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# Hotelling's Spatial Competition Reconsidered\*

Takatoshi Tabuchi<sup>†</sup>

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## Abstract

Oligopoly models are usually analyzed in the context of two firms anticipating that market outcomes would be qualitatively similar in the case of three or more firms. This is not an exception in the literature on Hotelling's location-then-price competition. In this paper, we show that the main findings in Hotelling's duopoly, brand bunching and the max-min principle of product differentiation no longer hold once three or more firms are allowed to enter the market. That is, oligopolists with three or more firms proliferate brands and neither maximize nor minimize product differentiation.

## 1 Introduction

Literature on the oligopoly theory is often confined to two firms for the sake of analytical convenience in the literature and with the expectation that the number of firms would not qualitatively affect oligopolistic markets much. However, in Hotelling's (1929) spatial competition, the difference in the equilibrium location between duopolies and oligopolies with three or more firms is commonly known. Hence, with regard to Hotelling's linear market, if two firms compete first in terms of location and then price, they will be located as far as possible; this is interpreted as maximum differentiation in characteristic space. On the other hand, Neven (1987) and Brenner (2005) demonstrate that if there are more than two firms in the market, they do not maximize differentiation. Therefore, some of

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the established results on Hotelling’s duopoly no longer hold when more than two firms are allowed to enter the market.

This paper aims to settle the two important debates in industrial organization—*brand proliferation* and the *max-min principle of product differentiation*—by considering an oligopoly with more than two firms. First, we tackle the established findings of Martinez-Giralt and Neven (1988) that duopolists do not open multiple outlets even if there are no fixed costs. The incentives to proliferate brands has long been attracting attention in the field of industrial organization. Schmalensee (1978) argued that incumbent firms may deter new entry by brand proliferation, whereas Judd (1985) showed that an incumbent firm withdraw some brands in order to avoid intense competition. It is true that Judd’s (1985) framework is general enough, but his conclusions are confined to duopoly. There is no guarantee that his conclusions hold for an oligopoly with more than two firms.

Second, we address the established findings of Tabuchi (1994), Ansari, Economides and Steckel (1997), and Irmen and Thisse (1998) that Hotelling was “almost right” because minimum differentiation (spatial concentration in a geographical space) can arise in equilibrium along all but one dimension. The question whether product differentiation is minimal or maximal has also been tackled in the field of industrial organization. We show that the differentiation is neither minimum nor maximum in the two-dimensional space once we extend the setting from two to three or more firms.

This paper deals with multi-outlet oligopoly in the context of location-then-price competition. Multi-outlet oligopoly can also be analyzed on the basis of different frameworks. First, it can be examined under location-then-quantity competition a la Cournot. However, we do not adopt this approach because equilibrium configurations do not necessarily fit the reality. Price competition in this paper yields segmented configurations, whereas the same does not hold for quantity competition according to Pal and Sarkar (2002). Under quantity competition, each firm opens outlets the way a monopolist would.

Second, rich equilibrium configurations might be obtained by assuming an elastic demand for the good. However, works such as Economides (1984) indicates that obtaining analytically meaningful results with elastic demand is too difficult. For example, despite the identical assumptions of the model with elastic demand, Wang and Yang (1999) and Rath and Zhao (2001) arrive at different equilibrium configurations when the reservation price is low.

Third, rich equilibrium configurations may also emerge by employing random utilities such as logit models in the consumers' choice of a firm. However, obtaining analytically meaningful and interesting results seems impossible when considering multiple outlets of more than two firms. We, therefore, focus on the traditional approach of Hotelling's location-then-price competition due to its mathematical tractability and because it provides a more accurate description of reality.

The general setting of Hotelling's location-then-price competition model is explained in the next section. We examine two kinds of consumer distributions. Section 3 considers the simplest one, where consumers are uniformly distributed over a circumference of a circle, and section 4 considers a more realistic but complicated distribution where consumers are uniformly distributed over a disk. We investigate both duopoly and triopoly with multiple outlets in these two sections. We then conduct a brief empirical analysis in section 5. Section 6 concludes the paper.

## 2 General Setting

Consumers are uniformly distributed over a convex set in  $\mathbb{R}$  or  $\mathbb{R}^2$ , which is a circumference of a circle or a disk. Each consumer purchases one unit of an identical good. There are  $n$  firms with 2 outlets at the most, which are located on the convex set. We ignore the cost of establishing outlets for analytical simplicity.<sup>1</sup> They produce a good without cost and sell it at the mill price. Consumers bear the transport cost for shopping, which is quadratic in distance  $x$  as given by  $tx^2$ .

The game in this paper is as follows. Each firm simultaneously determines the number and location of outlets in the first stage, and then they simultaneously select each mill price in the second stage. We know from Caplin and Nalebuff (1991) that there always exists a unique Nash price equilibrium in the second-stage price subgame under this setting, whereas there may be multiple equilibria in the first-stage location subgame. In this paper, we seek a sub-game perfect Nash equilibrium (SPNE).

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<sup>1</sup>Firms may establish more than two outlets when the additional costs of outlets are low. This may be likely in the case of a characteristic space such as the colors of clothes. We restrict our analysis to the case of two outlets because the overall results do not qualitatively differ much even if the firms are allowed to produce many varieties (i.e., establish many outlets).

Before examining the SPNE, define a Nash equilibrium in each stage. Because the maximum number of outlets is two, each firm has two strategies in each stage.

Nash price equilibrium in the second stage is a price system where no firm wants to change prices of its two outlets. More formally, it is defined by

$$\Pi_i(p_i^*, p_{-i}^*) \geq \Pi_i(p_i, p_{-i}^*) \quad \forall i = 1, \dots, n \quad (1)$$

where  $p_i \equiv (p_{i1}, p_{i2})$  is the price vector of firm  $i$ ,  $p_{-i}$  is the set of all price vectors except  $i$ , and  $p_{i1}$  and  $p_{i2}$  are the prices of firm  $i$ 's outlets 1 and 2.

The Nash location equilibrium in the first stage is a configuration where no firm wants either to relocate its outlets or to change their number. This is defined by

$$\Pi_i(x_i^*, x_{-i}^*) \geq \Pi_i(x_i, x_{-i}^*) \quad \forall x_{i1} \text{ and } i = 1, \dots, n \quad (2)$$

where  $x_i \equiv (x_{i1}, x_{i2})$  is the location vector of firm  $i$ ,  $x_{-i}$  is the set of all location vectors except  $i$ , and  $x_{i1}$  and  $x_{i2}$  are the locations of firm  $i$ 's outlets 1 and 2. The number of outlets of firm  $i$  is one if  $x_{i1} = x_{i2}$ , and two if  $x_{i1} \neq x_{i2}$ .

