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Keita Nakayama
Graduate school of Economics, University of Tokyo

Akihiko Takahashi
University of Tokyo

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Keita Nakayama and Akihiko Takahashi
Graduate school of Economics, The University of Tokyo

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Abstract

This paper proposes a new method to a bond portfolio problem in a multi-period setting. In particular, we apply a factor allocation approach to constructing the optimal bond portfolio in a class of multi-factor Gaussian yield curve models. In other words, we consider a bond portfolio problem in terms of a factors' allocation problem. Thus, we can obtain clear interpretation about the relation between the change in the shape of a yield curve and dynamic optimal strategy, which is usually hard to be obtained due to high correlations among individual bonds.

We first present a closed form solution of the optimal bond portfolio in a class of the multi-factor Gaussian term structure model. Then, we investigate the effects of various changes in the term structure on the optimal portfolio strategy through series of comparative statics.

1 Introduction

In recent years, the fixed income security market has grown rapidly and research for the trading strategies are getting sophisticated. However, there are few researches on dynamic optimal portfolio with term structure models. In particular, the changes in a yield curve shape are rarely reflected in the optimal portfolio strategy. Moreover, the optimal strategies for bond portfolio problems are usually hard to be interpreted because of high correlations among individual bonds. As is often reported in principal component analysis (PCA) for the change in term structure, most of the variations of spot yields with different maturities can be explained by three common factors.

Thus, to avoid high correlations, it seems better to consider portfolio problems not in terms of bonds but in terms of common factors. There are various term structure models to evaluate bonds and interest rate derivatives. By combining these models with portfolio optimization, especially, applying multi-factor yield curve models to optimal bond portfolio problems, we can analyze bond portfolio in terms of factors, which enables us to interpret optimal strategies intuitively. For example, because spot rates can be expressed as a linear combination of factors in a multi-factor Gaussian model, we can easily obtain factors exposures in a bond optimal portfolio.

Hence, we introduce a factor allocation approach. First, we decompose each bond's return into several factors' return and then the allocation of bonds can be converted into that of factors. Using this idea, we can easily analyze the effects of the parameter changes in a yield curve on the optimal portfolio.

In this paper, we use a multi-factor Gaussian term structure model with stochastic mean and power utility function for terminal wealth. Then, we derive a closed form solution of the optimal strategy for a dynamic bond portfolio problem using a result of Takahashi and Yoshida [2004]: they presented an explicit expression of optimal portfolio in a general Markovian setting based on Ocone and Karatzas [1991].

There are several previous works related to this theme. Sørensen [1999] and Brennan and Xia [2000] consider the dynamic portfolio optimization problem in a structure with a power utility, single/two-factor stochastic interest rate model and a constant market price of risk in the complete market. Sørensen [1999] uses a Vasicek one-factor model for discrete time optimal portfolio allocation utilizing the quasi-dynamic programming approach. Brennan and Xia [2000] investigates the optimal stock-bond mix along with a two-factor interest rate model and derive comparative statics with respect to the risk aversion. Liu [2006] solves the dynamic portfolio problem using a single factor model of interest rates in a general structure such as a stochastic interest rate, a stochastic market price of risk and a stochastic volatility. Most studies, however, have not focused on the term structure and the optimal bond portfolio.

Kobayashi, Takahashi and Tokioka [2005] mainly uses a general single-factor HJM model, stochastic market price of risk, stochastic volatility and the power utility. They first formulate a dynamic optimal bond portfolio problem based on the general setting utilizing the asymptotic expansion scheme and investigate the effect of the change in the market price of risk. Korn and Koziol [2006] uses multi-factor term structure models of the Vasicek type to analyze bond portfolio optimization with the static Mean-Variance approach.

We consider the dynamic bond portfolio problem and present a closed-form optimal strategy with a model including Sørensen [1999], Brennan and Xia [2000] and Korn and Koziol [2006] as special cases and our result includes the solution of Brennan and Xia [2000] in the case in which all securities consist of bonds. Furthermore utilizing the term structure model effectively, we propose the idea of a factor allocation: the bond portfolio problem is reinterpreted as the factor allocation problem, which enables us to investigate the relation between the term structure and the optimal bond portfolio clearly. In particular, we implement comparative statics in detail with respect to parameters which affect the shape of the term structure.

The organization of this paper is as follows. In Section 2, after we briefly introduce a dynamic optimization problem for bond portfolio in a class of Gaussian term structure models, we derive a closed form solution of optimal strategy. In Section 3, we implement series of comparative statics to investigate the effects of changes in the yield curve shape on the optimal portfolio strategy. Finally, Section 4 states conclusion.

2 DYNAMIC FACTOR ALLOCATION PROBLEM

In this section, we discuss the dynamic portfolio optimization under a multi-factor Gaussian term structure model with stochastic mean. First, we describe the dynamic portfolio problem combined with a multi-factor Gaussian model by employing the result from Takahashi and Yoshida [2004]. By specifying the processes of the state variable, the market price of risk and the instantaneous interest rate $r(t)$, we derive a closed form solution for optimal bond portfolio strategy under the general version of the multi-factor Gaussian term structure model with stochastic mean, which is one of our main contributions in this paper.

2.1 Dynamic Portfolio Problem in a General Markovian Setting

2.1.1 Description of the financial market

First, we describe the financial market. Assume the market is complete. Let $(\Omega,\ \mathscr{F},\ \{\mathscr{F}_t\}_{0\leq t\leq T<\infty},\ \mathscr{P})$ denote a probability space with a filtration and assume it satisfies usual conditions. T $(<\infty)$ denotes a fixed time horizon of the economy.

We suppose the instantaneous spot rate $r(t)$ to be $r(t) = r(\mathbf{X}_t)$, that is, it can be represented as some function of the m-dimensional state variable

$\mathbf{X}_t := (X_{1,t}, \ldots, X_{m,t})'$, whose variation is governed by:

$$d\mathbf{X}_t = \boldsymbol{V_0}(\mathbf{X}_t)\,dt + \sum_{j=1}^{m} \boldsymbol{V_j}(\mathbf{X}_t)\,d\hat{W}_{j,t} \quad \mathbf{X}_0 = \mathbf{x}, \tag{2.1}$$

here we use the notation of x' as the transpose of x. $\boldsymbol{V_j}(\cdot)$, $j = 0, 1, \ldots, m$ are $(m \times 1)$ column vectors respectively and set a $m \times m$ matrix as $\boldsymbol{V}(\cdot) := (\boldsymbol{V_1}(\cdot), \ldots, \boldsymbol{V_m}(\cdot))$. $\boldsymbol{V_j}(\cdot)$, $j = 0, 1, \ldots, m$ are some functions of $\mathbf{X}_t$ and satisfy the regularity conditions. $\hat{\mathbf{W}}_t := (\hat{W}_{1,t}, \hat{W}_{2,t}, \ldots, \hat{W}_{m,t})', 0 \leq t \leq T$ is a $\mathbf{R}^m$-valued Brownian motion whose components are independent Brownian motions defined on $(\Omega,\ \mathscr{F},\ \{\mathscr{F}_t\}_{0 \leq t \leq T < \infty},\ \mathscr{P})$. Let $P(t, T_i), i = 1, 2 \ldots, m$ and $P_0(t)$ denote the prices at time $t \in [0, T]$ of zero coupon bonds with the maturity T_i and that of money market account, respectively. Here we suppose the stochastic differential equation governing the movement of zero coupon bond's price, $P(t, T_i)$, and that of money market account as

$$\begin{aligned} dP(t, T_i) &= P(t, T_i)[\beta_i(t, \mathbf{X}_t)\,dt + \hat{\boldsymbol{\sigma}}_i(t)\,d\hat{\mathbf{W}}_t],\ P(0, T_i) = p_i,\ i = 1, 2 \ldots, m \\ dP_0(t) &= r(\mathbf{X}_t)P_0(t)\,dt, \quad P_0(0) = 1 \end{aligned} \tag{2.2}$$

where $\hat{\boldsymbol{\sigma}}_i(t) := (\hat{\sigma}_{i,1}(t), \ldots, \hat{\sigma}_{i,m}(t))$.

As we assume the financial market is complete, there uniquely exists a stochastic process, $\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t)$, which is given as the solution of the equation below:

$$\boldsymbol{\beta}(t, \mathbf{X}_t) - r(\mathbf{X}_t)\mathbf{1} = \hat{\boldsymbol{\sigma}}(t)\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t) \tag{2.3}$$

where $\boldsymbol{\beta}(t,\mathbf{X}_t)$ is a $(m \times 1)$ vector whose ith element is $\beta_i(t,\mathbf{X}_t)$, $\hat{\boldsymbol{\sigma}}(t)$ is a $(m \times m)$ matrix whose ith row is $\hat{\boldsymbol{\sigma}}_i(t)$ and $\mathbf{1}$ is a $(m \times 1)$ unit vector. This variable, $\hat{\boldsymbol{\theta}}(t,\mathbf{X}_t)$, is often referred as the market price of risk.

