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Generating a Target Payoff Distribution with the Cheapest Dynamic Portfolio: an Application to Hedge Fund Replication

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Abstract

This paper provides a new hedge fund replication method with the dynamic optimal portfolio. This is an extension of the methodology developed by Kat and Palaro (2005) and Papageorgiou, Remillard and Hocquard (2008) to multiple trading assets with both long and short positions. It is applied to the replication of CS/Tremont managed futures index, which performed very well even under the subprime and Lehman shocks. Empirical analyses show that the extension dramatically improves the replication performance in practice. Especially, the clone by multiple replicating tools enjoyed high returns under the credit crisis by controlling exposures efficiently, while the replica by the existing method incurred drawdown during the period. Moreover, the replication with an estimation method that reflects trend following strategy as the target brings correlated returns to the target.

*The internet appendix is available at http://ssrn.com/abstract=1555442

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1 Introduction

Recently, hedge fund replication products appeared in financial markets. Investment banks and asset management companies have launched such products one after another. Some of these institutions developed replication techniques collaborating with the pioneers in hedge fund research. (See, for example, Géhin (2007). The emergence and methods of hedge fund clones are described more in detail in Takahashi and Yamamoto (2010).) In addition, replication products overcome some demerits of hedge fund investing: high cost, low transparency, and low liquidity. The importance of transparency and liquidity has been recognized after subprime and Lehman shocks. Therefore, these products have gained increased attention. Wallerstein *et al.* (2010) examined the performances of existing hedge fund clone products.

As shown in Géhin (2007) and Wallerstein *et al.* (2010), many financial firms offer the clones of various investment strategies. Most of replication products are trying to replicate the performances of hedge funds in aggregate basis. In other words, their replication targets are aggregate hedge fund indices provided by HFR, CS/Tremont and so on. Others are trying to replicate the performances of some specific alternative investment strategies such as long/short equities, market-neutral, and so on. Replicators have been developing their own original methods by employing highly sophisticated models and statistical methods.

The methodologies for hedge fund replication can be categorized in the following three approaches: rule-based, factor-based, and distribution replicating approach. Rule-based approach mimics typical trading strategies that are employed by hedge funds. The method of Duarte *et al.* (2007) can also be regarded as rule-based fixed-income hedge fund clone techniques.

Factor-based clone tries to replicate risk exposures of the target fund. If this method succeeded, then the return of the clone tracked that of the target fund on month-to-month basis. Lo and Hasanhodzic (2007) and Fung and Hsieh (2007a, 2007b) studied hedge fund replication by this approach. The techniques of factor analysis for hedge funds that have been developed from the late 1990s such as that of Fung and Hsieh (1997, 2000, 2001) and Agarwal and Naik (2004) are directly applied to the replication.

Distribution replicating approach aims to replicate the distribution of hedge fund returns. This approach allows tracking error on month-to-month basis as long as a replicator delivers a similar return distribution to the replication target. This is a powerful approach when month-to-month replication with enough accuracy is difficult. Amin and Kat (2003) first tried the replication. However, an attractive character of the hedge fund returns is the low dependency on returns of traditional asset classes. Further, Kat and Palaro (2005) presented a modified method to replicate the dependence structure on the investor's existing portfolio, too. Kat and Palaro (2005) tried to replicate the return distribution of the target hedge fund and its dependence structure on an investor's existing portfolio through the dynamic trading of the investor's existing portfolio (proxied by a portfolio of stock index and bond futures) and another asset (replicating tool). Papageorgiou *et al.* (2008) proposed an alternative way to perform Kat-Palaro's replication methodology by utilizing a hedging scheme of options in an incomplete market.

Ideally, a clone is desired to track the return of the target almost completely like passive index funds, but it is of great difficult. We believe that replicators should choose an approach suited to their replication targets and satisfying investors' needs. If investors want to access to hedge funds' exposures on markets, the replicator should adopt factor-based approach. When investors want some return characteristics of an arbitrage strategy, the replicator should simply implement such a investment strategy based on some trading rules. If investors are attracted to hedge funds because of their distributional characteristics and low dependency on their own existing portfolio, distribution replication satisfies the needs.

Figure 1 shows the performances of CS/Tremont hedge fund indices by investment strategy from January 2007 to December 2009. Under the subprime and Lehman shock, most strategies incurred large drawdown, but managed futures funds enjoyed high returns. Therefore, this paper considers the replication of the managed futures index. Before that, what is managed futures?