### 3 Consumer distribution over a circumference of a circle

In this section, consumers are uniformly distributed over the circumference of a circle with a unit length, and in the next section, they are uniformly distributed over a disk.

#### 3.1 Duopoly

We first consider the benchmark case of two firms. Martinez-Giralt and Neven (1988) show that firms do not open multiple outlets in a duopoly and, further, that both firms establish one outlet at opposite ends of a diameter of a circle, which is unique up to rotation.<sup>2</sup>

Firms locate as far as possible from each other and do not open multiple outlets in order to mitigate the intense price competition. This implies that relaxing product differentiation is maximized in the characteristic space in the case of duopoly. In other

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<sup>2</sup>If consumers are uniformly distributed over a line segment, then firms locate at the opposite ends of the line segment (Neven, 1985).

words, under duopolistic location-then-price competition, price competition takes precedence over increasing the market reach.

## 3.2 Triopoly

There are three firms rather than two firms in the market. They can establish one or two outlets on the circumference of a circle. The location space is denoted by  $x \in [0, 1)$ .

If only one outlet is allowed in the case of triopoly, there exists a unique SPNE up to rotation given by  $(x_a, x_b, x_c) = (1/3, 2/3, 3/3)$ .<sup>3</sup> This is because the best locational reply of each firm in the first stage is shown to be a midpoint of the neighboring firms. Therefore, in the case of a single outlet, the principle of maximum differentiation holds both in a duopoly and triopoly.

However, in contrast to the case of a duopoly, we can show that firms have an incentive to open multiple outlets in the case of a triopoly. In order to see this, consider that firm  $a$  opens the second outlet. Then, we can easily show that establishing the second outlet at  $x_{a2} \in (1/3 - 0.041, 1/3 + 0.041)$  raises  $a$ 's profit. Insofar as a firm has an incentive to open multiple outlets, the single outlet triopoly  $(x_a, x_b, x_c) = (1/3, 2/3, 3/3)$  is not an SPNE.

It can be verified that there are five equilibrium candidates for spatial arrangements up to rotation

$$\left\{ \begin{array}{ll} \text{interlacing} & a_1 b_1 c_1 a_2 b_2 c_2 \\ \text{partial segmentation} & a_1 a_2 c_1 b_1 b_2 c_2 \\ \text{segmentation} & a_1 a_2 b_1 b_2 c_1 c_2 \\ \text{quasi-interlacing} & a_1 b_1 c_1 a_2 c_2 b_2 \\ \text{quasi-partial segmentation} & a_1 a_2 b_1 c_1 b_2 c_2 \end{array} \right.$$

Each spatial arrangement seems to yield the unique location of two outlets of each firm, obtained by using the damped Newton's method in Mathematica calculations with several initial values of locations. However, it can be shown that the last two arrangements are not Nash equilibria. quasi-interlacing is unstable because withdrawing one outlet of firm  $a$  raises its profit. Quasi-partial segmentation is unstable because relocating an outlet of

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<sup>3</sup>Brenner (2005) shows that the SPNE configuration of firms is given by  $(x_a^*, x_b^*, x_c^*) = (1/8, 4/8, 7/8)$  in the case of a unit line segment.

firm  $a$  to location, say,  $(c_1 + b_2)/2$  raises its profit. The remaining first three candidates are shown to be SPNE from the Appendix. Hence, we establish the following.

**Proposition 1** *The following configurations are SPNE:<sup>4</sup>*

(i) *interlacing*:  $(x_{a1}^*, x_{b1}^*, x_{c1}^*, x_{a2}^*, x_{b2}^*, x_{c2}^*) = (\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6})$  and  $\Pi_i^* = t/108$  for  $i = a, b, c$ .

(ii) *partial segmentation*:  $(x_{a1}^*, x_{a2}^*, x_{c1}^*, x_{b1}^*, x_{b2}^*, x_{c2}^*) = (\frac{3}{16}, \frac{5}{16}, \frac{8}{16}, \frac{11}{16}, \frac{13}{16}, \frac{16}{16})$  and  $\Pi_a^* = \Pi_b^* = 169t/12288 < 242t/12288 = \Pi_c^*$ .

(iii) *segmentation*:  $(x_{a1}^*, x_{a2}^*, x_{b1}^*, x_{b2}^*, x_{c1}^*, x_{c2}^*) = (\frac{1}{6}, \frac{1}{6} + r, \frac{3}{6}, \frac{3}{6} + r, \frac{5}{6}, \frac{5}{6} + r)$  and  $\Pi_i^* = (2 - 3r)t/54$  for  $i = a, b, c$ , where  $r = (11 - \sqrt{73})/18 \simeq 0.136$ .

The proof of Proposition 1 is contained in the Appendix. This proposition shows multiple equilibria in the first-stage location subgame, whereas there is a unique equilibrium in the second-stage price subgame. The interlacing configuration is socially optimum and the distances are maximized. However, this is not true for the other two configurations: firms neither maximize nor minimize product differentiation in characteristic competition.

Presume that the fourth firm enters and opens one outlet when the multi-outlet incumbents are located as in Proposition 1. Then, this profit maximizing firm would locate its outlet at  $x = 1/12$ ,  $1/6 + r/2$  and  $3/32$  in cases (i), (ii), and (iii), respectively. It can be shown that the profit of the entrant is the smallest ( $0.0014t$ ) in case (i) of interlacing configuration and largest ( $0.0031t$ ) in case (iii) of the segmentation. This is parallel to the profits of the incumbents, which are the smallest ( $0.0093t$ ) in the interlacing case and largest ( $0.0295t$ ) in the segmentation case. We may therefore state that *keen competition reduces the profits of incumbents but prevents the entry of new firms*.

If there exists a fixed cost for entry, which is slightly larger than  $0.0014t$ , then both the segmentation and the partial segmentation are vulnerable to the entry of the fourth firm, whereas the interlacing is not. In other words, *the interlacing configuration with keener competition is more likely to appear than the others*.