Thus we can rewrite (2.2) as:

$$\begin{aligned} dP(t,T_i) &= P(t,T_i)[(r(\mathbf{X}_t) + \hat{\boldsymbol{\sigma}}_i(t)\hat{\boldsymbol{\theta}}(t,\mathbf{X}_t))\,dt + \hat{\boldsymbol{\sigma}}_i(t)\,d\hat{\mathbf{W}}_t] \\ P(0,T_i) &= p_i,\ i = 1,2\ldots,m \end{aligned} \tag{2.4}$$

Given the opportunity set described in (2.2), investors will allocate their wealth among these $(m+1)$ assets. Let $\phi_i(t)$ denotes the number of units of asset i held at time t. From the budget constraint, the stochastic differential equation below should hold [1]:

$$d\mathscr{X}(t) = \boldsymbol{\phi}(t)'\,d\boldsymbol{P}(t) + (1 - \boldsymbol{\phi}(t)'\mathbf{1})\,dP_0(t) - c(t)dt, \quad \mathscr{X}(0) = x > 0 \tag{2.5}$$

where $\mathscr{X}(t)$ and $c(t)$ denote investors' total wealth and their non-negative consumption rate respectively, $\boldsymbol{\phi}(t)$ is a $(m \times 1)$ vector whose ith element is $\phi_i(t)$, $\mathbf{1}$ is a $(m \times 1)$ vector defined as $\mathbf{1} := (1,1,\ldots,1)'$, and $\boldsymbol{P}$ is a $(m \times 1)$ vector whose ith element is $P(t,T_i)$.

It is useful to replace $\boldsymbol{\phi}(t)$ from (2.5) by a new variable, $\boldsymbol{\pi}(t)$, a $(m \times 1)$ vector whose ith element is $\pi_i(t)$, which denotes the amount of money invested for ith asset at time t. Substituting for $dP(t,T_i)/P(t,T_i)$ from (2.4),

[1] For a detailed explanation of this budget constraint, see Merton [1971]

we can rewrite (2.5) as

$$\begin{aligned} d\mathscr{X}(t) &= [r(\mathbf{X}_t)\mathscr{X}(t) - c(t)]\,dt + \boldsymbol{\pi}(t)'[\hat{\boldsymbol{\sigma}}(t)\hat{\boldsymbol{\theta}}(t,\mathbf{X}_t)\,dt + \hat{\boldsymbol{\sigma}}(t)\,d\hat{\mathbf{W}}_t], \\ \mathscr{X}(0) &= x > 0, \end{aligned} \tag{2.6}$$

which is often referred as the budget-constraint dynamics.

Finally, we define $\mathscr{A}(x)$ as the set of stochastic processes $(\boldsymbol{\pi}, c)$ such that given $\mathscr{X}(0) = x > 0$, for all $t \in [0, T]$, $\mathscr{X}(t) \geq 0$ (a.s.).

2.1.2 Optimal portfolio problem

Here we analyze optimal portfolio selection. First, we specify optimal portfolio problem for investors:

Assumption 1 *We assume investors seek to maximize an objective function as below:*

$$\sup_{(\boldsymbol{\pi},c)\in\mathscr{A}(x)} \mathbf{E}^{\mathscr{P}}[U(\mathscr{X}(T))], \tag{2.7}$$

subject to the budget-constraint dynamics, (2.6), here $\mathbf{E}^{\mathscr{P}}[\cdot]$ *denotes the expectation operator under* $\mathscr{P}$ *and* U *denotes a utility function.*

Note that, with the objective function above, optimal consumption rule turn out to be $c(t) = 0$.

Furthermore, as assumed in Sørensen [1999], Brennan and Xia [2000], Kobayashi, Takahashi and Tokioka [2005], Liu [2006], We assume a utility function in (2.7) is specified as so-called power utility, that is:

Assumption 2 *We assume investors have a utility function as follows: for* $\delta < 1, \delta \neq 0$,

$$U(x) := \frac{x^\delta}{\delta}.$$

In this Markovian setting, Takahashi and Yoshida [2004] provides the following result based on Ocone and Karatzas [1991].

Theorem 2.1 *Under the same conditions as in Section 4 in Takahashi and Yoshida [2004], the optimal proportion of zero coupon bonds in wealth denoted by* $\boldsymbol{\pi}(t)'/\mathscr{X}(t)$ *are given as follows:*

$$\begin{aligned}\frac{\boldsymbol{\pi}(t)'}{\mathscr{X}(t)} &= \frac{1}{1-\delta}\hat{\boldsymbol{\theta}}(t,\mathbf{X}_t)'\hat{\boldsymbol{\sigma}}(t)^{-1} + \frac{\delta}{1-\delta}\frac{1}{\mathbf{E}_t^{\mathscr{P}}[(H_{t,T})^{-\delta/(1-\delta)}]}\times \\ &\times \mathbf{E}_t^{\mathscr{P}}\Bigg[(H_{t,T})^{-\delta/(1-\delta)}\Bigg(\int_t^T \partial r(\mathbf{X}_u)\,\mathbf{Y}_u\,\boldsymbol{V}(\mathbf{X}_t)\,du + \\ &+ \sum_{j=1}^m \int_t^T \partial\hat{\theta}_j(t,\mathbf{X}_u)\,\mathbf{Y}_u\,\boldsymbol{V}(\mathbf{X}_t)\,d\hat{W}_{j_u} + \\ &+ \sum_{j=1}^m \int_t^T \hat{\theta}_j(t,\mathbf{X}_u)\,\partial\hat{\theta}_j(t,\mathbf{X}_u)\,\mathbf{Y}_u\,\boldsymbol{V}(\mathbf{X}_t)\,du\Bigg)\Bigg]\hat{\boldsymbol{\sigma}}(t)^{-1}\end{aligned} \tag{2.8}$$

where $\mathbf{E}_t^{\mathscr{P}}[\cdot]$ *denotes the conditional expectation at time* t, $H_{t,T}$ *is defined by:*

$$H_{t,T} := \exp\Bigg(-\int_t^T \hat{\boldsymbol{\theta}}(t,\mathbf{X}_u)'\,d\hat{\mathbf{W}}_u - \frac{1}{2}\int_t^T |\hat{\boldsymbol{\theta}}(t,\mathbf{X}_u)|^2\,du - \int_t^T r(\mathbf{X}_u)\,du\Bigg),$$

and

$$\partial r(\mathbf{X}_u) = \left(\frac{\partial r}{\partial X_1}, \ldots, \frac{\partial r}{\partial X_m}\right)$$
$$\partial\hat{\theta}_j(t, \mathbf{X}_u) = \left(\frac{\partial\hat{\theta}_j}{\partial X_1}, \ldots, \frac{\partial\hat{\theta}_j}{\partial X_m}\right),$$

and $\mathbf{Y}_u$ *follows the* $(m \times m)$ *matrix valued stochastic differential equation: for* $u \in [t, T]$,

$$d\mathbf{Y}_u = \partial\boldsymbol{V_0}(\mathbf{X}_u)\mathbf{Y}_u\,du + \sum_{j=1}^{m}\partial\boldsymbol{V_j}(\mathbf{X}_u)\mathbf{Y}_u\,d\hat{W}_{j,u} \quad \mathbf{Y}_t = \boldsymbol{I} \tag{2.9}$$

where $\partial\boldsymbol{V_k}(\mathbf{X}_u)$ *is a* $(m\times m)$ *matrix such that* $\partial\boldsymbol{V_k}(\mathbf{X}_u) = (\partial V_k^i/\partial X_{j,u})_{1\le i,j\le m}$, V_k^i *denotes the* i*th element of* $\boldsymbol{V_k}$, $k = 0, \ldots, m$ *which appeared in (2.1) and* $\boldsymbol{I}$ *denotes* $(m \times m)$ *identity matrix.*

Proof. *See* Section *4.1* in Takahashi and Yoshida [2004]. □

In the next subsection, we will obtain a closed form solution of the optimal portfolio by specifying $\mathbf{X}_t$, $\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t)$ and $r(\mathbf{X}_t)$.

2.2 Optimal Portfolio in Multi-factor Gaussian model

In this subsection, we first specify the process of $\mathbf{X}_t$.

Assumption 3 *We assume the state variable* $\mathbf{X}_t$ *follows the process:*

$$\begin{aligned} dX_{1,t} &= \alpha_1(X_{2,t} + X_{3,t} + \cdots + X_{m,t} - X_{1,t})\,dt + \sigma_1\,dW_{1,t} \\ dX_{i,t} &= \alpha_i(\bar{X}_i - X_{i,t})\,dt + \sigma_i\,dW_{i,t}, \quad (i = 2, 3, \ldots m) \end{aligned} \tag{2.10}$$

where, $\alpha_1 > 0, \alpha_i \geq 0$*,* $\bar{X}_i \geq 0, i = 2, 3, \ldots m$*,* $\sigma_i \geq 0, i = 1, 2, \ldots m$ *are all constants and* $\mathbf{W}_t := (W_{1,t}, W_{2,t}, \ldots, W_{m,t})', 0 \leq t \leq T$ *is a* $\mathbf{R}^m$*-valued correlated Brownian motion defined on* $(\Omega,\ \mathscr{F},\ \{\mathscr{F}_t\}_{0 \leq t \leq T < \infty},\ \mathscr{P})$ *with* $dW_i\,dW_j = \rho_{ij}dt$ *(if* $i = j$*,* $\rho_{ij} \equiv 1$*). The correlated Brownian motion* $\mathbf{W}_t$ *can be expressed by using an independent Brownian motion* $\hat{\mathbf{W}}_t$ *as* $\mathbf{W}_t = \mathbf{C}\hat{\mathbf{W}}_t$*, where* $\mathbf{C}$ *is some lower triangular matrix obtained by Cholesky decomposition.*

Under this setting, (2.1) is reduced to

$$d\mathbf{X}_t = \boldsymbol{\alpha}(\bar{\mathbf{X}} - \mathbf{X}_t)\,dt + \boldsymbol{V}\,d\hat{\mathbf{W}}_t \quad \mathbf{X}_0 = \mathbf{x}, \tag{2.11}$$

where $\boldsymbol{\alpha}$ is a $(m \times m)$ diagonal matrix whose ith element is α_i, $\bar{\mathbf{X}} := (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_m)'$, $\bar{X}_1 := X_{2,t} + X_{3,t} + \cdots + X_{m,t}$, $\boldsymbol{V} := \boldsymbol{\sigma}\mathbf{C}$ and $\boldsymbol{\sigma}$ is a $(m \times m)$ diagonal matrix whose ith element is σ_i.

