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Macroeconomic Implications of Term Structures of Interest Rates under Stochastic Differential Utility with Non-Unitary EIS

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Abstract

Abstract: This paper proposes a continuous-time term-structure model under stochastic differential utility with non-unitary elasticity of intertemporal substitution (EIS, henceforth) in a representative-agent endowment economy with mean-reverting expectations on real output growth and inflation. Using this model, we make clear structural relationships among a term structure of real and nominal interest rates, utility form and underlying economic factors (in particular, inflation expectation). Notably, we show that, if (1) the EIS is less than one, (2) the agent is comparatively more risk-averse relative to time-separable utility, (3) short-term interest rates are pro-cyclical, and (4) the rate of expected inflation is negatively correlated with the rate of real output growth and its expected rate, then a nominal yield curve can have a low instantaneous riskless rate and an upward slope. *Keywords*: Stochastic differential utility; Non-unitary EIS; Term structure of interest rates; Inflation expectation. JEL codes: E43, G12.

1 Introduction

A term structure of interest rates plays a crucial role in practice. From a Macroeconomic perspective, investors and central banks obtain market information regarding future interest rates from bond yield curves. Also, from a Finance perspective, fixed-income markets trade a large amount of bonds and derivative securities sensitive to interest rates. The term structure of interest rates is used for pricing not only the bonds and the interest rate derivatives but also all other market securities.

Despite such importance of the term structure of interest rates, surprisingly, people know little about structural relationships among underlying economic factors, utility structure, and yield curves. From historical data, we know that, on average, a *nominal* yield curve slope up (Homer and Sylla (2005)). Based on standard term structure models such as Cox,

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Ingersoll, and Ross (1985), an upward-sloping *real* yield curve implies that, the *real* short-term rates should be counter-cyclical. On the other hand, from several empirical studies, we know that the *real* GDP growth rates are positively correlated with *nominal* short-term rates. How can we replicate the yield curves well by using structural economic factors?

To answer this question, there is a recent growing literature on the term structure of real and nominal interest rates using homothetic recursive-utility models. Notably, Piazzesi and Schneider (2006) predict in a recursive utility model in discrete time that, when inflation is bad news for consumption growth, the nominal yield curve slopes up, whereas the real yield curve slopes down. Also, Nakamura, Nakayama and Takahashi (2008) study a similar recursive utility model in continuous time and supports their results in a more rigorous way.[1] However, most of the previous recursive-utility models assume unitary elasticity of intertemporal substitution (EIS, henceforth) to achieve solvability for the analytical form of the yield curves. The assumption is restrictive in reality. For example, in those homothetic recursive utility models, the assumption results in a constant consumption/wealth ratio over time. According to several asset pricing papers, the result is not supported. Also, Chen, Favilukis and Ludvigson (2008) show empirically that EIS is different from one. Now, a question is raised: can a recursive-utility model with non-unitary EIS replicate an actual term structure of real/nominal interest rates better?

The purpose of this paper is to provide a framework to answer the question by constructing a continuous-time term structure model in environments with (i) stochastic differential utility (SDU, henceforth), a form of recursive utility in continuous time, with non-unitary EIS and (ii) mean-reverting expectations on the rates of inflation and real output growth.[2] Specifically, we find that, if (1) the EIS is strictly less than one, (2) the agent is comparatively more risk-averse relative to time-separable utility, (3) real short-term interest rates are pro-cyclical, and (4) the rate of expected inflation is negatively correlated with the rate of real output growth and its expected rate, then a nominal yield curve can have (i) a low instantaneous riskless rate and (ii) an upward slope. Intuitively, when the agent is characterized by the conditions (1) and (2), she has the motive of longing the instantaneous zero

[1] Their model replicates typical shapes of the term structures by controlling the structural parameters and the conditional variances/covariances of the state variables.

[2] We apply some theoretical results of SDU that are derived by Schroder and Skiadas (1999) and Skiadas (2007).

coupon bonds to mitigate income risk in the case that the expected income growth rate is positively correlated with the income growth rate (in which case the condition (3) holds true under some relevant conditions[3]). At the same time, she has the motive of shorting the nominal long-term bonds to mitigate income risk when the expected inflation rate is negatively correlated with the real output growth rate and its expected rate (that is, when the condition (4) holds true). This result resolves the risk-free rate puzzle, like Weil (1989) studies it in a discrete-time recursive utility model and, at the same time, produces an upward sloping nominal yield curve. Moreover, in this case, a higher level of the risk aversion results in a lower instantaneous riskless rate and a steeper upward-sloping nominal yield curve.

The main contributions of this paper are twofold. First, due to mathematical tractability of the continuous-time framework, this paper is successful in making clear relationships among the yield curves and some structural parameters of the economic environments and the utility form. In particular, this paper shows that non-unitary EIS and risk aversion to the uncertainty of future utility play a key role in determining the level and the slope of the real and nominal yield curves. Moreover, our paper probes more deeply into the effect of the expected inflation shock on the slope of the nominal yield curve. As a consequence, we show that, with regard to the role of monetary policy on the term structure of interest rates, higher credibility in price stability makes the upward-sloping nominal yield curve flatter.

Second, this paper is successful in solving numerically for yield curves. In general, it is difficult to examine quantitatively the term structure of interest rates under SDU with non-unitary EIS, because there is no closed-form solution of it. Against such difficulty, Hansen, Heaton, Roussanov and Lee (2008) derive the first-oder approximation around $\delta = 1$. In contrast, we obtain numerical results by using the regression-based Monte Carlo method of Gobet, Lemor and Warin (2005) for backward stochastic differential equations (BSDEs, henceforth).

In related literature, Duffie and Epstein (1992) and Duffie, Schroder and Skiadas (1997) look at a term structure model under SDU mainly with unitary EIS, and study the effect of the preference for the timing of resolution of uncertainty on the term structure. In contrast,

[3]For more details, see footnote 12 below.

this paper solves for real and nominal yield curves under SDU with non-unitary EIS, makes clear their relationships with the structural parameters of economic environments and the utility form, and draws macroeconomic implications from them.

This paper is organized as follows. The next section defines SDU. Section 3 sets up a real endowment economy under SDU and derives real yield curves. Section 4 extends the model into a nominal economy and derives nominal yield curves. Section 5 analyzes quantitatively the yield curves in relationships with macroeconomic factors and the utility form. The final section concludes. Several supplementary notes and proofs for theorems, propositions and lemmas are placed in Appendices.

2 Stochastic differential utility

This section defines stochastic differential utility (SDU, henceforth), a form of recursive utility in continuous time, of consumption. Time parameter is $t \in [0, T]$, where $T > 0$ is a given terminal time. Let $(\Omega, \{\mathcal{F}_t\}_{0 \le t \le T}, \mathbb{F}, P)$ denote a filtered probability space that satisfy the usual conditions. There are single non-storable consumption goods.

An agent consumes the consumption goods. A consumption process $\mathbf{c} = \{c_t;\, t \in [0, T]\}$ is assumed to be real-valued, non-negative and $\{\mathcal{F}_t\}_{0 \le t \le T}$-adapted, and satisfies some mathematical regularity conditions. The agent ranks her consumption plan $\mathbf{c}$ based on SDU of consumption $V_t(\mathbf{c})$ (we may also write simply V_t) for each $t \in [0, T]$, which is characterized by:

$$dV_t = -f(c_t, V_t)dt + \Sigma_t^\top dB_t; \quad V_T = 0. \tag{2.1}$$

B denotes a 2-dimensional Brownian motion defined on the probability space. The superscript $\top$ of a vector or a matrix represents its transpose. $f(c_t, V_t)$ is called a (normalized) aggregator. We focus attention on a particular form of the aggregator that is introduced by Schroder and Skiadas (1999): for constants α, β, δ,

$$f(c, v) \triangleq \begin{cases} (1+\alpha)\left\{\frac{c^{1-\delta}}{1-\delta}\,|v|^{\frac{\alpha}{1+\alpha}} - \beta v\right\} & (\text{if } \delta \neq 1), \\ (1+\alpha v)\left\{\log c - \frac{\beta}{\alpha}\log(1+\alpha v)\right\} & (\text{if } \delta = 1). \end{cases} \tag{2.2}$$

Call this type of the aggregator the Schroder-Skiadas (SS, henceforth) aggregator. We will use some theoretical results of Schroder and Skiadas (1999), without a further reference, in the remaining. For the details, see their paper. In Eq.(2.2), β denotes time preference. The reciprocal of δ (that is, $\frac{1}{\delta}$) denotes elasticity of intertemporal substitution (EIS, henceforth) where $\delta > 0$; non-unitary EIS corresponds to $\delta \neq 1$, whereas unitary EIS corresponds to $\delta = 1$. We will discuss the notion of α shortly below.

Put a parametric assumption:

Assumption 2.1 *$\beta \geq 0$ and* $\begin{cases} \alpha > -1 \quad \textit{and} \quad 1-\delta < \min\left\{1, \frac{1}{1+\alpha}\right\} & (\textit{if } \delta \neq 1), \\ \alpha \leq \beta \quad (\textit{if } \delta = 1). \end{cases}$

This assumption ensures the existence of a unique well-defined V_t for each consumption process $\mathbf{c}$. Moreover, V_0 is strictly increasing, concave, and homothetic in $\mathbf{c}$.

To interpret the parameter α intuitively, we take a monotonic transformation of the utility function V_t:

$$\hat{V}_t = \begin{cases} (V_t)^{\frac{1}{1+\alpha}} & \text{if} \quad \delta < 1; \\ -|V_t|^{\frac{1}{1+\alpha}} & \text{if} \quad \delta > 1; \\ \frac{1}{\alpha}\log(1+\alpha V_t) & \text{if} \quad \delta = 1. \end{cases} \tag{2.3}$$

Under the SS aggregator, the monotonically transformed utility process $\hat{V}_t$, which is ordinally equivalent to the original utility process V_t, is written as:

$$\hat{V}_t = \begin{cases} E_t\left[\int_t^T e^{-\beta(s-t)}\left(\frac{c_s^{1-\delta}}{1-\delta} + \frac{\alpha}{2}\hat{V}(s)^{-1}\left|\left|\sigma_{\hat{V}}(s)\right|\right|^2\right)ds\right] & (\text{when } \delta \neq 1), \\ E_t\left[\int_t^T e^{-\beta(s-t)}\left(\log c_s + \frac{\alpha}{2}\left|\left|\sigma_{\hat{V}}(s)\right|\right|^2\right)ds\right] & (\text{when } \delta = 1). \end{cases} \tag{2.4}$$

Note that standard time-separable utility, denoted by $\hat{V}_t^{(s)}$, is written as:

$$\hat{V}_t^{(s)} = \begin{cases} E_t\left[\int_t^T e^{-\beta(s-t)}\left(\frac{c_s^{1-\delta}}{1-\delta}\right)ds\right] & (\text{when } \delta \neq 1), \\ E_t\left[\int_t^T e^{-\beta(s-t)}\left(\log c_s\right)ds\right] & (\text{when } \delta = 1). \end{cases}$$

Thus, the utility $\hat{V}_t$ in Eq.(2.4) can be decomposed into two parts: (1) the time-separable utility ($\alpha = 0$) and (2) additional utility. In the second part, $||\sigma_{\hat{V}}(s)||^2$ stands for the uncertainty of the future utilities. When $\delta < 1$, since $\hat{V}_t > 0$ for all $t < T$,[4] the additional utility with $\alpha < 0$ causes an additional penalty for the uncertainty of the future utility, whereas the one with $\alpha > 0$ causes an additional reward for it. When $\delta < 1$, the agent is said to be *comparatively more risk-averse (relative to the time-separable utility)* if $-\alpha > 0$,

[4] For the proof of this claim, see appendices in Schroder and Skiadas (1999).

whereas the agent is said to be *comparatively less risk-averse (relative to the time-separable utility)* if $-\alpha < 0$.

When $\delta = 1$, $\hat{V}_t$ is not uniformly signed. Still, since the sign of the second part of the utility does not depend on the sign of $\hat{V}_t$, the same conclusions are valid when $\delta = 1$. On the other hand, when $\delta > 1$, the effect of the sign of α is reversed because $\hat{V}_t < 0$ for all $t < T$.

In either case of δ, the agent pays no attention to the uncertainty of the future utility when $\alpha = 0$; her utility, not only $\hat{V}$ but also V, is said to be time separable. When $\alpha \neq 0$, by contrast, the utility, not only $\hat{V}$ but also V, is said to be time-nonseparable.

3 Real yield curve

This section derives a real yield curve under SDU. We consider a representative-agent endowment economy. A representative agent is lived on $[0, T]$ and ranks a consumption plan based on the above-defined SDU.

There exist two state variables. The first state variable is the endowment of the consumption goods, denoted by e. The endowment process is exogenous and is governed by the following stochastic differential equation (SDE):

$$\frac{de_t}{e_t} = \mu_e(t)dt + \sigma_e^\top dB_t, \quad e_0 \in \mathbf{R}_+ \tag{3.1}$$

where $\mu_e(t) \triangleq \nu_t - \beta$ and $\sigma_e \in \mathbf{R}^{2\times 1}$ is a constant vector. ν_t stands for the expected endowment growth rate and is the second state variable, which is stochastic and, in particular, is mean-reverting:

$$d\nu_t = k(\bar{\nu} - \nu_t)dt + \sigma_\nu^\top dB_t, \quad \nu_0 \in \mathbf{R} \tag{3.2}$$

where $\bar{\nu}$ is a constant and $\sigma_\nu \in \mathbf{R}^{2\times 1}$ is a constant vector. $\bar{\nu}$ denotes the mean-reversion level of the expected endowment growth rate and k means the speed of the mean reversion.

From Skiadas (2007), under Assumption 2.1 and the above set-up, the equilibrium utility process V_t for $t \geq 0$ is well-defined for the following decoupled forward-backward stochastic differential equations (FBSDEs, henceforth):[5]

$$\begin{cases} \frac{de_t}{e_t} = \mu_e(t)dt + \sigma_e^\top dB_t, \quad e_0 \in \mathbf{R}_+, \\ d\nu_t = k(\bar{\nu} - \nu_t)dt + \sigma_\nu^\top dB_t, \quad \nu_0 \in \mathbf{R}, \\ dV_t = -f(e_t, V_t)dt + \Sigma_t^\top dB_t, \quad V_T = 0. \end{cases} \tag{3.3}$$

Assume that there is some function $J \in C^{1,2}([0,T] \times \mathbf{R}^2)$ such that $V_t = J(t, e_t, \nu_t)$. The equilibrium utility process $V_t = J(t, e_t, \nu_t)$ then satisfies:

$$dJ_t = -f(e_t, J_t)dt + \sigma_J(t)^\top dB_t. \tag{3.4}$$

where $\sigma_J \triangleq e\sigma_e \frac{\partial J}{\partial e} + \sigma_\nu \frac{\partial J}{\partial \nu}$. The derivation of Eq.(3.4) is shown in Appendix E.

In this equilibrium, a pricing kernel is written as:[6]

$$\pi_t = \exp\left\{ \int_0^t f_v(e_u, J_u)\, du \right\} f_c(e_t, J_t). \tag{3.5}$$

Note that $f_c(c,v) := \frac{\partial f(c,v)}{\partial c}$, $f_v(c,v) := \frac{\partial f(c,v)}{\partial v}$, and $f_{cv}(c,v) := \frac{\partial^2 f(c,v)}{\partial c \partial v}$ and so on. Under no arbitrage, the pricing kernel π_t satisfies the following equation, using an instantaneous riskless rate r_t and the market price of risk λ_t:

$$\pi_t = \exp\left\{ -\int_0^t r_u\, du \right\} \exp\left\{ -\int_0^t \lambda_u^\top\, dB_u - \frac{1}{2}\int_0^t ||\lambda_u||^2\, du \right\}.$$

Or, equivalently,

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \lambda_t^\top dB_t. \tag{3.6}$$

From Eq.(3.5) and Eq.(3.6), the market price of risk is specified as follows:

Lemma 3.1 *The market price of risk, λ_t, is given by*

$$\lambda_t = \sigma_e \left(-\frac{e f_{cc}^*}{f_c^*} \right) + \sigma_J(t) \left(-\frac{f_{cv}^*}{f_c^*} \right). \tag{3.7}$$

The superscript "$*$" of f and its partial derivatives denotes that they are evaluated at equilibrium values (that is, $c = e$ and $v = J$); that is, define $f^* \triangleq f(e, J)$, $f_c^* \triangleq f_c(e, J)$ and $f_v^* \triangleq f_v(e, J)$ in an abbreviated form.

[5] We can show the existence of the equilibrium in the endowment economy.

