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Abstract

This paper proposes a new approximation method of pricing barrier and average options under stochastic volatility environment by applying an asymptotic expansion approach. In particular, a high-order expansion scheme for general multi-dimensional diffusion processes is effectively applied. Moreover, the paper combines a static hedging method with the asymptotic expansion method for pricing barrier options. Finally, numerical examples show that the fourth or fifth-order asymptotic expansion scheme provides sufficiently accurate approximations under the λ-SABR and SABR models.

Keywords: barrier option, average option, knock-out option, stochastic volatility, static hedge, asymptotic expansion, λ-SABR model, SABR model

1 Introduction

Recently, it is a necessary and important task to evaluate exotic options such as barrier and average options based on calibration to liquid plain-vanilla option prices. Pricing vanilla options under some stochastic volatility model is a typical approach to calibration. However, to our knowledge closed-form solutions for exotic options' prices under stochastic volatility models are rarely obtained and hence pricing should usually rely on numerical approximation methods such as Monte Carlo methods or finite difference/element methods. For example, Itkin and Carr [17] solved barrier options using a PDE method and Ninomiya-Victoir [28] computed an average option using a Monte Carlo method. Straightforward application of those methods are time-consuming or/and produces only inaccurate estimates. Thus, we need to develop some sophisticated technique with those methods to satisfy requirements in practice. Alternatively, if we can obtain a closed-form formula that creates an accurate and fast-computing approximation, it is very useful.

This paper proposes an approximation method of pricing barrier and average options under stochastic volatility environment by applying an asymptotic expansion approach. In particular, a high-order expansion scheme for general multi-dimensional diffusion processes recently developed by Takahashi-Takehara-Toda [36] is effectively applied. Moreover, for pricing barrier options the paper combines the asymptotic expansion method with a static hedging method by Fink [9]. Numerical examples show that the fourth or fifth-order approximation of an asymptotic expansion scheme provides sufficiently fast and accurate approximations in practice under the λ-SABR model(Labordere [21]) and the SABR model(Hagan et.al. [15]); our method gives good approximations even in high volatility or/and high volatility on volatility situations, when it is usually difficult for numerical approximations to produce fast and accurate approximations.

For over a decade, static hedging techniques have been developed and investigated extensively for barrier type options. Bowie and Carr [2] and Carr, Ellis and Gupta [6] consider a static hedge method for barrier-type and lookback options by using *put call symmetry* (Carr [3]). Derman, Ergener and Kani [8] proposes the *calendar-spreads* method. Carr and Picron [7] presents a method for static hedging of timing risk which is applied to pricing barrier options. Carr and Chou [4], [5] shows a representation of any twice differentiable payoff function and then develops the so called *strike-spreads* method for static hedging of barrier, ratchet and lookback options under the Black-Scholes model.

Fink [9] generalizes the method of Derman, Ergener and Kani [8] for barrier options in a Heston's stochastic volatility. More recently, Nalholm and Poulsen [26] proposes a new technique for static hedging of barrier options under general asset dynamics, such as a jump-diffusion process with correlated stochastic volatility. Furthermore, Nalholm and Poulsen [25] examines the sensitivity of dynamic and static hedging methods for barrier options to model risk.

The asymptotic expansion is first applied to finance for evaluation of an average option that is a popular derivative in commodity markets. Kunitomo and Takahashi [18] and Takahashi [30] derive the approximation formulas for an average option by an asymptotic method based on log-normal approximations of an average price distribution when the underlying asset price follows a geometric Brownian motion(under the Black-Scholes model). Yoshida [42] applies a formula derived by the asymptotic expansion of certain statistical estimators for small diffusion processes. Thereafter, the asymptotic expansion have been applied to a broad class of problems in finance: See Takahashi [31], [32], Kunitomo and Takahashi [19], [20], Matsuoka, Takahashi and Uchida [23], Takahashi and Yoshida [37], [38], Muroi [24], and Takahashi and Takehara [33], [34], [35].

Although the asymptotic expansion method is applied to average options in [42], [30], [31], this paper is the first one that implements the expansion more than the second order and examines its numerical accuracy under stochastic volatility environment.

Moreover, to our best knowledge, other closed-form (approximation) formulas for barrier or average options under stochastic volatility environment have not been shown except Fouque, Papanicolaou and Sircar [10], [11] and Fouque and Han [12], [13]. They apply the singular perturbation method to pricing Barrier and Asian(average) options in a fast mean-reverting stochastic volatility model. See Yamamoto and Takahashi [40] for the accuracy of the approximation method; it shows through numerical experiments that the method provides sufficiently accurate option prices in a fast mean-reversion case of the volatility process while it does not in a non-fast mean-reversion case.

The organization of the paper is as follows: After a brief explanation of the asymptotic expansion in the next section, Section 3 introduces a new computation algorithm for the asymptotic expansion and derives an approximation formula for a density function of the underlying asset. Section 4 proposes an approximation method for pricing barrier options under stochastic volatility models by applying the asymptotic expansion with a static hedging method. It also provides numerical examples under the λ-SABR model. Section 5 applies the high-order expansion scheme to pricing average options and presents numerical examples under the SABR and λ-SABR models. Section 6 concludes. Finally, some error analysis is given in Appendix.

2 An Asymptotic Expansion in a Multi-dimensional Diffusion Process

This section briefly describes an asymptotic expansion method in a general multi-dimensional diffusion process. See Section 2 of [36] for the details.

Let (W, P) be the r-dimensional Wiener space. We consider a d-dimensional diffusion process $X_t^{(\epsilon)} = (X_t^{(\epsilon),1}, \cdots, X_t^{(\epsilon),d})$ which is the solution to the following stochastic differential equation:

$$\begin{aligned} dX_t^{(\epsilon),i} &= V_0^i(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V^i(X_t^{(\epsilon)})dW_t \quad (i = 1, \cdots, d) \\ X_0^{(\epsilon)} &= x_0 \in \mathbf{R}^d \end{aligned} \tag{1}$$

where $W = (W^1, \cdots, W^r)$ is a r-dimensional standard Wiener process, and $\epsilon \in (0, 1]$ is a known parameter.

Suppose that $V_0 = (V_0^1, \cdots, V_0^d) : \mathbf{R}^d \times (0, 1] \mapsto \mathbf{R}^d$ and $V = (V^1, \cdots, V^d)$: $\mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^r$ satisfy some regularity conditions.(e.g. V_0 and V are smooth functions with bounded derivatives of all orders.)

Next, suppose that a function $g : \mathbf{R}^d \mapsto \mathbf{R}$ to be smooth and all derivatives have polynomial growth orders. Then, a smooth Wiener functional $g(X_T^{(\epsilon)})$ has its asymptotic expansion;

$$g(X_T^{(\epsilon)}) \approx g_{0T} + \epsilon g_{1T} + \cdots$$

in L^p for every $p > 1$(or in $\mathbf{D}^\infty$) as $\epsilon \downarrow 0$. The coefficients in the expansion $g_{nT} \in \mathbf{D}^\infty (n = 0, 1, \cdots)$ can be obtained by Taylor's formula and represented based on multiple Wiener-Itô integrals. Here, $\mathbf{D}^\infty$ denotes the set of smooth Wiener functionals. See chapter V of Ikeda and Watanabe [16] for the detail.

Note that the leading term of the expansion g_{0T} is deterministic and expressed as

$$g_{0T} = g(X_T^{(0)}),$$

where $X_t^{(0)} = (X_t^{(0),1}, \cdots, X_t^{(0),d})$ is the solution of the ordinary differential equation:

$$\begin{aligned} dX_t^{(0),i} &= V_0^i(X_t^{(0)}, 0)dt \quad (i = 1, \cdots, d) \\ X_0^{(0)} &= x_0 \in \mathbf{R}^d. \end{aligned} \tag{2}$$

Next, normalize $g(X_T^{(\epsilon)})$ to

$$G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0, 1]$. Then,

$$G^{(\epsilon)} \approx g_{1T} + \epsilon g_{2T} + \cdots$$

in L^p for every $p > 1$(or in $\mathbf{D}^\infty$). Moreover, let

$$\hat{V}(x, t) = (\partial g(x))' [Y_T Y_t^{-1} V(x)]$$

where Y denotes the solution to the differential equation;

$$dY_t = \partial V_0(X_t^{(0)}, 0) Y_t dt; \; Y_0 = I_d.$$

Here, ∂V_0 denotes the $d \times d$ matrix whose (j, k)-element is $\partial_k V_0^j = \frac{\partial V_0^j(x,\epsilon)}{\partial x_k}$, V_0^j is the j-th element of V_0, and I_d denotes the $d \times d$ identity matrix.

Further, we make the following assumption:

$$\text{(Assumption 1)} \quad \Sigma_T = \int_0^T \hat{V}(X_t^{(0)}, t)\hat{V}(X_t^{(0)}, t)' dt > 0.$$

Note that g_{1T} follows a normal distribution with variance Σ_T; the density function of g_{1T} denoted by $f_{g_{1T}}(x)$ is given by

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{(x-C)^2}{2\Sigma_T}\right)$$

where

$$C = (\partial g(X_T^{(0)}))' \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X_t^{(0)}, 0) dt.$$

Hence, Assumption 1 means that the distribution of g_{1T} does not degenerate. In application, it is easy to check this condition in most cases.