Next, we put assumptions on $\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t)$ and $r(\mathbf{X}_t)$ as follows:

Assumption 4 *(1)We assume the market price of risk* $\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t)$ *is given by:*

$$\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t) := \hat{\boldsymbol{\theta}} = (\hat{\theta_1}, \ldots, \hat{\theta_m})',$$

where $\hat{\theta}_i,\ i = 1, 2, \ldots, m$ *are all constants.*

(2) We also assume the instantaneous spot rate $r(t)$ *is expressed as* $r(t) = r(\mathbf{X}_t) := X_{1,t}$.

Then, the dynamics of $\mathbf{X}_t$ under a risk neutral measure $\mathscr{Q}$ is given by:

$$d\mathbf{X}_t \quad = \quad \left[\boldsymbol{\alpha}(\bar{\mathbf{X}} - \mathbf{X}_t) - \boldsymbol{V}\hat{\boldsymbol{\theta}}\right] dt + \boldsymbol{V}\, d\hat{\mathbf{W}}^*{}_t \quad \mathbf{X}_0 = \mathbf{x}, \tag{2.12}$$

where $\hat{\mathbf{W}}_t^*$ denotes a $\mathbf{R}^m$-valued independent Brownian motion under $\mathscr{Q}$ as follows:

$$\hat{\mathbf{W}}_t^* = \hat{\mathbf{W}}_t + \hat{\boldsymbol{\theta}} t.$$

Furthermore, with a correlated Brownian motion, (2.12) can be rewritten as:

$$d\mathbf{X}_t \quad = \quad \left[\boldsymbol{\alpha}(\bar{\mathbf{X}} - \mathbf{X}_t) - \boldsymbol{\sigma}\boldsymbol{\theta}\right] dt + \boldsymbol{\sigma}\, d\mathbf{W}^*{}_t \quad \mathbf{X}_0 = \mathbf{x}, \tag{2.13}$$

where $\boldsymbol{\theta}$ is the market price of risk with respect to $\mathbf{W}_t$, and $\mathbf{W}_t^*$ represents a $\mathbf{R}^m$-valued correlated Brownian motion under $\mathscr{Q}$ given by:

$$\mathbf{W}_t^* = \mathbf{W}_t + \boldsymbol{\theta} t.$$

Note that $\boldsymbol{\theta}$ is obtained by $\boldsymbol{\theta} := \mathbf{C}\hat{\boldsymbol{\theta}}$ as $\mathbf{W}_t = \mathbf{C}\hat{\mathbf{W}}_t$. In the following discussion, we mainly use the independent Brownian motion $\hat{\mathbf{W}}_t^*$ except in the pricing of zero coupon bonds.

The model described in (2.13) belongs to the class of the multi-factor affine model and the general version of stochastic mean model including Vasicek[1977], Hull and White [1990], Balduzzi et al. [2000], He [2001] and Takahashi and Sato [2001] as special cases.

From Assumption 4 and (2.13), $r(u)$ is expressed by using a correlated Brownian motion under $\mathscr{Q}$:

$$\begin{aligned} r(u) &= e^{-\alpha_1(u-t)}X_{1,t} + \sum_{i=2}^{m}\left(\alpha_1 \int_t^u e^{-\alpha_i(s-t)-\alpha_1(u-s)}\,ds\right)X_{i,t} \\ &- \frac{\sigma_1\theta_1}{\alpha_1}(1-e^{-\alpha_1(u-t)}) \\ &+ \sum_{i=2}^{m}\left(\alpha_1 \int_t^u\int_t^s e^{-\alpha_i(s-\tau)-\alpha_1(u-s)}d\tau\,ds\right)(\alpha_i\bar{X}_i - \sigma_i\theta_i) \\ &+ \sigma_1\int_t^u e^{-\alpha_1(u-\tau)}\,dW^*_{1,\tau} \\ &+ \sum_{i=2}^{m}\sigma_i\alpha_1\int_t^u\int_\tau^u e^{-\alpha_i(s-\tau)-\alpha_1(u-s)}\,ds\,dW^*_{i,\tau} \end{aligned} \tag{2.14}$$

Under the no-arbitrage condition, the zero coupon price at time t with the maturity T_i, $P(t,T_i)$ is obtained by:

$$P(t,T_i) = E_t^{\mathscr{Q}}\left[\exp\left(-\int_t^{T_i} r(u)\,du\right)\right]$$

where $E_t^{\mathscr{Q}}[\cdot]$ is the conditional expectation operator under $\mathscr{Q}$ given information at time t. Then the bond price at time t with the maturity T_i is given by:

$$P(\mathbf{X}_t, T_i) := P(t, T_i) = \exp\left(a_{i,0}(\tau_i) + \sum_{j=1}^{m} a_{i,j}(\tau_i) X_{j,t}\right)$$

where $a_{i,0}(\tau_i)$ is some deterministic function with respect to $\tau_i := T_i - t$ and $a_{i,j}(\tau_i)$ is given by:

$$a_{i,1}(\tau_i) := -\frac{1 - e^{-\alpha_1 \tau_i}}{\alpha_1}, \quad a_{i,j}(\tau_i) := -\alpha_1 \int_t^T \int_t^u e^{-\alpha_i(s-t) - \alpha_1(u-s)} ds\, du.$$

From $It\hat{o}'s\ formula$, we obtain:

$$\begin{aligned} \frac{dP(\mathbf{X}_t, T_i)}{P(\mathbf{X}_t, T_i)} &= \sum_{j=1}^{m} a_{i,j}(\tau_i)\, dX_{j,t} + \frac{\partial P(\mathbf{X}_t, T_i)}{\partial t} / P(\mathbf{X}_t, T_i)\, dt \\ &+ \frac{1}{2} \sum_{k,j=1}^{m} a_{i,k}(\tau_i) a_{i,j}(\tau_i) \rho_{kj} \sigma_k \sigma_j\, dt. \end{aligned} \tag{2.15}$$

From the right hand side in (2.15), we can see that the return of each zero coupon bond can be decomposed into 3 terms: factor variation, carry and factor convexity. Especially, from the first term, we obtain factor exposure of the ith bond for each factor $j = 1, \ldots, m$ as $a_{i,j}(\tau_i)$. Here we define a $(m \times 1)$ vector, $\boldsymbol{a_i}(\tau_i)$, by $\boldsymbol{a_i}(\tau_i) := (a_{i,1}(\tau_i), a_{i,2}(\tau_i), \ldots, a_{i,m}(\tau_i))'$ and then the first term in (2.15) can be rewritten as $\boldsymbol{a_i}(\tau_i)' d\mathbf{X}_t$. Therefore, with (2.11), $\hat{\boldsymbol{\sigma}}_i(t)$ in (2.2) can be rewritten as $\hat{\boldsymbol{\sigma}}_i(t) = \boldsymbol{a_i}(\tau_i)' \boldsymbol{V}$. Finally, set factor exposure matrix $\boldsymbol{A}$ as $\boldsymbol{A} := (a_{i,j}(\tau_i))_{1 \le i,j \le m}$ then $\hat{\boldsymbol{\sigma}}$ appeared in (2.6) and (2.8) can be rewritten as

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{A}\boldsymbol{V}. \tag{2.16}$$

Then, by adding Assumption 3 and 4, the optimal portfolio strategy in Theorem 2.1 reduces to the following proposition:

Proposition 2.1 *Under the same conditions as in Theorem 2.1 and Assumption 3-4, the optimal proportion of zero coupon bonds are given as follows:*

$$\frac{\boldsymbol{\pi}(t)'}{\mathscr{X}(t)} = \frac{1}{1-\delta}\hat{\boldsymbol{\theta}}'\hat{\boldsymbol{\sigma}}^{-1} + \frac{\delta}{1-\delta}\mathscr{Y}'\,\boldsymbol{V}\hat{\boldsymbol{\sigma}}^{-1} \tag{2.17}$$

where $\mathscr{Y}$ is defined as below:

$$\mathscr{Y} := \begin{pmatrix} \frac{1-e^{-\alpha_1(T-t)}}{\alpha_1} \\ \\ \alpha_1 \int_t^T\int_t^u e^{-\alpha_2(s-t)-\alpha_1(u-s)}\,ds\,du \\ \vdots \\ \alpha_1 \int_t^T\int_t^u e^{-\alpha_m(s-t)-\alpha_1(u-s)}\,ds\,du \end{pmatrix} \tag{2.18}$$

Proof. $\partial\hat{\theta}_j(\mathbf{X}_u) \equiv 0$, $j = 1,\ldots,m$ follows from Assumption 4(1). As $\partial r(\mathbf{X}_u) = (1,0,\ldots,0)$ from Assumption 4(2), $\partial r(\mathbf{X}_u)\,\mathbf{Y}_u$ reduces to the first row of $\mathbf{Y}_u$, which is obtained by:

$$\left(e^{-\alpha_1(u-t)}, \alpha_1\int_t^u e^{-\alpha_2(s-t)-\alpha_1(u-s)}\,ds, \ldots, \alpha_1\int_t^u e^{-\alpha_m(s-t)-\alpha_1(u-s)}\,ds\right).$$