The results of Proposition 1 may be extended to a larger number of outlets. If each firm is allowed to open a maximum of three outlets, then we can verify that both interlacing

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<sup>4</sup>In the case of a line segment, we numerically confirm that interlacing  $(a_1b_1c_1a_2b_2c_2)$  and partial segmentation  $(a_1a_2c_1b_1b_2c_2)$  are SPNE, whereas other configurations are not.

$(x_{a1}^*, x_{b1}^*, x_{c1}^*, x_{a2}^*, x_{b2}^*, x_{c2}^*, x_{a3}^*, x_{b3}^*, x_{c3}^*)$  and segmentation  $(x_{a1}^*, x_{a2}^*, x_{a3}^*, x_{b1}^*, x_{b2}^*, x_{b3}^*, x_{c1}^*, x_{c2}^*, x_{c3}^*)$  are SPNE.

## 4 Consumer distribution over a disk

### 4.1 Duopoly

Geographical space is two-dimensional rather than one-dimensional, and the characteristic space is also multi-dimensional in reality. Therefore, it is of interest to consider a unit disk  $\{(x, y) : x^2 + y^2 \leq 1\}$  over which consumers are uniformly distributed keeping the other assumptions the same as those in the benchmark case.

Tabuchi (1994) shows that in the SPNE, two firms with one outlet each locate at the opposite sides of the circumference of a disk  $(x_a^*, y_a^*) = (-1, 0)$ ,  $(x_b^*, y_b^*) = (1, 0)$  up to rotation. This implies that firms maximize differentiation in one dimension ( $x$  axis), whereas they minimize differentiation in another dimension ( $y$  axis). This is the so-called max-min principle of product differentiation: firms maximize differentiation only in one dimension ( $x$  axis) while minimize differentiation in all other dimensions. This is also shown to hold for three dimensions by Ansari, Economides, and Steckel (1997) and for an arbitrary number of dimensions by Irmen and Thisse (1998).

### 4.2 Triopoly

What if there are three firms with one outlet each on the disk in section 4.1? Because the two-dimensional analysis with more than two firms is mathematically complicated, we seek a symmetric equilibrium. For this purpose, assume that the locations of firms  $a$ ,  $b$  and  $c$  are restricted to

$$(x_a, y_a) = (0, z_a), \quad (x_b, y_b) = (-\sqrt{3}z_b/2, -z_b/2), \quad (x_c, y_c) = (\sqrt{3}z_c/2, -z_c/2) \quad (3)$$

where  $z_i \in [0, 1]$  for  $i = a, b, c$  as depicted in Figure 1. Let  $p_i$  be the mill price of a good by firm  $i$ . Then, there are three market boundary lines

$$p_a + (x - x_a)^2 + (y - y_a)^2 = p_b + (x - x_b)^2 + (y - y_b)^2 \quad (4)$$

$$p_b + (x - x_b)^2 + (y - y_b)^2 = p_c + (x - x_c)^2 + (y - y_c)^2 \quad (5)$$

$$p_c + (x - x_c)^2 + (y - y_c)^2 = p_a + (x - x_a)^2 + (y - y_a)^2 \quad (6)$$



with a common intersection point  $(x_0, y_0)$  given by simultaneously solving two of the above three equations. Let  $(x_1, y_1)$  be the upper intersection point between (4) and the circumference of the unit circle,  $(x_2, y_2)$  is the lower intersection point between (5) and the circumference, and  $(x_3, y_3)$  is the upper intersection point between (6) and the circumference. Then, firm  $a$ 's demand  $D_a$  is determined by the area of the circular sector with vertices  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_3, y_3)$ ; firm  $b$ 's demand  $D_b$  by that with  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$ ; and firm  $c$ 's demand  $D_c$  by that with  $(x_0, y_0)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ .

Hence, the profit of firm  $i = a, b, c$  is given by

$$\Pi_i(p_a, p_b, p_c, z_a, z_b, z_c) = p_i D_i$$

Each firm  $i$  maximizes its profit with respect to its price  $p_i$ . Unlike the one-dimensional case, one cannot obtain a closed form solution of  $p_i$  from the first-order conditions:

$$\frac{\partial \Pi_i}{\partial p_i} = 0 \quad \text{for } i = a, b, c \quad (7)$$

However, we know from Caplin and Nalebuff (1991) that there always exists a unique Nash price equilibrium  $p_i^*$  in the second-stage price subgame because the demand is convex.

In the first-stage location subgame, firm  $i$  maximizes its profit with respect to  $z_i$  given the unique equilibrium prices  $p_i^*(z_a, z_b, z_c)$ . The first-order condition is

$$\begin{aligned} \frac{d\Pi_i}{dz_i} &= \frac{\partial \Pi_i}{\partial z_i} + \sum_{j=a,b,c} \frac{\partial \Pi_i}{\partial p_j} \frac{\partial p_j}{\partial z_i} \\ &= \frac{\partial \Pi_i}{\partial z_i} + \sum_{j \neq i} \frac{\partial \Pi_i}{\partial p_j} \frac{\partial p_j}{\partial z_i} = 0 \quad \text{for } i = a, b, c \end{aligned} \quad (8)$$

where the second equality is due to (7). The partial derivatives  $\partial p_j / \partial z_i$  can be obtained by applying the implicit function theorem to (7) as follows:

$$\begin{pmatrix} \frac{\partial p_a}{\partial z_i} \\ \frac{\partial p_b}{\partial z_i} \\ \frac{\partial p_c}{\partial z_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_a}{\partial p_a} & \frac{\partial f_a}{\partial p_b} & \frac{\partial f_a}{\partial p_c} \\ \frac{\partial f_b}{\partial p_a} & \frac{\partial f_b}{\partial p_b} & \frac{\partial f_b}{\partial p_c} \\ \frac{\partial f_c}{\partial p_a} & \frac{\partial f_c}{\partial p_b} & \frac{\partial f_c}{\partial p_c} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial f_a}{\partial z_i} \\ \frac{\partial f_b}{\partial z_i} \\ \frac{\partial f_c}{\partial z_i} \end{pmatrix}$$

where  $f_i \equiv \partial \Pi_i / \partial p_i$  is defined by (7).

The symmetric equilibrium candidate can be computed by solving (7) and (8) simultaneously and evaluating a symmetric solution  $z_i = z$  and  $p_i = p$  for  $i = a, b, c$ .