[6] The existence of the pricing kernel is ensured in our model. Also, see Skiadas (2007).

On the other hand, with regard to the spot rate r_t, since $r_t = -\mathcal{D}\pi_t/\pi_t$ in Eq.(3.6) where $\mathcal{D}\pi_t$ denotes the drift coefficient of π_t, we obtain the following lemma in a similar way to the above proof of Lemma 3.1:

Lemma 3.2 *The instantaneous riskless rate is given by:*

$$r_t = -f_v(e_t, J_t) - \frac{\mathcal{D}\, f_c(e_t, J_t)}{f_c(e_t, J_t)}. \tag{3.8}$$

Now, substitute the SS aggregator Eq.(2.2) into the above Eq.(3.7) and Eq.(3.8). We then obtain the following proposition:

Proposition 3.1 *Under the SS aggregator in Eq.*(2.2)*, the instantaneous riskless rate r_t and the market price of risk λ_t are specified in equilibrium as: when $\delta \neq 1$,*

$$\begin{aligned} r_t &= \beta + \delta\mu_e(t) - \frac{1}{2}\delta(1+\delta)\,||\sigma_e||^2 + \alpha\delta\frac{\sigma_e^\top \sigma_J(t)}{(1+\alpha)J_t} + \frac{\alpha}{2}\left|\left|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right|\right|^2 && (3.9)\\ &= \beta + \delta\mu_e(t) - \frac{\delta}{2}||\sigma_e||^2 - \frac{1}{2}||\lambda_t||^2 + \frac{\alpha(1+\alpha)}{2}\left|\left|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right|\right|^2 && (3.10)\\ \lambda_t &= \delta\sigma_e - \alpha\frac{\sigma_J(t)}{(1+\alpha)J_t}. && (3.11)\end{aligned}$$

When $\delta = 1$,

$$\begin{aligned} r_t &= \beta + \mu_e(t) - ||\sigma_e||^2 + \alpha\frac{\sigma_e^\top \sigma_J(t)}{(1+\alpha J_t)} && (3.12)\\ &= \beta + \mu_e(t) - \frac{1}{2}\,||\sigma_e||^2 - \frac{1}{2}||\lambda_t||^2 + \frac{1}{2}\alpha^2\left|\left|\frac{\sigma_J(t)}{(1+\alpha J_t)}\right|\right|^2 && (3.13)\\ \lambda_t &= \sigma_e - \alpha\frac{\sigma_J(t)}{(1+\alpha J_t)}. && (3.14)\end{aligned}$$

We provide the results with some intuitive interpretations as follows. First, with regard to the instantaneous riskless rate r_t, let us see Eq.(3.9) under $\delta \neq 1$ and Eq.(3.12) under $\delta = 1$. The first and second terms on the right hand side $\beta + \delta\mu_e(t) = (1-\delta)\beta + \delta\nu_t$ stands for the instantaneous return that the investor would demand if $\sigma_e = \sigma_\nu = \sigma_J(t) = 0$ (i.e., income is deterministic). These terms exist under $\alpha = 0$ (time separable utility) as well.

In the third term "$-\frac{1}{2}\frac{f^*_{ccc}}{f^*_c}\,||e_t\sigma_e||^2 = -\frac{\delta(1+\delta)}{2}\,||\sigma_e||^2$," f^*_{ccc} determines the sign of this term. In this model, $f^*_{ccc} > 0$ while $f^*_{cc} < 0$. Hence, the third term is negative. From an economic point of view, f^*_{ccc} implies prudence of the investor, that is, the strength of the investor's motive to make extra (i.e., precautionary) savings caused by future income being random rather than deterministic. Intuitively, when the investor is prudent (that is, $f^*_{ccc} > 0$), $-f^*_{cc}$ is decreasing in e; that is, the investor is more risk averse when her income level is lower. Accordingly, when her income is stochastic, the prudent investor has an incentive to hedge the downward income risk. Therefore, she demands bonds to hedge the risk, and can purchase them even when the riskless return is low; the equilibrium instantaneous riskless return is lowered. Note that in such stochastic income environments, this term exists when the utility is time separable (i.e., $\alpha = 0$) as well.

The fourth term "$-\frac{f^*_{ccv}}{f^*_c} e_t \sigma_e^\top \sigma_J(t) = \delta\alpha \frac{\sigma_e^\top \sigma_J(t)}{(1+\alpha)J_t}$" of Eq.(3.9) is specific to the time non-separable utility (i.e., $\alpha \neq 0$). With $\sigma_e^\top \sigma_J(t)$ given, the sign of the term depends on the sign of $-\frac{f^*_{ccv}}{f^*_c} e_t = \frac{\alpha\delta}{(1+\alpha)J_t}$. Recall that $f^*_c = (1+\alpha)(e_t)^{-\delta}|J_t|^{\frac{\alpha}{1+\alpha}}$ and $f^*_{ccv} = -\alpha\delta(e_t)^{-\delta-1}|J_t|^{-\frac{1}{1+\alpha}}\mathrm{sgn}(J_t)$. Since $\delta > 0$, $\alpha > -1$, and thus $f^*_c > 0$, the sign of this fourth term depends on the sign of $-f^*_{ccv}$, i.e., $\mathrm{sgn}(\alpha)\mathrm{sgn}(J_t)$. Recall that J_t is positive (negative) when $\delta < 1$ (when $\delta > 1$, respectively). Now, suppose $\sigma_e^\top \sigma_J(t) > 0$, that is, that when the endowment growth rate increases, the expected discounted utility increases (vice versa). When the agent is comparatively less risk-averse (that is, either when $\delta < 1$ and $\alpha > 0$ or when $\delta > 1$ and $\alpha < 0$), the fourth term is positive; it pushes up the equilibrium instantaneous riskless rate. An intuitive interpretation is as follows. Since $f^*_{ccv} < 0$, $-f^*_{cc}$ is increasing in J. Under $\sigma_e^\top \sigma_J(t) > 0$, when her income is lower, the investor becomes less risk-averse. Therefore, she has an incentive to sell (i.e., take a short position of) instantaneous zero-coupon bonds. On the other hand, suppose that the agent is comparatively more risk-averse (that is, either that $\delta > 1$ and $\alpha > 0$ or that $\delta < 1$ and $\alpha < 0$). Since $f^*_{ccv} > 0$, the effect of the fourth term is reversed. Also, when $\sigma_e^\top \sigma_J(t) < 0$, it is reversed. When $\delta = 1$, the interpretation of α is the same as in the case of $\delta < 1$ because, by construction, $(1 + \alpha J_t)$ is positive.

The fifth term "$-\frac{1}{2}\frac{f^*_{cvv}}{f^*_c}||\sigma_J(t)||^2 = \frac{\alpha}{2}\left|\left|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right|\right|^2$" of Eq.(3.9) is specific to the time non-separable utility (i.e., $\alpha \neq 0$) and non-unitary EIS (i.e., $\delta \neq 1$); the corresponding term does not exist in Eq.(3.12) under $\delta = 1$. Recall that $f^*_{cvv} = -\frac{\alpha}{(1+\alpha)}(e_t)^{-\delta}|J_t|^{\frac{-2-\alpha}{1+\alpha}}$. Accordingly, in the fifth term of Eq.(3.9), f^*_{cvv} is positive (negative) when $\alpha < 0$ (when $\alpha > 0$, respectively). Suppose $\alpha < 0$. Then, the marginal utility of the consumption f^*_c is convex in J. Therefore, due to Jensen's inequality, the utility uncertainty $||\sigma_J(t)||^2 > 0$ results in a higher level of the expected marginal utility of consumption (that is, the zero-coupon bond price) than in case that f_c^* is linear in J_t. Call this the "convexity effect" of the fifth term. Due to this effect, the equilibrium instantaneous riskless rate is lowered; the fifth term is negative. When $\alpha > 0$, the effect is reversed – call this the "concavity effect." Note that this effect of this term is independent of $\mathrm{sgn}(J)$. Notably, since $f_{cvv}^* = 0$ in case of $\delta = 1$, the term does not exist in the unitary-EIS case; it is specific to the case of $\delta \neq 1$ under the time nonseparable utility.

Next, with regard to the market price of risk λ, from Eq.(3.7),

$$\lambda_t = \sigma_e \left(-\frac{e f_{cc}^*}{f_c^*} \right) + \sigma_J(t) \left(-\frac{f_{cv}^*}{f_c^*} \right).$$

In the first term on the right hand side, $-\frac{e f_{cc}^*}{f_c^*} = \delta$ is relative risk aversion against the income risk under the part of the time separable utility. Hence, the first term stands for the risk price of income growth uncertainty. On the other hand, the second term $\sigma_J(t) \left(-\frac{f_{cv}^*}{f_c^*} \right)$ is specific to the time nonseparable utility (i.e., $\alpha \neq 0$). f_{cv}^* represents how a small change of J_t change the marginal utility of consumption f_c^*. Thus, in parallel to the the first term, $-\frac{f_{cv}^*}{f_c^*}$ means the additional risk price through J per one unit of $\sigma_J(t)$.

Let $P(t,s)$ and $R(t,s)$ denote time-t price of zero coupon bonds maturing at time s and the spot yields from time t to time s, respectively:

$$P(t,s) = E_t \left[\exp\left\{ -\int_t^s r_u du \right\} \exp\left\{ -\int_t^s \lambda_u^\top dB_u - \frac{1}{2} \int_t^s ||\lambda_u||^2 du \right\} \right], \tag{3.15}$$

$$R(t,s) = -\frac{1}{s-t} \log P(t,s). \tag{3.16}$$

From Eq.(3.15) and Eq.(3.16),

$$R(t,s) = -\frac{1}{s-t} \log E_t \left[\exp\left\{ -\int_t^s \left(r_u + \frac{1}{2} ||\lambda_u||^2 \right) du \right\} \exp\left\{ -\int_t^s \lambda_u^\top dB_u \right\} \right]. \tag{3.17}$$

From Eq.(3.10) and Eq.(3.13),

$$r_t + \frac{1}{2} ||\lambda_t||^2 = \begin{cases} \beta + \delta \mu_e(t) - \frac{\delta}{2} ||\sigma_e||^2 + \frac{\alpha(1+\alpha)}{2} \left|\left| \frac{\sigma_J(t)}{(1+\alpha)J_t} \right|\right|^2 & (\text{if } \delta \neq 1), \\ \beta + \mu_e(t) - \frac{1}{2} ||\sigma_e||^2 + \frac{1}{2} \alpha^2 \left|\left| \frac{\sigma_J(t)}{(1+\alpha J_t)} \right|\right|^2 & (\text{if } \delta = 1), . \end{cases}$$

Substituting these and Eq.(3.11) and Eq.(3.14) into Eq.(3.17),

$$R(t,s) = \begin{cases} -\frac{1}{s-t}\log E_t \left[\begin{array}{c} \exp\left\{ -\int_t^s \left(\beta + \delta\mu_e(u) - \frac{\delta}{2}||\sigma_e||^2 + \frac{\alpha(1+\alpha)}{2}\left|\left|\frac{\sigma_J(u)}{(1+\alpha)J_u}\right|\right|^2 \right) du \right\} \cdot \\ \exp\left\{ -\int_t^s \left(\delta\sigma_e - \alpha\frac{\sigma_J(u)}{(1+\alpha)J_u} \right)^\top dB_u \right\} \end{array} \right] & (\text{if } \delta \neq 1), \\ -\frac{1}{s-t}\log E_t \left[\begin{array}{c} \exp\left\{ -\int_t^s \left(\beta + \mu_e(u) - \frac{1}{2}||\sigma_e||^2 + \frac{1}{2}\alpha^2 \left|\left|\frac{\sigma_J(u)}{(1+\alpha J_u)}\right|\right|^2 \right) du \right\} \cdot \\ \exp\left\{ -\int_t^s \left(\sigma_e - \alpha\frac{\sigma_J(u)}{(1+\alpha J_u)} \right)^\top dB_u \right\} \end{array} \right] & (\text{if } \delta = 1). \end{cases} \tag{3.18}$$

When $\delta = 1$, we obtain a closed-form, analytical solution of the term structure of interest rates. The solution in the case of $T \to \infty$ is placed in Appendix A. For the derivation of it, see Nakamura, Nakayama, and Takahashi (2008). On the other hand, when $\delta \neq 1$, we obtain no explicit, analytical solution of the spot yields, because this model with $\delta \neq 1$ obtains no closed-form solution of J. Accordingly, we solve numerically the decoupled FBSDEs (3.3). For the details of our numerical method, see Appendix B.

4 Nominal yield curve

So far we have confined attention to the real economy. However, in practice, most fixed income products pay in nominal terms, not in real terms. A real zero coupon bond is a security that pays one unit of consumption goods at its maturity, whereas a nominal zero coupon bond pays one unit of currency at its maturity. This section investigates the nominal term structure by introducing a price index process.

First, set two additional state variables: the price index and its expected growth rate (that is, the expected inflation rate). In particular, the expected inflation process follows a mean-reversion process. Precisely, let N_t denote the price index process and ε is its expected inflation rate as follows:

$$\begin{aligned} \frac{dN_t}{N_t} &= \varepsilon_t dt + \sigma_n^\top dB_t, \quad N_0 \in \mathbf{R}_+ \\ d\varepsilon_t &= \theta(\bar{\varepsilon} - \varepsilon_t)dt + \sigma_\varepsilon^\top dB_t, \quad \varepsilon_0 \in \mathbf{R} \end{aligned}$$

where θ (the speed of the mean reversion) is a positive constant, $\bar{\varepsilon}$ (the mean-reversion level of the expected inflation rate) is a constant. Also, σ_n and $\sigma_\varepsilon \in \mathbf{R}^{4\times 1}$ are constant vectors. The processes defined in the previous subsections are modified appropriately.

The pricing equation is as follows: with regard to the nominal price of any asset $\hat{P}_t$,

$$\frac{\hat{P}_t}{N_t} = E_t\left[\frac{\pi_s}{\pi_t}\frac{\hat{P}_s}{N_s}\right].$$

Note that, for any variable x in real terms, $\hat{x}$ denotes the nominal value of x. In particular, with regard to the nominal bond that pays one unit of currency at maturity s.

$$\frac{\hat{P}_t}{N_t} = E_t\left[\frac{\pi_s}{\pi_t}\frac{1}{N_s}\right]. \tag{4.1}$$

Using Eq.(4.1), look at the role of the inflation factors (that is, the price index process N and the expected inflation process ε) in the equilibrium pricing. Decompose the right hand side of the equilibrium pricing formula Eq.(4.1) into two parts: the real pricing kernel $\frac{\pi_s}{\pi_t}$ and the real payoff at the maturity $\frac{1}{N_s}$. This model implicitly assumes that the agent maximizes his utility of real consumption, not of nominal one. In such economic circumstances, the real pricing kernel $\frac{\pi_s}{\pi_t}$ is the same as the one in the previous real economy.[7] In other words, the inflation factors influence the equilibrium price only through the real payoff, not through the real pricing kernel. A higher (lower) level of the price index depreciates (increases, respectively) the real value of the nominal payoff. Hence, when the inflation factors covariate more positively with the real pricing kernel, the price (the premium) of the nominal bond declines (increases, respectively).

Specifically, Let $\hat{P}(t,s)$ and $\hat{R}(t,s)$ denote time-t price of zero coupon bonds maturing at time s and the spot yields from time t to time s, respectively:

$$\hat{P}(t,s) = E_t\left[\exp\left\{-\int_t^s r_u du\right\}\exp\left\{-\int_t^s \lambda_u^\top dB_u - \frac{1}{2}\int_t^s ||\lambda_u||^2 du\right\}\cdot\frac{N_t}{N_s}\right], \tag{4.2}$$

$$\hat{R}(t,s) = -\frac{1}{s-t}\log\hat{P}(t,s). \tag{4.3}$$

From Eq.(4.2) and Eq.(4.3),

$$\hat{R}(t,s) = -\frac{1}{s-t}\log E_t\left[\exp\left\{-\int_t^s r_u du\right\}\exp\left\{-\int_t^s \lambda_u^\top dB_u - \frac{1}{2}\int_t^s ||\lambda_u||^2 du\right\}\cdot\frac{N_t}{N_s}\right].$$

[7]Suppose, on the contrary to our model, that the agent maximizes his utility of *nominal* consumption. Then, the inflation factors can influence the real pricing kernel.