Finally using $\boldsymbol{V}(\mathbf{X}_t) = \boldsymbol{V}$ and $\int_t^T \partial r(\mathbf{X}_u)\, \mathbf{Y}_u\, du = \mathscr{Y}'$, (2.8) turns out to be:

$$
\begin{aligned}
\frac{\boldsymbol{\pi}(t)'}{\mathscr{X}(t)} &= \frac{1}{1-\delta}\hat{\boldsymbol{\theta}}'\hat{\boldsymbol{\sigma}}^{-1} + \frac{\delta}{1-\delta}\frac{1}{\mathbf{E}_t^{\mathscr{P}}[(H_{t,T})^{-\delta/(1-\delta)}]} \times \\
&\times \mathbf{E}_t^{\mathscr{P}}\left[(H_{t,T})^{-\delta/(1-\delta)}\left(\int_t^T \partial r(\mathbf{X}_u)\, \mathbf{Y}_u\, \boldsymbol{V}\, du\right)\right]\hat{\boldsymbol{\sigma}}^{-1} \\
&= \frac{1}{1-\delta}\hat{\boldsymbol{\theta}}'\hat{\boldsymbol{\sigma}}^{-1} + \frac{\delta}{1-\delta}\frac{1}{\mathbf{E}_t^{\mathscr{P}}[(H_{t,T})^{-\delta/(1-\delta)}]} \times \\
&\times \mathbf{E}_t^{\mathscr{P}}[(H_{t,T})^{-\delta/(1-\delta)}]\left(\int_t^T \partial r(\mathbf{X}_u)\, \mathbf{Y}_u\, du\right)\boldsymbol{V}\hat{\boldsymbol{\sigma}}^{-1} \\
&= \frac{1}{1-\delta}\hat{\boldsymbol{\theta}}'\hat{\boldsymbol{\sigma}}^{-1} + \frac{\delta}{1-\delta}\mathscr{Y}'\, \boldsymbol{V}\hat{\boldsymbol{\sigma}}^{-1}.
\end{aligned}
$$

□

The first term on the right hand side of (2.17) represents mean-variance portfolio in a continuous-time setting: hence we call it MV term. The second term is specific to a multi-period setting and represents the intertemporal hedging demand defined by Merton [1971]: we call it IR-hedging term.

As observed in (2.15), the dynamics of a single bond's return can be decomposed into exposures to several factors' return. Thus the allocation to bonds is converted into the allocation to factors. While it is difficult to interpret the optimal portfolio in terms of bonds because of high correlation among them, we can interpret the optimal portfolio easier in terms of factors.

Therefore we consider factor exposure of the optimal strategy in Proposition 2.1. The process of the instantaneous return of the optimal bonds' portfolio can be described as follows:

$$
\begin{aligned}
\frac{d\mathscr{X}(t)}{\mathscr{X}(t)} &= \left(1-\sum_{i=1}^{m}\frac{\pi_i(t)}{\mathscr{X}(t)}\right)r(\mathbf{X}_t)dt + \sum_{i=1}^{m}\frac{\pi_i(t)}{\mathscr{X}(t)}\frac{dP(\mathbf{X}_t,T_i)}{P(\mathbf{X}_t,T_i)} \\
&= \left(1-\sum_{i=1}^{m}\frac{\pi_i(t)}{\mathscr{X}(t)}\right)r(\mathbf{X}_t)dt + \sum_{j=1}^{m}\left(\sum_{i=1}^{m}\frac{\pi_i(t)}{\mathscr{X}(t)}a_{i,j}(\tau_i)\right)dX_{j,t} \\
&+ \sum_{i=1}^{m}\frac{\pi_i(t)}{\mathscr{X}(t)}\frac{\partial P(\mathbf{X}_t,T_i)}{\partial t}/P(\mathbf{X}_t,T_i)\,dt + \frac{1}{2}\sum_{k,j=1}^{m}\sum_{i=1}^{m}\frac{\pi_i(t)}{\mathscr{X}(t)}a_{i,k}(\tau_i)a_{i,j}(\tau_i)\rho_{kj}\sigma_k\sigma_j\,dt.
\end{aligned}
\tag{2.19}
$$

The factor exposure to $X_{j,t}$ is given by the coefficient of $dX_{j,t}$ in the second term on the right hand side of (2.19). Then, we obtain the following corollary.

Corollary 2.1 *Factor exposure of the optimal portfolio strategy in Proposition 2.1 can be obtained as follows:*

$$
\frac{\boldsymbol{\pi}(t)'\mathbf{A}}{\mathscr{X}(t)} = \frac{1}{1-\delta}\hat{\boldsymbol{\theta}}'\boldsymbol{V}^{-1} + \frac{\delta}{1-\delta}\mathscr{Y}'. \tag{2.20}
$$

Proof. Using (2.16) and the result of Proposition 2.1, we can calculate as follows:

$$
\begin{aligned}
\frac{\boldsymbol{\pi}(t)'\mathbf{A}}{\mathscr{X}(t)} &= \frac{1}{1-\delta}\hat{\boldsymbol{\theta}}'\hat{\boldsymbol{\sigma}}^{-1}\mathbf{A} + \frac{\delta}{1-\delta}\mathscr{Y}'\,\boldsymbol{V}\hat{\boldsymbol{\sigma}}^{-1}\mathbf{A} \\
&= \frac{1}{1-\delta}\hat{\boldsymbol{\theta}}'\boldsymbol{V}^{-1}\mathbf{A}^{-1}\mathbf{A} + \frac{\delta}{1-\delta}\mathscr{Y}'\,\boldsymbol{V}\boldsymbol{V}^{-1}\mathbf{A}^{-1}\mathbf{A} \\
&= \frac{1}{1-\delta}\hat{\boldsymbol{\theta}}'\boldsymbol{V}^{-1} + \frac{\delta}{1-\delta}\mathscr{Y}'.
\end{aligned}
$$

□

Thus using this corollary, we can convert the optimal portfolio into its factor exposure and we can reinterpret the bond allocation as the factor allocation. This idea is useful for investigating the relationship between the change in term structure and that in the factor exposure of the optimal portfolio strategy.

In concluding this section, one final remark on Corollary 2.1 deserves mention. In this multi-factor model, the optimal allocation to each factor does not depend on the components of portfolio. That is, being current term structure equal, the factor allocation is also invariant even though the optimal bond allocation varies as we change component of the bond portfolio. Thus we can say that the essence of the portfolio problem lies in the factor allocation rather than the bond allocation. Once we obtain the optimal factor allocation from Corollary 2.1, then we can also get the optimal allocation for any bond portfolio through its factor exposure matrix. Therefore, we can see that the optimal allocation of bonds is essentially obtained by that of factors. When the term structure varies, the change will be reflected by the factor allocation directly and the bond portfolio will also change through the change of factor allocation. Therefore the effect of the term structure variation on the bond portfolio is indirect and thus it is easier to analyze factor allocation directly.

The factor allocation depends on parameters such as speed parameters of mean-reversion k_i, $i = 1, \ldots, m$ which are strongly related with the shape of the yield curve. One of our objectives is to analyze how the changes in the shape of yield curve affect the IR-hedging term and hence the whole portfolio strategy. To investigate this issue, we introduce a concrete 3-factor model in the next section and implement series of comparative statics.

3 Numerical Analysis

So far, we have outlined the dynamic portfolio problem with multi-factor model. In this section, we introduce a 3-factor model as an example of the multi-factor stochastic mean model in (2.12) and we implement series of comparative statics based upon the model. The result brings clear interpretation of the optimal portfolio, which is also one of our main contributions in this paper.

3.1 3-factor Model

In this subsection, we concentrate on the case where the risk-neutral dynamics of X_1, X_2 and X_3 are given by:

$$\begin{aligned} dX_1 &= [\alpha_1(X_2 + X_3 - X_1) - \theta_1\sigma_1]\,dt + \sigma_1\,dW_1^* \\ dX_2 &= [\alpha_2(\bar{X}_2 - X_2) - \theta_2\sigma_2]\,dt + \sigma_2\,dW_2^* \\ dX_3 &= [\alpha_3(\bar{X}_3 - X_3) - \theta_3\sigma_3]\,dt + \sigma_3\,dW_3^*. \end{aligned}$$

Here, we assume $\alpha_1 > \alpha_i, i = 2, 3, \alpha_2 > 0, \alpha_3 \geq 0,\ \theta_i \leq 0,\ i = 1, 2, 3$ and the other conditions are the same in the previous section.