Straightforward but tedious calculations yield a candidate for symmetric equilibrium as follows:

$$\begin{aligned} z_i^* &= \frac{2592+612\sqrt{3}-27\pi^2-2\sqrt{3}\pi^3}{24(432-\pi^2)} \approx 0.548 \\ p_i^* &= \frac{864\sqrt{3}+612\pi-9\sqrt{3}\pi^2-2\pi^3}{24(432-\pi^2)}\pi t \approx 0.993t \\ \Pi_i^* &= \frac{864\sqrt{3}+612\pi-9\sqrt{3}\pi^2-2\pi^3}{72(432-\pi^2)}\pi^2 t \approx 1.040t \end{aligned} \quad (9)$$

The sufficient conditions for Nash location equilibrium can be ascertained locally by checking the second-order conditions at  $z_i = z_i^*$  and  $p_i = p_i^*$ . However, one can check the sufficient conditions globally by numerical analysis in the following two steps. First, calculate  $p_a$  and  $p_b(=p_c)$  by numerically solving  $\partial\Pi_a/\partial p_a = 0$  and  $\partial\Pi_b/\partial p_b = 0$  in (7) for  $z_a = 0, 0.01, 0.02, \dots, 1$  given the values of  $z_b = z_c = z_i^*$ . Note that the unique price equilibrium is always guaranteed. Second, by plugging the equilibrium prices into  $\Pi_a$ , we can express  $\Pi_a$  as a function of  $z_a$  as drawn in Figure 2. Because the profit  $\Pi_a$  is obviously single-peaked at  $z_a = z_i^*$ , one can confirm that (9) is the unique SPNE under the location constraints (3).

Before interpreting the SPNE outcome, we should determine whether there are any other equilibria without the location constraints (3). One possible candidate for SPNE is axisymmetric location:

$$(x_a, y_a) = (-\widehat{z}, 0), \quad (x_b, y_b) = (0, 0), \quad (x_c, y_c) = (\widehat{z}, 0)$$

In order to verify this, assume that the locations of the three firms are restricted to the  $x$ -axis, such that

$$(x_a, y_a) = (z_a, 0), \quad (x_b, y_b) = (z_b, 0), \quad (x_c, y_c) = (z_c, 0)$$

where  $-1 \leq z_a \leq z_b \leq z_c \leq 1$ . The market boundaries are given by lines parallel to the  $y$ -axis. Conducting similar computations with axisymmetric location  $z_a = -z_c = -\widehat{z}$ , we have a unique candidate for SPNE:

$$\begin{aligned} (\widehat{z}_a, \widehat{z}_b, \widehat{z}_c) &\approx (-0.322, 0, 0.322) \\ (\widehat{p}_a, \widehat{p}_b, \widehat{p}_c) &\approx (0.319t, 0.215t, 0.319t) \\ (\widehat{\Pi}_a, \widehat{\Pi}_b, \widehat{\Pi}_c) &\approx (0.299t, 0.273t, 0.299t) \end{aligned}$$

However, this is not a SPNE because firm  $b$  has an incentive to relocate: if it moves from the center  $(x_b, y_b) = (0, 0)$  to the periphery  $(x_b, y_b) = (0, 1)$ , then its profit will rise from  $0.273t$  to  $0.439t$ . The last possibilities are asymmetric configurations, which

are unlikely to be equilibria given the uniform distribution of consumers. Hence, the symmetric configuration (9) seems to be the unique SPNE.

Several remarks are in order here. First, the equilibrium locations of the firms are inside the disk in the case of triopoly, whereas they are on the edges in the case of duopoly.<sup>5</sup> This suggests that triopolists locate themselves closer in order to gain the market area at a cost of tough price competition because location competition for market area is more important for their profits. On the other hand, duopolists locate themselves apart in order to relax price competition, which is more important for their profits than location competition.

Second, the optimum locations of firms can be determined by minimizing the sum of the transport costs. It can be readily shown that the optimal location is  $z_i^o = \sqrt{3}/\pi \approx 0.551 > 0.548 = z_i^*$ . That is, firms tend to locate themselves close to each other relative to the social optimum, implying that the location competition effect given by the first term in (8) is stronger than the price competition effect given by the second term in (8), as compared to the case of duopoly. Note, however, that three firms tend to locate themselves apart as relative to the social optimum in the linear case ( $x_2^o - x_1^o = 1/3 < 3/8 = x_2^* - x_1^*$ ). This implies that price competition is mitigated relative to location competition in the two-dimensional space. Put differently, location competition is important relative to price competition in the two-dimensional space because larger demand can be obtained by locating closer to the center in the two-dimensional space as compared to the one-dimensional space.

Third, because the equilibrium locations of firms are neither at the edges nor the center, the max-min principle of product differentiation, which is true in the case of duopoly (Tabuchi, 1994; Ansari, Economides and Steckel, 1997; Irmen and Thisse, 1998), no longer holds at all in the case of triopoly. The max-min principle of product differentiation itself is impossible for more than two firms and is nothing but an artifact for duopoly. Therefore, one may conclude that contrary to the case of duopoly, *Hotelling was not almost right*

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<sup>5</sup>Such different outcomes do not arise in the previous section because the space on the circumference of a circle is homogeneous. On the other hand, the space on a disk or a linear segment is heterogeneous because access to consumers differ between the center and peripheries. In other words, a closer location between three or more firms arises owing to the existence of centrality. Note that for the emergence of multiple outlets, three or more firms are needed, but centrality is not necessary, as we saw in section 3.2.

once there are three firms in the market.

Fourth, the qualitatively similar outcome can also be numerically confirmed in the case of quadropoly. The symmetric equilibrium candidate is unique and given by

$$\begin{aligned} z_i^* &= \frac{80+4\pi-\pi^2}{4\sqrt{2}(40-3\pi)} \approx 0.478 \\ p_i^* &= \frac{80+4\pi-\pi^2}{16(40-3\pi)} \pi t \approx 0.531t \\ \Pi_i^* &= \frac{80+4\pi-\pi^2}{64(40-3\pi)} \pi^2 t \approx 0.417t \end{aligned}$$

for  $i = a, b, c, d$ .<sup>6</sup> Observe that (i) the firms' locations are inside the disk:  $z_i^* < 1$ , (ii) firms tend to locate close as compared to the social optimum:  $z_i^o = 4\sqrt{2}/3\pi \approx 0.600 > 0.478 \approx z_i^*$ , and (iii) the max-min principle of product differentiation does not hold.