Let $\mathcal{S}$ be the real Schwartz space of rapidly decreasing $\mathbf{C}^\infty$-functions on $\mathbf{R}$ and $\mathcal{S}'$ be its dual space that is the space of the Schwartz tempered distributions. Next, take $\Phi \in \mathcal{S}'$. Then, by Watanabe theory(Watanabe [39], Yoshida [41]) a generalized Wiener functional $\Phi(G^{(\epsilon)})$ has an asymptotic expansion in $\mathbf{D}^{-\infty}$ as $\epsilon \downarrow 0$ where $\mathbf{D}^{-\infty}$ denotes the set of generalized Wiener functionals. See chapter V of Ikeda and Watanabe [16] for the detail. Hence, the expectation of $\Phi(G^{(\epsilon)})$ is expanded around $\epsilon = 0$ as follows: For $N = 0, 1, 2, \cdots$,

$$\mathbf{E}[\Phi(G^{(\epsilon)})] = \sum_{j=0}^{N} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} \int_{\mathbf{R}} \Phi(x) \sum_{k \in K_{j,m}} C^{j,m,k} (-1)^m \frac{\partial^m}{\partial x^m} \{\mathbf{E}\left[X^{j,m,k} \middle| g_{1T} = x\right] f_{g_{1T}}(x)\} dx + o(\epsilon^N) \tag{3}$$

where

$$\Phi^{(m)}(g_{1T}) = \left.\frac{\partial^m \Phi(x)}{\partial x^m}\right|_{x = g_{1T}}, \tag{4}$$

$$K_{j,m} = \left\{(k_1, \cdots, k_{j-m+1}); k_n \geq 0, \sum_{n=1}^{j-m+1} k_n = m, \sum_{n=1}^{j-m+1} n k_n = j\right\}, \tag{5}$$

$$X^{j,m,k} = \prod_{n=1}^{j-m+1} g_{(n+1)T}^{k_n}, \tag{6}$$

$$C^{j,m,k} = \prod_{n=1}^{j-m+1} \frac{m!}{k_1! \cdots k_{j-m+1}!}. \tag{7}$$

3 General Computational Scheme of Asymptotic Expansion

This section explains a general computational scheme of the asymptotic expansion developed by [36]. See Section 4 of [36] for the details. First, to compute conditional expectations $\mathbf{E}\left[X^{j,m,k} \middle| g_{1T} = x\right]$ in the right hand side of (3) we introduce the following lemma which can be derived from a property of Hermite polynomials and leads us to compute the unconditional expectations instead of the conditional ones.

Lemma 1 *Let (Ω, F, P) be a probability space. Suppose that $X \in L^2(\Omega, P)$ and Z is a random variable with Gaussian distribution with mean 0 and variance Σ. Then, the conditional expectation $E[X|Z = x]$ has the following expansion in $L^2(\mathbf{R}, \mu)$ where μ is the Gaussian measure on $\mathbf{R}$ with mean 0 and variance Σ:*

$$E[X|Z = x] = \sum_{n=0}^{\infty} a_n H_n(x; \Sigma) \tag{8}$$

where $H_n(x; \Sigma)$ is the Hermite polynomial of degree n which is defined as

$$H_n(x; \Sigma) = (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma}$$

and coefficients a_n are given by

$$a_n = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\frac{\partial^n}{\partial\xi^n}\bigg|_{\xi=0}\left\{e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}[e^{i\xi Z}X]\right\}. \tag{9}$$

(proof) See Lemma 4 of [36].

Here we define $\hat{g}_{1T}$ as

$$\hat{g}_{1T} = (\partial g(X_T^{(0)}))' \int_0^T [Y_T Y_t^{-1} V(X_t^{(0)})]dW_t = g_{1T} - C,$$

and define

$$Z_T^{\langle\xi\rangle} = \exp\{i\xi\hat{g}_{1T} + \frac{\xi^2}{2}\Sigma_T\}.$$

Then, from Lemma 1 and (3), we have the following expression of $\mathbf{E}[\Phi(G^{(\epsilon)})]$:

$$\mathbf{E}[\Phi(G^{(\epsilon)})] = \sum_{j=0}^{N}\epsilon^j \sum_{m=0}^{j}\frac{1}{m!}\int_{\mathbf{R}}\Phi(x)\sum_{k\in K_{j,m}} C^{j,m,k}(-1)^m\frac{\partial^m}{\partial x^m}\{\sum_{l=0}^{j+m}a_l^{j,m,k}H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}dx + o(\epsilon^N)$$

where

$$a_l^{j,m,k} = \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\frac{\partial^l}{\partial\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[X^{j,m,k}Z_T^{\langle\xi\rangle}]\right\}.$$

In particular, let Φ be the delta function at $x \in \mathbf{R}$, δ_x, we obtain the asymptotic expansion of density of $G^{(\epsilon)}$:

$$\begin{aligned} f_{G^{(\epsilon)}}(x) &= \mathbf{E}[\delta_x(G^{(\epsilon)})] \\ &= \sum_{j=0}^{N}\epsilon^j\sum_{m=0}^{j}\sum_{k\in K_{j,m}}\sum_{l=0}^{j+m}\frac{a_l^{j,m,k}C^{j,m,k}}{m!}(-1)^m\frac{\partial^m}{\partial x^m}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\} + o(\epsilon^N). \end{aligned} \tag{10}$$

3.1 Asymptotic Expansion of Density Function

This subsection summarizes a general computational method for the asymptotic expansion of the density function (10) developed by [36]. In particular, we show that coefficients in the expansion is obtained through a system of ordinary differential equations that is solved easily, and derive a concrete expression of the expansion up to ϵ^2-order. Due to limitation of space, some equations necessary for the concrete expression of the expansion are omitted. See Section 4.1 of [36] for the full expressions.

First, the equation (10) is wrote down more explicitly up to ϵ^2-order:

$$\begin{aligned} f_{G^{(\epsilon)}}(x) &= a_0^{0,0,(0)}H_0(x-C;\Sigma_T)f_{g_{1T}}(x) \\ &\quad +\epsilon\left\{\sum_{l=0}^{2}a_l^{1,1,(1)}(-1)\frac{\partial}{\partial x}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}\right\} \\ &\quad +\epsilon^2\left\{\sum_{l=0}^{3}a_l^{2,1,(0,1)}(-1)\frac{\partial}{\partial x}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}\right. \\ &\quad \left.+\frac{1}{2}\sum_{l=0}^{4}a_l^{2,2,(2,0)}\frac{\partial^2}{\partial x^2}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}\right\} + o(\epsilon^2), \end{aligned}$$

where coefficients $a_l^{j,m,k}$ are given by

$$a_l^{0,0,(0)} = \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\frac{\partial^l}{\partial\xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[Z_T^{\langle\xi\rangle}]\right\}$$

$$
\begin{aligned}
a_l^{1,1,(1)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\left.\frac{\partial^l}{\partial\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[g_{2T}Z_T^{\langle\xi\rangle}]\right\}\\
a_l^{2,1,(0,1)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\left.\frac{\partial^l}{\partial\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[g_{3T}Z_T^{\langle\xi\rangle}]\right\}\\
a_l^{2,2,(2,0)} &= \frac{1}{l!}\frac{1}{(i\Sigma_T)^l}\left.\frac{\partial^l}{\partial\xi^l}\right|_{\xi=0}\left\{\mathbf{E}[g_{2T}^2Z_T^{\langle\xi\rangle}]\right\}
\end{aligned}
\tag{11}
$$

To compute the unconditional expectations in (11), they derive ordinary differential equations (ODEs) of the components of the expectations.

We summarize the result of [36] as the following theorem (in the followings, for simplicity, it is assumed that V_0 doesn't depend on ϵ, and write $V_0(x,\epsilon)$ as $V_0(x)$) : See Section 4.1 of [36] for the details of derivation and the full expressions.

Theorem 1 *The asymptotic expansion of the density of $G^{(\epsilon)}$ up to ϵ^2-order is given by*

$$
f_{G^{(\epsilon)}}(x) = f_{g_{1T}}(x) + \epsilon\left\{\sum_{l=1}^{3} C_{1l}H_l(x;\Sigma_T)\right\}f_{g_{1T}}(x) + \epsilon^2\left\{\sum_{l=1}^{6} C_{2l}H_l(x;\Sigma_T)\right\}f_{g_{1T}}(x) + o(\epsilon^2).
$$

where

$$
\begin{aligned}
&C_{1l} = \Sigma_T a_{l-1}^{1,1,(1)},\\
&C_{21} = \Sigma_T a_0^{2,1,(0,1)}, \quad C_{2l} = \Sigma_T a_{l-1}^{2,1,(0,1)} + \frac{1}{2}\Sigma_T^2 a_{l-2}^{2,2,(2,0)} (l \geq 2).
\end{aligned}
$$

$a_l^{j,m,k}$ are given by (11), and expectations in (11) are obtained as

$$
\begin{aligned}
\mathbf{E}[g_{2T}Z_T^{\langle\xi\rangle}] &= \frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j g(X_T^{(0)})\eta_{2,2}^{i,j}(t;\xi) + \frac{1}{2}\sum_{i=1}^{d}\partial_i g(X_T^{(0)})\eta_{2,1}^{i}(t;\xi)\\
\mathbf{E}[g_{3T}Z_T^{\langle\xi\rangle}] &= \frac{1}{6}\sum_{i,j,k=1}^{d}\partial_i\partial_j\partial_k g(X_T^{(0)})\eta_{3,3}^{i,j,k}(t;\xi) + \frac{1}{2}\sum_{i,j=1}^{d}\partial_i\partial_j g(X_T^{(0)})\eta_{3,2}^{i,j}(t;\xi)\\
&\quad + \frac{1}{6}\sum_{i=1}^{d}\partial_i g(X_T^{(0)})\mathbf{E}[A_{3T}^i Z_T^{\langle\xi\rangle}],\\
\mathbf{E}[g_{2T}^2 Z_T^{\langle\xi\rangle}] &= \frac{1}{4}\sum_{i,j,k,l=1}^{d}\partial_i\partial_j g(X_T^{(0)})\partial_k\partial_l g(X_T^{(0)})\eta_{4,4}^{i,j,k,l}(t;\xi)\\
&\quad + \frac{1}{2}\sum_{i,j,k=1}^{d}\partial_i\partial_j g(X_T^{(0)})\partial_i g(X_T^{(0)})\eta_{4,3}^{i,j,k}(t;\xi)\\
&\quad + \frac{1}{4}\sum_{i,j=1}^{d}\partial_i g(X_T^{(0)})\partial_j g(X_T^{(0)})\eta_{4,2}^{i,j}(t;\xi)
\end{aligned}
$$

where $\eta_{j,m}$ are obtained as the solutions to the following system of ODEs:

$$
\begin{aligned}
\frac{d}{dt}\eta_{1,1}^{j}(t;\xi) &= (i\xi)\hat{V}(X_t^{(0)},t)V^j(X_t^{(0)})' + \sum_{j'=1}^{d}\eta_{1,1}^{j'}(t;\xi)\partial_{j'}V_0^j(X_t^{(0)})\\
\frac{d}{dt}\eta_{2,1}^{j}(t;\xi) &= 2(i\xi)\sum_{j'=1}^{d}\eta_{1,1}^{j'}(t;\xi)\hat{V}(X_t^{(0)},t)\partial_{j'}V^j(X_t^{(0)})'\\
&\quad + \sum_{j'=1}^{d}\eta_{2,1}^{j'}(t;\xi)\partial_{j'}V_0^j(X_t^{(0)}) + \sum_{j'=1}^{d}\sum_{k'=1}^{d}\eta_{2,2}^{j',k'}(t;\xi)\partial_{j'}\partial_{k'}V_0^j(X_t^{(0)})
\end{aligned}
$$

$$\frac{d}{dt}\eta_{2,2}^{j,k}(t;\xi) = (i\xi)\left\{\eta_{1,1}^{k}(t;\xi)\hat{V}(X_t^{(0)},t)V^j(X_t^{(0)})' + \eta_{1,1}^{j}(t;\xi)\hat{V}(X_t^{(0)},t)V^k(X_t^{(0)})'\right\}$$
$$+V^j(X_t^{(0)})V^k(X_t^{(0)})' + \sum_{j'=1}^{d}\sum_{k'=1}^{d}\eta_{2,2}^{j',k'}(t;\xi)\partial_{j'}V_0^j(X_t^{(0)})\partial_{k'}V_0^k(X_t^{(0)}). \tag{12}$$