Applying Ito's formula to $\frac{1}{N_t}$ simply,

$$\begin{aligned} d\left(\frac{1}{N_t}\right) &= -\frac{1}{(N_t)^2}dN_t + \frac{1}{N_t}\left|\left|\sigma_n\right|\right|^2 dt \\ &= \frac{1}{N_t}\left\{\left(-\varepsilon_t + \left|\left|\sigma_n\right|\right|^2\right)dt - \sigma_n^\top dB_t\right\} \end{aligned}$$

Thus,

$$\begin{aligned} \hat{R}(t,s) &= -\frac{1}{s-t}\log E_t\left[\begin{array}{c} \exp\left\{-\int_t^s r_u du\right\}\exp\left\{-\int_t^s \lambda_u^\top dB_u - \frac{1}{2}\int_t^s \left|\left|\lambda_u\right|\right|^2 du\right\}\cdot \\ \exp\left\{-\int_t^s\left(\varepsilon_u - \left|\left|\sigma_n\right|\right|^2\right)du\right\}\exp\left\{-\int_t^s \sigma_n^\top dB_u - \frac{1}{2}\int_t^s \left|\left|\sigma_n\right|\right|^2 du\right\} \end{array}\right] \\ &= -\frac{1}{s-t}\log E_t\left[\begin{array}{c} \exp\left\{-\int_t^s\left(r_u + \varepsilon_u - \left|\left|\sigma_n\right|\right|^2 - \sigma_n^\top\lambda_u\right)du\right\}\cdot \\ \exp\left\{-\int_t^s\left(\lambda_u + \sigma_n\right)^\top dB_u - \frac{1}{2}\int_t^s \left|\left|\lambda_u + \sigma_n\right|\right|^2 du\right\} \end{array}\right] \\ &= -\frac{1}{s-t}\log E_t\left[\begin{array}{c} \exp\left\{-\int_t^s \hat{r}_u du\right\}\cdot \\ \exp\left\{-\int_t^s \hat{\lambda}_u^\top dB_u - \frac{1}{2}\int_t^s \left|\left|\hat{\lambda}_u\right|\right|^2 du\right\} \end{array}\right] \qquad (4.4) \end{aligned}$$

where $\hat{r}_t$ and $\hat{\lambda}_t$ are defined as:

$$\hat{r}_t = r_t + \varepsilon_t - \left|\left|\sigma_n\right|\right|^2 - \sigma_n^\top\lambda_t, \qquad (4.5)$$

$$\hat{\lambda}_t = \lambda_t + \sigma_n. \qquad (4.6)$$

From an analogue of Eq.(3.17), $\hat{r}_t$ and $\hat{\lambda}_t$ can be interpreted as a nominal instantaneous riskless rate and the nominal market price of risk, respectively.

Similarly to the arguments regarding the real yield curve in the previous section, when $\delta = 1$, we obtain a closed-form, analytical solution of the term structure of interest rates (see Appendix A), whereas, when $\delta \neq 1$, we numerically solve for the yield curve. For the numerical method, see Appendix B.

5 Quantitative analysis: Macroeconomic implications

In this section, we draw macroeconomic implications from quantitative results regarding the real and the nominal yield curves under the SDU characterized by the SS aggregator. We confine attention to the case of non-unitary EIS ($\delta \neq 1$), which takes on the value either of

0.5 or 1.5 in our numerical analyses.[8] This magnitude of EIS (i.e., $1/\delta$) that we focus on is similar to the ones chosen by Bansal and Yaron (2004) and Hansen, Heaton, Roussanov and Lee (2008). In fact, it is consistent with a large previous empirical literature.[9] With regard to α, we set $\alpha \in \{0,\ 0.9\}$ when $\delta = 0.5$ and $\alpha \in \{0,\ 3,\ 9\}$ when $\delta = 1.5$. Note that we restrict α to be non-negative, although α can take negative values. This is because, with α fixed at a positive value, by changing from $\delta = 0.5$ into $\delta = 1.5$, the role of α is reversed; when $\delta = 0.5$, the agent is comparatively less risk-averse, whereas when $\delta = 1.5$, she is comparatively more risk-averse. Therefore, our model is rich enough under the restriction of $\alpha \geq 0$. Set $\beta = 0.01$.

Under the above-specified underlying economic structures, we take a sufficiently large value of T; we set $T = 400$ years. The instantaneous riskless rate and the spot yields are evaluated at time $t = 0$. We set the parameters and the variance and covariance matrices of the state variables as in Table 1.

For simplicity, we assume that the correlation between inflation and real factors are zero. By doing so, we confine attention to the effect of the expected inflation on a nominal term structure of interest rates.

Table 1: Set of Parameters

For $\delta = 0.5$.

β	0.01	α	0 or 0.9
$\|\|\sigma_e\|\|$	0.05	$\|\|\sigma_n\|\|$	0.1
ν_0	0.03	ε_0	0.03
$\bar{\nu}$	0.03	$\bar{\varepsilon}$	0.03
k	0.5	θ	0.1
$\|\|\sigma_\nu\|\|$	0.02	$\|\|\sigma_\varepsilon\|\|$	0.02
$\rho_{e\nu}$	0.5 or -0.5	$\rho_{e\varepsilon}, \rho_{\nu\varepsilon}$	0.5 or -0.5
		$\rho_{en}, \rho_{\nu n}, \rho_{\varepsilon n}$	0

For $\delta = 1.5$.

β	0.01	α	0, 3 or 9
$\|\|\sigma_e\|\|$	0.05	$\|\|\sigma_n\|\|$	0.05
ν_0	0.03	ε_0	0.03
$\bar{\nu}$	0.03	$\bar{\varepsilon}$	0.03
k	1	θ	0.05
$\|\|\sigma_\nu\|\|$	0.02	$\|\|\sigma_\varepsilon\|\|$	0.01
$\rho_{e\nu}$	0.5 or -0.5	$\rho_{e\varepsilon}, \rho_{\nu\varepsilon}$	0.5 or -0.5
		$\rho_{en}, \rho_{\nu n}, \rho_{\varepsilon n}$	0

5.1 Level of the yield curve

Let us examine the real instantaneous riskless rate at time $t = 0$, which we regard as the level of the yield curve in real terms. In Eq.(3.9), with δ fixed and α changed, only the fourth term $\alpha\delta\frac{\sigma_e^\top \sigma_J(t)}{(1+\alpha)J_t}$ and the fifth term $\frac{1}{2}\alpha\left|\left|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right|\right|^2$ can be changed. Look at the case of

[8] Under the SS aggregator, the utility form with the unitary EIS is not a limit of the one with the non-unitary EIS as $\delta \to 1$, because of the technical reasons.

[9] For the reference, see Bansal and Yaron (2004).

Table 2: The values of the covariance processes at time 0: $\delta = 0.5,\ \alpha = 0.9$.

	$\frac{\sigma_e^\top \sigma_J}{(1+\alpha)J_t}$	$\frac{\sigma_\nu^\top \sigma_J}{(1+\alpha)J_t}$	$\frac{\sigma_n^\top \sigma_J}{(1+\alpha)J_t}$	$\frac{\sigma_\varepsilon^\top \sigma_J}{(1+\alpha)J_t}$	$\left\|\left\|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right\|\right\|^2$
$\rho_{e\nu} = 0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = 0.5$	0.00171	0.000631	-0.0000000827	0.000438	0.00145
$\rho_{e\nu} = 0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = -0.5$	0.00171	0.000631	-0.0000000827	-0.000438	0.00145
$\rho_{e\nu} = -0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = 0.5$	0.000726	0.000144	0.0000000598	0.000435	0.000489
$\rho_{e\nu} = -0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = -0.5$	0.000726	0.000144	0.0000000598	-0.000435	0.000489

$\delta = 0.5$. Set $\alpha \in \{0,\ 0.9\}$. From the numerical results (Table 2), the covariance between the endowment growth rate and the utility in equilibrium (i.e., $\sigma_e^\top \sigma_J(t)$) is positive at time $t = 0$; $\frac{\sigma_e^\top \sigma_J(t)}{(1+\alpha)J_t}$ is 0.00171 (when $\rho_{e\nu} = 0.5$) and 0.000726 (when $\rho_{e\nu} = -0.5$) at time 0. When $\alpha > 0$ and $\delta < 1$ (that is, the agent is comparatively less risk-averse), the investor possesses the motive of shorting the bonds; the fourth term is positive. Also, the fifth term is positive due to the concavity effect. Accordingly, a positive level of α pushes up the level of yield curve in comparison with the case of the time separable utility ($\alpha = 0$). Moreover, a higher level of α causes a higher level of the motive of shorting the bonds in the fourth term and a higher level of the concavity effect in the fifth term. Therefore, a higher α results in a higher level of the instantaneous riskless rate. In our examples, the correlation between the endowment growth rate and the expected endowment growth rate ($\rho_{e\nu}$) takes on the value either of 0.5 or -0.5. From Figure 2, when $\alpha = 0.9$, $\rho_{e\nu} > 0$ results in a slightly higher level of the yield curve in comparison with $\rho_{e\nu} < 0$. The reason is as follows. From Table 2, both $\left|\left|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right|\right|^2$ and $\frac{\sigma_e^\top \sigma_J(t)}{(1+\alpha)J_t}$ are larger when $\rho_{e\nu} > 0$ than when $\rho_{e\nu} < 0$ at time $t = 0$.[10] Accordingly, the fourth term and the fifth term both are larger when $\rho_{e\nu} = 0.5$ than when $\rho_{e\nu} = -0.5$; the yield curve is lifted up higher when $\rho_{e\nu} = 0.5$. With a higher level of α, the lift is higher.

Next, look at the case of $\delta = 1.5$. We take $\alpha \in \{0,\ 3,\ 9\}$. From the numerical results (Table 3), the covariance between the endowment growth rate and the utility in equilibrium (i.e., $\sigma_e^\top \sigma_J(t)$) is positive at time $t = 0$, as in the case of $\delta = 0.5$; since $J < 0$, $\frac{\sigma_e^\top \sigma_J(t)}{(1+\alpha)J_t}$ is -0.00146 (when $\rho_{e\nu} = 0.5$) and -0.00100 (when $\rho_{e\nu} = -0.5$) for $\alpha = 3$ and -0.00140 (when $\rho_{e\nu} = 0.5$) and -0.00101 (when $\rho_{e\nu} = -0.5$) for $\alpha = 9$ at time 0. When $\alpha > 0$ and $\delta > 1$ (that is, the agent is comparatively more risk-averse), the fourth term $\alpha\delta\frac{\sigma_e^\top \sigma_J(t)}{(1+\alpha)J_t}$ is negative, implying that the investor possesses the motive of holding the bonds. This is a contrast to

[10]Note that these results are consistent with the case of $\delta = 1$ that is shown in Appendix A.

Table 3: The values of the covariance processes at time 0: $\delta = 1.5$.

		$\frac{\sigma_e^\top \sigma_J}{(1+\alpha)J_t}$	$\frac{\sigma_\nu^\top \sigma_J}{(1+\alpha)J_t}$	$\frac{\sigma_n^\top \sigma_J}{(1+\alpha)J_t}$	$\frac{\sigma_\varepsilon^\top \sigma_J}{(1+\alpha)J_t}$	$\left\|\left\|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right\|\right\|^2$
$\alpha = 3$	$\rho_{e\nu} = 0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = 0.5$	-0.00146	-0.000437	0.0000000333	-0.000170	0.000309
	$\rho_{e\nu} = 0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = -0.5$	-0.00146	-0.000437	0.0000000333	0.000170	0.000309
	$\rho_{e\nu} = -0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = 0.5$	-0.00100	0.0000523	0.0000000307	-0.000174	0.000158
	$\rho_{e\nu} = -0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = -0.5$	-0.00100	0.0000523	0.0000000307	0.000174	0.000158
$\alpha = 9$	$\rho_{e\nu} = 0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = 0.5$	-0.00140	-0.000417	0.0000000318	-0.000163	0.000282
	$\rho_{e\nu} = 0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = -0.5$	-0.00140	-0.000417	0.0000000318	0.000163	0.000282
	$\rho_{e\nu} = -0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = 0.5$	-0.00101	0.0000528	0.0000000310	-0.000175	0.000160
	$\rho_{e\nu} = -0.5,\ \rho_{e\varepsilon}, \rho_{\nu\varepsilon} = -0.5$	-0.00101	0.0000528	0.0000000310	0.000175	0.000160

the above case of $\delta = 0.5$. The fifth term $\frac{1}{2}\alpha \left|\left|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right|\right|^2$ is positive due to the concavity effect, as in the case of $\delta = 0.5$. A higher level of α causes a higher level of the motive of purchasing the bonds in the fourth term and a higher level of the concavity effect of the fifth term. Since these two effects are opposite, the total effect is uncertain analytically. In the numerical examples (Figure 1), $\left|\left|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right|\right|^2$ is approximately $0.00016 \sim 0.00031$ at time 0. Accordingly, the fourth term is stronger than the fifth term; the level of the yield curve is lower when $\alpha = 9$ than when $\alpha = 3$, in either case of $\rho_{e\nu} = 0.5$ or $\rho_{e\nu} = -0.5$. More precisely, $\frac{\sigma_e^\top \sigma_J(t)}{(1+\alpha)J_t}$ is smaller when $\rho_{e\nu} = 0.5$ than when $\rho_{e\nu} = -0.5$. This is consistent with the case of $\delta = 1$ that is shown in Appendix A, as discussed in footnote 10. Following the logic, we can guess that, when $\rho_{e\nu}$ is lower than -0.5, $\frac{\sigma_e^\top \sigma_J(t)}{(1+\alpha)J_t}$ could be larger; moreover, it could be positive. The total effect of the fourth term and the fifth term could then be positive. If so, on the contrary to the above numerical result, the level of the yield curve would be higher when $\alpha > 0$ than when $\alpha = 0$. On the other hand, by contrast, when $\rho_{e\nu} > 0$ and $\alpha > 0$, the level of the yield curve is low.

Next, with regard to the level of the nominal yield curve, from Eq.(4.5), the difference between the real instantaneous riskless rate and the nominal one is $\hat{r}_t - r_t = \varepsilon_t - ||\sigma_n||^2 - \sigma_n^\top \lambda_t$. With regard to the far right term, from Eq.(3.11), $\sigma_n^\top \lambda_t = \sigma_n^\top \left(\delta\sigma_e - \alpha\frac{\sigma_J(t)}{(1+\alpha)J_t}\right)$. Due to the parametric assumption of zero correlation between the inflation rate and the real factors and between the inflation rate and the expected inflation rate, $\sigma_n^\top \lambda_t$ is negligible (Table 2, Table 3). Accordingly, α does not influence the difference $\hat{r}_t - r_t$. In these numerical examples, the nominal instantaneous riskless rate at time $t = 0$ is higher than the real one by approximately 2.00% when $\delta = 0.5$ and 2.75% when $\delta = 1.5$ (Figure 1, Figure 2).