Managed futures funds seek for attractive investment opportunities in futures markets all over the world. They are employing dynamic trading strategies including leverage and short sales to exploit them. Many of them are trend follower utilizing technical analysis and system trading. In the past financial crises, most managed futures funds earned profits by capturing turning points of trends quickly and taking short positions. Then, what is the suited approach to the replication of such a strategy?

First, suppose rule-based approach is adopted. Then, a replicator mimics typical trading rules employed by managed futures funds on hearing basis. As just mentioned, many managed futures funds adopt system trading based on their own trading rules. Therefore, a rule-based managed futures replication product end up with a fund that has no difference from an original managed

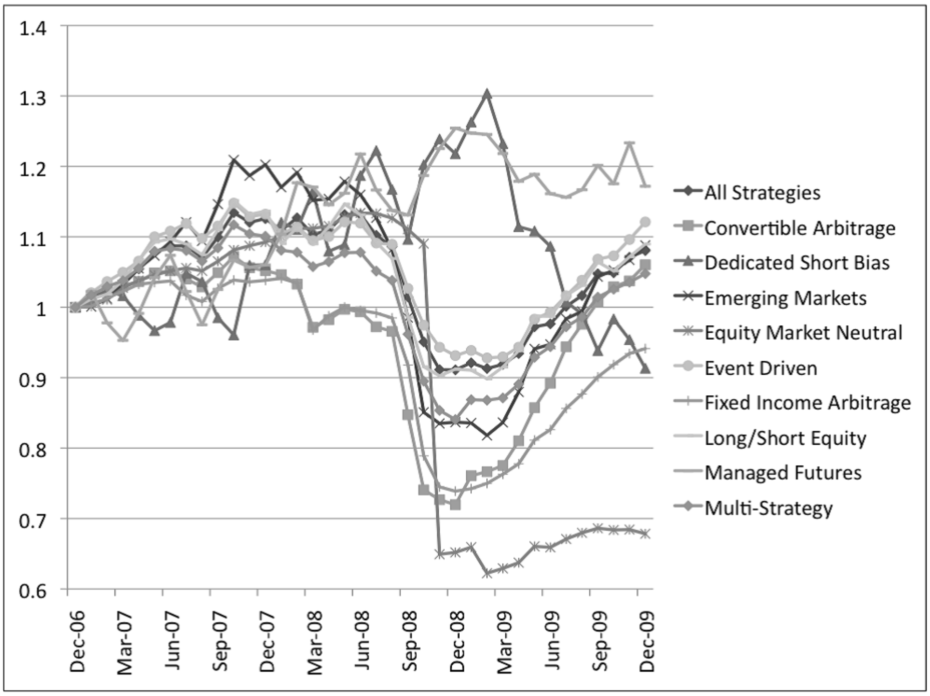

Figure 1: **Performances of CS/Tremont Hedge Fund Indices by Investment Strategy (Jan 2007-Dec 2009).** This figure shows the growth of net asset values of CS/Tremont hedge fund indices by investment strategy in US dollar basis from January 2007 to December 2009. The initial net asset values for all the strategies are normalized to one at the end of December 2006. The hedge fund indices are downloadable from the homepage of CS/Tremont or Bloomberg. This paper uses the data downloaded from the homepage for the analysis.

futures fund.

Next, how about factor-based approach? If we choose this approach, our first task is finding factors that drive the return of the managed futures index. However, it is difficult to find factors that explain the returns of managed futures funds. (See, for example, Hakamada *et al.* (2007).) Therefore, its factor-based replication is very tough or impossible.

Therefore, distribution replication becomes a powerful approach. As shown in Figure 1, managed futures performed very well under the credit crisis. Thus, their dependence structures on investors' stock and bond portfolios would also attract investors. The methodology proposed by Kat and Palaro (2005) can replicate this character in theory. Figure 2 shows the performances of CS/Tremont managed futures index and its clone based on their methodology. Under Lehman shock, the replica incurred drawdown, while the replication target earned profits. The reason of this failure is that the method can trade only one replicating tool with only long position. Mainly, the no-short sales constraint induced the drawdown. It is considered that many managed futures funds captured the change of market trend and took short positions. In addition, they invest in global futures markets, while the clone can trade only one replicating tool. This restriction is also too restrictive.

