned} T_1 &= \sqrt{m_n} \left[\frac{\left(\frac{1}{l_n} \sum_{k=n+1-l_n}^{n} z_{kg}^2 - \sigma_{gg}^{(x)} \right) - \left(\frac{1}{m_n} \sum_{k=1}^{m_n} z_{kg}^2 - \sigma_{gg}^{(x)} \right)}{\sigma_{gg}^{(x)} + \left(\frac{1}{m_n} \sum_{k=1}^{m_n} z_{kg}^2 - \sigma_{gg}^{(x)} \right)} \right] \\ &= -\sqrt{m_n} \left[\frac{\frac{1}{m_n} \sum_{k=1}^{m_n} z_{kg}^2 - \sigma_{gg}^{(x)}}{\sigma_{gg}^{(x)}} \right] + \frac{\sqrt{m_n}}{\sqrt{l_n}} \sqrt{l_n} \left[\frac{\frac{1}{l_n} \sum_{k=n+1-l_n}^{n} z_{kg}^2 - \sigma_{gg}^{(x)}}{\sigma_{gg}^{(x)}} \right] + o_p(1) . \end{aligned}$$

Because of the condition $0 < \alpha < \beta < 1$, we have $m_n / l_n \to 0$ as $n \to \infty$ and then the second term converges to 0 in probability. The first term converges to $N(0, 2)$ by Theorem 1 and thus we have the result.

Q.E.D.

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Appendix : Tables and Figures

In this Appendix we gather Tables and Figures, which we have mentioned in Section 3 and Section 4.

Table 3.1 : Estimation of realized volatility :

Case I ($a_0 = 1, a_1 = a_2 = 0$)

5000	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI
True	2.000E-04	2.000E-06		2.000E-04	2.000E-07	
Mean	2.06E-04	2.01E-06	2.02E-02	2.00E-04	2.11E-07	2.20E-03
SD	5.26E-05	9.64E-08	4.96E-04	5.13E-05	9.84E-09	5.08E-05
True	2.000E-04	2.000E-08		2.000E-04	2.000E-09	
Mean	2.00E-04	3.02E-08	4.00E-04	2.01E-04	1.23E-08	2.20E-04
SD	5.35E-05	1.37E-09	7.83E-06	5.24E-05	5.66E-10	4.57E-06
20000	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI
True	2.000E-04	2.000E-06		2.000E-04	2.000E-07	
Mean	2.03E-04	2.01E-06	8.02E-02	2.01E-04	2.02E-07	8.20E-03
SD	4.01E-05	5.39E-08	9.76E-04	4.03E-05	5.46E-09	1.01E-04
True	2.000E-04	2.000E-08		2.000E-4	2.000E-09	
Mean	2.00E-04	2.25E-08	1.00E-03	2.01E-04	4.55E-09	2.80E-04
SD	4.03E-05	6.17E-10	1.18E-05	3.90E-05	1.17E-10	2.77E-06

Note : In Table 3.1, $\hat{\sigma}_x^2$ and $\hat{\sigma}_v^2$ correspond to the estimates for the variances $\mathbf{\Sigma}_x$ (3.2) and $\mathbf{\Sigma}_v$, respectively. Mean and SD are the sample mean and the standard deviation of the SIML estimator in the simulation. HI stands for the historical volatility.

Table 3.2 : Estimation of realized volatility :

Case II ($a_0 = 1, a_1 = 1, a_2 = 1$)

5000	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI
True	3.67E-04	2.000E-06		3.667E-04	2.000E-07	
Mean	3.67E-04	2.02E-06	2.04E-02	3.67E-04	2.19E-07	2.37E-03
SD	9.80E-04	9.58E-08	4.89E-04	1.03E-04	1.08E-08	5.71E-05
True	2.000E-04	2.000E-08		2.000E-04	2.000E-09	
Mean	3.67E-04	3.896E-08	5.66E-04	3.62E-04	2.09E-08	3.87E-04
SD	1.02E-04	1.90E-09	1.22E-05	9.74E-05	1.02E-09	7.97E-06
20000	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI
True	3.667E-04	2.000E-06		3.667E-04	2.000E-07	
Mean	3.63E-04	2.00E-06	8.04E-02	3.66E-04	2.05E-07	8.36E-03
SD	7.48E-05	5.40E-08	9.71E-04	7.74E-09	5.37E-09	9.87E-05
True	3.667E-04	2.000E-06		3.667E-04	2.000E-07	
Mean	3.68E-04	2.46E-08	1.17E-03	3.63E-04	6.66E-09	4.47E-04
SD	7.65E-05	6.55E-10	1.35E-05	7.49E-05	1.81E-10	4.62E-06