Figure 1: $\delta = 1.5$ Yield curves

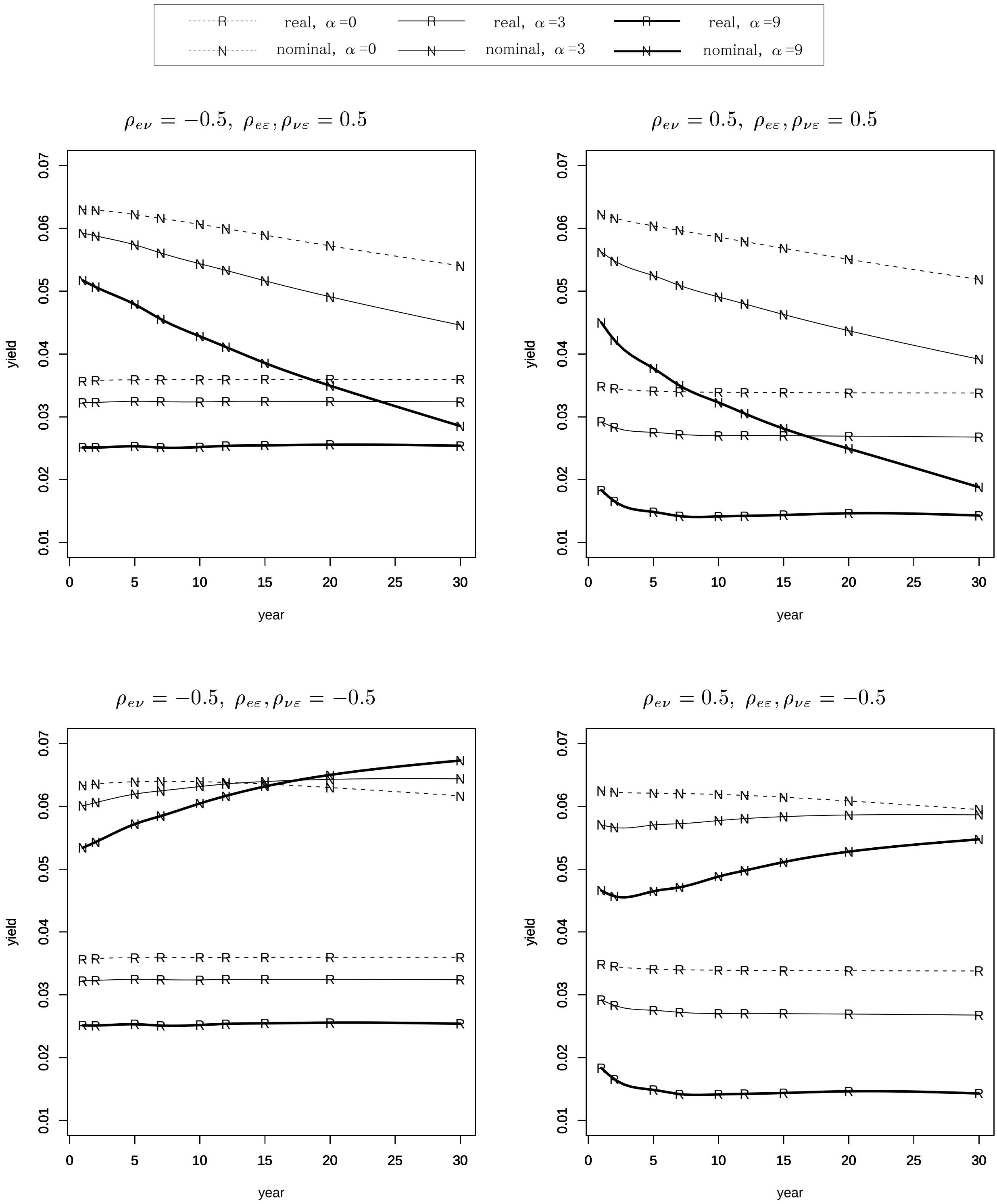
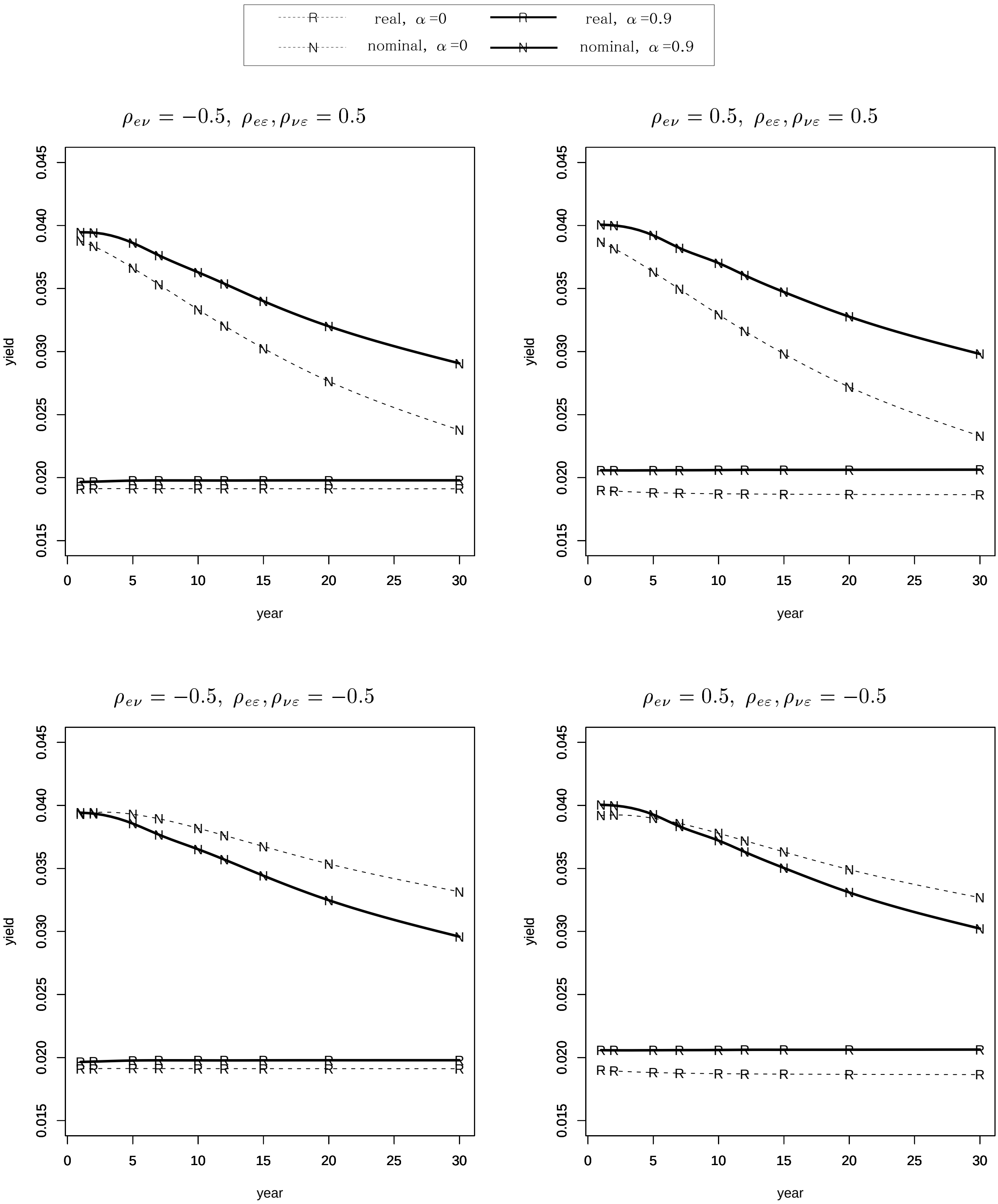


Figure 2: $\delta = 0.5$ Yield curves



5.2 Slope of the yield curve

With the slope of the yield curve, from Eq.(3.18),

$$\begin{aligned} R(t,s) &= -\frac{1}{s-t}\log E_t \left[\begin{array}{c} \exp\left\{ -\int_t^s \left(\beta + \delta\mu_e(u) - \frac{\delta}{2}||\sigma_e||^2 + \frac{\alpha(1+\alpha)}{2} \left|\left| \frac{\sigma_J(u)}{(1+\alpha)J_u} \right|\right|^2 \right) du \right\} \cdot \\ \exp\left\{ -\int_t^s \left(\delta\sigma_e - \alpha \frac{\sigma_J(u)}{(1+\alpha)J_u} \right)^\top dB_u \right\} \end{array} \right] \\ &= -\frac{1}{s-t}\log E_t \left[\exp \left(\begin{array}{l} -\int_t^s \left(\begin{array}{l} (1-\delta)\beta + \delta\nu_u - \frac{\delta}{2}||\sigma_e||^2 \\ +\frac{\alpha(1+\alpha)}{2} \left|\left| \frac{\sigma_J(u)}{(1+\alpha)J_u} \right|\right|^2 \end{array} \right) du \\ -\int_t^s \left(\delta\sigma_e - \alpha\frac{\sigma_J(u)}{(1+\alpha)J_u} \right)^\top dB_u \end{array} \right) \right]. \end{aligned} \tag{5.1}$$

To obtain some intuitive understandings of this equation, we impose two simplifications. First, set $k = 0$: that is, $\nu_u = \nu_t + \int_t^u \sigma_\nu^\top dB_w$. Second, $\frac{\sigma_J(t)}{(1+\alpha)J_t}$ is replaced by a constant matrix, denoted by $\tilde{\sigma}_J$. The reasoning for these two treatments will be discussed below. Let $R^a(t,s)$ denote the spot yield that is approximated based on these two simplifications.

$$R^a(t,s) = r_t^a - \delta^2 ||\sigma_\nu||^2 \frac{(s-t)^2}{6} - \delta^2\sigma_e^\top\sigma_\nu \frac{(s-t)}{2} + \alpha\delta\sigma_\nu^\top\tilde{\sigma}_J\frac{(s-t)}{2}. \tag{5.2}$$

where

$$r_t^a := \lim_{s\downarrow t} R^a(t,s) = R^a(t,t) = \delta\nu_t + (1-\delta)\beta - \frac{\delta(1+\delta)}{2}||\sigma_e||^2 + \frac{\alpha}{2}||\tilde{\sigma}_J||^2 + \alpha\delta\sigma_e^\top\tilde{\sigma}_J.$$

That is, r_t^a represents this instantaneous riskless rate and stands for the level of the yield curve. Call this approximation *normality approximation*. For the details of the derivation of this normality approximation, see Appendix C.

Eq.(5.2) provides us with some intuitive interpretations of the results in real terms. However, Eq.(5.2) is not exactly the spot yield that we are solving for numerically, in the sense that it is obtained on the assumptions of (1) $k = 0$ and of (2) $\frac{\sigma_J(t)}{(1+\alpha)J_t}$ being replaced by the constant matrix $\tilde{\sigma}_J$. With regard to the first assumption, we can guess that a higher level of the speed ($k > 0$) results in a flatter yield curve than in the case of zero speed of the mean reversion ($k = 0$). On the other hand, with regard to the second assumption, under our parametric assumptions in Table 1, $\left|\left|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right|\right|^2$ is constant over the first 30 years, which period of time we are focusing on (Figure 3).

We can conjecture that this is because our EIS parameter is relatively close to unity.

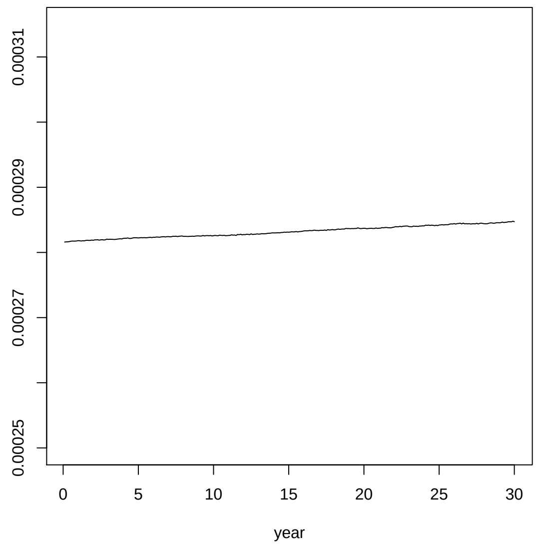


Figure 3: The mean value of $\left|\left|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right|\right|^2$ over first 30 years in the case of $\delta = 1.5,\ \alpha = 9,\ \rho_{e\nu} = 0.5$ and $\rho_{e\varepsilon} = \rho_{\nu\varepsilon} = -0.5$. The other parameters are set as Table 1. The value is calculated as sample mean in the Monte-Carlo simulation. In each time-step, the (standard deviation)/(mean) ratio is 5×10^{-2} at most. It is small enough to regard the process as deterministic.

Accordingly, the second simplification is a pertinent approximation under our parametric examples.[11] In short, Eq.(5.2) is a useful tool to study our numerical results in an intuitive way.

Look at our numerical results regarding the slope of the real yield curve based on Eq.(5.2) in more details. With δ fixed, let us examine the effect of α on the slope of the real yield curve. As a benchmark case, look at the case of the time separable utility ($\alpha = 0$). First, $-\delta^2 \left|\left|\sigma_\nu\right|\right|^2 \frac{(s-t)^2}{6}$ is derived from $\delta \int_t^u \sigma_\nu^\top dB_w$ in Eq.(C.1) shown in Appendix C. From Eq.(3.9), we know that $\int_t^u \sigma_\nu^\top dB_w$ represents the accumulation of the uncertainty of the equilibrium instantaneous riskless rate from time t to time u under the normality approximation. Accordingly, the term $-\delta^2 \left|\left|\sigma_\nu\right|\right|^2 \frac{(s-t)^2}{6}$ corresponds to expected discounting. By the Jensen's inequality, this term is negative – call this effect the "convexity effect of the expected discounting." Therefore, $-\delta^2 \left|\left|\sigma_\nu\right|\right|^2 \frac{(s-t)^2}{6}$ pushes down the slope of the yield curve; the downward effect is increasing by square in maturity length.

Second, when $\sigma_e^\top \sigma_\nu > 0$,[12] the term $-\delta^2 \sigma_e^\top \sigma_\nu \frac{(s-t)}{2}$ reduces the slope of the curve. In other

[11] Note that Eq.(5.2) is not necessarily be a good approximation in general.

[12] From Eq.(3.9), σ_ν corresponds, at least partly, to the volatility of the equilibrium instantaneous riskless rate. Moreover, if $\frac{\sigma_J(t)}{(1+\alpha)J_t}$ is characterized by a constant matrix, then $\delta\sigma_\nu$ is equal to it. Thus, $\sigma_e^\top \sigma_\nu > 0$ means that real interest rates are pro-cyclical.

words, under $\sigma_e^\top \sigma_\nu > 0$, the long-term bonds play a role of hedging income risk, because holding the bonds mitigates the uncertainty of the future utility. Therefore, the investor is willing to purchase the bonds at a lower rate of return; the term $-\delta^2 \sigma_e^\top \sigma_\nu \frac{(s-t)}{2}$ is negative. Call this effect a hedging effect. On the other hand, when $\sigma_e^\top \sigma_\nu < 0$, the effect is reversed. Note that the effect of this term is linear in maturity length.

In total, when $\rho_{e\nu} = 0.5$, both the first and the second effects are negative; the yield curve is sloping down. In addition, the slope should be more steeply under $\delta = 1.5$ than under $\delta = 0.5$. On the other hand, when $\rho_{e\nu} = -0.5$, the first effect and the second effect are opposite. The total effect is uncertain analytically. Still, the first effect is getting bigger by square in maturity length, whereas the second effect is linear in maturity length. Even although the yield curve may be sloping up (i.e., the second effect may be overwhelming the first one) in short maturity length, it can slope down (the first effect is overwhelming the second one) at longer maturity. Regardless of such detailed arguments, however, the real yield curve under $\alpha = 0$ is almost flat in either case of δ because we are setting a high level of the speed of the mean reversion of the expected endowment growth rate ($k = 0.5$ for $\delta = 0.5$ and $k = 1$ for $\delta = 1.5$).

Next, look at the case of the time-nonseparable utility ($\alpha \neq 0$). In this case, the term $\alpha\delta\sigma_\nu^\top \tilde{\sigma}_J \frac{(s-t)}{2}$ is effective; that is, this term is specific to the time nonseparable utility. In parallel to the arguments regarding the level of the yield curves in the previous subsection, when $\alpha > 0$, the term $\alpha\delta\sigma_\nu^\top \tilde{\sigma}_J \frac{(s-t)}{2}$ stands for the motive of shorting (holding) the long-term bonds when $\sigma_\nu^\top \tilde{\sigma}_J > 0$ (when $\sigma_\nu^\top \tilde{\sigma}_J < 0$, respectively). When $\sigma_\nu^\top \tilde{\sigma}_J > 0$ (when $\sigma_\nu^\top \tilde{\sigma}_J < 0$), the motive of shorting (holding) the long-term bonds pushes up (pushes down, respectively) the slope of the yield curve. With a higher level of α, the effect of this term is increasing. Note that, when $\alpha < 0$, the effect is reversed. Also, the effect is bigger when $\delta = 1.5$ than when $\delta = 0.5$. However, as discussed above, we set a high level of the speed of the mean reversion of the expected endowment growth rate. The real yield curve is almost flat under $\alpha \neq 0$ as well. Still, when $\delta = 1.5$ and $\rho_{e\nu} = 0.5$, we can observe in Figure 1 that the yield curve is sloping down slightly in short maturity area. This is because (1) the negative effect of the term $\alpha\delta\sigma_\nu^\top \tilde{\sigma}_J \frac{(s-t)}{2} < 0$ is added to the negative effects of the convexity effect $-\delta^2 \|\sigma_\nu\|^2 \frac{(s-t)^2}{6}$ and the hedging effect $-\delta^2 \sigma_e^\top \sigma_\nu \frac{(s-t)}{2}$, (2) all those negative effects are amplified when δ is bigger, and (3) there is a small effect of the mean reversion in short maturity area. The downward slope in such a short-maturity area is steeper when α is higher.

Note that, whereas we set such a high speed of the mean reversion of the expected endowment process, we set a lower speed θ of the mean reversion of th expected inflation process. By doing so, we can confine more attention to the effect of the expected inflation on the nominal yield curve.