Due to limitation of space, the remaining equations are omitted. See Proposition 2 in Section 4.1 of [36] for the full expressions.

Note that each ODE in (12) does not involve any higher order terms, and only lower or the same order terms appear in the right hand side of the ODE. So, one can easily solve (analytically or numerically) the system of ODEs and evaluate expectations. Indeed, for example, consider the following two-dimensional diffusion process with parameter $\epsilon \in (0,1]$ which is known as λ-SABR Model(e.g. Labordere [21]):

$$\begin{aligned} dS_t^{(\epsilon)} &= \mu S_t^{(\epsilon)}dt + \epsilon\sigma_t^{(\epsilon)}(S_t^{(\epsilon)})^{\beta}dW_t^1, \\ d\sigma_t^{(\epsilon)} &= \lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\nu_1\sigma_t^{(\epsilon)}dW_t^1 + \epsilon\nu_2\sigma_t^{(\epsilon)}dW_t^2. \end{aligned} \tag{13}$$

Here, $\beta \in [0,1]$ is a constant, $W = (W^1, W^2)$ is a two dimensional Brownian motion and $\nu_1 = \rho\nu$, $\nu_2 = (\sqrt{1-\rho^2})\nu$ where ν is a positive constant and $\rho \in [-1,1]$.

To compute an option price on S, we need the density function of S whose asymptotic expansion is given by (10) with setting $g(S,\sigma) = S$. Then the corresponding differential equations up to the second order are given by

$$\begin{aligned} \frac{d}{dt}\eta_{1,1}^{S}(t;\xi) &= (i\xi)(S_t^{(0)})^{2\beta}(\sigma_t^{(0)})^2 + \mu\eta_{1,1}^{S}(t;\xi), \\ \frac{d}{dt}\eta_{1,1}^{\sigma}(t;\xi) &= (i\xi)\nu_1(S_t^{(0)})^{\beta}(\sigma_t^{(0)})^2 - \lambda\eta_{1,1}^{\sigma}(t;\xi), \\ \frac{d}{dt}\eta_{2,1}^{S}(t;\xi) &= 2(i\xi)\beta(S_t^{(0)})^{2\beta-1}(\sigma_t^{(0)})^2\eta_{1,1}^{S}(t;\xi) + 2(i\xi)(S_t^{(0)})^{2\beta}\sigma_t^{(0)}\eta_{1,1}^{\sigma}(t;\xi) + \mu\eta_{2,1}^{S}(t;\xi), \end{aligned}$$

where $S_t^{(0)} = S_0e^{\mu t}$ and $\sigma_t^{(0)} = e^{-\lambda t}(\sigma_0 - \theta) + \theta$. Since these equations are linear and have hierarchical structure, one can easily integrate them as

$$\begin{aligned} \eta_{1,1}^{S}(t;\xi) &= (i\xi)\int_0^t e^{\mu(t-t_1)}(S_{t_1}^{(0)})^{2\beta}(\sigma_{t_1}^{(0)})^2dt_1, \\ \eta_{1,1}^{\sigma}(t;\xi) &= (i\xi)\int_0^t e^{-\lambda(t-t_1)}\nu_1(S_{t_1}^{(0)})^{\beta}(\sigma_{t_1}^{(0)})^2dt_1, \\ \eta_{2,1}^{S}(t;\xi) &= 2(i\xi)^2\int_0^t\int_0^{t_1} e^{\mu(t-t_2)}\beta(S_{t_1}^{(0)})^{2\beta-1}(\sigma_{t_1}^{(0)})^2(S_{t_2}^{(0)})^{2\beta}(\sigma_{t_2}^{(0)})^2dt_2dt_1 \\ &\quad +2(i\xi)^2\int_0^t\int_0^{t_1} e^{\mu(t-t_1)-\lambda(t_1-t_2)}(S_{t_1}^{(0)})^{2\beta}\sigma_{t_1}^{(0)}\nu_1(S_{t_2}^{(0)})^{\beta}(\sigma_{t_2}^{(0)})^2dt_2dt_1. \end{aligned}$$

Integrals appeared in the right hand side can be analytically evaluated, but the expressions are lengthy and hence omitted. Other higher order terms can be easily integrated in the similar manner.

Then, the asymptotic expansion of the density function of $G^{(\epsilon)} = \frac{S_T^{(\epsilon)} - S_T^{(0)}}{\epsilon}$ can be expressed as

$$f_{G^{(\epsilon)}}(x) \approx f_{g_{1T}}(x) + \epsilon C_{13}H_3(x;\Sigma_T)f_{g_{1T}}(x) + \cdots \tag{14}$$

where

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}}\exp\left(-\frac{x^2}{2\Sigma_T}\right)$$

with

$$\Sigma_T = \int_0^T e^{2\mu(T-t)}(S_t^{(0)})^{2\beta}(\sigma_t^{(0)})^2dt$$

and

$$
\begin{aligned}
C_{13} &= \frac{1}{\Sigma_T^3}\int_0^T\int_0^{t_1} e^{\mu(T-t_2)}\beta(S_{t_1}^{(0)})^{2\beta-1}(\sigma_{t_1}^{(0)})^2(S_{t_2}^{(0)})^{2\beta}(\sigma_{t_2}^{(0)})^2 dt_2 dt_1 \\
&\quad + \frac{1}{\Sigma_T^3}\int_0^T\int_0^{t_1} e^{\mu(T-t_1)-\lambda(t_1-t_2)}(S_{t_1}^{(0)})^{2\beta}\sigma_{t_1}^{(0)}\nu_1(S_{t_2}^{(0)})^{\beta}(\sigma_{t_2}^{(0)})^2 dt_2 dt_1.
\end{aligned}
$$

Note that, since the first term of the expansion (14) is a normal density function, we can see that the asymptotic expansion method approximates the density function by a normal density function and its correction terms.

Moreover, we can provide some interpretations to the corrected terms of the expansion as follows: The ϵ-order adjusts the approximated distribution partially to the skewness of implied volatilities because in the coefficient of ϵ, the correlation parameter ρ between the underlying asset price and its volatility appears for the first time. This is observed in Equation (13) where C_{13} includes $\nu_1 = \rho\nu$. Also, the ϵ^2-order adjusts the approximated distribution partially to the smile, that is a fat tail of the true distribution of the asset price because full parameters of volatility on volatility (both ν_1 and ν_2) appear in the coefficient of ϵ^2, though the equation is not reported in the paper due to its lengthy expression.

3.2 Asymptotic Expansion of Option Prices

This subsection applies the asymptotic expansion to option pricing. We consider the plain vanilla option on the underlying asset $g(X_T^{(\epsilon)})$ whose dynamics is given by (1).

For example, an asymptotic expansion up to $\epsilon^{(N+1)}$ of a call option price at time 0 with maturity T and strike price K where $K = g(X_T^{(0)}) - \epsilon y$ for arbitrary $y \in \mathbf{R}$ is given by

$$
C(K,T) = \epsilon P(0,T)\int_{-y}^{\infty}(x+y)f_{G^{(\epsilon)},N}(x)dx + o(\epsilon^{(N+1)}).
$$

Here, $P(0,T)$ denotes the price at time 0 of a zero coupon bond with maturity T and $f_{G^{(\epsilon)},N}$ is the normal asymptotic expansion of density of $G^{(\epsilon)}$ up to ϵ^N-th order given by (10):

$$
f_{G^{(\epsilon)},N}(x) = \sum_{j=0}^{N}\epsilon^j\sum_{m=0}^{j}\sum_{k\in K_{j,m}}\sum_{l=0}^{j+m}\frac{a_l^{j,m,k}C^{j,m,k}}{m!}(-1)^m\frac{\partial^m}{\partial x^m}\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\}.
$$

In particular, by Theorem 1, an asymptotic expansion up to ϵ^3 of a call option price at time 0 with maturity T and strike price K where $K = g(X_T^{(0)}) - \epsilon y$ for arbitrary $y \in \mathbf{R}$ is expressed as

$$
\begin{aligned}
C(K,T) &= \epsilon P(0,T)\int_{-y}^{\infty}(x+y)f_{g_{1T}}(x)dx \\
&\quad +\epsilon^2 P(0,T)\int_{-y}^{\infty}(x+y)\left\{\sum_{l=1}^{3}C_{1l}H_l(x;\Sigma_T)\right\}f_{g_{1T}}(x)dx \\
&\quad +\epsilon^3 P(0,T)\int_{-y}^{\infty}(x+y)\left\{\sum_{l=1}^{6}C_{2l}H_l(x;\Sigma_T)\right\}f_{g_{1T}}(x)dx + o(\epsilon^3).
\end{aligned}
\tag{15}
$$

Remark that integrals appeared in the right hand side can be calculated by the following formulas related to the Hermite polynomial, which leads to a closed-form approximation formula for option prices.

$$
\begin{aligned}
\int_{-y}^{\infty}H_k(x;\Sigma)f_{g_{1T}}(x)dx &= \Sigma H_{k-1}(-y;\Sigma)f_{g_{1T}}(y)\ (k\geq 1), \\
\int_{-y}^{\infty}xH_k(x;\Sigma)f_{g_{1T}}(x)dx &= -\Sigma y H_{k-1}(-y;\Sigma)f_{g_{1T}}(y) \\
&\quad +\Sigma^2 H_{k-2}(-y;\Sigma)f_{g_{1T}}(y)\ (k\geq 2)
\end{aligned}
$$

For example, if the underlying asset follows λ-SABR model (13), an approximate price of a call option on $S^{(\epsilon)}$ at time 0 with maturity T and strike $K = S_T^{(0)} - \epsilon y$ up to ϵ^2-order is given by

$$\begin{aligned} C(K,T) &= \epsilon P(0,T)\left(\Sigma_T f_{g_{1T}}(y) + yN\left(\frac{y}{\sqrt{\Sigma_T}}\right)\right) \\ &\quad -\epsilon^2 P(0,T) C_{13}\Sigma_T^2 y f_{g_{1T}}(y) + o(\epsilon^2), \end{aligned} \tag{16}$$

where $f_{g_{1T}}$, Σ_T and C_{13} are given by (14) and $N(x)$ represents a cumulative distribution function of a standard normal distribution. Note that $C_{11} = C_{12} = 0$ in this case.