Note : In Table 3.2, $\hat{\sigma}_x^2$ and $\hat{\sigma}_v^2$ correspond to the estimates for the variances $\mathbf{\Sigma}_x$ (3.2) and $\mathbf{\Sigma}_v$, respectively. Mean and SD are the sample mean and the standard deviation of the SIML estimator in the simulation. HI stands for the historical volatility.

Table 3.3 : Estimation of realized volatility :

Case III ($a_0 = 1, a_1 = -1, a_2 = 1$)

5000	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI
True	1.667E-04	2.000E-06		1.677E-04	2.000E-07	
Mean	1.70E-04	2.01E-06	2.02E-02	1.68E-04	2.09E-07	2.17E-03
SD	4.48E-05	9.05E-08	4.80E-04	4.22E-05	9.71E-09	5.16E-05
True	1.667E-04	2.000E-08		1.667E-04	2.000E-09	
Mean	1.70E-04	2.86E-08	3.67E-04	1.69E-04	1.06E-10	1.87E-04
SD	4.79E-05	1.34E-09	7.67E-06	4.29E-05	4.88E-10	3.76E-06
20000	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI
True	1.667E-04	2.000E-06		1.667E-04	2.000E-07	
Mean	1.71E-04	2.00E-06	8.02E-02	1.66E-04	2.02E-07	8.17E-03
SD	3.41E-05	5.55E-08	1.01E-03	3.30E-05	5.21E-09	9.66E-05
True	1.667E-04	2.000E-8		1.667E-04	2.000E-09	
Mean	1.66E-05	2.21E-08	9.66E-04	1.66E-04	4.12E-09	2.47E-04
SD	3.14E-05	6.00E-10	1.10E-05	3.11E-05	1.11E-10	2.56E-06

Note : In Table 3.3, $\hat{\sigma}_x^2$ and $\hat{\sigma}_v^2$ correspond to the estimates for the variances $\mathbf{\Sigma}_x$ in (3.2) and $\mathbf{\Sigma}_v$, respectively. Mean and SD are the sample mean and the standard deviation of the SIML estimator in the simulation. HI stands for the historical volatility.

Table 3.4 : Estimation of realized volatility :

Case IV ($a_0 = 3, a_1 = -3, a_2 = 1$)

5000	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI
True	3.67E-06	2.000E-06		3.67E-04	2.000E-07	
Mean	3.71E-04	2.02E-06	2.04E-02	3.701E-04	2.19E-07	2.37E-03
SD	9.69E-05	9.43E-08	4.79E-04	1.000E-04	1.02E-08	5.55E-05
True	3.67E-04	2.000E-09		3.667E-04	2.000E-09	
Mean	3.70E-04	3.88E-08	5.66E-04	3.71E-04	2.08E-08	3.87E-04
SD	1.05E-05	1.87E-10	1.18E-06	1.03E-04	1.02E-09	8.26E-06
20000	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI
True	3.67E-06	2.000E-06		3.67E-04	2.000E-07	
Mean	3.73E-04	2.01E-06	8.04E-02	3.71E-04	2.05E-07	8.37E-03
SD	8.08E-05	5.38E-08	9.86E-04	7.62E-05	5.49E-09	9.70E-05
True	3.67E-04	2.000E-09		3.667E-04	2.000E-09	
Mean	3.66E-04	2.46E-08	1.17E-03	3.67E-05	6.65E-09	4.47E-04
SD	7.55E-05	6.88E-10	1.26E-05	7.60E-05	1.82E-10	4.48E-06

Note : In Table 3.4, $\hat{\sigma}_x^2$ and $\hat{\sigma}_v^2$ correspond to the estimates for the variances $\mathbf{\Sigma}_x$ in (3.4) and $\mathbf{\Sigma}_v$, respectively. Mean and SD are the sample mean and the standard deviation of the SIML estimator in the simulation. HI stands for the historical volatility.