From such a perspective, examine the slope of the yield curve in nominal terms. From Eq. (4.4),

$$
\begin{aligned}
\hat{R}(t,s) &= -\frac{1}{s-t} \log E_t \left[\begin{array}{c} \exp\left\{ -\int_t^s \left(r_u + \varepsilon_u - \|\sigma_n\|^2 - \sigma_n^\top \lambda_u \right) du \right\} \cdot \\ \exp\left\{ -\int_t^s (\lambda_u + \sigma_n)^\top dB_u - \frac{1}{2}\int_t^s \|\lambda_u + \sigma_n\|^2 du \right\} \end{array} \right] \\
&= -\frac{1}{s-t} \log E_t \left[\exp \left(\begin{array}{c} -\int_t^s \left(\begin{array}{c} (1-\delta)\beta + \delta\nu_u - \frac{\delta}{2}\|\sigma_e\|^2 \\ +\alpha\frac{\delta}{2}\left\| \frac{\sigma_J(u)}{(1+\alpha)J_u} \right\|^2 + \varepsilon_u - \frac{1}{2}\|\sigma_n\|^2 \end{array} \right) du \\ -\int_t^s \left(\delta\sigma_e - \alpha\frac{\sigma_J(u)}{(1+\alpha)J_u} + \sigma_n \right)^\top dB_u \end{array} \right) \right].
\end{aligned}
$$

Like the above arguments in real terms, impose three simplifications: (1) $k=0$, (2) $\theta=0$, and (3) $\frac{\sigma_J(t)}{(1+\alpha)J_t}$ is replaced by the constant matrix $\tilde{\sigma}_J$. The approximated nominal spot yields, denoted by $\hat{R}^a(t,s)$, are written as:

$$
\hat{R}^a(t,s) = \left[\begin{array}{l} R^a(t,s) + (\hat{r}_t^a - r_t^a) \\ -\|\sigma_\varepsilon\|^2 \frac{(s-t)^2}{6} - \delta\sigma_\nu^\top \sigma_n \frac{(s-t)}{2} - \delta\sigma_\nu^\top \sigma_\varepsilon \frac{(s-t)^2}{6} \\ -\delta\sigma_e^\top \sigma_\varepsilon \frac{(s-t)}{2} + \alpha\sigma_\varepsilon^\top \tilde{\sigma}_J \frac{(s-t)}{2} - \sigma_n^\top \sigma_\varepsilon \frac{(s-t)}{2} \end{array} \right]. \tag{5.3}
$$

where $\hat{r}_t^a \triangleq r_t^a + (\varepsilon_t - \|\sigma_n\|^2 - \sigma_n^\top \lambda_t)$ denotes the corresponding nominal instantaneous riskless rate. From an analogue of Eq.(4.5), the term $\hat{r}_t^a - r_t^a = \varepsilon_t - \|\sigma_n\|^2 - \sigma_n^\top \lambda_t$ represents the level difference between the real and the nominal yield curves. Note that a higher level of the speed of the mean reversion $\theta > 0$ results in a flatter yield curve than this equation Eq.(5.3) shows. For the details of the derivation of this normality approximation, see Appendix C.

In our numerical examples, we set no correlation between the inflation rate and the real factors and between the inflation rate and the expected inflation rate. Accordingly,

$$\hat{R}^a(t,s) = \left[\begin{array}{l} R^a(t,s) + (\hat{r}^a_t - r^a_t) \\ -\left|\left|\sigma_\varepsilon\right|\right|^2 \frac{(s-t)^2}{6} - \delta\sigma_e^\top \sigma_\varepsilon \frac{(s-t)}{2} - \delta\sigma_\nu^\top \sigma_\varepsilon \frac{(s-t)^2}{6} + \alpha\sigma_\varepsilon^\top \tilde{\sigma}_J \frac{(s-t)}{2} \end{array} \right]. \tag{5.4}$$

where $\hat{r}^a_t - r^a_t = \varepsilon_t - \left|\left|\sigma_n\right|\right|^2 + \alpha\sigma_n^\top \tilde{\sigma}_J$. We can conjecture that $\sigma_n^\top \tilde{\sigma}_J$ is negligible as discussed above.

Examine Eq.(5.4), basically in parallel to the above arguments in real terms. First, $-\left|\left|\sigma_\varepsilon\right|\right|^2 \frac{(s-t)^2}{6}$ stands for a nominal convexity effect regarding the expected discounting. Second, $-\delta\sigma_e^\top \sigma_\varepsilon \frac{(s-t)}{2}$ and $-\delta\sigma_\nu^\top \sigma_\varepsilon \frac{(s-t)^2}{6}$ represent the motive of holding (shorting) the nominal long-term bonds when $\sigma_e^\top \sigma_\varepsilon > 0$ ($\sigma_e^\top \sigma_\varepsilon < 0$) and $\sigma_\nu^\top \sigma_\varepsilon > 0$ ($\sigma_\nu^\top \sigma_\varepsilon < 0$), respectively. Third, the far right term $\alpha\sigma_\varepsilon^\top \tilde{\sigma}_J \frac{(s-t)}{2}$ with $\alpha > 0$ means the motive of shorting (holding) the long-term bonds when $\sigma_\varepsilon^\top \tilde{\sigma}_J > 0$ (when $\sigma_\varepsilon^\top \tilde{\sigma}_J < 0$, respectively). Note that, for $\alpha < 0$, the effect of this term is reversed.

Now, we investigate our numerical results regarding the slope of the nominal yield curves in the following four cases (that is, $\{\delta,\ \rho_{e\varepsilon}\} \in \{\{0.5,\ 1.5\} \times \{0.5,\ -0.5\}\}$), based on Eq.(5.4). Recall that, for simplicity, we have set $\rho_{e\varepsilon} = \rho_{\nu\varepsilon}$. Look at the case of $\delta = 0.5$. See Figure 2. Suppose $\rho_{e\varepsilon} = \rho_{\nu\varepsilon} = 0.5$. First, the nominal convexity effect is negative. Second, $-\delta\sigma_e^\top \sigma_\varepsilon \frac{(s-t)}{2}$ and $-\delta\sigma_\nu^\top \sigma_\varepsilon \frac{(s-t)^2}{6}$ represent the motive of holding the nominal long-term bonds; these terms are both negative. Therefore, when $\alpha = 0$, the slope of the yield curve is definitely negative. Third, $\sigma_\varepsilon^\top \tilde{\sigma}_J$ is positive in this case, from Table 2.[13] Thus, the far right term $\alpha\sigma_\varepsilon^\top \tilde{\sigma}_J \frac{(s-t)}{2} > 0$ represents the motive of shorting the long-term bonds. In the numerical results, in total, the slope is negative when $\alpha > 0$, because the far right term is relatively weak. Also, for a higher level of α, the slope is flatter, because the positive effect of the far right term is larger.

Suppose $\rho_{e\varepsilon} = \rho_{\nu\varepsilon} = -0.5$. See Figure 2 again. First, the nominal convexity effect is negative. Second, $-\delta\sigma_e^\top \sigma_\varepsilon \frac{(s-t)}{2}$ and $-\delta\sigma_\nu^\top \sigma_\varepsilon \frac{(s-t)^2}{6}$ represent the motive of shorting the nominal long-term bonds; these terms are both positive. Third, $\sigma_\varepsilon^\top \tilde{\sigma}_J$ is negative in this case, from Table 2. Thus, the far right term $\alpha\sigma_\varepsilon^\top \tilde{\sigma}_J \frac{(s-t)}{2} < 0$ represents the motive of holding the long-term bonds. In the numerical results, when $\alpha = 0$, the slope of the yield curve is

[13] This result is consistent with the one in the case of $\delta = 1$ as in Appendix A. Such consistency holds true of the following three cases as well.

negative, because the first convexity effect is relatively strong. In addition, when $\alpha = 0.9$, the slope is also negative, since the far right term $\alpha\sigma_\varepsilon^\top \tilde{\sigma}_J \frac{(s-t)}{2} < 0$. Also, for a higher level of α, the slope is steeper, because the negative effect of the far right term is larger.

Next, look at the case of $\delta = 1.5$. See Figure 1. Suppose $\rho_{e\varepsilon} = \rho_{\nu\varepsilon} = 0.5$. First, the nominal convexity effect always pushes down the slope. Second, when $\rho_{e\varepsilon} = \rho_{\nu\varepsilon} = 0.5$, $-\delta\sigma_e^\top \sigma_\varepsilon \frac{(s-t)}{2}$ and $-\delta\sigma_\nu^\top \sigma_\varepsilon \frac{(s-t)^2}{6}$ represent the motive of holding the nominal long-term bonds; theses terms are both negative. Third, $\sigma_\varepsilon^\top \tilde{\sigma}_J < 0$ in this case, from Table 3. Therefore, for $\alpha > 0$, the far right term $\alpha\sigma_\varepsilon^\top \tilde{\sigma}_J \frac{(s-t)}{2}$ represents the motive of holding the long-term bonds; the term is negative. In total, since all these terms are negative, the slope is definitely negative. The negative slope is steeper when $\alpha = 9$ than when $\alpha = 3$, because the far right term shows the stronger motive of holding the long-term bonds.

On the other hand, look at the case of $\rho_{e\varepsilon} = \rho_{\nu\varepsilon} = -0.5$. See Figure 1 again. First, the nominal convexity effect is the same as above. Second, the effects of $-\delta\sigma_e^\top \sigma_\varepsilon \frac{(s-t)}{2}$ and $-\delta\sigma_\nu^\top \sigma_\varepsilon \frac{(s-t)^2}{6}$ are reversed in comparison with the above case. That is, they stand for the motive of shorting the nominal long-term bonds; these terms are both positive. Third, $\sigma_\varepsilon^\top \tilde{\sigma}_J$ is positive in this case, from Table 3. Accordingly, the far right term $\alpha\sigma_\varepsilon^\top \tilde{\sigma}_J \frac{(s-t)}{2} > 0$ represents the motive of shorting the long-term bonds. The total effect is uncertain analytically. In our numerical examples, the slope is almost positive when $\alpha > 0$ because the convexity effect is overwhelmed by the far right term, although the slope is slightly negative when $\alpha = 0$. In addition, for a higher level of $\alpha > 0$, the slope is steeper due to the far right term.

Focusing on the last case, together with the result regarding the level of the yield curve, we find that, when $\alpha > 0$, the level of the nominal yield curve (i.e., the instantaneous riskless rate) is lower than when $\alpha = 0$, whereas the slope is almost positive when $\alpha > 0$. In particular, when $\rho_{e\nu} > 0$ and $\alpha > 0$, the level tends to be low. Thus, the case resolves the risk-free rate puzzle and, at the same time, results in an upward slope of the yield curve. This result is consistent with actual nominal yield curves.[14]

[14]When $\alpha < 0$ and $\delta < 1$, we may obtain similar results to replicate a nominal yield curves, because the agent is comparatively more risk averse in the parametric situation as well. However, since δ is relatively small, the effect of the risk aversion on the yield curve tends to be small. Accordingly, it may be difficult to achieve the above desirable results, since α is restricted to be larger than -1 by Assumption 2.1.

6 Conclusion

This paper constructs a continuous-time term structure model in environments with (i) SDU with non-unitary EIS and (ii) mean-reverting expectations on the inflation and the real output growth. With regard to future work, we will apply this model to an empirical analysis. Also, we will explore a numerical method of solving BSDEs with non-Lipschitz conditions in order to deal with the EIS that is sufficiently away from unity.

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A Factor Decomposition of yield curves in case of $\delta = 1$

When $\delta = 1$, we obtain closed-form solutions for the real/nominal term structure of interest rates. In this appendix, we place some results of the factor decompositions of a term structure of real/nominal interest rates in the case when $T \to \infty$. For the derivation of them, see Nakamura, Nakayama, and Takahashi (2008).

Let the superscript s denote the value under the time separable utility (i.e., $\alpha = 0$). Then,

$$
\begin{aligned}
r_t^s &= \nu_t - ||\sigma_e||^2 \\
r_t &= r_t^s + (\mathit{Additional\ endowment\ shock}) \\
R^s(t,s) &= (\mathit{Real\ expectations}) + (\mathit{Separable\ utility's\ real\ term\ premium}) + (\mathit{Real\ convexity\ effect}) \\
R(t,s) &= R^s(t,s) + (\mathit{Additional\ endowment\ shock}) + (\mathit{Additional\ expected\ endowment\ shock}) \\
\hat{r}_t^s &= r_t^s + \varepsilon_t + (\mathit{Nominal\ risk\ aversion}) \\
\hat{r}_t &= \hat{r}_t^s + (\mathit{Additional\ endowment\ shock}) + (\mathit{Additional\ inflation\ shock}) \\
\hat{R}^s(t,s) &= R^s(t,s) + (\mathit{Expected\ inflation\ rate}) + (\mathit{Nominal\ risk\ aversion}) \\
&+ (\mathit{Separable\ utility's\ nominal\ term\ premium}) + (\mathit{Nominal\ convexity\ effect}) \\
\hat{R}(t,s) &= \hat{R}^s(t,s) + (\mathit{Additional\ endowment\ shock}) + (\mathit{Additional\ inflation\ shock}) \\
&+ (\mathit{Additional\ expected\ endowment\ shock}) + (\mathit{Additional\ expected\ inflation\ shock}).
\end{aligned}
$$

where

$$
\begin{aligned}
TES &\triangleq \frac{1}{\beta}\left(\sigma_e + \frac{\sigma_\nu}{k+\beta}\right) \\
(\textit{Real expectations}) &\triangleq r_t^s + (\bar{\nu} - \nu_t)\left(1 - \frac{1-e^{-k(s-t)}}{k(s-t)}\right) \\
(\textit{Separable utility's real term premium}) &\triangleq -\frac{\sigma_\nu^\top \sigma_e}{k}\left(1 - \frac{1-e^{-k(s-t)}}{k(s-t)}\right) \\
(\textit{Real convexity effect}) &\triangleq -\frac{||\sigma_\nu||^2}{2k^2}\left(1 - 2\frac{1-e^{-k(s-t)}}{k(s-t)} + \frac{1-e^{-2k(s-t)}}{2k(s-t)}\right) \\
(\textit{Additional endowment shock}) &\triangleq \frac{\alpha}{\beta}\left(||\sigma_e||^2 + \frac{\sigma_e^\top \sigma_\nu}{k+\beta}\right) \\
(\textit{Additional expected endowment shock}) &\triangleq \frac{1}{k}\frac{\alpha}{\beta}\left(\sigma_\nu^\top \sigma_e + \frac{||\sigma_\nu||^2}{k+\beta}\right)\left(1 - \frac{1-e^{-k(s-t)}}{k(s-t)}\right) \\
(\textit{Expected inflation rate}) &\triangleq E_t\left[\frac{1}{s-t}\log\frac{N_s}{N_t}\right] + \frac{||\sigma_n||^2}{2} \\
(\textit{Nominal risk aversion}) &\triangleq -||\sigma_n||^2 - \sigma_n^\top \sigma_e \\
(\textit{Separable utility's nominal term premium}) &\triangleq -\frac{\sigma_\varepsilon^\top \sigma_e}{\theta}\left(1 - \frac{1-e^{-\theta(s-t)}}{\theta(s-t)}\right) \\
&- \left[\frac{\sigma_n^\top \sigma_\varepsilon}{\theta}\left(1 - \frac{1-e^{-\theta(s-t)}}{\theta(s-t)}\right) + \frac{\sigma_n^\top \sigma_\nu}{k}\left(1 - \frac{1-e^{-k(s-t)}}{k(s-t)}\right)\right] \\
(\textit{Nominal convexity effect}) &\triangleq -\frac{||\sigma_\varepsilon||^2}{2\theta^2}\left(1 - 2\frac{1-e^{-\theta(s-t)}}{\theta(s-t)} + \frac{1-e^{-2\theta(s-t)}}{2\theta(s-t)}\right) \\
&- \frac{\sigma_\varepsilon^\top \sigma_\nu}{k\theta}\Bigg(1 - \frac{1-e^{-k(s-t)}}{k(s-t)} - \frac{1-e^{-\theta(s-t)}}{\theta(s-t)} \\
&\qquad + \frac{1-e^{-k(s-t)}}{k(s-t)}\frac{1-e^{-\theta(s-t)}}{\theta(s-t)}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
(\textit{Additional inflation shock}) &\triangleq \frac{\alpha}{\beta}\left(\sigma_n^\top \sigma_e + \frac{\sigma_n^\top \sigma_\nu}{k+\beta}\right) \\
(\textit{Additional expected inflation shock}) &\triangleq \frac{1}{\theta}\frac{\alpha}{\beta}\left(\sigma_\varepsilon^\top \sigma_e + \frac{\sigma_\varepsilon^\top \sigma_\nu}{k+\beta}\right)\left(1 - \frac{1-e^{-\theta(s-t)}}{\theta(s-t)}\right)
\end{aligned}
$$

Note that, in this paper, *TES* stands for "*Total endowment shock*."