Finally, consider whether triopolists on a disk have an incentive to open multiple outlets. In order to simplify the analysis, fix the locations of firms  $b$  and  $c$  as

$$(x_b, y_b) = (-\sqrt{3}z_i^*/2, -z_i^*/2), \quad (x_c, y_c) = (\sqrt{3}z_i^*/2, -z_i^*/2)$$

and split the outlet of firm  $a$  horizontally as

$$(x_{a1}, y_{a1}) = (-e, z_i^*), \quad (x_{a2}, y_{a2}) = (e, z_i^*)$$

where  $z_i^* \approx 0.548$  is given by (9).

Then, it can be verified that

$$\frac{d\Pi_a}{de} = \frac{\partial\Pi_a}{\partial e} + \sum_{j=b,c} \frac{\partial\Pi_a}{\partial p_j} \frac{\partial p_j}{\partial e} \approx 0.09t > 0$$

which implies that firm  $a$  has an incentive to split its outlet into two. Due to the locational symmetry, each firm has an incentive to establish multiple outlets.

## 5 Average distance

Figure 3 illustrates the locations of convenience stores for 2009 inside the red circle with radius one kilometer around Shinjuku station in Tokyo. There are five firms—Seven-Eleven, Lawson, FamilyMart, am/pm, and CircleKSunkus—with 14, 13, 12, 19, and 15

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<sup>6</sup>There is another location equilibrium candidate: firms  $a$ ,  $b$ , and  $c$  locate symmetrically about the center, and firm  $d$  locates at the center; distances from the location of firm  $d$  and the remaining firms are  $z \approx 0.465$ . However, this is not an SPNE because firm  $d$  can raise its profit by relocating to an edge that is farther away from the competitors.

stores, respectively. Similar location configurations are observed around Shibuya station in Tokyo, Sakae station in Nagoya, and Nanba station in Osaka.

There is no doubt that convenience stores are typically multi-outlet spatial oligopolists. Although convenience stores sell a wide variety of products, their selections of varieties are quite similar. Therefore, goods are more or less homogeneous, which suggests that Hotelling's spatial competition may apply. Convenience stores compete in the number and location of outlets. They normally sell goods at regular prices. However, they often compete in price in terms of, for example, discounts on plastic bottles of water.

In order to see whether the locations of convenience stores are interlaced or segmented, we compute the ratio of the average distance between outlets belonging to the same firm,  $d_s$ , to that belonging to different firms,  $d_d$ . We refer to  $d_s/d_d$  as the degree of mixing. By measuring the distances between convenience stores around Shinjuku, Shibuya, Sakae, and Nanba, we obtain the degree of mixing  $d_s/d_d = 0.99, 0.98, 1.04,$  and  $1.03,$  respectively.

Whether the values close to 1 imply interlacing or segmentation can be determined by computing the degree of mixing in each configuration in Proposition 1 in section 3.2. It can be readily shown that the degree of mixing is the largest ( $d_s/d_d = 2$ ) in the case of interlacing configuration because  $d_s = 1/2$  and  $d_d = 1/4$ . It is an intermediate value ( $d_s/d_d = 0.8$ ) in the case of partial segmentation and the smallest value ( $d_s/d_d = 0.2$ ) in the case of segmentation.

Because the values of the degree of mixing are intermediate, the spatial configurations of convenience stores around the four big stations in Japan are neither interlaced nor segmented. They are mid-way between the two configurations.

## 6 Conclusion

We revisited Hotelling's location-then-price competition by considering uniform distributions of consumers over a circumference of a circle and a disk in order to settle the debates on brand proliferation and max-min principle of product differentiation.

Comparing the market outcomes between duopoly and oligopoly with three or more firms, we have shown that firms proliferate brands in the latter but not in the former, and that firms neither maximize nor minimize product differentiation in the latter but not in the former. We may therefore conclude that duopoly substantially differs from oligopoly

with three or more firms.

We also conducted a brief empirical analysis in order to find the difference between theory and reality. Computing the degree of mixing, we have shown that the spatial configurations of convenience stores near big stations in Japan are partially segmented.

However, much work remains to be done in order to fully settle the above two debates. Our analysis has dismissed the sequential entry of firms. If firm entry is sequential under perfect foresight (Prescott and Visscher, 1977) in the setting of section 3.2, then market segmentation is an SPNE, which ensures higher profits for all firms according to preliminary simulations. We have also dismissed endogenizing the number of outlets, which depends on the fixed costs of entry. These market outcomes are perhaps more complicated but, nevertheless, richer and may better fit the reality.

## Appendix: Proof of Proposition 1

Firm  $i = a, b$ , and  $c$  establishes outlets  $i_1$  and  $i_2$  at  $x = x_{i1}, x_{i2}$ . The number of outlets of firm  $i$  is 1 if  $x_{i1} = x_{i2}$  and 2 if  $x_{i1} \neq x_{i2}$ . Suppose outlets of firm  $i$  are located such that  $x_j < x_{i1} < x_k$  and  $x_l < x_{i2} < x_m$ , where  $j, k, l, m$  could be  $i_1$  or  $i_2$ . Then, the full prices of the good in the visiting outlets  $i$  and  $j$  are equal at location  $\hat{x}_{ab}$  of marginal consumers:

$$p_{i1} + (\hat{x}_{ij} - x_{i1})^2 = p_j + (x_j - \hat{x}_{ij})^2$$

which leads to the market boundary between outlets  $i$  and  $j$

$$\hat{x}_{ij} = \frac{p_j - p_{i1}}{2(x_j - x_{i1})} + \frac{x_j + x_{i1}}{2}$$

Marginal consumers at  $\hat{x}_{ij}$  are indifferent between visiting outlets  $i$  and  $j$ . The other market boundaries are similarly computed. Then, the profit of firm  $i$  is defined as

$$\Pi_i = p_{i1}(\hat{x}_{ik} - \hat{x}_{ij}) + p_{i2}(\hat{x}_{il} - \hat{x}_{im})$$

Because  $\Pi_i$  is quadratic and concave in  $p_{i1}$  and  $p_{i2}$ , the first-order condition is linear in  $p_{i1}$  and  $p_{i2}$ , ensuring that the unique equilibrium prices are explicitly obtained. Plugging the equilibrium prices into the profits, they can be expressed as functions of locations  $x_{a1}, x_{a2}, x_{b1}, x_{b2}, x_{c1}$  and  $x_{c2}$ . Solving the first-order conditions with respect to locations yields the necessary conditions for SPNE.