Amenc *et al.* (2008) pointed out the following two shortcomings of the existing distribution replication methodology. The first problem is that "the success of the method is merely related to a possible replication of long-horizon returns, with no success in replicating their time-series properties." Secondly, they said "A serious concern also remains over the robustness of the results, in particular those related to the difference in average returns with respect to the choice of the reserve asset (that is referred to as replicating tool in this paper) and sample period. As a result, the the investor is finally left with the question of selecting/designing a well-diversified strategic and/or tactical benchmark that could be used as a reserve asset so as to generate the highest risk-adjusted return." How to overcome these shortcomings?

This article extends the distribution replication methodology developed by Kat and Palaro (2005) and Papageorgiou *et al.* (2008). The new method can trade multiple assets, and take both of long and short positions. Therefore, it reflects mentioned managed futures funds' investment behavior. When multiple replicating tools are tradable, there are infinitely many payoffs that have the target distribution. This paper proposes to choose the cheapest one among them. The cheapest payoff is replicated by the dy-

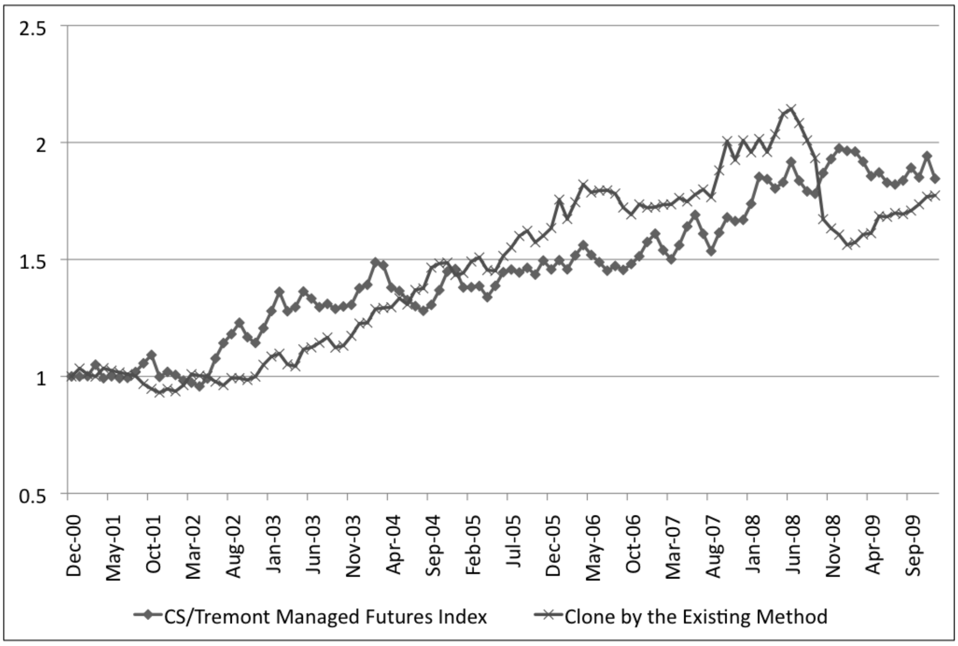

Figure 2: **Replication Performance of the Existing Distribution Replication Method for CS/Tremont Managed Futures Index.** This figure shows the growth of the net asset values ($) of CS/Tremont managed futures index and its clone by the existing method from January 2001 to December 2009. The replication is based on Kat and Palaro (2005) and Papagerogiou *et al.* (2008). The initial net asset values are normalized to one at the end of December 2000.

namic trading strategy of the investor's portfolio and replicating tools. The cheapest payoff is supported by utility maximization theory. It can be shown that the dynamic trading strategy generating the cheapest payoff for some distribution maximizes a von Neumann-Morgenstern utility. Dybvig (1988) derived this result in equally probable finite space setting. See the internet appendix for the proof in the continuous-time framework.

As will be shown, by the extension, the replication performance is dramatically improved in practice. Especially, the clone by the new method performed very well even under the subprime and Lehman shocks. In other words, the extension resolves the second concern that was pointed out by Amenc *et al.* (2008). Moreover, the replication with an estimation method that reflects trend following strategy as the target brings correlated returns to the target to some extent. By reflecting the investment strategy of the target in estimation of the trading assets' price processes, the first problem that was pointed out by Amenc *et al.* (2008) is relieved. In sum, the extension to multiple assets with long and short positions and the estimation procedure with reflecting the investment strategy of the target overcome the two shortcomings of the existing distribution replication method.