B Numerical method

The outline of our numerical method is as follows. We use directly the transformed utility $\hat{V}$ in Eq.(2.4), instead of the utility V, for our numerical approximation. We derive numerically equilibrium utility and equilibrium volatility, denoted by (U, σ_U), that are corresponding to $(\hat{V}, \sigma_{\hat{V}})$. We then transform these numerically obtained (U, σ_U) back into (J, σ_J) by using the inverse transformation of Eq.(2.3) and substitute them into the formulation of the yield curves that is derived in Section 3 and Section 4.

We can characterize the equilibrium transformed utility process U in continuous time, which will be solved for numerically below, by imposing $c = e$:

$$\begin{cases} \frac{de_t}{e_t} = \mu_e(t)dt + \sigma_e^\top dB_t, \quad e_0 \in \mathbf{R}_+, \\ d\nu_t = k(\bar{\nu} - \nu_t)dt + \sigma_\nu^\top dB_t, \quad \nu_0 \in \mathbf{R}, \\ dU_t = -g(e_t, U_t, \sigma_U(t))dt + \sigma_U(t)^\top dB_t, \quad U_T = 0. \end{cases} \tag{B.1}$$

where

$$g(e, U, \sigma_U) = \begin{cases} \frac{e^{1-\delta}}{1-\delta} - \beta U + \frac{\alpha}{2}(U)^{-1}||\sigma_U||^2 \quad \text{if} \quad \delta > 0 \text{ and } \delta \neq 1, \\ \log e - \beta U + \frac{\alpha}{2}||\sigma_U||^2 \quad \text{if} \quad \delta = 1. \end{cases} \tag{B.2}$$

As in Appendix A, when $\delta = 1$, we have achieved an explicit (closed-form), analytical solution to it. On the other hand, when $\delta \neq 1$, we do not obtain any such solution, although there exists a unique solution to Eq.(B.1) under Assumption 2.1. Thus, we analyze numerically the case of $\delta \neq 1$ by applying the method of Gobet, Lemor and Warin (2005) as follows. The approximation method consists of three steps.

B.1 Discretization

We first introduce some notations and definitions as follows. For a σ-algebra $\mathcal{F}$, $\mathbf{L}^2(\mathcal{F})$ is the space of square integrable, $\mathcal{F}$-measurable, possibly multidimensional, random variables. Let $\{t_n = nh = nT/N\}_{n=0}^N$ denote discretized times where $h > 0$ and $N \in \mathbf{N}$ are the length of each time step and the number of time steps respectively. N is set to be sufficiently large. Let an $\mathbf{R}$-valued sequence $\{e_{t_n}\}_{n=0}^N$ denote a sequence of the discretized-time version of the original process $\{e_t\}_{0 \leq t \leq T}$. With regard to the other processes, define similarly: $\{B_{t_n}\}_{n=0}^N$, $\{\nu_{t_n}\}_{n=0}^N$, $\{\sigma_{J,t_n}\}_{n=0}^N$, $\{J_{t_n}\}_{n=0}^N$, $\{\mu_{t_n}\}_{n=0}^N$, $\{\sigma_{U,t_n}\}_{n=0}^N$, and $\{U_{t_n}\}_{n=0}^N$. Also, $\Delta B_{t_i} \triangleq B_{t_{i+1}} - B_{t_i}$.

With regard to the approximation of the forward processes e_t and ν_t, we use a standard scheme. In this appendix, we omit the description of the scheme. Assume, instead, that the approximated forward processes, denoted by $\{\tilde{e}_{t_n}\}_{n=0}^N$ and $\{\tilde{\nu}_{t_n}\}_{n=0}^N$, are obtained. By doing so, we can focus our discussion on the backward approximation of the utility process.

Look at the original process that is characterized by the BSDE:

$$dU_t = -g(e_t, U_t, \sigma_U(t))dt + \sigma_U(t)^\top dB_t \quad \text{with}\, U_T = 0\, \text{given}.$$

By discretizing the continuous time $[0, T]$ into discretization times $\{t_n\}_{n=0}^N$, consider the following procedure of approximating $\{U_{t_n}, \sigma_U(t_n)\}$ in a backward manner:

1. Set $\tilde{U}_{t_N} = U_{t_N} = 0$.
2. For $n = N - 1$ to 0, compute

$$(\tilde{U}_{t_n}, \tilde{\sigma}_U(t_n)) \quad = \quad \arg \min_{(U,\, \sigma_U) \in \mathbf{L}^2(\mathcal{F}_{t_n})} E\left[\tilde{U}_{t_{n+1}} - U + g(\tilde{e}_{t_n}, U, \sigma_U)h - \sigma_U^\top \Delta B_{t_n}\right]^2. \tag{B.3}$$

3. Output $\{\tilde{U}_{t_n}\}_{n=0}^N$, $\{\tilde{\sigma}_U(t_n)\}_{n=0}^N$.

Therefore, an optimal solution $(\tilde{U}_{t_n}, \tilde{\sigma}_U(t_n))$ of the problem (B.3) can be characterized by

$$\tilde{\sigma}_{U,l}(t_n) \quad = \quad \frac{E_{t_n}\left[\tilde{U}_{t_{n+1}} \Delta B_{l,t_n}\right]}{h}, \text{ for } l = 1, \cdots, d, \tag{B.4}$$

$$\tilde{U}_{t_n} \quad = \quad E_{t_n}\left[\tilde{U}_{t_{n+1}}\right] + g\left(\tilde{e}_{t_n}, \tilde{U}_{t_n}, \tilde{\sigma}_U(t_n)\right) h \tag{B.5}$$

where the subscript l denotes the dimension of $\sigma_U(t)$.

B.2 Regression

Second, we replace the conditional expectation that appears in Eq.(B.4)-Eq.(B.5) by an $\mathbf{L}^2$ projection on the space generated by a finite number of functions of $X_t = (e_t, \nu_t)$ (call them function bases), because $\{U_t\}_{0 \le t \le T}$ and $\{\sigma_U(t)\}_{0 \le t \le T}$ are Markov processes. We derive a solution combining the projection on the function bases and I Picard iterations. The integer I is a fixed parameter. Assume that the integer I is sufficiently large that the iterations result in reaching at a fixed point, if any.

More specifically, since $\{U_t\}_{0 \le t \le T}$ and $\{\sigma_U(t)\}_{0 \le t \le T}$ are Markov processes, U_t and $\sigma_U(t)$ can be expressed as functions of the state variables e_t and ν_t for each t. This logic is similar to the recent computation method of American option pricing (e.g., Clément, Lamberton, and Protter (2002)). Define a sequence of measurable real-valued functions defined on the state space as $\mathbf{p}_{l,n}(x_{t_n}) \triangleq (p_{l,n,1}(x_{t_n}), \cdots, p_{l,n,K}(x_{t_n}))^\top$ for $n \in \{0, 1, \cdots, N-1\}$, and $l \in \{0, 1, \cdots, d\}$ and a finite integer K, satisfying the following conditions:

Assumption B.1 *For each $n \in \{0, 1, \cdots, N-1\}$ and $l \in \{0, 1, \cdots, d\}$, the sequence of $\{p_{l,n,k}(X_{t_n})\}_{k\in\{1,2,\cdots,K\}}$ is total in $\mathbf{L}^2(\mathcal{F}_{t_n})$.*

Assumption B.2 *For each $n \in \{0, 1, \cdots, N-1\}$ and $l \in \{0, 1, \cdots, d\}$,*

$$\text{If } \sum_{k=1}^{K} \lambda_k p_{l,n,k}(X_{t_n}) = 0 \,\text{a.s.}, \text{ then } \lambda_k = 0 \text{ for } k \in \{1, 2, \cdots, K\}.$$

Notice that the subscript $l = 0$ of a variable represents that the variable is corresponding to U_t. Call the sequence of $\{p_{l,n,k}(X_{t_n})\}_{k\in\{1,2,\cdots,K\}}$ *function bases*. Note that K stands for the finite number of the function bases that generate the vector space. Recall that d denotes the dimension of the Brownian motion B. We then approximate the conditional expectation with respect to X_{t_n} by the orthogonal projection on the space generated by the function bases $\{p_{l,n,k}(X_{t_n})\}_{k\in\{1,2,\cdots,K\}}$. For example, we may take indicator functions of the state variables e_t and ν_t as the function bases.

This approximation corresponds to the linear regression of U_{t_n} and $\sigma_U(t_n)$ to the function bases in each time step:

$$U_{t_n} \approx \mathbf{b}_{0,n}^\top \mathbf{p}_{0,n}(e_{t_n}, \nu_{t_n}), \quad \sigma_{U,1}(t_n) \approx \mathbf{b}_{1,n}^\top \mathbf{p}_{1,n}(e_{t_n}, \nu_{t_n}), \quad \cdots, \quad \sigma_{U,d}(t_n) \approx \mathbf{b}_{d,n}^\top \mathbf{p}_{d,n}(e_{t_n}, \nu_{t_n}),$$

where $\mathbf{b}_{l,n}$ is a coefficient vector $(b_{l,n,1}, \cdots, b_{l,n,K})^\top$ for each $l = 0, 1, \cdots, d$ and each n. In particular, we set the following form of $U(t, e, \nu)$: for $\delta > 0$ and $\delta \neq 1$,

$$U(t, e, \nu) = q(t)\frac{e^{1-\delta}}{1-\delta} + m(t)\nu + n(t).$$

where $q(t),\ m(t),\ n(t)$ are deterministic functions only of time t. That is,

$$p_{l,n,1}(e, \nu) = \frac{e^{1-\delta}}{1-\delta}, \quad p_{l,n,2}(e, \nu) = \nu, \quad p_{l,n,3}(e, \nu) = 1, \quad \text{for } l = 0, 1, 2, \text{ and } n = 0, \cdots, N-1.$$

The linearity of the function $U(t, e, \nu)$ in $\frac{e^{1-\delta}}{1-\delta}$ and ν is the reason why we have taken the transformation of the utility from V to U in this numerical analysis.

The above procedure is then rewritten as:

1. Set $\tilde{U}_{t_N} = 0$.

2. For $n = N-1$ to 0,

   (a) Set $\mathbf{b}_{l,n}^0 = \mathbf{0} \in \mathbf{R}^K$ for $l = 0, \cdots, d$.

   (b) For $i = 1$ to I, compute

$$\begin{aligned}\{\mathbf{b}_{l,n}^i\}_{l=0}^d \quad = \quad & \arg \min_{\{\mathbf{b}_{l,n}\}_{l=0}^d} E\Big[\tilde{U}_{t_{n+1}} - \mathbf{b}_{0,n}^\top \mathbf{p}_{0,n}(\tilde{e}_{t_n}, \tilde{\nu}_{t_n}) \\ & + g\left(\tilde{e}_{t_n}, \mathbf{b}_{0,n}^{i-1\top} \mathbf{p}_{0,n}(\tilde{e}_{t_n}, \tilde{\nu}_{t_n}), \cdots, \mathbf{b}_{d,n}^{i-1\top} \mathbf{p}_{d,n}(\tilde{e}_{t_n}, \tilde{\nu}_{t_n})\right) h \\ & + \sum_{l=1}^d \mathbf{b}_{l,n}^\top \mathbf{p}_{l,n}(\tilde{e}_{t_n}, \tilde{\nu}_{t_n}) \Delta B_{l,t_n}\Big]^2,\end{aligned}$$

   (c) Compute

$$\tilde{U}_{t_n} = \mathbf{b}_{0,n}^{I\top} \mathbf{p}_{0,n}(\tilde{e}_{t_n}, \tilde{\nu}_{t_n}),\ \tilde{\sigma}_{U,1}(t_n) = \mathbf{b}_{1,n}^{I\top} \mathbf{p}_{1,n}(\tilde{e}_{t_n}, \tilde{\nu}_{t_n}),\ \cdots,\ \tilde{\sigma}_{U,d}(t_n) = \mathbf{b}_{d,n}^{I\top} \mathbf{p}_{d,n}(\tilde{e}_{t_n}, \tilde{\nu}_{t_n})$$

3. Output $\{\tilde{U}_{t_n}\}_{n=0}^N$, $\{\tilde{\sigma}_U(t_n)\}_{n=0}^N$.

B.3 Monte Carlo procedure

Finally, we evaluate numerically the expectation operation in the above procedure by a Monte-Carlo procedure. Let M denote the number of Monte-Carlo simulations. M is set to be sufficiently large. Let the set of $\mathbf{R}$-valued sequences $\{\{e_{t_n}^m\}_{n=0}^N\}_{m=1}^M$ denote a sequence of the discretized-time version of the original process $\{e_t\}_{0 \le t \le T}$. With regard to the other processes, define similarly: $\{\{B_{t_n}^m\}_{n=0}^N\}_{m=1}^M$, $\{\{\nu_{t_n}^m\}_{n=0}^N\}_{m=1}^M$, $\{\{\sigma_{J,t_n}^m\}_{n=0}^N\}_{m=1}^M$, $\{\{J_{t_n}^m\}_{n=0}^N\}_{m=1}^M$, $\{\{\mu_{t_n}^m\}_{n=0}^N\}_{m=1}^M$, $\{\{\sigma_{U,t_n}^m\}_{n=0}^N\}_{m=1}^M$, and $\{\{U_{t_n}^m\}_{n=0}^N\}_{m=1}^M$. Also, $\Delta B_{t_i}^m \triangleq B_{t_{i+1}}^m - B_{t_i}^m$.

To summarize, our algorithm is:

Algorithm

1. Set $\tilde{U}_{t_N}^m = 0$ for all m.

2. For $k = N-1$ to 0:

   (a) Set $\mathbf{b}_{l,n}^0 = \mathbf{0} \in \mathbf{R}^K$ for $l = 0, \cdots, d$.

(b) For $i = 1$ to I, compute

$$
\begin{aligned}
\{\mathbf{b}_{l,n}^{i}\}_{l=0}^{d} &= \arg\min_{\{\mathbf{b}_{l,n}\}_{l=0}^{d}} \frac{1}{M}\sum_{m=1}^{M}\Big[\tilde{U}_{t_{n+1}}^{m} - \mathbf{b}_{0,n}^{\top}\mathbf{p}_{0,n}\left(\tilde{e}_{t_n}^{m}, \tilde{\nu}_{t_n}^{m}\right) \\
&\quad + hf\left(\tilde{e}_{t_n}, \tilde{\nu}_{t_n}, \mathbf{b}_{0,n}^{i-1\top}\mathbf{p}_{0,n}(\tilde{e}_{t_n}^{m}, \tilde{\nu}_{t_n}^{m}), \cdots, \mathbf{b}_{d,n}^{i-1\top}\mathbf{p}_{d,n}(\tilde{e}_{t_n}^{m}, \tilde{\nu}_{t_n}^{m})\right) \\
&\quad - \sum_{l=0}^{d}\mathbf{b}_{l,n}^{\top}\mathbf{p}_{l,n}(\tilde{e}_{t_n}^{m}, \tilde{\nu}_{t_n}^{m})\Delta B_{l,n}^{m}\Big]
\end{aligned}
$$

(c) Compute for $m = 1, \cdots, M$:

$$
\begin{aligned}
\tilde{U}_{t_n}^{m} &= \rho_{0,n}\left(\mathbf{b}_{0,n}^{I\top}\mathbf{p}_{0,n}(\tilde{e}_{t_n}^{m}, \tilde{\nu}_{t_n}^{m})\right), \\
\tilde{\sigma}_{U,1}^{m}(t_n) &= \rho_{1,n}\left(\mathbf{b}_{1,n}^{I\top}\mathbf{p}_{1,n}(\tilde{e}_{t_n}^{m}, \tilde{\nu}_{t_n}^{m})\right), \cdots, \tilde{\sigma}_{U,d}^{m}(t_n) = \rho_{d,n}\left(\mathbf{b}_{d,n}^{I\top}\mathbf{p}_{d,n}(\tilde{e}_{t_n}^{m}, \tilde{\nu}_{t_n}^{m})\right)
\end{aligned}
$$

3. Output $\{\{\tilde{U}_{t_n}^{m}\}_{0\le n\le N}\}_{1\le m\le M}$ and $\{\{\tilde{\sigma}_{Ut_n}^{m}\}_{0\le n\le N}\}_{1\le m\le M}$.

where $\rho_{l,n} : \mathbf{R} \to \mathbf{R}$, $l = 0, \cdots, d$ are truncation functions, which are introduced to exclude outliers. We omit the details of the truncation functions (see Gobet, Lemor and Warin (2005)). From Eq.(2.3) and Ito's formula, for each m and each n, $\tilde{J}_{t_n}^{m}, \tilde{\sigma}_{J,t_n}^{m}$ are obtained. In our numerical examples, we set $M = 70,000$, $N = 4,000$, and $I = 10$.