As a result, we are able to compute option prices very fast due to the closed-form approximation formula. This closed-form formula is very useful not only for pricing options but also for calibrations: Using the formula with some optimization algorithm, one can find the model parameters that fit the market prices in much more efficient way than Monte Carlo simulations. For an example of calibration with asymptotic expansion method, see Section 3.3.3 in p.17 of Takahashi and Takehara [35].

Moreover, Appendix examines how different choices of ϵ and y affect the approximation errors of option prices given their multiplication, ϵy is fixed, that is a strike price $K = S_T^{(0)} - \epsilon y$ is fixed.

4 Pricing Barrier Option

This section applies an asymptotic expansion scheme explained in the previous section with a static hedging method by Fink [9] to approximate the value of barrier options. First, we construct a portfolio of plain-vanilla and digital options that approximates the value of a barrier option. Especially, we show that in addition to plain-vanilla options, digital options are very useful in static hedging for an in-the-money knock-out call option. Then, the approximate values of the portfolio of plain-vanilla and digital options are computed by the asymptotic expansion scheme.

4.1 Static Hedge

First, we briefly describe the static hedging method used in this section.

The payoff of an in-the-money knock-out call with maturity T, strike K and barrier B is expressed as

$$(S_T - K)^+ 1_{\{M_T < B\}},$$

where S_t denotes the underlying asset price at t, $M_t := \max\{S_u; 0 \le u \le t\}$ and B is a constant such that $B > K$(and $B > S_0$). On the other hand, the payoff of an out-of-the-money knock-out call with maturity T, strike K and barrier B is expressed as

$$(S_T - K)^+ 1_{\{Q_T > B\}},$$

where $Q_t := \min\{S_u; 0 \le u \le t\}$ and B is a constant such that $B < K$ (and $B < S_0$).

Hereafter, $C(t,T,K,v)$ denotes the price of a plain-vanilla call option at t with maturity T, strike K and time-t volatility v, and $D(t,T,K,v)$ denotes the price of a digital option at t with maturity T, strike K and time-t volatility v. Note that the payoff of the digital is given by 1 if $S_T \ge K$ and 0 otherwise.

In the following, we describe the procedure of our static hedging method.

- First of all, for replication of the value of the barrier option at maturity when the barrier is not hit until the maturity T, we long one unit of a plain-vanilla call option with maturity T and strike K.

  In addition, for an in-the-money knock-out call we may short $\alpha(B-K)$ (where $\alpha = 1$ or $\alpha = 2$) units of a digital option with maturity T and strike B to replicate the value when the barrier is hit just before the maturity. We describe this point by using the following example.

- (Example)
  In-the-money knock-out call with $K = 90$ and $B = 100$:

A portfolio for static hedging of this option:
(a) long one unit of a plain-vanilla call option with $K = 90$.

(b) short 10 units when $\alpha = 1$ or 20 units when $\alpha = 2$ of a digital option with $K = 100$.

(c) a portfolio of call options with $K \geq 100$ explained below.

Suppose that the barrier is hit just before the maturity.

The values of (a), (b) and (c) are given as follows:

(a) about 10,

(b) about 5 when $\alpha = 1$, or 10 when $\alpha = 2$ (the value of a digital option at ATM just before the maturity is about a half of its payoff.),

(c) about 0.

When $\alpha = 1$, the replication error is reduced to about half of the error for the replication without digital options.

When $\alpha = 2$, the replication error is reduced to about 0. However, remark that the error shows up when the barrier is hit at maturity.

In numerical examples of the next subsection, we will show that using a digital option for static hedging of an in-the-money knock-out call is effective. In particular, comparing result of using digital options with that of using only European options, we will see that the number of time points(t_i, $i = 1, \cdots, N$ below) used for static hedging can be decreased substantially to achieve sufficient accuracy in practice, which leads to speed up the computation dramatically.

- Next, we fix some $t_1 (< T)$, $T_1 (\in (t_1, T])$ and $v_1, v_2, \cdots, v_m$that are volatility levels at t_1 used for static hedging.

  Then, we consider the case when the barrier is hit at t_1.

  We choose plain-vanilla call options with maturity T_1 so that the total value combined with $C(t_1, T, K, v_i) - \alpha(B - K)D(t_1, T, K, v_i)$ is 0 when the volatility at t_1 is $v_j (j = 1, \cdots, m)$. Note that their strikes are chosen above or equal to the barrier B so that they expire out-of-the-money if the barrier is not hit until T_1.

  Thus, at t_1 we choose x_{1j} $(j = 1, \cdots, m)$ units of plain vanilla options with strikes $K = B + \gamma_j$ and maturity T_1 where $\gamma_j \geq 0$ $(j = 1, \cdots, m)$ are given constants that are different each other and α is 0 for out-of-the-money knock-out call options and 0, 1 or 2 for in-the-money knock-out call options.

  In other words, we solve the following system of linear equations with respect to x_{1j} $(j = 1, \cdots, m)$.

$$\begin{cases} C(t_1, T, K, v_1) - \alpha(B - K)D(t_1, T, K, v_1) + \sum_{j=1}^{m} x_{1j} C(t_1, T_1, B + \gamma_j, v_1) = 0 \\ \vdots \\ C(t_1, T, K, v_m) - \alpha(B - K)D(t_1, T, K, v_m) + \sum_{j=1}^{m} x_{1j} C(t_1, T_1, B + \gamma_j, v_m) = 0 \end{cases}$$

- Next, we fix $t_2 (< t_1)$, $T_2 (\in (t_2, t_1])$ and $v_1, v_2, \cdots, v_m$ that are volatility levels at t_2 used for static hedging.

  Then, we consider the case when the barrier is hit at t_2.

  We choose plain-vanilla call options with maturity T_2 so that the total value combined with $C(t_1, T, K, v_i) - \alpha(B - K)D(t_1, T, K, v_i) + \sum_{j=1}^{m} x_{1j} C(t_1, T_1, B + \gamma_j, v_1)$ is 0 when the volatility at t_2 is $v_j (j = 1, \cdots, m)$. Their strikes are chosen above or equal to the barrier B so that they expire out-of-the-money if the barrier is not hit until T_2.

  In the same way as before, at t_2 we choose x_{2j} $(j = 1, \cdots, m)$ units of plain vanilla call options with strikes $K = B + \gamma_j$ and maturity T_2 where $\gamma_j \geq 0$ $(j = 1, \cdots, m)$ are given constants and α is 0 for out-of-the-money knock-out call options and 0, 1 or 2 for in-the-money knock-out call options.

In other words, we solve the following system of linear equations with respect to x_{2j} $(j = 1, \cdots, m)$.

$$\begin{cases} C(t_2, T, K, v_1) - \alpha(B-K)D(t_2, T, K, v_1) \\ \qquad\qquad + \sum_{j=1}^{m} x_{1j} C(t_2, T_1, B+\gamma_j, v_1) + \sum_{j=1}^{m} x_{2j} C(t_2, T_2, B+\gamma_j, v_1) = 0 \\ \qquad\qquad\qquad \vdots \\ C(t_2, T, K, v_m) - \alpha(B-K)D(t_2, T, K, v_m) \\ \qquad\qquad + \sum_{j=1}^{m} x_{1j} C(t_2, T_1, B+\gamma_j, v_m) + \sum_{j=1}^{m} x_{2j} C(t_2, T_2, B+\gamma_j, v_m) = 0 \end{cases}$$

- In the same manner, a portfolio of plain-vanilla call options for static hedging of a barrier option is recursively determined towards time 0 at prespecified time points $T = t_0 > t_1 > t_2 > \cdots > t_N = 0$.

  Hence, an approximate value at $t = 0$ of the barrier option is obtained by the value of the portfolio at $t = 0$.

Note that when the values of plain-vanilla and digital options can not be analytically obtained, our asymptotic expansion scheme introduced in the previous section is very useful in practice for both constructing a portfolio for static hedging and computing the initial values of the portfolio that is an approximate value of a target barrier option because our scheme can provide an accurate and fast-computing approximation. The next subsection demonstrates the effectiveness through numerical examples.

4.2 Numerical Examples

This subsection shows numerical examples that investigate accuracy of our approximation method. In particular, we take λ-SABR Model(e.g. Labordere [21]) for the underlying asset model and test our method for both out-of-the-money and in-the-money knock-out call options.

In the λ-SABR Model, the dynamics of the underlying asset price S is given as follows:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma(t)S(t)^{\beta} dW^1(t), \\ d\sigma(t) &= \lambda(\theta - \sigma(t))dt + \nu_1 \sigma(t) dW^1(t) + \nu_2 \sigma(t) dW^2(t). \end{aligned} \tag{17}$$

Here, μ is a constant, $\beta \in [0, 1]$ is a constant, $W = (W^1, W^2)$ is a two dimensional Brownian motion and $\nu_1 = \rho\nu,\ \nu_2 = (\sqrt{1-\rho^2})\nu$ where ν is a positive constant and $\rho \in [-1, 1]$.