This paper is outlined as follows. The following section briefly reviews the existing distribution replication methodology. Section 3 extends the method to multiple replicating tools with long and short positions. Then, section 4 applies the new method to the replication of CS/Tremont managed futures index. Finally, section 5 concludes this paper.

2 Review of existing distribution replication methodology

Let us review the distribution replication methodology developed by Kat and Palaro (2005) and Papageorgiou *et al.* (2008). Consider an investor who has been investing in traditional assets such as stocks and bonds, and plans to invest in a hedge fund. Assume that he is attracted to the hedge fund because of its return distribution and the dependence structure on his portfolio. Kat and Palaro (2005) proposed to replicate the return distribution of the hedge fund and its dependence structure on the investor's existing portfolio by the dynamic trading strategy of the investor's portfolio (proxied by a portfolio of stock index and bond futures) and another asset (replicating tool).

Let S^0 be a risk-free asset, S^1 be the investor's existing portfolio, and S^2 be a replicating tool. Assume that S^1 is also tradable. Let 0 and T be the start and terminal dates of the investment, respectively. (For example, they are assumed to be the start and end of the month.) Although Papageorgiou *et al.* (2008) assumed an incomplete market, this paper assume a complete market. Suppose the financial assets follow stochastic differential equations (SDEs)

$$\begin{aligned} dS_t^0 &= r_t S_t^0 dt, \\ dS_t^1 &= \mu_t^1 S_t^1 dt + \sigma_t^{11} S_t^1 dW_t^1, \\ dS_t^2 &= \mu_t^2 S_t^2 dt + \sigma_t^{21} S_t^2 dW_t^1 + \sigma_t^{22} S_t^2 dW_t^2, \end{aligned}$$

where W_t^1 and W_t^2 are independent Wiener processes, and r_t, μ_t^i, and σ_t^{ij} $(i, j = 1, 2)$ satisfy some measurability and integrability conditions. Normalize initial asset values so that $S_0^0 = S_0^1 = S_0^2 = 1$.

To replicate the joint return distribution of the target hedge fund and the investor's existing portfolio, we need to perform the following steps. First, stochastic processes $\{S_t^1\}_{t=0}^T$ and $\{S_t^2\}_{t=0}^T$ are inferred. Following that, the joint distribution of $R_T^1 = \log S_T^1$ and $R_T^2 = \log S_T^2$ is also obtained. Let R_T be the random variable that represents the log return of the target hedge fund. Second, the joint distribution of the investor's portfolio and hedge fund returns $\left(R_T^1,\, R_T\right)$ is estimated. This is the target joint distribution to replicate. Third, the payoff function, which transforms the joint distribution of $\left(R_T^1, R_T^2\right)$ to that of $\left(R_T^1, R_T\right)$, is determined. Finally, the payoff is priced and replicated through the dynamic trading of S^1 and S^2.

In the first step (inference of $\{S_t^1\}_{t=0}^T$ and $\{S_t^2\}_{t=0}^T$), Kat and Palaro (2005) and Papageorgiou *et al.* (2008) modeled them by using Gaussian and non-Gaussian distributions such as Gaussian mixtures, and selected the best fitted one.

As hedge funds exhibit skewness and fat-tails, and are non-linearly related to traditional asset classes, in the second step (estimation of $(R_T^1,\, R_T)$), Kat and Palaro (2005) proposed to model R_T^1 and R_T separately, and then connect them by a copula. For hedge fund returns, it is desirable to use the distribution class that can capture its skewness and fat-tails. For example, Kat and Palaro (2005) and Papageorgiou *et al.* (2008) used Gaussian, Student-t, Gaussian mixture and Johnson distributions. The Johnson distribution class is often used for the analysis of non-normal behavior of hedge fund returns as seen in Kaplan and Knowles (2004), Pérez (2004), and Passow (2005). Some copulas can capture the asymmetric dependence structure flexibly. For example, the Clayton copula has more dependence in the lower tail than in the upper tail. This allows for the copula to capture the strong dependence in bear markets and weak dependence in bull markets. For example, Mitchell and Pulvino (2005) showed that risk arbitrage funds tend to exhibit a similar dependence structure. See, for example, Joe (1997) or Nelsen (1999) for the introduction to copula.