Now, we have obtained the approximations of those original processes for some very small h. From an analogue of Eq.(3.18), define:

$$
\tilde{R}(0,t_n) \triangleq -\frac{1}{t_n}\log\frac{1}{M}\sum_{m=1}^{M}\left[\begin{array}{c}\exp\left\{-\sum_{i=0}^{n}\left(\beta + \delta\tilde{\mu}_{e,t_i}^{m} - \frac{\delta}{2}||\sigma_e||^2 + \frac{\alpha(1+\alpha)}{2}\left|\left|\frac{\tilde{\sigma}_{J,t_i}^{m}}{(1+\alpha)\tilde{J}_{t_i}^{m}}\right|\right|^2\right)h\right\}\cdot \\ \exp\left\{-\sum_{i=0}^{n}\left(\delta\sigma_e - \alpha\frac{\tilde{\sigma}_{J,t_i}^{m}}{(1+\alpha)\tilde{J}_{t_i}^{m}}\right)^{\top}\Delta B_{t_i}^{m}\right\}\end{array}\right],
$$

where $\tilde{\mu}_{e,t_i}^{m} \triangleq \tilde{\nu}_{t_i}^{m} - \beta$. For $0 \le n \le N$ and for the small h, define $\tilde{R}(0,t_n)$ as our discrete-time version of the real spot yields at t_n. We can calculate the nominal yield curve similarly.

B.4 Appendix to Appendix B

There is one caveat. Gobet, Lemor and Warin (2005) assume the Lipschitz condition:

Assumption B.3 *The driver g satisfies the following continuity estimate:*

$$
|g(e,U,\sigma_U) - g(e',U',\sigma_U')| \le C(|e-e'| + |U-U'| + |\sigma_U - \sigma_U'|)
$$

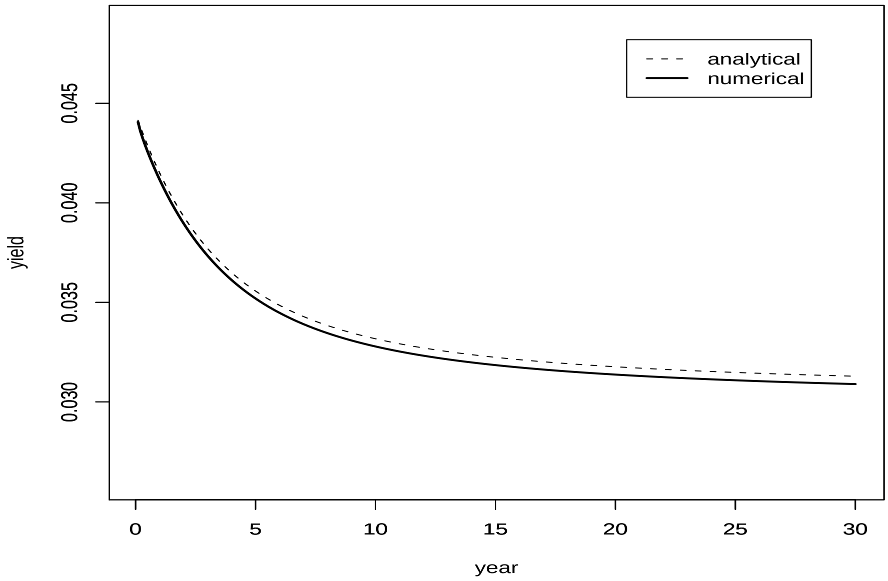

Figure 4: The analytical value and the numerical value of the real yield curves in the case of $\delta = 1,\ \alpha = -1,\ ||\sigma_e|| = 0.01,\ ||\sigma_\nu|| = 0.01,\ \rho_{e\nu} = 0.2,\ \beta = 0.03$ and $k = 0.5$. To draw the graph, we use the analytical form obtained in Nakamura, Nakayama and Takahashi (2008). The numerical value is calculated as sample mean in the Monte-Carlo simulation.

for any $(e, U, \sigma_U), (e', U', \sigma'_U) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^2$.

This assumption is sufficient to ensure the existence of a unique solution (e, ν, U, σ_U) to Eq.(B.1). On the other hand, from Eq.(B.2), our model does not satisfy the Lipschitz condition of the driver g with respect to σ_U.

Still, our numerical method works well. We can conjecture the reason with the following two points. First, let us look at the case of $\delta = 1$, for the reference. We have the analytical solution of the yield curve, which is shown as in Appendix A. At the same time, by setting $U(t, e, \nu) = q(t)\log e + m(t)\nu + n(t)$ in the second procedure of the above algorithm, we obtain a numerical solution in the case of $\delta = 1$. We can then compare the numerical solution with the analytical solution. The comparison shows that the difference is 4 basis points, at most; this is quite small relative to the level of the spot yields. Hence, the discrete-time approximation replicates the analytical results well; our numerical method performs well under our parametric assumptions in the case of $\delta = 1$ (Figure 4).

Second, in the case of $\delta \neq 1$, we show in Section 5 that the numerical results are well-explained under the normality approximation. In fact, there, $\left|\left|\frac{\sigma_J(t)}{(1+\alpha)J_t}\right|\right|^2$ is constant over the first 30 years, which period of time we are focusing on (see the above-mentioned Figure 3). We can then conjecture that the numerical method works well, like in the case of $\delta = 1$, because δ is set to be not far away from unity in our numerical analyses.

C Normality Approximation of yield curves

We impose two simplifications. First, set $k = 0$: that is, $\nu_u = \nu_t + \int_t^u \sigma_\nu^\top dB_w$. In real terms, substituting this into Eq.(5.1),

$$R(t,s) = -\frac{1}{s-t}\log E_t\left[\exp\left(\begin{array}{l} -\int_t^s \left(\begin{array}{l} \delta\int_t^u \sigma_\nu^\top dB_w + \left\{\delta\nu_t + (1-\delta)\beta - \frac{\delta}{2}||\sigma_e||^2\right\} \\ +\frac{\alpha(1+\alpha)}{2}\left|\left|\frac{\sigma_J(u)}{(1+\alpha)J_u}\right|\right|^2 \end{array}\right) du \\ -\int_t^s \left(\delta\sigma_e - \alpha\frac{\sigma_J(u)}{(1+\alpha)J_u}\right)^\top dB_u \end{array}\right)\right]$$

Note:

$$\int_t^s \left(\delta\int_t^u \sigma_\nu^\top dB_w\right) du = \int_t^s \sigma_\nu^\top \left(\int_u^s \delta\, dw\right) dB_u = \delta\int_t^s (s-u)\sigma_\nu^\top dB_u.$$

Hence,

$$\begin{aligned} R(t,s) &= -\frac{1}{s-t}\log E_t\left[\exp\left(\begin{array}{l} -\int_t^s \left(\delta\int_t^u \sigma_\nu^\top dB_w\right) du - \int_t^s \left(\delta\nu_t + (1-\delta)\beta - \frac{\delta}{2}||\sigma_e||^2\right) du \\ -\int_t^s \left(\frac{\alpha(1+\alpha)}{2}\left|\left|\frac{\sigma_J(u)}{(1+\alpha)J_u}\right|\right|^2\right) du \\ -\int_t^s \left(\delta\sigma_e - \alpha\frac{\sigma_J(u)}{(1+\alpha)J_u}\right)^\top dB_u \end{array}\right)\right] \\ &= -\frac{1}{s-t}\log E_t\left[\exp\left(\begin{array}{l} -\delta\int_t^s (s-u)\sigma_\nu^\top dB_u - (s-t)\left\{\delta\nu_t + (1-\delta)\beta - \frac{\delta}{2}||\sigma_e||^2\right\} \\ -\int_t^s \left(\frac{\alpha(1+\alpha)}{2}\left|\left|\frac{\sigma_J(u)}{(1+\alpha)J_u}\right|\right|^2\right) du \\ -\int_t^s \left(\delta\sigma_e - \alpha\frac{\sigma_J(u)}{(1+\alpha)J_u}\right)^\top dB_u \end{array}\right)\right] \\ &= \left[\begin{array}{l} \delta\nu_t + (1-\delta)\beta - \frac{\delta}{2}||\sigma_e||^2 \\ -\frac{1}{s-t}\log E_t\left[\exp\left(\begin{array}{l} -\frac{\alpha(1+\alpha)}{2}\int_t^s \left|\left|\frac{\sigma_J(u)}{(1+\alpha)J_u}\right|\right|^2 du \\ +\int_t^s \left(-\delta\left\{\sigma_e + (s-u)\sigma_\nu\right\} + \alpha\frac{\sigma_J(u)}{(1+\alpha)J_u}\right)^\top dB_u \end{array}\right)\right] \end{array}\right]. \end{aligned}$$

Second, impose another simplification: $\frac{\sigma_J(t)}{(1+\alpha)J_t}$ is replaced by a constant matrix, denoted by $\tilde{\sigma}_J$. Applying Ito's formula to this,

$$
\begin{aligned}
R^a(t,s) &= \left[\begin{array}{l} \delta\nu_t + (1-\delta)\beta - \frac{\delta}{2}||\sigma_e||^2 \\ -\frac{1}{s-t}\log\left[\exp\left(\begin{array}{l} -\frac{\alpha(1+\alpha)}{2}\int_t^s ||\tilde{\sigma}_J||^2 du \\ +\frac{1}{2}\int_t^s \left\{\begin{array}{l} \delta^2||\sigma_e||^2 + (s-u)^2\delta^2 ||\sigma_\nu||^2 \\ +\alpha^2 ||\tilde{\sigma}_J||^2 - 2\alpha\delta\sigma_e^\top\tilde{\sigma}_J \\ +2(s-u)\delta\sigma_\nu^\top(\delta\sigma_e - \alpha\tilde{\sigma}_J) \end{array}\right\} du \end{array}\right)\right] \end{array}\right] \\
&= \left[\begin{array}{l} \delta\nu_t + (1-\delta)\beta - \frac{\delta(1+\delta)}{2}||\sigma_e||^2 + \frac{\alpha}{2}||\tilde{\sigma}_J||^2 + \alpha\delta\sigma_e^\top\tilde{\sigma}_J \\ -\frac{1}{s-t}\left(\begin{array}{l} \frac{\delta^2}{2}||\sigma_\nu||^2 \int_t^s (s-u)^2 du + \delta^2\sigma_e^\top\sigma_\nu\int_t^s(s-u)du \\ -\alpha\delta\sigma_\nu^\top\tilde{\sigma}_J\int_t^s(s-u)du \end{array}\right) \end{array}\right].
\end{aligned}
$$

Since $\int_t^s (s-u)du = \frac{(s-t)^2}{2}$ and $\int_t^s (s-u)^2 du = \frac{(s-t)^3}{3}$,

$$
\begin{aligned}
R^a(t,s) &= \left[\begin{array}{l} \delta\nu_t + (1-\delta)\beta - \frac{\delta(1+\delta)}{2}||\sigma_e||^2 + \frac{\alpha}{2}||\tilde{\sigma}_J||^2 + \alpha\delta\sigma_e^\top\tilde{\sigma}_J \\ -\frac{1}{s-t}\left(\begin{array}{l} \frac{\delta^2}{2}||\sigma_\nu||^2\frac{(s-t)^3}{3} + \delta^2\sigma_e^\top\sigma_\nu\frac{(s-t)^2}{2} \\ -\alpha\delta\sigma_\nu^\top\tilde{\sigma}_J\frac{(s-t)^2}{2} \end{array}\right) \end{array}\right] \\
&= \left[\begin{array}{l} \delta\nu_t + (1-\delta)\beta - \frac{\delta(1+\delta)}{2}||\sigma_e||^2 + \frac{\alpha}{2}||\tilde{\sigma}_J||^2 + \alpha\delta\sigma_e^\top\tilde{\sigma}_J \\ -\delta^2||\sigma_\nu||^2\frac{(s-t)^2}{6} - \delta^2\sigma_e^\top\sigma_\nu\frac{(s-t)}{2} + \alpha\delta\sigma_\nu^\top\tilde{\sigma}_J\frac{(s-t)}{2} \end{array}\right].
\end{aligned}
$$

Note that, as $s \downarrow t$,

$$
\begin{aligned}
\lim_{s\downarrow t} R^a(t,s) &= R^a(t,t) \\
&= \delta\nu_t + (1-\delta)\beta - \frac{\delta(1+\delta)}{2}||\sigma_e||^2 + \frac{\alpha}{2}||\tilde{\sigma}_J||^2 + \alpha\delta\sigma_e^\top\tilde{\sigma}_J \\
&=: r_t^a.
\end{aligned}
$$

That is, r_t^a represents this instantaneous riskless rate and stands for the level of the yield curve. Thus,

$$
R^a(t,s) = r_t^a - \delta^2||\sigma_\nu||^2\frac{(s-t)^2}{6} - \delta^2\sigma_e^\top\sigma_\nu\frac{(s-t)}{2} + \alpha\delta\sigma_\nu^\top\tilde{\sigma}_J\frac{(s-t)}{2}. \quad \text{(C.1)}
$$

In nominal terms, the approximated nominal spot yields, denoted by $\hat{R}^a(t,s)$, are written as:

$$\hat{R}^a(t,s) = -\frac{1}{s-t}\log E_t\left[\exp\left(\begin{array}{l} -\int_t^s \left\{\begin{array}{l} (1-\delta)\beta + \delta\left(\nu_t + \int_t^u \sigma_\nu^\top dB_w\right) \\ -\frac{\delta}{2}||\sigma_e||^2 + \alpha\frac{\delta}{2}\,||\tilde{\sigma}_J||^2 \\ +\left(\varepsilon_t + \int_t^u \sigma_\varepsilon^\top dB_w\right) - \frac{1}{2}\,||\sigma_n||^2 \end{array}\right\} du \\ -\int_t^s \left(\delta\sigma_e - \alpha\tilde{\sigma}_J + \sigma_n\right)^\top dB_u \end{array}\right)\right]$$

$$= \left[\begin{array}{l} (1-\delta)\beta + \delta\nu_t - \frac{\delta}{2}||\sigma_e||^2 + \alpha\frac{\delta}{2}\,||\tilde{\sigma}_J||^2 + \varepsilon_t - \frac{1}{2}\,||\sigma_n||^2 \\ -\frac{1}{s-t}\log E_t\left[\exp\left\{\begin{array}{l} -\int_t^s \delta(s-u)\sigma_\nu^\top dB_u - \int_t^s (s-u)\sigma_\varepsilon^\top dB_u \\ -\int_t^s \left(\delta\sigma_e - \alpha\tilde{\sigma}_J\right)^\top dB_u - \int_t^s \left(\sigma_n\right)^\top dB_u \end{array}\right\}\right] \end{array}\right].$$

Applying Ito's formula again,

$$\hat{R}^a(t,s) = \left[\begin{array}{l} R^a(t,s) + (\hat{r}_t^a - r_t^a) \\ -\,||\sigma_\varepsilon||^2\,\frac{(s-t)^2}{6} - \delta\sigma_\nu^\top\sigma_n\frac{(s-t)}{2} - \delta\sigma_\nu^\top\sigma_\varepsilon\frac{(s-t)^2}{6} \\ -\delta\sigma_e^\top\sigma_\varepsilon\frac{(s-t)}{2} + \alpha\sigma_\varepsilon^\top\tilde{\sigma}_J\frac{(s-t)}{2} - \sigma_n^\top\sigma_\varepsilon\frac{(s-t)}{2} \end{array}\right].$$

where $\hat{r}_t^a \triangleq r_t^a + (\varepsilon_t - ||\sigma_n||^2 - \sigma_n^\top\lambda_t)$ denotes the nominal instantaneous riskless rate.

D Optimal consumption/portfolio choice

This appendix solves a maximization problem of the representative agent to obtain the optimal utility process characterized by Eq.(3.3) and the equilibrium pricing kernel Eq.(3.5). Our solution method is basically according to the standard method of stochastic controls (Ma and Yong (1999)). This is also a direct application of Skiadas (2007).