In numerical examples, the following three cases are tested for options in static hedging:

1. Only European Call Options(E)
2. European Call Options - [$(B - K)$ units of a Digital Option] (E-D)
3. European Call Options - [$2(B - K)$ units of a Digital Option] (E-DD)

Next, we explain the setup of this numerical experiment.

- The initial price of the underlying asset: $S(0) = 100$
- The maturity of a barrier option: $T = 0.05$, 1 or 2.
- The drift of the underlying asset price process: $\mu = 0$
- The interval of calendar spreads, that is $\Delta t_i = t_{i-1} - t_i (i = 1, \cdots, N)$ is 0.01 for $T = 0.05$ and $\Delta t_i = T/20$ or $\Delta t_i = T/5$ for $T = 1, 2$.
- The maturities of options in a static hedging portfolio: $T_i = t_{i-1} (i = 1, \cdots, N)$.
- For volatility levels v_i used for static hedging, strike prices of plain-vanilla options in static hedging, strike and barrier prices of barrier options and the other parameters, see Table 1-4.

Benchmark prices of target barrier options and their standard errors are obtained by Monte Carlo simulations with the following setup. (For the detail of the extrapolation method used in the simulations, see Gobet [14] for example.)

Table 1: Volatilities $(\eta(t) = \left(\theta + (\sigma(0) - \theta)\, e^{-\lambda t}\right))$

v_1	v_2	v_3	v_4	v_5
$\frac{\eta(t)}{(1+3\nu\sqrt{T})}$	$\frac{\eta(t)}{(1+\nu\sqrt{T})}$	$\eta(t)$	$(1+\nu\sqrt{T})\eta(t)$	$(1+3\nu\sqrt{T})\eta(t)$

Table 2: Strike Prices of Plain-Vanilla Options in Static Hedging (Strike Price = Barrier Price+γ_i)

	γ_1	γ_2	γ_3	γ_4	γ_5
$T = 2$ $\Delta t_i = 0.1$	0	2.5	5	7.5	10
$T = 2$ $\Delta t_i = 0.4$	0	5	10	15	20
$T = 1$ $\Delta t_i = 0.1$	0	2	4	6	8
$T = 1$ $\Delta t_i = 0.2$	0	3	6	9	12
$T = 0.05$ $\Delta t_i = 0.01$	0	0.5	1	1.5	2

Table 3: Strike and Barrier Prices of Barrier Options (I-IV, IX:out-of-the-money knock-out, V-VIII, X:in-the-money knock-out)

	Strike	Barrier
I	95	85
II	105	85
III	95	90
IV	105	90
V	95	115
VI	105	115
VII	95	110
VIII	105	110
IX	100	98
X	100	102

Table 4: Parameters

	$\sigma(0)$	λ	θ	ν	ρ	β	T
i	1	1.2	1	0.3	-0.5	0.5	1
ii	0.2	1.2	0.1	0.6	-0.5	1	1
iii	1	1.2	1	0.3	-0.5	0.5	2
iv	0.2	1.2	0.1	0.6	-0.5	1	2
v	2	1.2	1	0.6	-0.5	0.5	1
vi	0.1	1.2	0.1	0.3	-0.5	1	1
vii	2	1.2	1	0.6	-0.5	0.5	2
viii	0.1	1.2	0.1	0.3	-0.5	1	2
ix	1	1.2	1	0.3	-0.5	0.5	0.05
x	0.2	1.2	0.1	0.6	-0.5	1	0.05
xi	2	1.2	1	0.6	-0.5	0.5	0.05
xii	0.1	1.2	0.1	0.3	-0.5	1	0.05

- Number of trials: 20,000,000
- Extrapolation method with
  1000 and 2000 time steps for cases i, iii, vi, viii(small volatility cases)

  2000 and 4000 time steps for cases ii, iv, v, vii(large volatility cases)

  100 and 200 time steps for cases ix, xii(small volatility and short maturity cases)

  200 and 400 time steps for cases x, xi(large volatility and short maturity cases)

Finally, the approximate values of portfolios of plain-vanilla and digital options are computed by the fifth-order asymptotic expansion scheme. Note that since the corresponding system of ordinary differential equations is solved analytically and hence a closed-form approximation formula for pricing the options is obtained, these approximate values are computed in a few seconds (about 10^{-1} seconds for $\Delta t_i = T/5$ and about 1.7 seconds for $\Delta t_i = T/20$ for computing the values of the portfolios in Intel Xeon 5160 processor at 3.00GHz.) On the other hand, if Monte Carlo simulations are applied in construction of static hedging portfolios, it is very time-consuming and hence is not useful in practice.

Tables 5, 6, 7 and 8 show the results. Generally, the method provides good approximations of barrier option prices. In particular, even when the number of time-steps for the static hedging is only five, our approximation method still works very well(see Table 7 and 8): Especially, Table 8 shows that use of digital options with the asymptotic expansion improves accuracies of approximations for in-the-money knock-out call option prices. This substantial reduction of time-steps makes computation speed faster, which implies an advantage of our method in practice.

Table 5: Out-of-the-money knock out: $\Delta t_i = T/20$ for $T = 1, 2$, $\Delta t_i = 0.01$ for $T = 0.05$

		MC(s.e.)	E	Diff E	Plain Vanilla			MC(s.e.)	E	Diff E	Plain Vanilla
I	i	6.997 (0.004)	7.000	0.004	7.033	III	i	6.690 (0.004)	6.694	0.004	7.033
	ii	8.728 (0.006)	8.729	0.001	9.247		ii	7.644 (0.005)	7.640	-0.004	9.247
	iii	8.288 (0.005)	8.291	0.004	8.543		iii	7.442 (0.005)	7.442	0.000	8.543
	iv	9.630 (0.007)	9.634	0.004	10.743		iv	8.050 (0.007)	8.061	0.011	10.743
	v	8.745 (0.005)	8.748	0.003	9.354		v	7.636 (0.005)	7.629	-0.007	9.354
	vi	6.954 (0.004)	6.960	0.005	6.984		vi	6.675 (0.004)	6.681	0.006	6.984
	vii	9.630 (0.006)	9.625	-0.005	10.893		vii	8.036 (0.006)	8.036	0.000	10.893
	viii	8.252 (0.005)	8.261	0.008	8.465		viii	7.455 (0.005)	7.448	-0.007	8.465
II	i	1.934 (0.002)	1.938	0.004	1.940	IV	i	1.909 (0.002)	1.908	-0.001	1.940
	ii	4.007 (0.004)	4.008	0.001	4.153		ii	3.660 (0.004)	3.661	0.001	4.153
	iii	3.429 (0.003)	3.435	0.005	3.473		iii	3.239 (0.003)	3.239	0.000	3.473
	iv	5.286 (0.005)	5.285	-0.001	5.705		iv	4.603 (0.005)	4.610	0.007	5.705
	v	3.935 (0.004)	3.937	0.003	4.107		v	3.587 (0.004)	3.584	-0.003	4.107
	vi	1.972 (0.002)	1.977	0.005	1.979		vi	1.950 (0.002)	1.954	0.005	1.979
	vii	5.190 (0.005)	5.181	-0.009	5.660		vii	4.508 (0.005)	4.505	-0.002	5.660
	viii	3.490 (0.003)	3.496	0.007	3.528		viii	3.313 (0.003)	3.310	-0.002	3.528
						IX	ix	0.858 (0.001)	0.855	-0.003	0.892
							x	1.321 (0.001)	1.317	-0.005	1.758
							xi	1.315 (0.001)	1.313	-0.002	1.759
							xii	0.859 (0.001)	0.858	-0.001	0.892

Table 6: In-the-money knock out: $\Delta t_i = T/20$ for $T = 1, 2$, $\Delta t_i = 0.01$ for $T = 0.05$

		MC(s.e.)	E	E-D	E-DD	Diff E	Diff E-D	Diff E-DD	Plain Vanilla
V	i	4.572 (0.003)	4.559	4.562	4.566	-0.013	-0.010	-0.006	7.033
	ii	2.523 (0.002)	2.534	2.525	2.516	0.010	0.001	-0.008	9.247
	iii	2.880 (0.002)	2.885	2.882	2.878	0.005	0.002	-0.001	8.543
	iv	1.659 (0.002)	1.668	1.659	1.650	0.010	0.001	-0.008	10.743
	v	2.721 (0.002)	2.738	2.728	2.718	0.017	0.007	-0.004	9.354
	vi	4.338 (0.003)	4.335	4.337	4.338	-0.003	-0.002	-0.000	6.984
	vii	1.806 (0.002)	1.817	1.807	1.797	0.011	0.001	-0.009	10.893
	viii	2.669 (0.002)	2.681	2.676	2.671	0.011	0.007	0.002	8.465
VI	i	0.689 (0.001)	0.684	0.686	0.687	-0.005	-0.003	-0.002	1.940
	ii	0.383 (0.001)	0.395	0.391	0.386	0.012	0.007	0.003	4.153
	iii	0.434 (0.001)	0.436	0.434	0.432	0.002	0.001	-0.001	3.473
	iv	0.246 (0.001)	0.256	0.251	0.247	0.010	0.005	0.001	5.705
	v	0.433 (0.001)	0.446	0.441	0.436	0.013	0.008	0.003	4.107
	vi	0.629 (0.001)	0.629	0.629	0.630	-0.000	0.000	0.001	1.979
	vii	0.281 (0.001)	0.291	0.286	0.281	0.010	0.005	0.000	5.660
	viii	0.383 (0.001)	0.388	0.386	0.383	0.005	0.003	0.000	3.528
VII	i	2.368 (0.002)	2.370	2.368	2.367	0.002	0.000	-0.001	7.033
	ii	1.018 (0.001)	1.028	1.022	1.016	0.009	0.003	-0.003	9.247
	iii	1.177 (0.001)	1.188	1.185	1.183	0.010	0.008	0.006	8.543
	iv	0.618 (0.001)	0.627	0.622	0.616	0.009	0.004	-0.001	10.743
	v	1.075 (0.001)	1.085	1.078	1.072	0.009	0.003	-0.004	9.354
	vi	2.257 (0.002)	2.249	2.254	2.260	-0.008	-0.002	0.003	6.984
	vii	0.655 (0.001)	0.662	0.657	0.651	0.008	0.002	-0.004	10.893
	viii	1.117 (0.001)	1.125	1.123	1.120	0.009	0.006	0.004	8.465
VIII	i	0.114 (0.000)	0.116	0.115	0.115	0.001	0.001	0.000	1.940
	ii	0.046 (0.000)	0.052	0.050	0.048	0.005	0.003	0.002	4.153
	iii	0.053 (0.000)	0.055	0.054	0.053	0.002	0.002	0.001	3.473
	iv	0.028 (0.000)	0.032	0.030	0.028	0.004	0.002	0.000	5.705
	v	0.052 (0.000)	0.057	0.055	0.053	0.005	0.003	0.001	4.107
	vi	0.104 (0.000)	0.101	0.103	0.105	-0.003	-0.001	0.001	1.979
	vii	0.031 (0.000)	0.035	0.033	0.031	0.004	0.002	0.000	5.660
	viii	0.047 (0.000)	0.050	0.049	0.048	0.002	0.002	0.001	3.528
X	ix	0.121 (0.000)	0.138	0.130	0.122	0.017	0.009	0.001	0.892
	x	0.023 (0.000)	0.030	0.030	0.029	0.007	0.007	0.006	1.758
	xi	0.023 (0.000)	0.025	0.024	0.024	0.002	0.002	0.001	1.759
	xii	0.119 (0.000)	0.133	0.127	0.120	0.014	0.007	0.000	0.892