After estimating the parameters and selecting models of price processes of the trading assets and the joint return distribution of the target hedge fund and the investor's existing portfolio, a payoff function should be determined. Since the joint payoff distribution is replicated by the dynamic trading of investor's portfolio S^1 and replicating tool S^2, they created a payoff by a function of S^1_T and S^2_T. They found function $\tilde{g}$ that satisfies the following equation:

$$\mathbb{P}\Big(R^1_T \leq a,\ \tilde{g}\Big(R^1_T, R^2_T\Big) \leq b\Big) = \mathbb{P}(R^1_T \leq a, R_T \leq b) \ \text{ for any } a, b,$$

or equivalently,

$$\mathbb{P}\Big(\tilde{g}\Big(R^1_T, R^2_T\Big) \leq b\big|R^1_T = a\Big) = \mathbb{P}(R_T \leq b|R^1_T = a) \ \text{ for any } a, b.$$

Then, $\tilde{g}(\cdot,\cdot)$ is given by

$$\tilde{g}(a, b) = F^{-1}_{R_T|a}(F_{R^2_T|a}(b)),$$

where $F_{R_T|a}$ and $F_{R^2_T|a}$ are the conditional distribution functions of R_T and R^2_T under $R^1_T = a$. As a function of the asset prices, the payoff function is represented as

$$\hat{g}(S^1_T, S^2_T) = \exp\big\{\tilde{g}\big(\log S^1_T, \log S^2_T\big)\big\}. \tag{1}$$

If one obtained the payoff function, the replicating strategy encounters the same problem with pricing and hedging of derivatives. The dynamic replicating strategy is given by the *delta*-hedging strategy of the payoff $\hat{g}(S^1_T, S^2_T)$. If the initial cost for the trading strategy is less (more) than one, then the target payoff can be realized by a lower (higher) cost. The remaining (shortage of) money is invested (funded) in the risk-free asset. This means that the shape of the probability density function can be replicated, but the mean return is higher (lower) than the target fund by the difference of the initial cost. In this case, the replicating tool does (does not) include greater investment opportunity than the target hedge fund. Note that the payoff function $\hat{g}(\cdot, \cdot)$ is an increasing function with respect to the second argument. Then, the *delta*-hedging strategy never takes a short position for S^2. In pp. 17-18 of Kat and Palaro (2005), the authors claim that users of this method should choose replicating tool S^2 that includes the positive expected return factor uncorrelated to the return of the investor's portfolio. Then, the long position for S^2 is justified. Therefore, the choice of a replicating tool is crucial.

As just described, this methodology can replicate the shape of the probability density function, but cannot fit the mean. If you found a greater investment opportunity than the target fund, the mean return would be superior and vice versa. Therefore, the usage of only one asset is restrictive. Papageorgiou *et al.* (2008) synthesized multiple assets to create one replicating tool by equal-weighing, but there would be inefficiencies in the *ad hoc* fixed weighted portfolio. The extension of the investment universe would bring in higher mean returns.

3 Extension to Multiple Assets with Long and Short Positions

This section extends the methodology described in the previous section to multiple assets and both of long and short positions. Let S^0 be a risk-free asset, S^1 be the investor's existing portfolio, and $S^2, \cdots, S^n$ be risky assets (replicating tools). Suppose that the price processes of the financial risky assets $\{S_t^i\}_{t=0}^T$ $(i = 1, \cdots, n)$ satisfy SDEs

$$dS_t^i = \mu_t^i S_t^i dt + \sum_{j=1}^{i} \sigma_t^{ij} S_t^i dW_t^j \quad (i = 1, \cdots, n), \tag{2}$$

where μ_t^i, and σ_t^{ij} satisfy some measurability and integrability conditions for any $1 \leq j \leq i \leq n$. All of the initial asset values are normalized, so that $S_0^0 = \cdots = S_0^n = 1$. The following notations by n-dimensional vectors and a $n \times n$ matrix are introduced. $S_t = (S_t^1, \cdots, S_t^n)'$, $\mu_t = (\mu_t^1, \cdots, \mu_t^n)'$,

$\vec{1} = (1, \cdots, 1)'$, and

$$\sigma_t = \begin{pmatrix} \sigma_t^{11} & & O \\ \vdots & \ddots & \\ \sigma_t^{n1} & \cdots & \sigma_t^{nn} \end{pmatrix},$$

where $'$ represents transposition so that S_t, μ_t and $\vec{1}$ are column vectors. Suppose that σ_t is invertible almost surely. Then, there exists the unique market price of risk

$$\theta_t = \sigma_t^{-1}(\mu_t - r_t\vec{1}).$$

In other words, the financial market is complete. The financial market is denoted by $\mathcal{M} = (r, \mu, \sigma)$.