There are five SPNE candidates up to rotation and permutation:

$$\left\{ \begin{array}{ll} \text{interlacing} & (x_{a1}, x_{b1}, x_{c1}, x_{a2}, x_{b2}, x_{c2}) = \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6}\right) \\ \text{segmentation} & (x_{a1}, x_{a2}, x_{b1}, x_{b2}, x_{c1}, x_{c2}) = \left(\frac{1}{6}, \frac{1}{6} + r, \frac{3}{6}, \frac{3}{6} + r, \frac{5}{6}, \frac{5}{6} + r\right) \\ \text{partial segmentation} & (x_{a1}, x_{a2}, x_{c1}, x_{b1}, x_{b2}, x_{c2}) = \left(\frac{3}{16}, \frac{5}{16}, \frac{8}{16}, \frac{11}{16}, \frac{13}{16}, \frac{16}{16}\right) \\ \text{quasi-partial segmentation} & (x_{a1}, x_{b1}, x_{c1}, x_{c2}, x_{a2}, x_{b2}) = (r_1, r_2, r_3, 1 - r_3, 1 - r_2, 1 - r_1) \\ \text{quasi-interlacing} & (x_{a1}, x_{b1}, x_{c1}, x_{b2}, x_{a2}, x_{c2}) = \left(r_4, r_5, \frac{1}{2}, \frac{1}{2} + r_4, \frac{1}{2} + r_5, 1\right) \end{array} \right.$$

where  $r = (11 - \sqrt{73})/18 \simeq 0.136$ ,  $r_1 \simeq 0.120$ ,  $r_2 \simeq 0.313$ ,  $r_3 \simeq 0.462$ ,  $r_4 \simeq 0.182$ , and  $r_5 \simeq 0.318$ . In the following sections, the first three candidates are shown to be SPNE, whereas the last two are not.

## A.1 Interlacing

Because this is a perfectly symmetric configuration, it is obvious that the first-order conditions of the location competition are met. For indicating SPNE, we show that  $(x_{a1}, x_{a2}) = (1/6, 4/6)$  is the maximizer of  $\tilde{\Pi}_a(x_{a1}, x_{a2})$  in the intervals of  $x_{a1} \in [0, 2/6]$  and  $x_{a2} \in [3/6, 5/6]$  given  $(x_{b1}, x_{c1}, x_{b2}, x_{c2}) = (2/6, 3/6, 5/6, 6/6)$ .

Let  $\tilde{\Pi}_i$  be the profit of firm  $i$  after plugging the equilibrium prices solved in the second stage. Because the Nash location equilibrium conditions in the first stage are  $d\tilde{\Pi}_i/dx_{i1} = d\tilde{\Pi}_i/dx_{i2} = 0$  for  $i = a, b, c$ , we have

$$\frac{d\tilde{\Pi}_a}{dx_{a1}} - \frac{d\tilde{\Pi}_a}{dx_{a2}} = \frac{g(x_{a1}, x_{a2})(2x_{a2} - 2x_{a1} - 1)}{h(x_{a1}, x_{a2})} = 0 \quad (1)$$

where

$$\begin{aligned} g(x_{a1}, x_{a2}) \equiv & 108241920x_{a2}^3x_{a1}^7 - 216483840x_{a2}^2x_{a1}^7 + 136525824x_{a2}x_{a1}^7 - 26873856x_{a1}^7 + 36870930432x_{a2}^6x_{a1}^6 \\ & - 147483721728x_{a2}^5x_{a1}^6 + 235828537344x_{a2}^4x_{a1}^6 - 192013576704x_{a2}^3x_{a1}^6 + 83907332352x_{a2}^2x_{a1}^6 - 18819385344x_{a2}x_{a1}^6 \\ & + 1715452416x_{a1}^6 - 36870930432x_{a2}^6x_{a1}^5 + 149258889216x_{a2}^5x_{a1}^5 - 241745762304x_{a2}^4x_{a1}^5 + 199335002112x_{a2}^3x_{a1}^5 \\ & - 88030672128x_{a2}^2x_{a1}^5 + 19878916032x_{a2}x_{a1}^5 - 1823454336x_{a1}^5 + 5385222144x_{a2}^6x_{a1}^4 - 23020194816x_{a2}^5x_{a1}^4 \\ & + 39280267008x_{a2}^4x_{a1}^4 - 33944479488x_{a2}^3x_{a1}^4 + 15571376640x_{a2}^2x_{a1}^4 - 3624103200x_{a2}x_{a1}^4 + 347479024x_{a1}^4 \\ & + 108241920x_{a2}^7x_{a1}^3 + 2732673024x_{a2}^6x_{a1}^3 - 11952914688x_{a2}^5x_{a1}^3 + 19689827328x_{a2}^4x_{a1}^3 - 16320934080x_{a2}^3x_{a1}^3 \\ & + 7224779904x_{a2}^2x_{a1}^3 - 1625057936x_{a2}x_{a1}^3 + 144034880x_{a1}^3 - 54120960x_{a2}^7x_{a1}^2 + 31041792x_{a2}^6x_{a1}^2 + 551311488x_{a2}^5x_{a1}^2 \\ & - 1409446080x_{a2}^4x_{a1}^2 + 1451289696x_{a2}^3x_{a1}^2 - 727687440x_{a2}^2x_{a1}^2 + 173715240x_{a2}x_{a1}^2 - 15964948x_{a1}^2 + 1223424x_{a2}^7x_{a1} \\ & - 187612416x_{a2}^6x_{a1} + 763132032x_{a2}^5x_{a1} - 1264670400x_{a2}^4x_{a1} + 1080729904x_{a2}^3x_{a1} - 500962800x_{a2}^2x_{a1} \\ & + 119644672x_{a2}x_{a1} - 11463652x_{a1} + 798336x_{a2}^7 - 15372864x_{a2}^6 + 46260096x_{a2}^5 - 56503376x_{a2}^4 + 32255720x_{a2}^3 \\ & - 7983148x_{a2}^2 + 475172x_{a2} + 69217 \\ h(x_{a1}, x_{a2}) \equiv & 18(3024x_{a2}^2x_{a1}^2 - 4032x_{a2}x_{a1}^2 + 1056x_{a1}^2 - 1008x_{a2}^2x_{a1} + 1368x_{a2}x_{a1} - 368x_{a1} - 204x_{a2}^2 + 268x_{a2} - 71)^3 \end{aligned}$$

Because  $h(x_{a1}, x_{a2})$  is always positive, the numerator of (1) should be zero for Nash location equilibrium. We show  $g(x_{a1}, x_{a2}) \neq 0$  below and conclude that  $2x_{a2} - 2x_{a1} - 1 = 0$  holds.