A consumption process $\mathbf{c} = \{c_t;\ t \in [0,T]\}$ is assumed to be real-valued, non-negative and $\{\mathcal{F}_t\}_{0\le t\le T}$-adapted, and satisfies some mathematical regularity conditions for a utility function to be well-defined. The set of the consumption processes $\mathbf{c}$ is denoted by $\mathcal{C}$. Let $\mathcal{H}$ denote the Hilbert space of every $x \in \mathcal{L}(\mathbf{R})$ such that $E_0\left[\int_0^T (x_t)^2\,dt + (x_T)^2\right] < \infty$ with the inner product $(x|y) \triangleq E_0\left[\int_0^T x_t y_t\,dt + x_T y_T\right]$ for $x, y \in \mathcal{H}$. Assume that $\mathcal{C}$ is in $\mathcal{H}$ and is convex. The agent ranks her consumption plan $\mathbf{c} \in \mathcal{C}$ based on SDU of consumption $V_t(\mathbf{c})$ (we may also write simply V_t) for each $t \in [0,T]$, which is characterized by:

$$dV_t = -f(c_t, V_t, \Sigma_t)dt + \Sigma_t^\top dB_t; \quad V_T = 0.$$

Assume that $f(c, V, \Sigma)$ is differentiable in c, V, Σ. Also, we assume the existence of a unique well-defined V_t for each consumption process $\mathbf{c} \in \mathcal{C}$ and, moreover, that V_0 is strictly increasing, concave, and homothetic in $\mathbf{c}$. We can also write V_t as follows:

$$V_t = E_t \left[\int_t^T f(c_s, V_s, \Sigma_s) ds \right].$$

There are security markets that consist of $(m+1)$ securities whose prices are denoted by $(P_0, S_1, \cdots, S_m) \in \mathbf{R}^{m+1}$. Specifically, the markets are characterized by the following excess return processes:

$$\begin{aligned} \frac{dP_0(t)}{P_0(t)} &= r_t dt; \quad P_0(0) = 1, \\ dR_t &= \left(\frac{dS_1(t)}{S_1(t)}, \frac{dS_2(t)}{S_2(t)}, \cdots, \frac{dS_m(t)}{S_m(t)} \right)^\top \\ &= \mu_R(t)\, dt + \sigma_R(t)^\top dB_t; \quad R_0 = \bar{r}. \end{aligned}$$

Assume that there is no arbitrage in the markets, that is, there is some m-dimensional vector process $\eta = \{\eta_t;\ 0 \le t \le T\}$ such that

$$\mu_R = \sigma_R^\top \eta.$$

In addition, to guarantee completeness, we assume that σ_R is nonsingular for a.e.-t, a.s. We define the pricing kernel $\pi = \{\pi_t;\ 0 \le t \le T\}$ by

$$\frac{d\pi_t}{\pi_t} = -r_t\, dt - \eta_t^\top dB_t.$$

Let $\psi_t \in \mathbf{R}^m$ (for each t) denote a time-t allocation rate on the risky securities, and $\psi \triangleq \{\psi_t;\ 0 \le t \le T\}$. The wealth process held by the agent is characterized by the following stochastic differential equiation:

$$dW_t = (r_t W_t - c_t)dt + W_t \psi_t^\top dR_t; \quad W_0 = w.$$

For notational convenience, define $\sigma_W(t) = W_t \sigma_R(t) \psi_t$. Thus,

$$dW_t = (r_t W_t - c_t + \eta^\top \sigma_W(t))dt + \sigma_W(t)^\top dB_t; \quad W_0 = w.$$

Now, we solve a maximization problem of the agent with respect to $(\mathbf{c}, \psi)$. A pair $(\mathbf{c}, \psi)$ is said to be optimal if $\mathbf{c}$ maximizes his expected utility when $(\mathbf{c}, \psi)$ is financed. The maximization problem is defined as: for all t,

$$J_t = \max_{\mathbf{c},\psi} V_t(\mathbf{c})$$

subject to

$$dW_t = (r_t W_t - c_t + \eta^\top \sigma_W(t))dt + \sigma_W(t)^\top dB_t; \quad W_0 = w.$$

Fix a pair $(\mathbf{c}, \psi)$. For notational convenience, define $s \triangleq (c, V, \Sigma) \in S$ corresponding to the pair $(\mathbf{c}, \psi)$ where S denotes a convex subset of some Euclidean space X. The pair $(\mathbf{c}, \psi)$ and the corresponding s satisfy the following two conditions.

Condition D.1 *For a pair $(\mathbf{c}, \psi)$ and the corresponding s,*

$$(f_c, f_V, f_\Sigma) \in \partial f(s) \triangleq \left\{ \delta \in X;\, f(s+h) \leq f(s) + \delta^\top h,\, \forall h \text{ subject to } s + h \in S \right\} \tag{D.1}$$

where $f_x \triangleq \frac{\partial f}{\partial x}$ for $x = c, V, \Sigma$.

In convex analysis, for a concave function f, the set defined on the right-hand side of Eq.(D.1) is called the supergradient of the function f at s.

Next, define a process $\mathcal{E}_t(f_V, f_\Sigma)$ that is characterized by the following SDE:

$$\frac{d\mathcal{E}_t(f_V, f_\Sigma)}{\mathcal{E}_t(f_V, f_\Sigma)} = f_v dt + f_\Sigma dB_t;\; \mathcal{E}_0(f_V, f_\Sigma) = 1.$$

As some integrability restriction,

Condition D.2 *For the pair $(\mathbf{c}, \psi)$,*

$$E_t \left[\sup_{0 \leq t \leq T} \mathcal{E}_t(f_V, f_\Sigma)^2 \right] < \infty.$$

In the remaining, we omit the subscript time t unless it causes any confusion. Recall the assumption that the set of the consumption processes $\mathcal{C}$ is convex.

Lemma D.1 *Under Conditions D.1 and D.2, for the pair $(\mathbf{c}, \psi)$ and the corresponding s and for any x such that $x \in \mathcal{H}$ and $\mathbf{c} + x \in \mathcal{C}$ and $\mathbf{c} + x$ is feasible,*

$$V_0(\mathbf{c} + x) \leq V_0(\mathbf{c}) + (\pi | x).$$

where $\pi = \mathcal{E}(f_V, f_\Sigma) f_c$ *that is assumed to belong to* $\mathcal{H}$*.*

Proof. Set $x \in \mathcal{H}$ and $\mathbf{c} + x \in \mathcal{C}$ and $\mathbf{c} + x$ is feasible. Define

$$\begin{aligned} \delta &\triangleq V(\mathbf{c} + x) - V(\mathbf{c}) \\ \Delta &\triangleq \Sigma(\mathbf{c} + x) - \Sigma(\mathbf{c}) \\ p &\triangleq f(c, V, \Sigma) + f_c x + f_V \delta + f_\Sigma \Delta \end{aligned}$$

From Condition D.1, $p \geq 0$. By direct algebra,

$$d\delta = dV(\mathbf{c} + x) - dV(\mathbf{c}).$$

Since, by construction,

$$dV = -f(c, V, \Sigma)dt + \Sigma dB; \quad V_T = 0,$$

we have

$$d\delta = -(f_c x + f_V \delta + f_\Sigma \Delta - p)dt + \Delta^\top dB; \quad \delta_T = f_c x_T - p_T = 0.$$

Therefore,

$$d(\mathcal{E}\delta) = (-\mathcal{E} f_c + \mathcal{E} p)dt + \cdots dB$$

Since, by Condition D.2, $E_t \left[\sup_{0 \leq t \leq T} (\mathcal{E}_t(f_V, f_\Sigma))^2\right] < \infty$,

$$\begin{aligned} \mathcal{E}_0 \delta_0 &= E_0 \left[\int_0^T (\mathcal{E} f_c x - \mathcal{E} p)dt + \mathcal{E} f_c x_T - \mathcal{E} p_T \right] \\ &= (\mathcal{E} f_c | x) - (\mathcal{E} | p). \end{aligned}$$

Since $\mathcal{E}_0 = 1$,

$$\delta_0 = V(\mathbf{c} + x) - V(\mathbf{c}) \leq (\pi | x).$$

The desired result is obtained. ■

Impose the following additional condition:

Condition D.3 *For the pair* $(\mathbf{c}, \psi)$*,* $E_0 [\sup_t \pi_t W_t] < \infty$*.*

Then,

Lemma D.2 *Under Condition D.3, for the pair* $(\mathbf{c}, \psi)$ *and for any* x *such that* $x \in \mathcal{H}$ *and* $\mathbf{c} + x \in \mathcal{C}$ *and* $\mathbf{c} + x$ *is feasible,*

$$(\pi|x) \leq 0 \quad \textit{with equality when } x = 0.$$

Proof. By construction,

$$d(\pi W) = \pi c dt + \cdots dB.$$

By Condition D.3,

$$\pi_0 W_0 = E_0 \left[\int_0^T \pi_s c_s \, ds + \pi_T c_T \right] = (\pi|\mathbf{c}).$$

For any x such that $x \in \mathcal{H}$ and $\mathbf{c} + x \in \mathcal{C}$ and $\mathbf{c} + x$ is feasible,

$$\pi_0 W_0 \geq (\pi|\mathbf{c} + x)$$

Therefore, $(\pi|x) \leq 0$. ■

Now, we obtain our main theoretical result in this appendix:

Proposition D.1 *Suppose Conditions D.1, D.2, D.3, and* $\pi = \mathcal{E}(f_V, f_\Sigma) f_c \in \mathcal{H}$. *Then a pair* $(\mathbf{c}, \psi)$ *is optimal.*

Proof. By Lemma D.1 and Lemma D.2, we obtain the result directly. ■

Finally, we specify the utility function form and market structure and solve for an equilibrium $(\mathbf{c}, \psi)$ and pricing kernel explicitly using market clearing conditions. In fact, the optimization with respect to ψ requires explicit specification of the market structure characterized by (μ_R, σ_R). To be consistent with the model in the main text, assume that there exists a single risky security with positive net supply. That is, $m = 1$. Also, there is a single riskless asset with zero supply, whose price is P_0. The excess return process of the risky security (with some initial investment w_1 given) is characterized by:

$$W_1 dR_1 = (W_1 \mu_{R_1} + e)dt + W_1 \sigma_{R_1}^\top dB; \quad W_1(0) = w_1 > 0$$

where W_1 denotes the amount of the investment in the risky security. Recall that e is the endowment of the consumption goods and its process is characterized by Eq.(3.1) and Eq.(3.2). Since the utility is strictly increasing and the consumption good is perishable, $\psi = (1, 0, \cdots, 0)^\top$ holds true to clear the markets in equilibrium, that is, all is invested in the risky security in equilibrium. Therefore, in equilibrium,

$$\begin{aligned} c_t &= e_t \quad \text{for all } t \\ W(0) &= w_1 \\ dW &= W(r + \mu_{R_1})dt + W\sigma_{R_1}^\top dB. \end{aligned}$$

Thus, the optimal utility process is characterized by Eq.(3.3). The endowment works as a dividend, all of which is consumed at each instant.

We here focus on the particular type (i.e., Skiadas-Schroder type) of SDU that is used in the above main text:

$$dV_t = -f(c_t, V_t)dt + \Sigma_t^\top dB_t; \quad V_T = 0,$$

where, for constants α, β, δ,

$$f(c, v) \triangleq \begin{cases} (1+\alpha)\left\{\frac{c^{1-\delta}}{1-\delta}|v|^{\frac{\alpha}{1+\alpha}} - \beta v\right\} & (\text{if } \delta \neq 1), \\ (1+\alpha v)\left\{\log c - \frac{\beta}{\alpha}\log(1+\alpha v)\right\} & (\text{if } \delta = 1). \end{cases}$$

Put parametric assumptions:

$$\beta \geq 0 \text{ and } \begin{cases} \alpha > -1 \quad \text{and} \quad 1-\delta < \min\left\{1, \frac{1}{1+\alpha}\right\} & (\text{if } \delta \neq 1), \\ \alpha \leq \beta \quad (\text{if } \delta = 1). \end{cases}$$

The assumptions ensure the existence of a unique well-defined V_t for each consumption process $\mathbf{c} \in \mathcal{C}$. Moreover, V_0 is strictly increasing, concave, and homothetic in $\mathbf{c}$.

Since $f_\Sigma = 0$ in this formulation, from Lemma D.1,

$$\pi_t = \exp\left\{\int_0^t f_v(e_u, J_u)\, du\right\} f_c(e_t, J_t).$$

We obtain Eq.(3.5).

E Proofs of theorems

E.1 Proof of Eq.(3.4)

Proof: From Eq.(2.1), $V_t + \int_0^t f(c_u, V_u)\, du$ *is a martingale:*

$$V_t + \int_0^t f(c_u, V_u)\, du = E_t\left[\int_0^T f(c_u, V_u)\, du\right].$$

Since the drift of the process of $V_t + \int_0^t f(c_u, V_u)\, du$ *must be zero, by applying Ito's formula to* $V_t = J(t, X_t)$*,*

$$0 = f(e, J) + \partial_t J + \partial_x J\, b + \frac{1}{2}\,\mathrm{tr}\left\{a\,\partial_{xx} J\, a^\top\right\}; \quad J(T, X_T) = 0, \tag{E.1}$$

where $X_t \triangleq (e_t, \nu_t)$*,* $a_t \triangleq (e_t\sigma_e, \sigma_\nu)^\top$*,* $b_t \triangleq (e_t\mu_e(t), k(\bar{\nu} - \nu_t))^\top$*,* $\partial_t J := \frac{\partial J}{\partial t}$*,* $\partial_x J := \frac{\partial J}{\partial x}$*, and* $\partial_{xx} J := \frac{\partial^2 J}{\partial x^2}$*. Hence, we obtain Eq.*(3.4)*.* ■

E.2 Proof of Lemma 3.1

Proof: Applying Ito's formula to π_t *in Eq.*(3.5)*,*

$$\frac{d\pi_t}{\pi_t} = f_v(e_t, J_t)dt + \frac{df_c(e_t, J_t)}{f_c(e_t, J_t)}. \tag{E.2}$$

Comparing Eq.(3.6) *with Eq.*(E.2)*, we see that* λ_t *must be the diffusion coefficient of* $-\frac{df_c}{f_c}$*. Then an application of Ito's formula to* f_c *leads to Eq.*(3.7)*.* ■

E.3 Proof of Proposition 3.1

Proof: Consider the case of $\delta \neq 1$*. Substitute Eq.*(2.2) *into Eq.*(3.7) *and Eq.*(3.8)*. We need to identify* f_v^* *and* $\dfrac{df_c^*}{f_c^*}$*. With regard to the market price of risk* λ*, substituting* $-\dfrac{f_{cc}^*}{f_c^*} = \dfrac{\delta}{e}$ *and* $-\dfrac{f_{cv}^*}{f_c^*} = -\dfrac{\alpha}{(1+\alpha)J_t}$ *into Eq.*(3.7)*, we obtain Eq.*(3.11)*. Next, with regard to the instantaneous riskless rate* r_t*, applying Ito's formula to* f_c^**,*

$$\frac{df_c^*}{f_c^*} = \frac{f_{cc}^*}{f_c^*}de + \frac{f_{cv}^*}{f_c^*}dJ + \frac{1}{2}\left(\frac{f_{ccc}^*}{f_c^*}(de)^2 + 2\frac{f_{ccv}^*}{f_c^*}(de)(dJ) + \frac{f_{cvv}^*}{f_c^*}(dJ)^2\right).$$

Hence,

$$
\begin{aligned}
r_t &= -f_v^* - \frac{\mathcal{D}\, f_c^*}{f_c^*} \\
&= -f_v^* - \left\{ \frac{f_{cc}^*}{f_c^*}\, \mathcal{D}\, e_t + \frac{f_{cv}^*}{f_c^*}\, \mathcal{D}\, J_t + \frac{1}{2} \left(\frac{f_{ccc}^*}{f_c^*}\, ||e_t \sigma_e||^2 + 2 \frac{f_{ccv}^*}{f_c^*} e_t \sigma_e^\top \sigma_J(t) + \frac{f_{cvv}^*}{f_c^*}\, ||\sigma_J(t)||^2 \right) \right\} \\
&= \beta + \delta \mu_e(t) - \frac{1}{2} \left(\frac{f_{ccc}^*}{f_c^*}\, ||e_t \sigma_e||^2 + 2 \frac{f_{ccv}^*}{f_c^*} e_t \sigma_e^\top \sigma_J(t) + \frac{f_{cvv}^*}{f_c^*}\, ||\sigma_J(t)||^2 \right) \\
&= \beta + \delta \mu_e(t) - \frac{\delta(1+\delta)}{2}\, ||\sigma_e||^2 + \delta \alpha \frac{\sigma_e^\top \sigma_J(t)}{(1+\alpha) J_t} + \frac{\alpha}{2} \left|\left| \frac{\sigma_J(t)}{(1+\alpha) J_t} \right|\right|^2 .
\end{aligned}
$$

Eq.(3.9) *and Eq.*(3.10) *are obtained. The proof for* $\delta = 1$ *is similar.* ■