Then $r(u)$, $\boldsymbol{V}$ and $\mathscr{Y}$ in this model are expressed by using a correlated Brownian motion under $\mathscr{Q}$:

$$
\begin{aligned}
r(u) &= e^{-\alpha_1(u-t)}X_{1,t} + \frac{\alpha_1}{\alpha_1-\alpha_2}(e^{-\alpha_2(u-t)} - e^{-\alpha_1(u-t)})X_{2,t} \\
&+ \frac{\alpha_1}{\alpha_1-\alpha_3}(e^{-\alpha_3(u-t)} - e^{-\alpha_1(u-t)})X_{3,t} \\
&- (1-e^{-\alpha_1(u-t)})\frac{\sigma_1\theta_1}{\alpha_1} \\
&+ \left\{(1-e^{-\alpha_1(u-t)}) - \frac{\alpha_1}{\alpha_1-\alpha_2}(e^{-\alpha_2(u-t)} - e^{-\alpha_1(u-t)})\right\}(\bar{X}_2 - \frac{\sigma_2\theta_2}{\alpha_2}) \\
&+ \left\{(1-e^{-\alpha_1(u-t)}) - \frac{\alpha_1}{\alpha_1-\alpha_3}(e^{-\alpha_3(u-t)} - e^{-\alpha_1(u-t)})\right\}(\bar{X}_3 - \frac{\sigma_3\theta_3}{\alpha_3}) \\
&+ \sigma_1\int_t^u e^{-\alpha_1(u-s)}\,dW^*_{1,s} \\
&+ \sigma_2\frac{\alpha_1}{\alpha_1-\alpha_2}\int_t^u\left(e^{-\alpha_2(u-s)} - e^{-\alpha_1(u-s)}\right)dW^*_{2,s} \\
&+ \sigma_3\frac{\alpha_1}{\alpha_1-\alpha_3}\int_t^u\left(e^{-\alpha_3(u-s)} - e^{-\alpha_1(u-s)}\right)dW^*_{3,s}
\end{aligned}
\qquad \text{(when } \alpha_3 \neq 0\text{)} \quad (3.1)
$$

$$
\begin{aligned}
r(u) = {} & e^{-\alpha_1(u-t)} X_{1,t} + \frac{\alpha_1}{\alpha_1-\alpha_2}(e^{-\alpha_2(u-t)} - e^{-\alpha_1(u-t)}) X_{2,t} + (1-e^{-\alpha_1(u-t)}) X_{3,t} \\
& - (1-e^{-\alpha_1(u-t)}) \frac{\sigma_1\theta_1}{\alpha_1} \\
& + \left\{ (1-e^{-\alpha_1(u-t)}) - \frac{\alpha_1}{\alpha_1-\alpha_2}(e^{-\alpha_2(u-t)} - e^{-\alpha_1(u-t)}) \right\} (\bar{X}_2 - \frac{\sigma_2\theta_2}{\alpha_2}) \\
& - \left\{ (u-t) - \frac{1-e^{-\alpha_1(u-t)}}{\alpha_1} \right\} \sigma_3\theta_3 \qquad \text{(when } \alpha_3\text{=0)} \qquad (3.2) \\
& + \sigma_1 \int_t^u e^{-\alpha_1(u-s)}\, dW^*_{1,s} \\
& + \sigma_2 \frac{\alpha_1}{\alpha_1-\alpha_2} \int_t^u \left(e^{-\alpha_2(u-s)} - e^{-\alpha_1(u-s)} \right) dW^*_{2,s} \\
& + \sigma_3 \int_t^u \left(1 - e^{-\alpha_1(u-s)} \right) dW^*_{3,s}
\end{aligned}
$$

$$
\boldsymbol{V} = \begin{pmatrix} \sigma_1 & & \mathbf{0} \\ & \sigma_2 & \\ \mathbf{0} & & \sigma_3 \end{pmatrix} \begin{pmatrix} 1 & & \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & \\ \rho_{13} & x & y \end{pmatrix}
$$

where

$$
\begin{aligned}
x &:= \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} \\
y &:= \sqrt{\frac{1-(\rho_{12}^2+\rho_{23}^2+\rho_{13}^2) + 2\rho_{12}\rho_{23}\rho_{13}}{1-\rho_{12}^2}}.
\end{aligned}
$$

$$\mathscr{Y} := \begin{pmatrix} \frac{1-e^{-\alpha_1(T-t)}}{\alpha_1} \\ \\ \frac{\alpha_1}{\alpha_1-\alpha_2}\left[\frac{1-e^{-\alpha_2(T-t)}}{\alpha_2} - \frac{1-e^{-\alpha_1(T-t)}}{\alpha_1}\right] \\ \\ \frac{\alpha_1}{\alpha_1-\alpha_3}\left[\frac{1-e^{-\alpha_3(T-t)}}{\alpha_3} - \frac{1-e^{-\alpha_1(T-t)}}{\alpha_1}\right] \end{pmatrix} \quad (\text{when } \alpha_3 \neq 0) \tag{3.3}$$

$$\mathscr{Y} := \begin{pmatrix} \frac{1-e^{-\alpha_1(T-t)}}{\alpha_1} \\ \\ \frac{\alpha_1}{\alpha_1-\alpha_2}\left[\frac{1-e^{-\alpha_2(T-t)}}{\alpha_2} - \frac{1-e^{-\alpha_1(T-t)}}{\alpha_1}\right] \\ \\ (T-t) - \frac{1-e^{-\alpha_1(T-t)}}{\alpha_1} \end{pmatrix} \quad (\text{when } \alpha_3 = 0) \tag{3.4}$$

3.2 Properties of the 3-factor model

In this subsection, we briefly review properties of the 3-factor model introduced above. First, we check the characteristics of factors and parameters such as α_i and θ_i. Here we present the comparative statics of the 3-factor model with $\theta_1 = \theta_2 = \alpha_3 = 0$ which will be used for the analysis in the next subsection.

As explained by He [2001], X_1 captures the short rate controlled by the central bank, X_2 represents the movements of the yield curve slope and X_3 tracts the movements of the long-term interest rate. We investigate how the change of each factor affects the term structure. In Figure $1-3$, the horizontal axis and the vertical axis represent spot rate maturities and changes in spot rates when each factor X_i increases by 20 basis points (bps) respectively. From Figure $1-3$, we can confirm X_1 affects mainly the short sector in the term structure and X_2 and X_3 affect mainly the middle and the long term sectors respectively.

The positive constants α_1 and α_2 control the speed of mean-reversion of X_1 and X_2 respectively. The faster (slower) the reversion is, the shorter (longer) the shock to the economic system (or new information) stays in the market, and therefore the steeper (flatter) the yield curve becomes. Thus, α_1 and α_2 affect the slope of the short - middle term sector and that of the middle - long term sector respectively. We investigate how the changes of parameters in the model affect the term structure in Figure $4-7$ where the horizontal axis and the vertical axis represent spot rate maturities and changes in spot rates respectively. We can confirm above from Figure 4, Figure 5 and Figure 6 which show the effects of each parameter's change. Figure 7 shows the increase of the market price of risk pushes up the spot rates of the long term sector.

3.3 Comparative Statics

Next we show the results from series of comparative statics based on this model. From Corollary 2.1, we can see how the changes in the shape of yield curve, investor's preference and so on affect the portfolio strategy. The relative risk aversion (RRA given by $1 - \delta$) affects both. The MPR θ_i, the volatility σ_i and the correlation ρ_{ij} affect only the MV term. Investment Horizon T and the speed of mean-reversion α_i affect only the IR-hedging term.

From Corollary 2.1, we can also see the hedging demand for X_1 and X_2 will converge to 0 as α_1, $\alpha_2 \to \infty$. It is well-known that when the state variables are deterministic, the dynamic parts will disappear. As α_1, $\alpha_2 \to \infty$, X_1 and X_2 become almost deterministic because these two factors follow mean-reverting processes. Therefore the hedge demand for X_1 and X_2 will disappear. We can observe the same result for X_3 in (3.3) as α_1, $\alpha_3 \to \infty$. On the other hand, the hedging demand for X_3 in (3.4) is the increasing function of investment horizon T because when $\alpha_3 = 0$, X_3 does not have a mean-reversion property.

Based upon these analysis, we implement series of comparative statics. First, the initial parameters are reported in Table 1. We use estimates of these parameters from He [2001] and set the components of the portfolio, the investment horizon and the RRA as in Table 2. The optimal portfolio strategy and its factor allocation in this setting are given in Table 3.

Two points deserve mention. First, the IR-hedging term is not negligible and affects the whole strategy. Second, although the allocation to the long-term bond (30-year bond) is quite small in the MV term because of its high volatility, its allocation in the IR-hedging term is not small because it is necessary to hedge mainly against X_3. This is important because if we use

Mean-Variance scheme only, the position will be a simple spread strategy without 30-year bond which cannot hedge against yield curve risks other than the spread risk. Thus our approach is more preferable and practical because we can hedge against several yield curve risks.

Next, we shift several parameters to investigate further details of properties of this model. The results are shown in Figure 8 - 17 where we take the values of each parameter in a horizontal axis and the portfolio weight for each bond or the allocation to each factor in a vertical axis, respectively. First, we see the effect of the variation of α_1. The results are shown in Figure 8 and Figure 9. From these results, we can see several points. First, as mentioned before, the effects on each bond's allocation are very complicated and unclear because of high correlations among bonds: from Figure 8, it is difficult to see how the change of α_1 affects. In contrast, Figure 9 shows a clear message: there is no change in the MV term of factor allocation and thus only the IR-hedging term can reflect the steepening and flattening. As α_1 becomes larger, the allocation to X_1 in the IR-hedging term decreases. This is because the larger α_1 becomes, the faster X_1 reverts and hence X_1 becomes less uncertain. Therefore, the steepening (flattening) in the short term sector makes investors reduce (increase) the hedging position to X_1, and increase (reduce) hedging positions against X_2 and X_3 because these factors become riskier than X_1.

From Figure 11 we can see the same is true for α_2: the increase of α_2 decreases the exposure to X_2. Moreover, this case shows the advantage of the factor allocation approach. It is difficult to interpret how α_2 affects each bond from Figure 10. On the other hand, from Figure 11, we can interpret the effect intuitively as follows: steepening (flattening) in the middle term sector makes investors reduce (increase) only their hedge exposures to X_2 while exposures to X_1 and X_3 stay constant.

Next we consider the effect of the change of θ_3. In Figure 12, the factor allocation in the IR-hedging term unchanges while the exposures to X_2 and X_3 in the MV term change. The decrease of θ_3 makes the instantaneous expected return of X_3 increase and therefore X_3 becomes more attractive in Mean-Variance basis. Thus the exposure to X_3 increase as θ_3 decreases. Moreover, because the correlation between X_2 and X_3 is set to be negative, investors can increase the exposure to X_3 further by using the exposure to X_2 for hedging. As a result, the exposure to X_2 also increases. We can confirm this from Table 4. The exposure to X_3 with $\rho_{23} \neq 0$ is larger than that with $\rho_{23} = 0$.