Table 3.5 : Estimation of realized volatility :

Case V (Stochastic Volatility)

5000	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI
True	4.22E-04	2.000E-06		4.22E-04	2.000E-07	
Mean	4.23E-04	2.02E-06	2.05E-02	4.23E-04	2.22E-07	2.42E-03
SD	1.07E-04	9.61E-08	4.93E-04	1.11E-04	1.02E-08	5.39E-05
True	4.22E-04	2.000E-08		4.22E-04	2.000E-09	
Mean	4.23E-04	4.18E-08	6.23E-04	4.20E-04	2.37E-08	4.42E-04
SD	1.09E-04	1.97E-09	1.45E-05	1.09E-04	1.24E-09	1.09E-05
20000	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI	$\hat{\sigma}_x^2$	$\hat{\sigma}_v^2$	HI
True	4.22E-04	2.000E-06		4.22E-04	2.000E-07	
Mean	4.25E-04	2.00E-06	8.04E-02	4.19E-04	2.06E-07	8.42E-03
SD	8.01E-05	5.56E-08	9.87E-04	8.10E-05	5.50E-09	9.84E-05
True	4.22E-04	2.000E-08		4.22E-04	2.000E-9	
Mean	4.21E-04	2.54E-08	1.22E-03	4.19E-04	7.37E-09	5.02E-04
SD	8.26E-05	6.67E-10	1.39E-05	7.97E-05	2.05E-10	6.27E-06

Note : In Table 3.5, $\hat{\sigma}_x^2$ and $\hat{\sigma}_v^2$ correspond to the estimates for the variances $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_v$ when we have the stochastic volatility model of (3.3) and (3.4). Mean and SD are the sample mean and the standard deviation of the SIML estimator in the simulation. HI stands for the historical volatility.

Table 3.6 : SIML and Realized Kernel methods (n=10000, H=3)

	$\hat{\sigma}_x^2$(SIML)	$\hat{\sigma}_v^2$(SIML)	$\hat{\sigma}_{RK}^2$(RK)
True Value	$\sigma_x^2 = 1.0$	$\sigma_v^2 = 0.01$	$\sigma_x^2 = 1.0$
Mean	1.0157	0.010	0.9908
SD	0.2285	3.44E-04	0.9986
	$\hat{\sigma}_x^2$(SIML)	$\hat{\sigma}_v^2$(SIML)	$\hat{\sigma}_{RK}^2$(RK)
True Value	$\sigma_x^2 = 1.0$	$\sigma_v^2 = 0.001$	$\sigma_x^2 = 1.0$
Mean	1.0051	0.010	1.006
SD	0.2337	3.55E-05	0.1216
	$\hat{\sigma}_x^2$(SIML)	$\hat{\sigma}_v^2$(SIML)	$\hat{\sigma}_{RK}^2$(RK)
True Value	$\sigma_x^2 = 1.0$	$\sigma_v^2 = 0.0001$	$\sigma_x^2 = 1.0$
Mean	1.0151	1.26E-04	0.9993
SD	0.2257	4.48E-06	0.0371

Note : In Tables 3.6 and 3.7, $\hat{\sigma}_x^2$ and $\hat{\sigma}_v^2$ correspond to the SIML estimates for the variances $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_v$ while $\hat{\sigma}_{RK}^2$ corresponds to the Realized Kernel estimate of σ_x^2, which is based one Bandorff-Nielsen et al. (2008).

Table 3.7 : SIML and Realized Kernel methods (n=10000, H=10)