In complete market $\mathcal{M}$, the unique state price density process is given by

$$H_t = \exp\left\{ -\int_0^t r_u du - \frac{1}{2}\int_0^t ||\theta_u||^2 du - \int_0^t \theta'_u dW_u \right\}.$$

The no-arbitrage price of any $\mathcal{F}_T$-measurable payoff X is given by $x = E[H_T X]$. X can be replicated by a dynamic trading of the financial assets with initial cost x. (See, for example, Karatzas and Shreve (1998).)

When multiple replicating tools are tradable, there are infinitely many payoffs that have a target distribution. This paper proposes to choose the cheapest one among them. Dybvig (1988) showed that the cheapest way to attain a given payoff is by allocating terminal wealth in the reverse order of the state price density in an equally probable finite state complete market. The internet appendix proves that this claim is also valid in our setting under some conditions. In the following, the cheapest payoffs that attain the same marginal distribution and the same joint distribution with the investor's existing portfolio as the target hedge fund are presented.

First, let us see the cheapest payoff that has the same marginal distribution as the target hedge fund. Let ξ be the payoff of the target hedge fund (i.e. $R_T = \log \xi$). For convenience, the minus logarithm state price density process L_t is introduced:

$$L_t = -\log H_t = \int_0^t r_u du + \frac{1}{2}\int_0^t ||\theta_u||^2 du + \int_0^t \theta'_u dW_u. \tag{3}$$

Since Arrow-Debreu securities are tradable in complete market, this paper propose to create a payoff using H_T (or L_T). Let F_ξ and F_{L_T} denote the distribution functions of ξ and L_T, respectively. Assume that F_ξ is invertible function. If X is defined as follows, X has the same distribution with ξ:

$$X = f(L_T), \tag{4}$$

where

$$f(l) = F_\xi^{-1}\big(F_{L_T}(l)\big).$$

Since F_ξ^{-1} and F_{L_T} are increasing, payoff X increases in L_T, and therefore it is reverse order of state price density H_T. Thus, X is the cheapest payoff among the payoffs that have the same distribution as ξ.

The choice of the cheapest payoff for some marginal distribution can be justified theoretically. It can be shown that the cost minimization for a marginal payoff distribution is equivalent to a von Neumann-Morgenstern utility maximization. Dybvig (1988) proved this claim in the equally probable finite state setting. The internet appendix proves this claim in continuous-time framework. This assertion ensures that our method is applicable to not only hedge fund replication but also dynamic portfolio optimization in investment management.

Next, let us create the cheapest payoff that has the same joint distribution with the investor's portfolio as the target hedge fund. Denote the conditional distribution functions of ξ and L_T under condition $S_T^1 = s$ by $F_{\xi|s}$ and $F_{L_T|s}$ respectively. Assume that $F_{\xi|s}$ is invertible function for any s. If X is defined as follows, (S_T^1, X) has the same joint distribution with (S_T^1, ξ):

$$X = g(S_T^1, L_T), \tag{5}$$

where

$$g(s,l) = F_{\xi|s}^{-1}(F_{L_T|s}(l)).$$

Since $F_{\xi|s}^{-1}$ and $F_{L_T|s}$ are increasing, payoff X increases in L_T, and therefore it is reverse order of state price density H_T. Thus, X is the cheapest payoff among the payoffs that have the same joint distribution with the investor's existing portfolio as ξ. See the internet appendix for the proof.

Let us see the dynamic portfolio that replicates the cheapest payoffs. Let π_t^i $(i = 0, \cdots, n)$ represent the money amount invested in asset i at time t. n-dimensional vector π_t is defined by $\pi_t = (\pi_t^1, \cdots, \pi_t^n)'$, which denotes the portfolio of risky assets. Let x be the initial cost required to realize the cheapest payoff X_T for some payoff distribution. The initial cost x is invested in the financial assets by a dynamic self-financing trading strategy to generate payoff X_T. In other words, the portfolio value at time t, X_t, satisfies

$$X_t = \pi_t^0 + \pi_t'\vec{1},$$

for any t. In a differential form, this is

$$dX_t = r_t X_t dt + \pi_t'(\mu_t - r_t\vec{1})dt + \pi_t'\sigma_t dW_t.$$

The dynamic portfolio can be obtained for the case of Markovian coefficients concretely. (See the internet appendix.) Martingale method with Malliavin calculus easily gives us the dynamic portfolio generating the payoffs. See, for example, Karatzas and Shreve (1998) for the basics of martingale method, and Nualart (2006) for the introduction to Malliavin calculus. As for the application of Malliavin calculus to the dynamic optimal portfolio, see for example, Ocone and Karatzas (1991), Detemple *et al.* (2003, 2008), and Takahashi and Yoshida (2004). This paper assumes that r, μ and σ are deterministic functions of time t. Then, the dynamic replicating portfolios are represented much simpler.