From Corollary 2.1, we can confirm $\sigma_i, i = 1, 2, 3$ affect only the MV term. We can see this from Figure 13 and Figure 14. The increase of σ_i will reduce only the allocation to X_i and others stay constant.

Next we can refer to Figure 15 and Figure 16 to see the effect of the investment horizontal. Two points deserve mention. First, the MV term has no horizontal effect and this effect is specific to the dynamic term. Second, from Figure 15 the effect on the allocation to bonds is not monotone. The increase of the investment horizon increases the uncertainty in the future but at the same time it also reduces the duration of each bond and that makes uncertainty decrease. Which of them is dominant depends on circumstances. On the other hand, the effect on the factor allocation is monotone increasing from Figure 16. This is simply because uncertainty becomes larger as the investment horizon increases.

Finally, we consider the effect of the variation of the RRA. The results are given in Figure 17. When the $RRA = 1$, the investor has a log utility and the optimal strategy is the same with that of Mean-Variance approach. From Figure 17 we can see several points. First, as for X_1 the MV term has no exposure to X_1 therefore the IR-hedging term is dominant. If the $RRA < 1$, the investor is less risk averse and takes an aggressive position, which corresponds to positive exposures in the IR-hedging term. If the $RRA > 1$, on the other hand, the investor is more risk averse and takes a cautious position: the opposite position with the MV term.

We summarize the results of the comparative statics in Table 5.

4 Conclusion

In this paper, we analyzed the dynamic fixed-income portfolio optimization with a multi-factor Gaussian yield curve model. The main results obtained in this paper are summarized as follows.

First, we combined the dynamic portfolio optimization and a multi-factor Gaussian term structure model to obtain an analytical expression of the optimal bond allocation: it enables us to examine the model parameters' effects efficiently.

Second, by introducing the idea of factor allocation, we can easily interpret optimal portfolio which is usually hard to be understood by considering the portfolio in terms of bonds.

Third, we investigate how the change in the term structure affects the optimal portfolio through series of comparative statics.

References

[1] Balduzzi, D., Foresi, S. and Sundaram, R., "Stochastic Mean Models of the Term Structure of Interest Rates," pp. 126-161, in *Advanced Fixed Income Valuation Tools*, ed. N. Jegadeesh and B. Tuckman, New York: John Wiley & Sons, Inc. 2000

[2] Brennan, M., and Xia, Y., "Stochastic Interest Rates and Bond-Stock Mix," *European Finance Review* 4 (2000), 197-210

[3] He, H., "Modeling Term Structures of Swap Spreads," 2001, Discussion Paper

[4] Hull, J., and White, A., "Pricing Interest Rate Derivative Securities," *Review of Financial Studies*, 3, No.4 (1990) 573-592

[5] Kobayashi, T, Takahashi A. and Tokioka N., “Dynamic Optimality of Yield Curve Strategies,” *International Review of Finance*, Vol. **4**, Issue 1-2 (2003), 49-78

[6] Korn, O., and Koziol C., “Bond Portfolio Optimization: A Risk-Return Approach,” *Journal of Fixed Income* **15** (2006), 48-60

[7] Liu, J., “Portfolio Selection in Stochastic Environments,” *Review of Financial Studies*, (2006) forthcoming

[8] Merton, C., “Optimum consumption and portfolio rules in a continuous-time model,” *Journal of Economic Theory* **3** (1971), 247-257

[9] Ocone, D. and Karatzas, I., “A generalized Clark representation formula, with application to optimal portfolio,” *Stochastics and Stochastics Reports* **34** (1991), 187-220

[10] Sørensen, C., “Dynamic asset allocation and fixed income management,” *Journal of Financial and Quantitative Analysis* 34 (1999), 513-532

[11] Takahashi, A. and Sato, S., ”A Monte Carlo Filtering Approach for Estimating the Term Structure of Interest Rates,” *Annals of the Institute of Statistical Mathematics*, Vol.53 (2001), 50-62

[12] Takahashi, A., and Yoshida N., “An Asymptotic Expansion Scheme for Optimal Investment Problems,” *Statistical Inference for Stochastic Processes* **7:** (2004), 153-188

[13] Vasicek, A., "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5 (1977), 177-188

Figure 1: X_1 effect:+20 bps

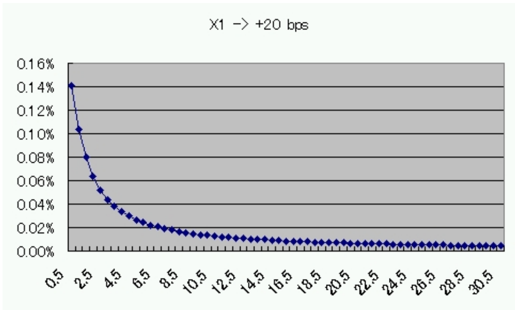


Figure 2: X_2 effect:+20 bps

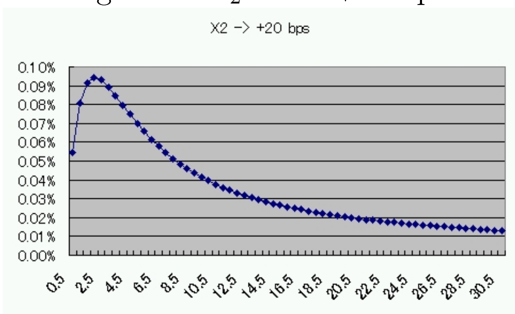


Figure 3: X_3 effect:+20 bps

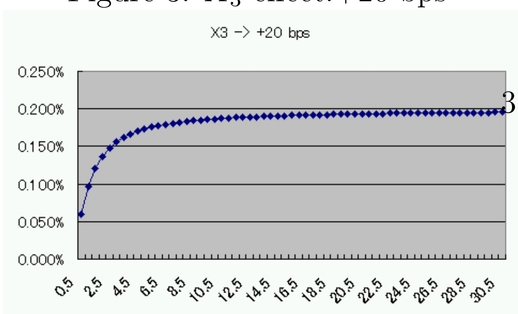


Figure 4: α_1 effect: +0.5

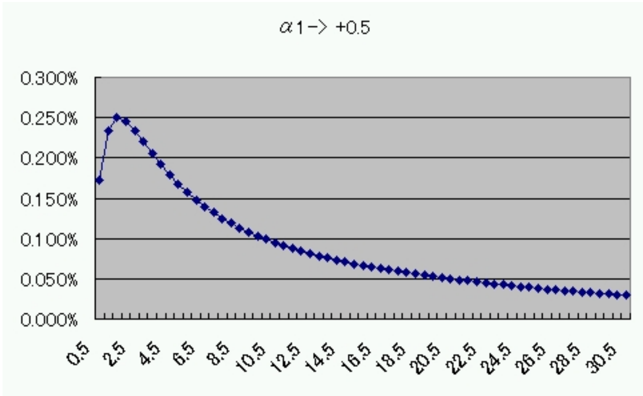


Figure 5: α_2 effect:+0.5, $X_2(0) < 0$

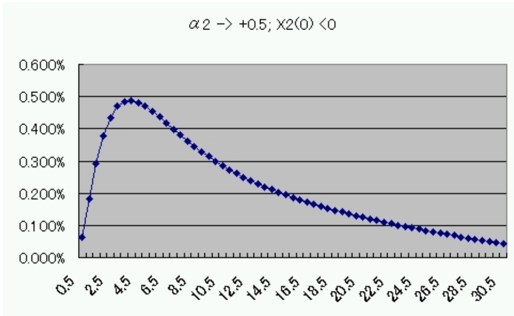


Figure 6: α_2 effect:+0.5, $X_2(0) > 0$

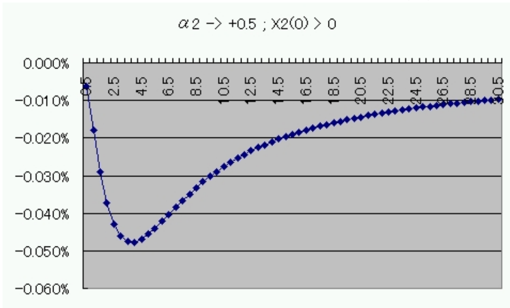


Figure 7: θ_3 effect:-0.7

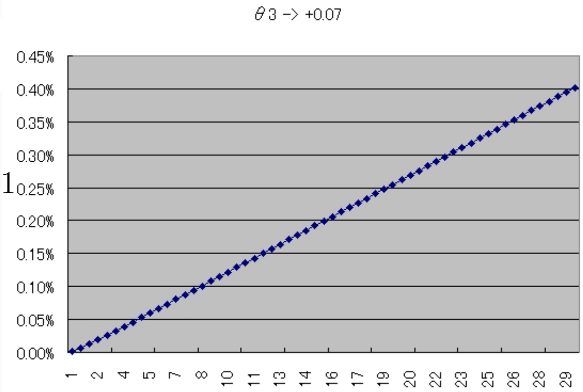


Table 1: Model Parameter Set

α_1	1.50	σ_1	0.50%
α_2	0.50	σ_2	1.50%
α_3	0	σ_3	1.25%
θ_1	0	ρ_{12}	0
θ_2	0	ρ_{13}	0
θ_3	-0.125	ρ_{23}	-0.3