Because  $g(x_{a1}, x_{a2})$  is the 7th order polynomial with respect to  $x_{a1}$ , the sign of  $\partial^6 g / \partial x_{a1}^6$  does not change. Because  $\partial^6 g(0, x_{a2}) / \partial x_{a1}^6 < 0$  and  $\partial^6 g(2/6, x_{a2}) / \partial x_{a1}^6 < 0$ , we have  $\partial^6 g / \partial x_{a1}^6 < 0$  for all  $x_{a1} \in [0, 2/6]$ . Hence, the sign of  $\partial^3 g / \partial x_{a1}^3$  changes a maximum of three times. However, because  $\partial^3 g(0, x_{a2}) / \partial x_{a1}^3 < 0$  and  $\partial^3 g(2/6, x_{a2}) / \partial x_{a1}^3 > 0$ , the sign of  $\partial^3 g / \partial x_{a1}^3$  changes exactly once from negative to positive. Hence, the sign of  $\partial g / \partial x_{a1}$  changes a maximum of three times. Nevertheless, because  $\partial g(0, x_{a2}) / \partial x_{a1} > 0$  and  $\partial g(2/6, x_{a2}) / \partial x_{a1} < 0$ , the sign of  $\partial g / \partial x_{a1}$  changes exactly once from positive to negative. However, because  $g(0, x_{a2}) > 0$  and  $g(2/6, x_{a2}) > 0$ ,  $g(x_{a1}, x_{a2}) > 0$  holds for all  $x_{a1} \in [0, 2/6]$ .

Therefore, (1) implies  $x_{a2} = x_{a1} + 1/2$ . Substituting this into  $d\tilde{\Pi}_a/dx_{a1} = 0$ , we can show that  $(x_{a1}, x_{a2}) = (1/6, 4/6)$  is the unique solution of  $d\tilde{\Pi}_a/dx_{a1} = d\tilde{\Pi}_a/dx_{a2} = 0$ . Furthermore, we can verify the second-order conditions for local maximum. Putting these results together, we arrive at  $(x_{a1}, x_{a2}) = (1/6, 4/6)$  as the global maximizer of  $\Pi_a^*(x_{a1}, x_{a2})$ . Finally, we can verify that withdrawing one of the two outlets decreases the profit of firm  $a$ .

Moreover, because the same can be applicable for  $i = b, c$ ,  $(x_{a1}, x_{b1}, x_{c1}, x_{a2}, x_{b2}, x_{c2}) = (1/6, 2/6, 3/6, 4/6, 5/6, 6/6)$  is the unique interlacing configuration that is an SPNE up to permutation. By plugging these values, into prices and profits, we have the equilibrium prices and profits given in Proposition 1.

## A.2 Partial segmentation

The first-order conditions of the location competition are readily confirmed. Similar to the interlacing case, we show that  $(x_{a1}, x_{a2}) = (3/16, 5/16)$  is the maximizer of  $\tilde{\Pi}_a$  for all  $0 \leq x_{a1} \leq x_{a2} \leq 8/16$  given  $(x_{c1}, x_{b1}, x_{b2}, x_{c2}) = (8/16, 11/16, 13/16, 16/16)$  and that  $(x_{c1}, x_{c2}) = (8/16, 16/16)$  is the maximizer of  $\tilde{\Pi}_c$  in the intervals of  $x_{c1} \in [0, 3/16]$  and  $x_{c2} \in [5/16, 11/16]$  given  $(x_{a1}, x_{a2}, x_{b1}, x_{b2}) = (3/16, 5/16, 11/16, 13/16)$

Computing  $d\tilde{\Pi}_a/dx_{a1} + d\tilde{\Pi}_a/dx_{a2} = 0$ , we get an expression similar to (1). Further, by identifying the derivatives up to the 4th order, we can show that  $(x_{a1}, x_{a2}) = (3/16, 5/16)$



is the unique solution of  $d\tilde{\Pi}_a/dx_{a1} = d\tilde{\Pi}_a/dx_{a2} = 0$ , and that the second-order conditions for local maximum are satisfied.

Likewise,  $(x_{c1}, x_{c2}) = (8/16, 16/16)$  is shown to be the unique solution of  $d\tilde{\Pi}_c/dx_{c1} = d\tilde{\Pi}_c/dx_{c2} = 0$ , and that the second-order conditions for local maximum are satisfied. We can also verify that dropping one of the two outlets decreases the profit of firm  $c$ .

Hence,  $(x_{a1}, x_{a2}, x_{c1}, x_{b1}, x_{b2}, x_{c2}) = (3/16, 5/16, 8/16, 11/16, 13/16, 16/16)$  is the unique partial segmentation configuration that is an SPNE up to permutation. The equilibrium prices and profits given in Proposition 1 are also shown.

### A.3 Segmentation

The first-order conditions of the location competition can be easily shown. We show that  $(x_{a1}, x_{a2}) = (1/6, 1/6 + r)$  is the maximizer of  $\tilde{\Pi}_a$  for all  $-1/6 + r \leq x_{a1} \leq x_{a2} \leq 3/6$  given  $(x_{b1}, x_{b2}, x_{c1}, x_{c2}) = (3/6, 3/6 + r, 5/6, 5/6 + r)$ .

Computing  $d\tilde{\Pi}_a/dx_{a1} + d\tilde{\Pi}_a/dx_{a2} = 0$ , we have an expression similar to (1). By identifying the derivatives up to the 4th order, we can similarly show that  $(x_{a1}, x_{a2}) = (1/6, 4/6)$  is the unique solution of  $d\tilde{\Pi}_a/dx_{a1} = d\tilde{\Pi}_a/dx_{a2} = 0$ , and that the second-order conditions for local maximum are satisfied. Hence,  $(x_{a1}, x_{a2}, x_{b1}, x_{b2}, x_{c1}, x_{c2}) = (1/6, 1/6 + r, 3/6, 3/6 + r, 5/6, 5/6 + r)$  is the unique segmentation configuration that is an SPNE up to permutation. The equilibrium prices and profits given in Proposition 1 are similarly shown.