Proposition 1 *Assume that r, μ and σ are deterministic functions of time t. Then, in a complete market $\mathcal{M}$, the dynamic portfolio generating payoff $f(L_T)$ is given by*

$$\pi_t^M = \sigma'(t)^{-1}\phi_t^M, \tag{6}$$

where

$$\phi_t^M = \frac{\theta(t)}{H_t}E_t[H_T f'(L_T)]. \tag{7}$$

The portfolio that attains payoff $g(S_T^1, L_T)$ is given by

$$\pi_t^J = \sigma'(t)^{-1}\phi_t^J, \tag{8}$$

where $\phi_t^J = (\phi_t^{J1}, \cdots, \phi_t^{Jn})$ is given by

$$\phi_t^{J1} = \frac{\theta^1(t)}{H_t}E_t[H_T g_2(S_T^1, L_T)] + \frac{\sigma^{11}(t)}{H_t}E_t[H_T g_1(S_T^1, L_T)S_T^1], \tag{9}$$

$$\phi_t^{Ji} = \frac{\theta^i(t)}{H_t}E_t[H_T g_2(S_T^1, L_T)], \text{ for } i = 2, \cdots, n, \tag{10}$$

where g_i $(i = 1, 2)$ represents the partial derivative of g with respect to the i-th argument.

Portfolios π_t^M and π_t^J are obtained if the conditional expectations in equations (7), (9) and (10) are evaluated.

The interpretations for the optimal portfolio constituent factors are as follows. As for π_t^M, $\frac{1}{H_t}E_t[H_T f'(L_T)]$ is the present value of the sensitivity of the terminal payoff to the change of L_T. This quantity corresponds to *delta* in the option theory. The volume of the risky asset portfolio increases in this quantity. This factor contributes to generating the target distribution. In addition, the replicating strategy allocates the wealth to tradable assets according to the market price of risk $\theta^i(t)$. Through this operation, the cheapest strategy is realized. The difference of π_t^J from π_t^M is the second term in equation (9). This is the present value of the sensitivity of the terminal payoff to the change of W_T^1. This term contributes to the generation of the dependence structure on the investor's existing portfolio.

For the case of constant coefficients, the computational burden to obtain the dynamic replicating portfolio does not increase in the number of replicating tools. To get the dynamic portfolio for the marginal distribution, the conditional expectation in equation (7) needs to be evaluated. Then, all we need is the distribution of L_T under the information at time 0 and t. Since L_T follows one-dimensional Gaussian distribution, the conditional expectation can be numerically computed by the Monte Carlo simulations or one-dimensional numerical integration. To obtain the dynamic portfolio for the joint distribution, we need conditional expectations in equations (9) and (10). They can be numerically computed by the Monte Carlo simulations or two-dimensional numerical integrations. This is because all we need are the joint distributions of (W_T^1, L_T) under the information at time 0 and t. Since L_t is given by equation (3), L_T is represented as

$$L_T = \int_0^T r(t)dt + \frac{1}{2}\int_0^T ||\theta(t)||^2 dt + \int_0^T \theta^1(t)dW_t^1 + \sum_{i=2}^n \int_0^T \theta^i(t)dW_t^i.$$

The distributions of $\sum_{i=2}^n \int_0^T \theta^i(t)dW_t^i$ under the information at time 0 and t are Gaussian distributions with means 0 and $\sum_{i=2}^n \int_0^t \theta^i(u)dW_u^i$ and variances $\sum_{i=2}^n \int_0^T \{\theta^i(u)\}^2 du$ and $\sum_{i=2}^n \int_t^T \{\theta^i(u)\}^2 du$, respectively. Therefore, the joint distributions of (W_T^1, L_T) can be described by two-dimensional Gaussian distributions. Thus, the extension to multiple replicating tools does not bring any disadvantage in the computation for the dynamic replicating portfolios.