Table 7: Out-of-the-money knock out: $\Delta t_i = T/5$ for $T = 1, 2$

		MC(s.e.)	E	Diff E	Plain Vanilla			MC(s.e.)	E	Diff E	Plain Vanilla
I	i	6.997 (0.004)	7.000	0.004	7.033	III	i	6.690 (0.004)	6.695	0.005	7.033
	ii	8.728 (0.006)	8.731	0.003	9.247		ii	7.644 (0.005)	7.644	-0.000	9.247
	iii	8.288 (0.005)	8.291	0.004	8.543		iii	7.442 (0.005)	7.443	0.001	8.543
	iv	9.630 (0.007)	9.634	0.004	10.743		iv	8.050 (0.007)	8.061	0.011	10.743
	v	8.745 (0.005)	8.750	0.005	9.354		v	7.636 (0.005)	7.633	-0.004	9.354
	vi	6.954 (0.004)	6.960	0.006	6.984		vi	6.675 (0.004)	6.682	0.006	6.984
	vii	9.630 (0.006)	9.625	-0.005	10.893		vii	8.036 (0.006)	8.036	0.000	10.893
	viii	8.252 (0.005)	8.260	0.008	8.465		viii	7.455 (0.005)	7.449	-0.006	8.465
II	i	1.934 (0.002)	1.938	0.004	1.940	IV	i	1.909 (0.002)	1.912	0.003	1.940
	ii	4.007 (0.004)	4.002	-0.005	4.153		ii	3.660 (0.004)	3.658	-0.002	4.153
	iii	3.429 (0.003)	3.434	0.004	3.473		iii	3.239 (0.003)	3.238	-0.001	3.473
	iv	5.286 (0.005)	5.280	-0.006	5.705		iv	4.603 (0.005)	4.607	0.004	5.705
	v	3.935 (0.004)	3.928	-0.006	4.107		v	3.587 (0.004)	3.580	-0.007	4.107
	vi	1.972 (0.002)	1.977	0.005	1.979		vi	1.950 (0.002)	1.954	0.005	1.979
	vii	5.190 (0.005)	5.175	-0.015	5.660		vii	4.508 (0.005)	4.503	-0.005	5.660
	viii	3.490 (0.003)	3.496	0.007	3.528		viii	3.313 (0.003)	3.310	-0.003	3.528

Table 8: In-the-money knock out: $\Delta t_i = T/5$ for $T = 1, 2$

		MC(s.e.)	E	E-D	E-DD	Diff E	Diff E-D	Diff E-DD	Plain Vanilla
V	i	4.572 (0.003)	4.634	4.602	4.571	0.062	0.030	-0.002	7.033
	ii	2.523 (0.002)	2.576	2.548	2.520	0.053	0.025	-0.003	9.247
	iii	2.880 (0.002)	2.970	2.926	2.883	0.091	0.047	0.003	8.543
	iv	1.659 (0.002)	1.734	1.697	1.660	0.081	0.044	0.007	10.743
	v	2.721 (0.002)	2.794	2.761	2.729	0.072	0.040	0.008	9.354
	vi	4.338 (0.003)	4.401	4.372	4.343	0.063	0.034	0.005	6.984
	vii	1.806 (0.002)	1.888	1.846	1.804	0.082	0.040	-0.002	10.893
	viii	2.669 (0.002)	2.756	2.716	2.675	0.087	0.046	0.006	8.465
VI	i	0.689 (0.001)	0.721	0.705	0.690	0.032	0.016	0.001	1.940
	ii	0.383 (0.001)	0.419	0.405	0.391	0.035	0.021	0.007	4.153
	iii	0.434 (0.001)	0.478	0.456	0.434	0.045	0.023	0.001	3.473
	iv	0.246 (0.001)	0.286	0.267	0.249	0.040	0.021	0.003	5.705
	v	0.433 (0.001)	0.473	0.457	0.441	0.041	0.024	0.008	4.107
	vi	0.629 (0.001)	0.662	0.647	0.632	0.033	0.018	0.003	1.979
	vii	0.281 (0.001)	0.325	0.304	0.283	0.044	0.023	0.002	5.660
	viii	0.383 (0.001)	0.426	0.405	0.385	0.043	0.022	0.002	3.528
VII	i	2.368 (0.002)	2.435	2.405	2.376	0.067	0.038	0.008	7.033
	ii	1.018 (0.001)	1.064	1.045	1.027	0.046	0.027	0.008	9.247
	iii	1.177 (0.001)	1.241	1.214	1.186	0.064	0.036	0.009	8.543
	iv	0.618 (0.001)	0.664	0.642	0.620	0.046	0.024	0.002	10.743
	v	1.075 (0.001)	1.127	1.106	1.085	0.051	0.030	0.009	9.354
	vi	2.257 (0.002)	2.323	2.296	2.268	0.066	0.039	0.011	6.984
	vii	0.655 (0.001)	0.702	0.678	0.654	0.048	0.024	-0.000	10.893
	viii	1.117 (0.001)	1.177	1.150	1.123	0.060	0.033	0.006	8.465
VIII	i	0.114 (0.000)	0.136	0.126	0.116	0.021	0.011	0.002	1.940
	ii	0.046 (0.000)	0.062	0.055	0.049	0.015	0.009	0.003	4.153
	iii	0.053 (0.000)	0.072	0.063	0.054	0.019	0.010	0.001	3.473
	iv	0.028 (0.000)	0.043	0.035	0.028	0.015	0.007	0.000	5.705
	v	0.052 (0.000)	0.068	0.061	0.054	0.016	0.009	0.002	4.107
	vi	0.104 (0.000)	0.124	0.115	0.106	0.020	0.011	0.002	1.979
	vii	0.031 (0.000)	0.047	0.039	0.031	0.016	0.008	0.000	5.660
	viii	0.047 (0.000)	0.066	0.057	0.048	0.019	0.010	0.001	3.528

5 Pricing Average Options

This section applies the high-order expansion scheme described in Section 3 to pricing average options.

In particular, we describe the method using numerical examples under the λ-SABR and SABR models.

5.1 Average Options under λ-SABR and SABR Models

We consider the average European call and put options under the λ-SABR model (17) with interest rate=0% for simplicity. In particular, when $\lambda = 0$ the model becomes the SABR model. Further, we define

$$S_A^{(\epsilon)}(t) = \int_0^t S^{(\epsilon)}(u)du.$$

Then, the average European call option price with strike K and maturity T can be written as

$$C_A^{(\epsilon)}(K,T) = \mathbf{E}\left[\max\left\{\frac{1}{T}S_A^{(\epsilon)}(T) - K, 0\right\}\right].$$

Thus, if we consider the following three-dimensional diffusion process, we can easily see that it is a special case of (1) and the general method explained in Section 3 can be applied:

$$\begin{aligned}
dS_A^{(\epsilon)}(t) &= S^{(\epsilon)}(t)dt, \\
dS^{(\epsilon)}(t) &= \epsilon\sigma^{(\epsilon)}(t)(S^{(\epsilon)}(t))^\beta dW_t^1, \\
d\sigma^{(\epsilon)}(t) &= \lambda(\theta - \sigma^{(\epsilon)}(t))dt + \epsilon\nu_1\sigma^{(\epsilon)}(t)dW_t^1 + \epsilon\nu_2\sigma^{(\epsilon)}(t)dW_t^2
\end{aligned} \tag{18}$$

with $S_A^{(\epsilon)}(0) = 0$, $S^{(\epsilon)}(0) = S_0$ and $\sigma^{(\epsilon)}(0) = \sigma$.