Table 2: Portfolio Parameter Set

Investment Horizon	1
RRA	4

Zero Coupon Bonds	2
	7
	30

Table 3: Initial Result

	MV		**IR Hedge**			**Total**
2	0.882		0.937		2	1.820
7	-0.836	+	-0.353	=	7	-1.189
30	-0.002		0.045		30	0.043
					Cash	0.673
					Total	1

factor allocation

	MV	**IR Hedge**	**Total**
X_1	0.0000	-0.3884	-0.3884
X_2	0.7661	-0.3027	0.4635
X_3	4.1570	-0.3616	3.7954

Figure 8: The effect of α_1 variation: bonds

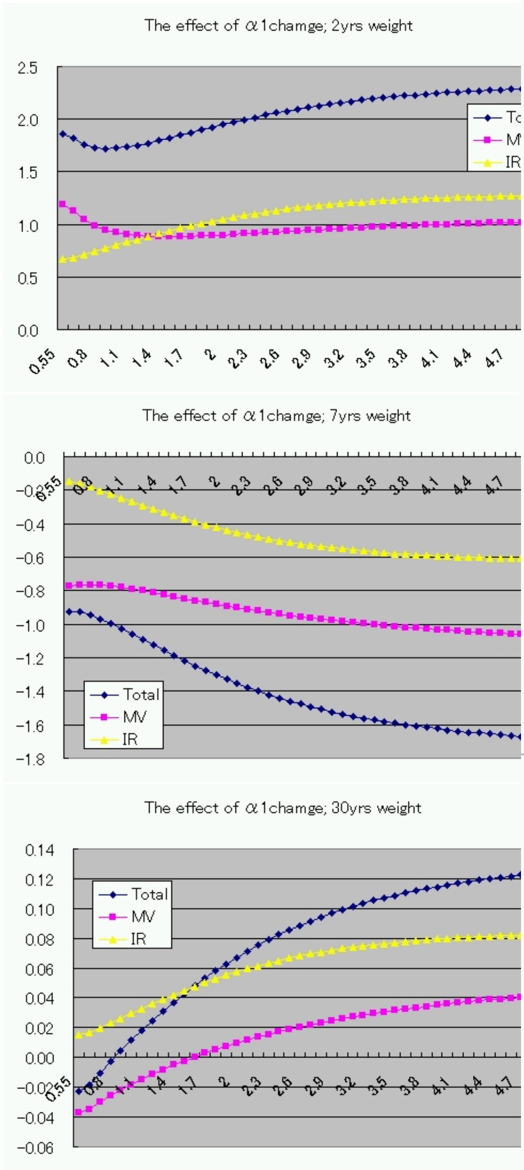


Figure 9: The effect of α_1 variation: factor allocation

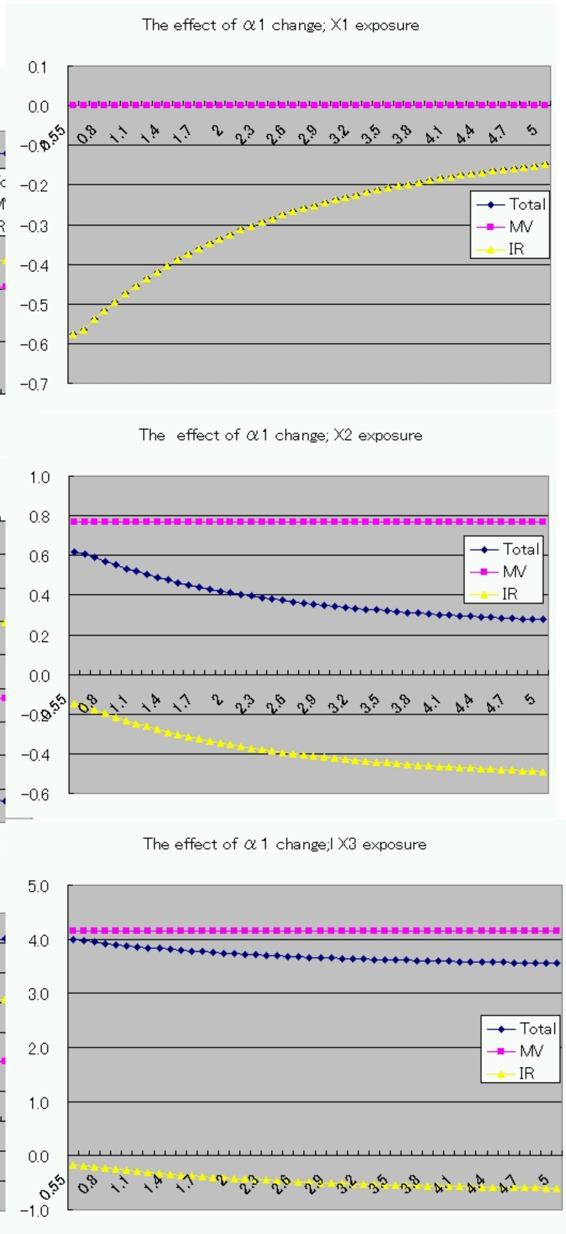


Figure 10: The effect of α_2 variation: bonds

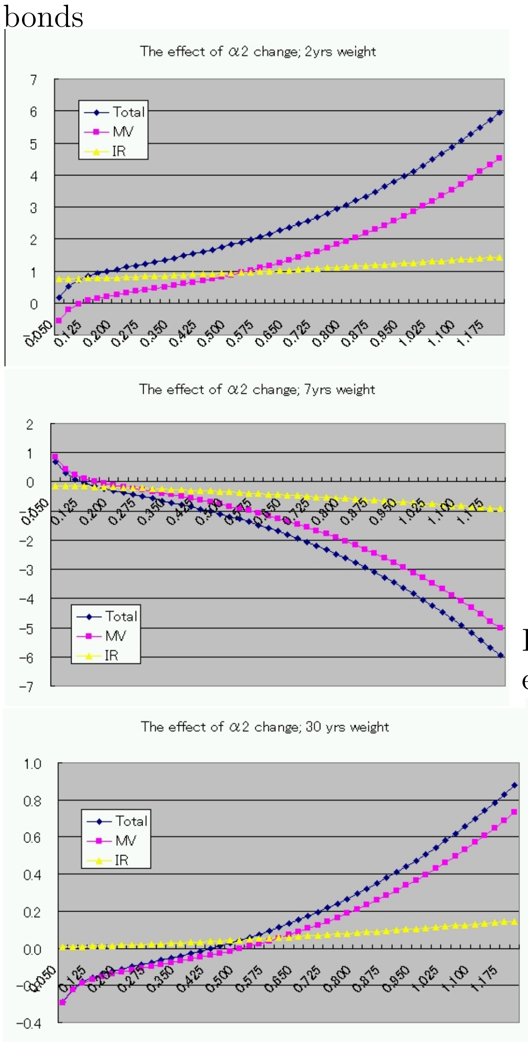


Figure 11: The effect of α_2 variation: exposure to X_2

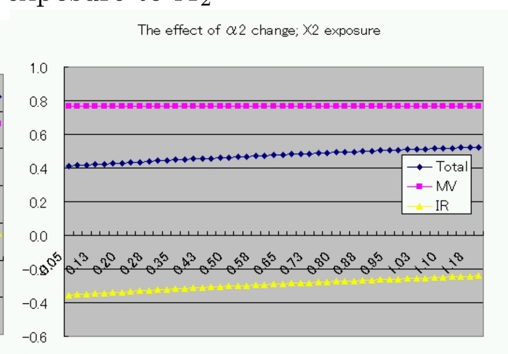
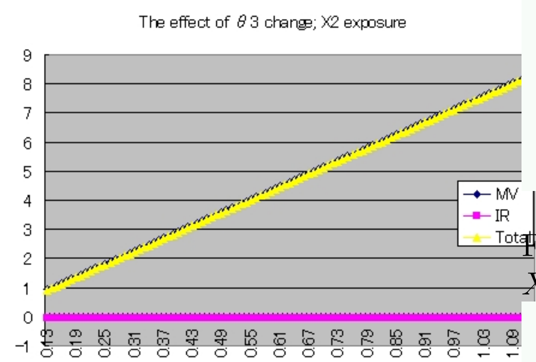


Figure 12: The effect of θ_3 variation: factor allocation



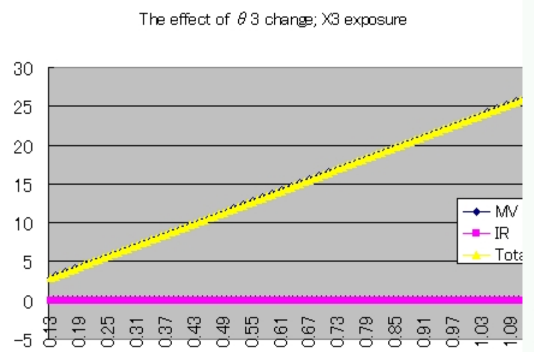


Figure 13: The effect of σ_2 variation: X_2

Figure 14: The effect of σ_3 variation: X_3

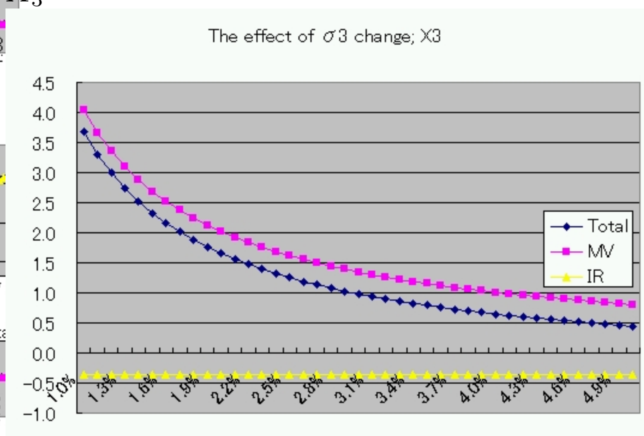


Table 4: θ_3 and ρ_{23}

θ_3	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Factor Allocation							
MV term							
X_2 ($\rho_{23} = -0.38$)	1.1103	1.6655	2.2207	2.7758	3.3310	3.8862	4.4413
X_3 ($\rho_{23} = -0.38$)	6.0246	9.0369	12.0492	15.0615	18.0738	21.0861	24.0984
X_2 ($\rho_{23} = 0$)	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
X_3 ($\rho_{23} = 0$)	5.1546	7.7320	10.3093	12.8866	15.4639	18.0412	20.6186
X_2 ($\rho_{23} = 0.38$)	-1.1103	-1.6655	-2.2207	-2.7758	-3.3310	-3.8862	-4.4413
X_3 ($\rho_{23} = 0.38$)	6.0246	9.0369	12.0492	15.0615	18.0738	21.0861	24.0984