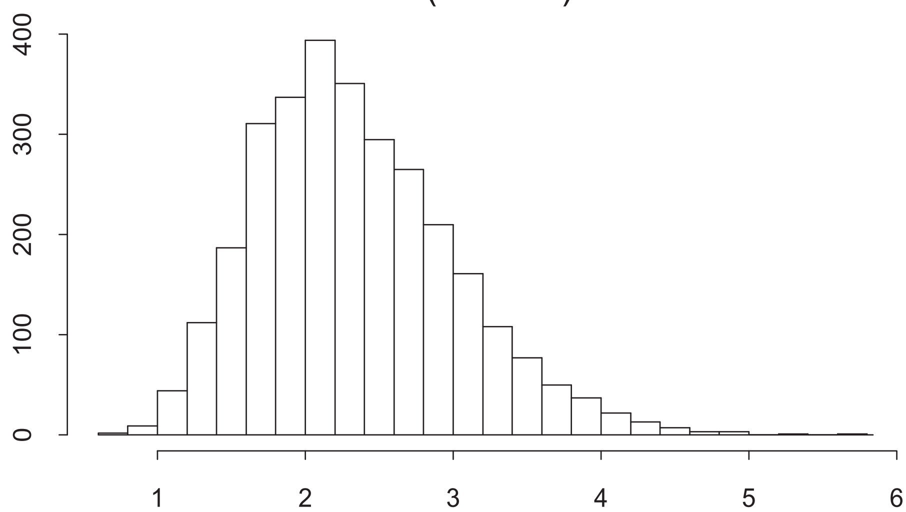
	$\hat{\sigma}_x^2$(SIML)	$\hat{\sigma}_v^2$(SIML)	$\hat{\sigma}_{RK}^2$(RK)
True Value	$\sigma_x^2 = 1.0$	$\sigma_v^2 = 0.03$	$\sigma_x^2 = 1.0$
Mean	1.0748	0.0299	0.9987
SD	0.2681	0.0011	0.6444
	$\hat{\sigma}_x^2$(SIML)	$\hat{\sigma}_v^2$(SIML)	$\hat{\sigma}_{RK}^2$(RK)
True Value	$\sigma_x^2 = 1.0$	$\sigma_v^2 = 0.001$	$\sigma_x^2 = 1.0$
Mean	0.9911	0.0010	1.0009
SD	0.2257	3.72E-05	0.057
	$\hat{\sigma}_x^2$(SIML)	$\hat{\sigma}_v^2$(SIML)	$\hat{\sigma}_{RK}^2$(RK)
True Value	$\sigma_x^2 = 1.0$	$\sigma_v^2 = 0.0001$	$\sigma_x^2 = 1.0$
Mean	1.0077	3.55E-05	0.9991
SD	0.2297	1.22E-06	0.0445

Table 4.1 : Estimation of Realized Volatility :

	$\hat{\sigma}_x^2$	HI
1s	4.085E-05	4.946E-04
5s	3.994E-05	2.601E-04
10s	4.990E-05	1.764E-04
30s	3.551E-05	9.449E-05
60s	4.550E-05	6.964E-05

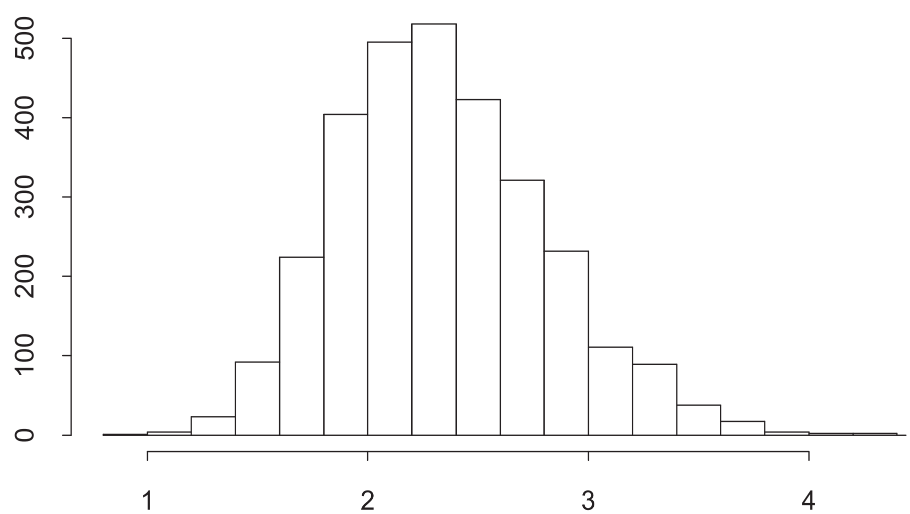
Note : In Table 4.1, $\hat{\sigma}_x^2$ corresponds to the variance estimate of $\mathbf{\Sigma}_x$ and HI stands for the historical volatility.

Figure 3.1 Histogram of the SIML estimators (n=5000)



σ_v^2=0.0001, $\sigma(0)^2$=1, a_0=5, a_1=-16, a_2=16

Figure 3.2 Histogram of the SIML estimators (n=30000)



σ_v^2=0.0001, $\sigma(0)^2$=1, a_0=5, a_1=-16, a_2=16