The corresponding differential equations up to the second order are given by

$$\begin{aligned}
\frac{d}{dt}\eta_{1,1}^S(t;\xi) &= (i\xi)(S_t^{(0)})^\beta\sigma_t^{(0)}\hat{V}(t), \\
\frac{d}{dt}\eta_{1,1}^\sigma(t;\xi) &= (i\xi)\nu_1\sigma_t^{(0)}\hat{V}(t) - \lambda\eta_{1,1}^\sigma(t;\xi), \\
\frac{d}{dt}\eta_{2,1}^{S_A}(t;\xi) &= \eta_{2,1}^S(t;\xi), \\
\frac{d}{dt}\eta_{2,1}^S(t;\xi) &= 2(i\xi)\beta(S_t^{(0)})^{\beta-1}\sigma_t^{(0)}\hat{V}(t)\eta_{1,1}^S(t;\xi) + 2(i\xi)(S_t^{(0)})^\beta\hat{V}(t)\eta_{1,1}^\sigma(t;\xi),
\end{aligned}$$

where $S_t^{(0)} = S_0$, $\sigma_t^{(0)} = e^{-\lambda t}(\sigma - \theta) + \theta$ and

$$\hat{V}(t) = (T-t)(S_t^{(0)})^\beta\sigma_t^{(0)}.$$

Then, the asymptotic expansion of the density function of $\tilde{G}^{(\epsilon)} = \frac{S_{AT}{}^{(\epsilon)} - S_{AT}{}^{(0)}}{\epsilon}$ can be obtained as

$$f_{\tilde{G}^{(\epsilon)}}(x) \approx f_{g_{1T}}(x) + \epsilon\tilde{C}_{13}H_3(x;\tilde{\Sigma}_T)f_{g_{1T}}(x) + \cdots \tag{19}$$

where

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\tilde{\Sigma}_T}}\exp\left(-\frac{x^2}{2\tilde{\Sigma}_T}\right)$$

with

$$\tilde{\Sigma}_T = \int_0^T \hat{V}(t)^2 dt$$

and

$$\begin{aligned}
\tilde{C}_{13} &= \frac{1}{\tilde{\Sigma}_T^3}\int_0^T\int_0^{t_1}\int_0^{t_2}\beta(S_{t_2}^{(0)})^{\beta-1}\sigma_{t_2}^{(0)}\hat{V}(t_2)(S_{t_3}^{(0)})^\beta\sigma_{t_3}^{(0)}\hat{V}(t_3)dt_3dt_2dt_1 \\
&\quad + \frac{1}{\tilde{\Sigma}_T^3}\int_0^T\int_0^{t_1}\int_0^{t_2}e^{-\lambda(t_2-t_3)}(S_{t_2}^{(0)})^\beta\hat{V}(t_2)\nu_1\sigma_{t_3}^{(0)}\hat{V}(t_3)dt_3dt_2dt_1.
\end{aligned}$$

As in the plain vanilla case described in Section 3.1, integrals appeared in the coefficients of the expansion can be analytically evaluated, but the expressions are lengthy and hence omitted. Moreover, by a similar calculation to Section 3.2, we have the following closed-form approximation formula for the average European call option up to ϵ^2:

$$\begin{aligned} C_A^{(\epsilon)}(K,T) &= \epsilon P(0,T)\left(\frac{\tilde{\Sigma}_T}{T} f_{g_{1T}}(y) + \frac{y}{T} N\left(\frac{y}{\sqrt{\tilde{\Sigma}_T}}\right)\right) \\ &\quad -\epsilon^2 P(0,T)\frac{\tilde{C}_{13}\tilde{\Sigma}_T^2 y}{T} f_{g_{1T}}(y) + o(\epsilon^2), \end{aligned} \tag{20}$$

where $y = \frac{TS_0 - TK}{\epsilon}$ and $P(0,T)$ denotes the price at time 0 of a zero coupon bond with maturity T .

5.2 Numerical Examples

This subsection provides some numerical examples of our asymptotic expansion method for pricing average options under the λ-SABR and SABR models to see the effectiveness of the higher order asymptotic expansions. Further, as a special case of the SABR model, we apply our method to the constant volatility case (Black-Scholes model) and compare approximation accuracies of our method with those of other approximation methods.

5.2.1 Constant Volatility Case

First, we apply our method to the constant volatility case (the Black-Scholes model) which is obtained by setting $\lambda = \nu_i = 0 (i = 1,2)$ and $\beta = 1$ in (18). Then, the asymptotic expansion of the density function (19) can be simplified as

$$f^{BS}_{\tilde{G}^{(\epsilon)}}(x) \approx f^{BS}_{g_{1T}}(x) + \epsilon \tilde{C}^{BS}_{13} H_3(x; \tilde{\Sigma}^{BS}_T) f^{BS}_{g_{1T}}(x) + \cdots,$$

where

$$f^{BS}_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\tilde{\Sigma}^{BS}_T}} \exp\left(-\frac{x^2}{2\tilde{\Sigma}^{BS}_T}\right)$$

with

$$\begin{aligned} \tilde{\Sigma}^{BS}_T &= \int_0^T (T-t)^2\sigma^2 S_0^2 dt = \frac{1}{3}\sigma^2 S_0^2 T^3, \\ \tilde{C}^{BS}_{13} &= \frac{1}{(\tilde{\Sigma}^{BS}_T)^3}\int_0^T\int_0^{t_1}\int_0^{t_2}(T-t_2)(T-t_3)\sigma^4 S_0^3 dt_3 dt_2 dt_1 = \frac{1}{5}\sigma^2 S_0 T^2. \end{aligned} \tag{21}$$

A closed-form approximation formula to the average European call option under the Black-Scholes model can be obtained by replacing $\tilde{\Sigma}_T$ and $\tilde{C}_{13}$ by $\tilde{\Sigma}^{BS}_T$ and $\tilde{C}^{BS}_{13}$ respectively in (20).

In the Black-Scholes case, unlike the stochastic volatility cases, there are several approximation methods for pricing an average option. Here we compare approximation accuracies of our asymptotic expansion method with those of these existing methods.

We consider the average European call option under the Black-Scholes model. We calculate approximated prices of average options by the asymptotic expansion method up to the fifth order and we also calculate approximated prices by the moment matching method given by Levy[22] and by the lower bound for average options given by Nielsen and Sandmann[27].

In the numerical examples, ϵ is set to be one and other parameters are given in Table 9.

Benchmark values are computed by Monte Carlo simulations. We use the second order scheme given by Ninomiya-Victoir[28] as a discretization scheme with 128 time steps for case i, and with 256 time steps for case ii and iii respectively. We adopt Mersenne-twister as a random number generating engine, and generate 5×10^7 paths with antithetic sampling in each simulation. We calculate the lower bound given by Nielsen and Sandmann with 1024 time steps.

Table 9: Parameters for the Black-Scholes Models

Case	$S(0)$	σ	T
i	100	0.3	1
ii	100	0.3	2
iii	100	0.5	2

Table 10: Approximation errors for average call options under Black-Scholes model.

Case	Strike(C/P)	MC(s.e.)	Levy (Diff.)	N-S (Diff.)	A.E.(Difference) 1st	2nd	3rd	4th	5th
i	70 Call	30.081 (0.002)	0.026	-0.001	0.212	-0.065	-0.007	0.001	0.000
	90 Call	12.667 (0.001)	0.082	0.001	0.363	0.012	-0.002	-0.001	-0.001
	100 Call	6.896 (0.001)	0.031	0.003	0.014	0.014	-0.001	-0.001	-0.001
	110 Call	3.367 (0.001)	-0.031	0.002	-0.336	0.015	0.000	-0.001	-0.001
	130 Call	0.622 (0.000)	-0.054	-0.001	-0.329	-0.051	0.008	0.000	-0.001
ii	70 Call	30.555 (0.002)	0.126	-0.002	0.751	-0.080	-0.026	-0.002	0.001
	90 Call	14.993 (0.002)	0.169	0.003	0.582	0.043	-0.001	0.002	0.002
	100 Call	9.729 (0.002)	0.092	0.005	0.043	0.043	0.001	0.001	0.002
	110 Call	6.067 (0.001)	0.000	0.004	-0.491	0.048	0.004	0.001	0.002
	130 Call	2.168 (0.001)	-0.103	-0.001	-0.862	-0.031	0.022	-0.002	0.001
iii	70 Call	33.179 (0.004)	0.568	-0.016	2.319	0.081	-0.063	-0.006	0.002
	90 Call	20.639 (0.004)	0.536	-0.006	1.134	0.186	-0.014	0.000	0.003
	100 Call	16.095 (0.003)	0.415	-0.003	0.192	0.192	0.000	0.000	0.003
	110 Call	12.509 (0.003)	0.271	-0.003	-0.736	0.212	0.013	-0.001	0.002
	130 Call	7.542 (0.002)	0.008	-0.008	-2.045	0.193	0.050	-0.006	0.002

Benchmark prices by Monte Carlo simulations and their standard errors are given in Table 10. Also, approximation errors of the moment matching method(Levy), the lower bound given by Nielsen and Sandmann(N-S) and of our asymptotic expansions are reported in Table 10.

From the results above, asymptotic expansions almost always improve the accuracy of the approximation as the order of expansion increases and the forth or fifth order asymptotic expansion have smaller or equal approximation errors to those of other methods. Further, as seen in the next subsection, our method can be extended in the same framework to the stochastic volatility case where these other methods cannot be applied.

5.2.2 Stochastic Volatility Case

Next, we consider the stochastic volatility case such as λ-SABR/SABR model described in (18).

In the following numerical example, approximated prices by the asymptotic expansion method are calculated up to the fourth order for the λ-SABR model and up to the fifth order for the SABR model respectively. Note that all the solutions to differential equations are obtained analytically. Benchmark values are computed by Monte Carlo simulations. ϵ is set to be one and other parameters used in the test are given in Table 11 for the λ-SABR case (i, ii and iii) and the SABR case (iv, v and vi).

In Monte Carlo simulations for benchmark values, we use Euler-Maruyama scheme as a discretization scheme with extrapolation method with 256 and 512 time steps for case i, ii, iv, v and with 512 and 1024 time steps for case iii and vi respectively. In each simulation, we generate 5×10^7 paths with antithetic sampling.

Results are in Table 12 for the λ-SABR case and in Table 13 for the SABR case respectively. Since the solution to the system of ordinary differential equations is solved analytically, computing time for the asymptotic expansions is less than 10^{-3} seconds which is much shorter than that for the Monte Carlo

Table 11: Parameters for the λ-SABR models

Case	$S(0)$	β	$\sigma(0)$	λ	θ	ν	ρ	T
i	100	1.0	0.3	1.0	0.3	0.3	-0.5	1
ii	100	1.0	0.3	1.0	0.3	0.6	-0.5	1
iii	100	1.0	0.3	1.0	0.3	0.3	-0.5	2
iv	100	1.0	0.5	0	-	0.5	-0.5	1
v	100	0.5	3.0	0	-	0.3	-0.5	1
vi	100	1.0	0.5	0	-	0.5	-0.5	2

simulations.