Figure 15: The effect of *Investment Horizon* variation: bonds

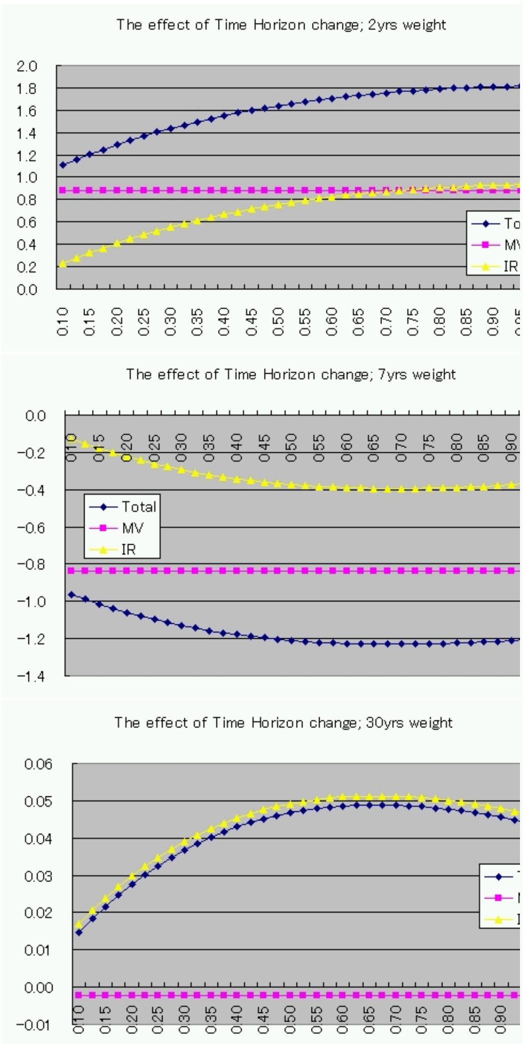


Figure 16: The effect of *Investment Horizon* variation: factor allocation

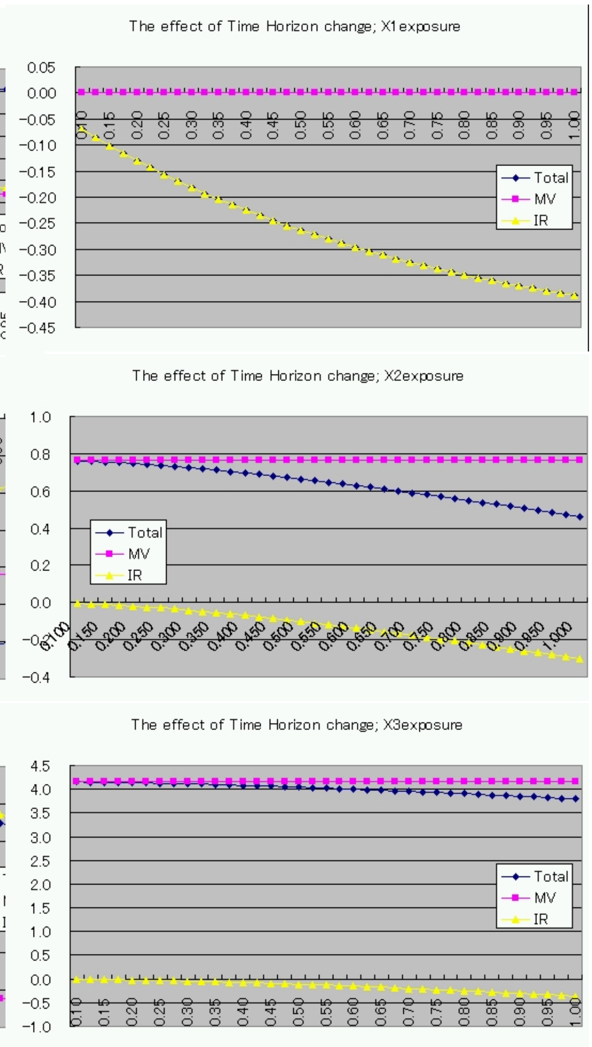


Figure 17: The effect of RRA variation: factor allocation

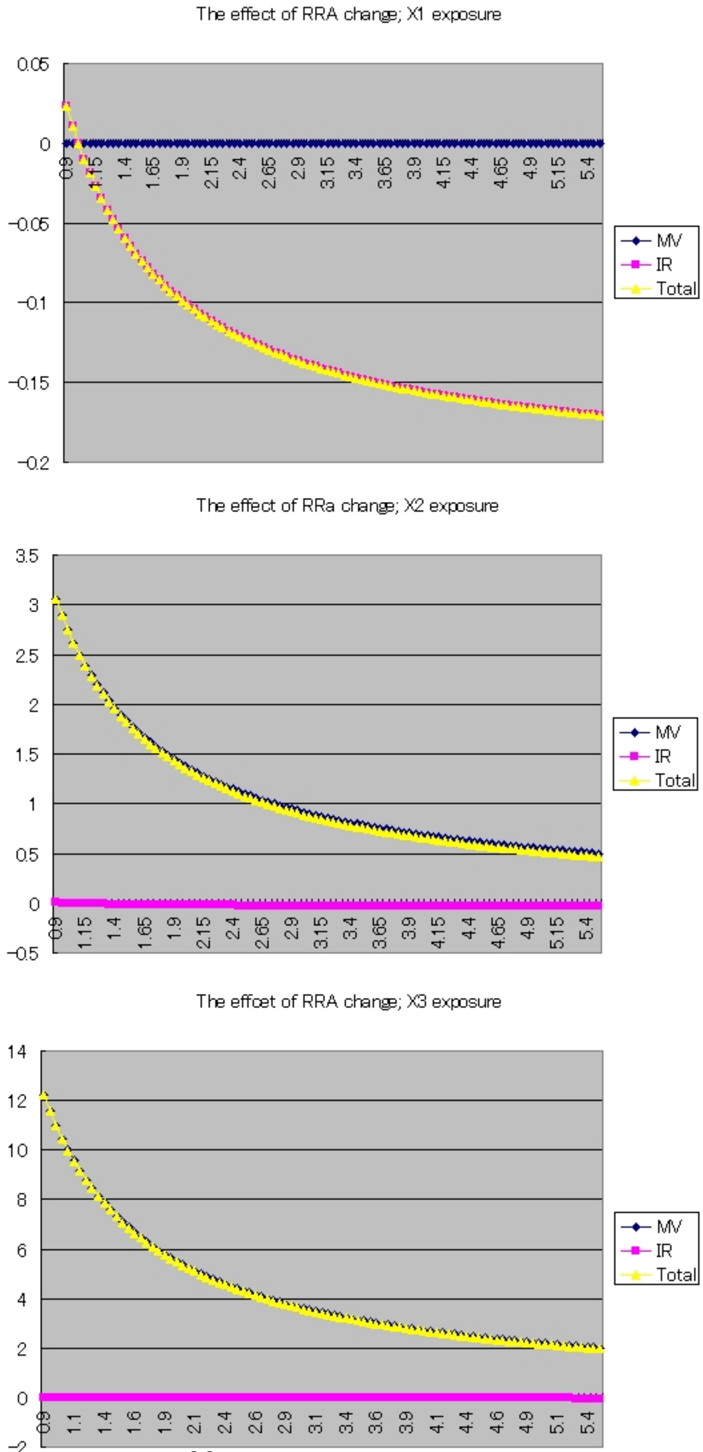


Table 5: The summary of effects

Parameter	The effect on the portfolio strategy (the factor allocation)	
	MV term	*IR Hedge term*
Term Structure		
α_1	——	$\alpha_1 \uparrow \Rightarrow X_1 \downarrow$, X_2 & $X_3 \uparrow$
α_2	——	$\alpha_2 \uparrow \Rightarrow X_2 \downarrow$
θ_3	$\theta_3 \uparrow \Rightarrow X_3 \uparrow$ $\rho_{23} < 0 \Rightarrow X3 \upuparrows$, $X_2 \uparrow$	——
σ_1	——	——
σ_2	$\sigma_2 \uparrow \Rightarrow X_2 \downarrow$	——
σ_3	$\sigma_3 \uparrow \Rightarrow X_3 \downarrow$	——
Investment strategy		
T (Investment Horizon)	——	T $\uparrow \Rightarrow X_1, X_2, X_3 \uparrow$
RRA (Risk Preference)	RRA > 1, RRA $\uparrow \Rightarrow X_2$ & $X_3 \downarrow$	RRA > 1, RRA $\uparrow \Rightarrow X_1$, X_2, $X_3 \uparrow$