### A.4 quasi-interlacing

Calculating the first-order conditions of location equilibrium for arrangement  $a_1 b_1 c_1 a_2 c_2 b_2$  by the damped Newton's method in Mathematica, we get  $(x_{a1}, x_{b1}, x_{c1}, x_{b2}, x_{a2}, x_{c2}) = (r_4, r_5, 1/2, 1/2 + r_4, 1/2 + r_5, 1)$ . However, this is not a Nash equilibrium because the profit of firm  $b$  increases upon withdrawing its first outlet.

### A.5 quasi-partial segmentation

Calculating the first-order conditions of location equilibrium for arrangement  $a_1 a_2 b_1 c_1 b_2 c_2$  by the damped Newton's method command in Mathematica, we have  $(x_{a1}, x_{b1}, x_{c1}, x_{c2}, x_{a2}, x_{b2}) =$

$(r_1, r_2, r_3, 1 - r_3, 1 - r_2, 1 - r_1)$ . However, this is not a Nash equilibrium because the profit of firm  $c$  increases upon relocating its first outlet from  $x_{c1} = r_3$  to  $x_{c1} = 0$ .

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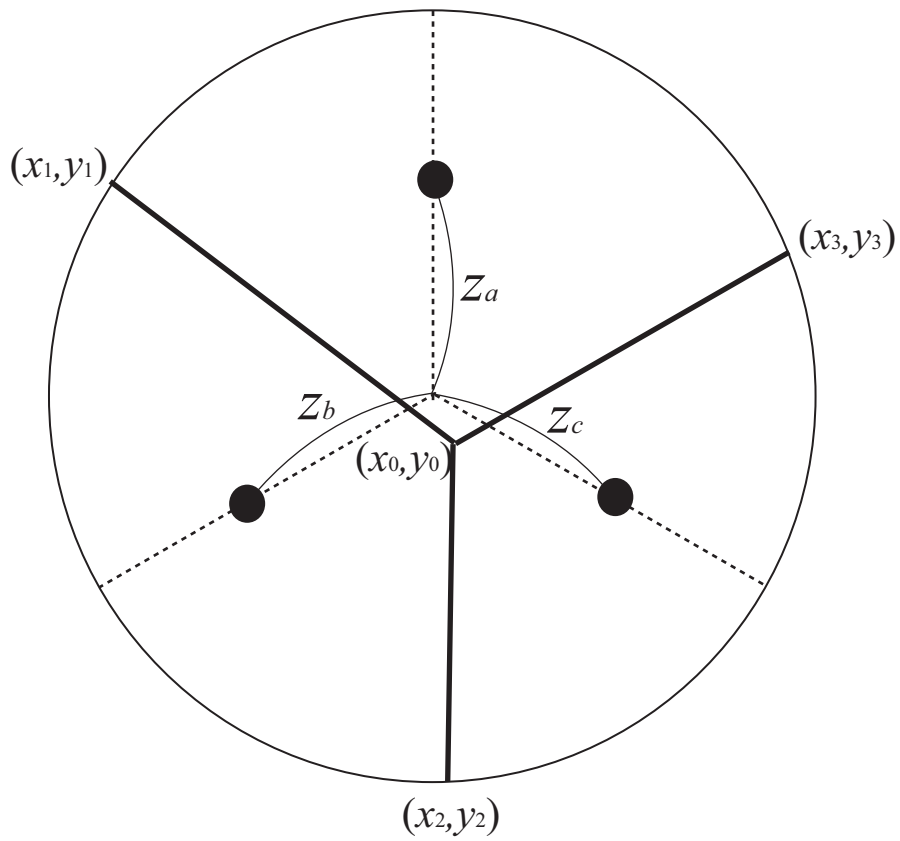


Figure 1: Three single-store firms on a disk

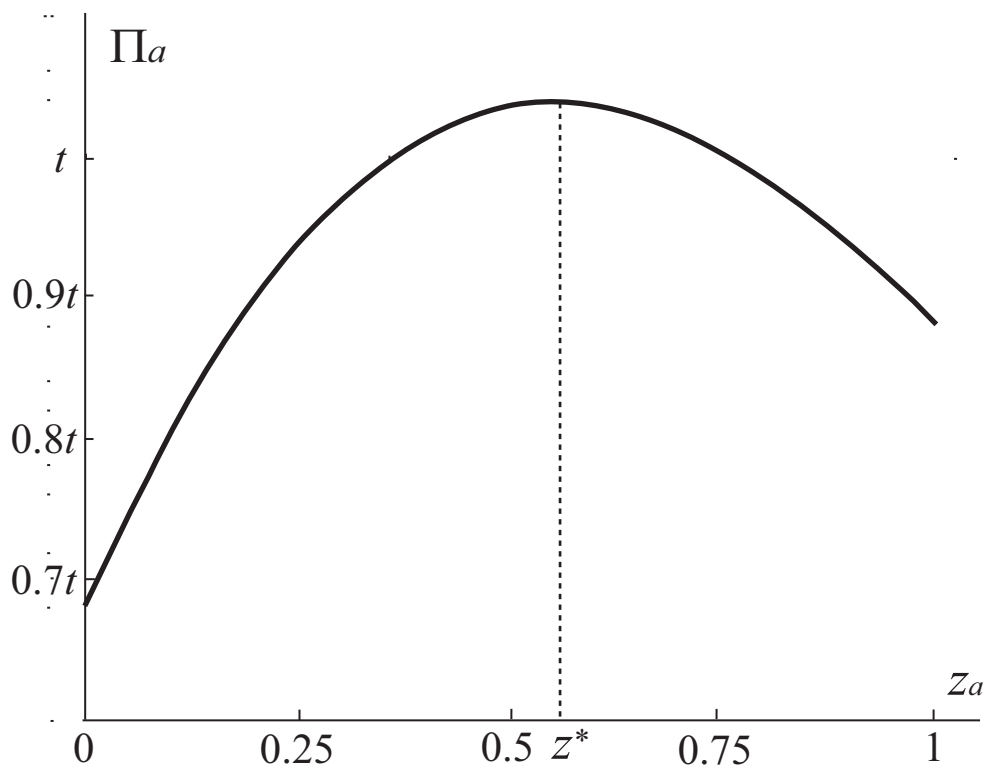


Figure 2: Firm  $a$ 's profit  $\Pi_a$  as a function of  $Z_a$



Figure 3: Convenience stores near Shinjuku station (blue: Seven-Eleven, red: Lawson, green: FamilyMart, yellow: am/pm, light blue: CircleKSunkus)