Table 12: Asymptotic expansions for average options under the λ-SABR model up to the fourth order

			A.E.(Difference)			
Case	Strike(C/P)	MC	1st	2nd	3rd	4th
i	50 Put	0.000 (0.000)	0.009	-0.010	0.003	-0.001
	80 Put	0.804 (0.000)	0.261	0.011	0.004	0.003
	100 Call	6.873 (0.001)	0.036	0.036	0.005	0.005
	120 Call	1.306 (0.000)	-0.240	0.010	0.004	0.005
	150 Call	0.046 (0.000)	-0.036	-0.017	-0.004	0.000
ii	50 Put	0.005 (0.000)	0.005	-0.001	0.007	-0.001
	80 Put	0.988 (0.000)	0.078	0.002	0.030	0.007
	100 Call	6.886 (0.001)	0.024	0.024	0.007	0.007
	120 Call	1.183 (0.000)	-0.117	-0.042	-0.014	0.009
	150 Call	0.035 (0.000)	-0.025	-0.020	-0.012	-0.004
iii	50 Put	0.024 (0.000)	0.162	-0.076	-0.001	0.001
	80 Put	2.251 (0.001)	0.609	0.060	0.004	0.003
	100 Call	9.685 (0.002)	0.088	0.088	0.001	0.001
	120 Call	3.348 (0.001)	-0.488	0.061	0.005	0.006
	150 Call	0.495 (0.000)	-0.309	-0.071	0.004	0.002

From the results above, in each case the higher order asymptotic expansion almost always improves the accuracy of approximation by the lower expansions. In particular, the higher order asymptotic expansions effectively approximate the prices in long-term cases or high-volatility of volatility (ν) cases in which the lower order asymptotic expansions can not approximate the prices well.

Finally, we remark that in the asymptotic expansion method the approximate density functions are expressed as a product of polynomials and the Gaussian density function: Because these polynomial-based approximation functions have wavy forms, higher order approximation sometimes provides worse approximation to the density at particular values (and to the option prices at particular strikes) than lower ones as seen in Table 12 and 13. However, on average absolute differences decrease as higher order correction terms are included.

Table 13: Asymptotic expansions for average options under the SABR model up to the fifth order

Case	Strike(C/P)	MC(s.e.)	A.E.(Difference)				
			1st	2nd	3rd	4th	5th
iv	50 Put	0.137 (0.000)	0.351	-0.034	0.027	-0.014	-0.012
	80 Put	3.496 (0.001)	0.679	0.136	0.038	0.014	0.002
	100 Call	11.359 (0.002)	0.158	0.158	0.020	0.020	0.007
	120 Call	4.623 (0.001)	-0.448	0.096	-0.001	0.023	0.011
	150 Call	0.964 (0.001)	-0.476	-0.091	-0.029	0.013	0.015
v	50 Put	0.008 (0.000)	0.002	0.002	0.003	0.001	0.000
	80 Put	1.054 (0.000)	0.012	0.012	0.013	0.005	0.004
	100 Call	6.897 (0.001)	0.013	0.013	0.007	0.007	0.006
	120 Call	1.070 (0.000)	-0.004	-0.004	-0.003	0.005	0.003
	150 Call	0.012 (0.000)	-0.002	-0.002	-0.002	0.000	-0.000
vi	50 Put	0.854 (0.000)	1.324	0.170	0.132	-0.020	-0.067
	80 Put	6.883 (0.001)	1.321	0.454	0.120	0.049	-0.020
	100 Call	15.824 (0.003)	0.463	0.463	0.073	0.073	0.002
	120 Call	8.713 (0.002)	-0.509	0.357	0.023	0.093	0.024
	150 Call	3.339 (0.001)	-1.162	-0.008	-0.046	0.106	0.060

6 Conclusions

This paper proposed a new approximation method for pricing barrier and average options under stochastic volatility environment by applying an asymptotic expansion approach which enabled us to calculate high-order expansions easily.

For pricing barrier options, it combined an asymptotic expansion scheme with a static hedging method by Fink [9]. Applying an asymptotic expansion method to approximation of the values for the hedging option portfolio, it obtained a closed-from approximation formula for barrier options which can be easily calculated. Through the numerical examples under the λ-SABR model, it was shown that using the fifth-order asymptotic expansion scheme, our method had sufficient accuracy of the approximation. Also, numerical examples showed that using digital options had advantages in both approximation accuracy and computational speed for the valuation of in-the-money knock out options.

For average options, to our knowledge, this paper is the first one that implements the fourth and fifth order asymptotic expansion under the λ-SABR and SABR models and examines its numerical accuracy. Numerical experiments showed that the higher order expansions generally approximated the prices of the average options in longer-term or/and higher-volatility of volatility cases better than the lower order expansions.

Finally, the proposed method is general enough to be applied to other multi-dimensional diffusion models such as local-stochastic-volatility models [29] including the quadratic Heston model [1]. Thus, comparison of various models for pricing barrier and average options can be implemented based on the calibration to liquid plain-vanilla options in actual markets, which is one of our main research topics in the next step.

Appendix A Effect of Different Choices of ϵ and y on the Approximation Errors

This appendix examines the effect of different choices of ϵ and y on the approximation errors of the option pricing formula (15), given their multiplication ϵy is fixed; that is, a strike price $K = S_T^{(0)} - \epsilon y$ is fixed.

In particular, we compare the approximation errors of European call option prices for different values of ϵ under the following λ-SABR model:

$$\begin{aligned} dS_t^{(\epsilon)} &= \epsilon \sigma_t^{(\epsilon)} (S_t^{(\epsilon)})^\beta dW_t^1, \\ d\sigma_t^{(\epsilon)} &= \lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon \nu_1 \sigma_t^{(\epsilon)} dW_t^1 + \epsilon \nu_2 \sigma_t^{(\epsilon)} dW_t^2. \end{aligned}$$

In the numerical example, we compute approximate prices and errors by the asymptotic expansion method with $\epsilon = 1, 0.5, 0.25$. For each level of ϵ, y is chosen so that a strike price K is fixed: namely, $y = (S_T^{(0)} - K)/\epsilon$ where $S_T^{(0)} = S_0$ in this example. Other parameters in the λ-SABR model are given in Table 14.

Table 14: Parameters for the λ-SABR models

Parameter	S_0	β	$\sigma(0)$	λ	θ	ν	ρ	T
	100	1.0	0.3	1.0	0.3	0.3	-0.5	10

Table 15 shows benchmark prices(MC) obtained by a Monte Carlo simulation with 5×10^7 paths and approximation errors of the asymptotic expansion method(A.E.) up to ϵ^3-order against the benchmark prices. Also, Figure 1 shows the errors of the ϵ^3-order approximation.

It is observed that for each strike price the approximation error decreases as ϵ becomes smaller.

Table 15: Approximation Errors

Strike(K)		50	70	80	90	100	110	120	130	150
Call/Put		Put	Put	Put	Put	Call	Call	Call	Call	Call
$\epsilon = 1$										
$y = (S_T^{(0)} - K)/\epsilon$		50	30	20	10	0	-10	-20	-30	-50
MC		9.014	18.255	23.781	29.810	36.275	33.122	30.302	27.774	23.455
A.E.	1st	8.973	6.470	4.904	3.248	1.572	-0.065	-1.616	-3.049	-5.468
(Difference)	2nd	1.479	1.556	1.536	1.535	1.572	1.648	1.752	1.865	2.025
	3rd	-0.328	-0.183	-0.131	-0.076	-0.019	0.037	0.085	0.127	0.218
$\epsilon = 0.5$										
$y = (S_T^{(0)} - K)/\epsilon$		100	60	40	20	0	-20	-40	-60	-100
MC		1.196	5.279	8.792	13.297	18.723	14.968	11.919	9.464	5.935
A.E.	1st	2.367	2.308	1.790	1.046	0.201	-0.625	-1.337	-1.876	-2.373
(Difference)	2nd	-0.103	0.194	0.214	0.204	0.201	0.217	0.238	0.238	0.097
	3rd	-0.123	-0.023	-0.011	-0.004	0.002	0.008	0.013	0.022	0.078
$\epsilon = 0.25$										
$y = (S_T^{(0)} - K)/\epsilon$		200	120	80	40	0	-40	-80	-120	-200
MC		0.012	0.615	2.019	4.868	9.435	5.654	3.221	1.756	0.470
A.E.	1st	0.138	0.549	0.621	0.423	0.027	-0.363	-0.580	-0.593	-0.320
(Difference)	2nd	-0.095	-0.032	0.018	0.029	0.027	0.031	0.023	-0.012	-0.087
	3rd	0.000	-0.010	-0.002	0.001	0.002	0.003	0.004	0.009	0.008

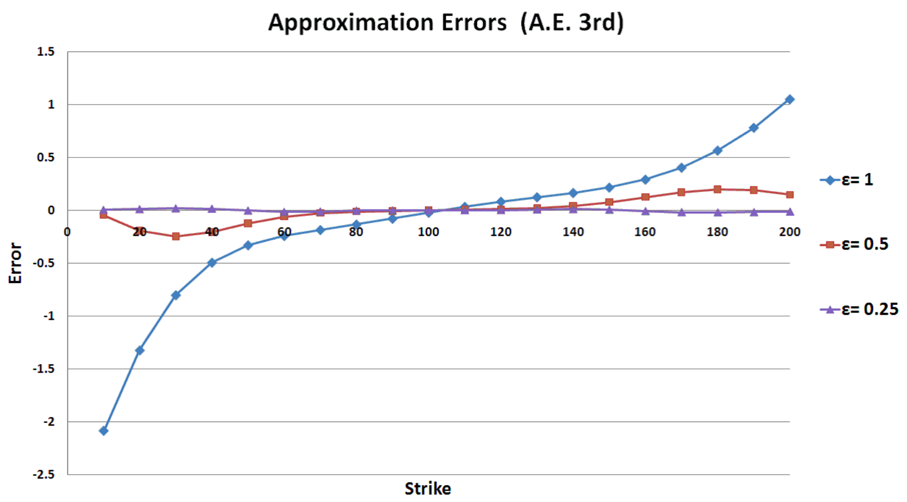

Figure 1: Approximation Errors of 3rd-order asymptotic expansion with $\epsilon = 1, 0.5, 0.25$

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