4 Replication of CS/Tremont Managed Futures Index

Let us apply the new method to the replication of CS/Tremont managed futures index. The monthly log return of the index is replicated. The replication performance is examined on in-sample and out-of-sample basis. This paper uses the following investor's existing portfolio and risky assets. Assume that the investor's existing portfolio is composed of 50% Japanese stocks and 50% Japanese government bonds (JGB). Since these assets are traded dynamically, TOPIX futures and long-term JGB futures were used as the proxies. Both of them are listed on the Tokyo Stock Exchange. The S&P 500 futures, NYMEX WTI crude oil futures, COMEX gold futures, and JPY against USD spot currency are used as replicating tools. It is considered that managed futures funds invest in these assets. For the purpose of comparison, the replication result of payoff (1) (i.e. the clone by the existing replication method) is also shown. Here, the replicating tool is the equally weighted portfolio of the four assets used by our method. All the data are obtained from Bloomberg. The log returns on futures are calculated by rolling the front contract. The front contract is rolled on the last trading day of the maturity month. Our base currency is USD because that of the target index is also USD. Since TOPIX and JGB futures are denominated in JPY, a currency hedge is applied. Accordingly, the log returns of these assets are adjusted by the difference between the interest rates of USD and JPY. Libor rates are used for the interest rates. This paper uses the same statistical methodology as Papageorgiou *et al.* (2008). (Details are described in the working paper version of the paper.)

4.1 In-sample Analysis

In this subsection, let us replicate the target index in an ideal situation where the models and parameters of the target hedge fund return and trading risky assets' price processes are known and there are no transaction costs. The replication performance in the period from January 2001 to December 2009 is examined. First, using the data of CS/Tremont managed futures index in this period, estimate the monthly log return of the replication target. The method is same as Papageorgiou *et al.* (2008). The best-fitted model is chosen from a Gaussian mixture with m regimes (m = 1, 2, 3, 4, 5) and

Johnson unbounded distribution. Here, we get the best-fitted model: mixture of two Gaussian distribution $N(-0.37\%, 3.14\%^2)$ and $N(4.60\%, 1.07\%^2)$ with probabilities 81% and 19%.

Next, the copula model between the monthly log returns of the replication target and the investor's portfolio is estimated. The best-fitted copula model is selected from the Gaussian, Clayton, Frank and Gumbel. Here, we get the best-fitted model: Gaussian copula with correlation -0.06.

Then, the stochastic processes of trading risky assets are estimated. In this example, assume that they follow log-normal process. In other words, all of the coefficients of (2) are constant. The parameters are estimated by maximum likelihood. The parameter estimates and correlation matrix of trading assets are described in Tables 1 and 2 respectively. The correlation between the investor's portfolio and the single replicating tool is 0.26.

Finally, the payoffs for the marginal and joint distributions ((4) and (5)) are replicated by the dynamic replicating portfolios (6) and (8) respectively. The portfolio is rebalanced on daily basis. For the purpose of comparison, payoff (1) is also replicated. This is the clone by the existing methodology.

Figure 3 shows the performances of the target index and replicating strategies. The replication by single replicating tool incurred large drawdown under Lehman shock. This failure is crucial in the replication of a managed futures index, because one of attractive characters of the strategy is high performance under financial crises. Our new methods succeeded in overcoming the failure, and performed very well. They outperformed the target index. Here, note that no transaction cost is assumed and that the performance of the target index is after deduction of management and performance fees, while those of replications are not. In out-of-sample analysis in the next subsection, we take into account transaction costs and fees. The marginal distribution clone outperformed the joint distribution replica. This is because replicating strategy (6) is the cheapest strategy for the target payoff distribution. Trading strategy (8) requires additional cost to replicate the same dependence structure on the investor's portfolio as the target index.

Table 3 shows the summary statistics of the target and replicated log returns. The target index exhibits a little negative skew and negative excess kurtosis. The clones by multiple assets succeeded in reproduce these characters, while the single replicating tool brought very high kurtosis because of the negative tail event. As for dependency on the investor's existing portfolio, the target index has correlation -0.10. Since the replicating tools have low correlations with the investor's portfolio, the marginal distribution clone has correlation -0.02. The joint distribution replica resulted in correlation -0.11. The correlation with the investor's portfolio was successfully replicated by paying some costs. The clone by single replicating tool has high correlation 0.47, because it incurred large loss in Lehman shock. Two sample Kolmogorov-Smirnov tests were performed to test whether the replications generated the same return distribution as the target index. As already mentioned, the mean returns are different from the target by initial costs of the replicating strategy. Therefore, the tests were implemented for demeaned time-series data. The p-values are 0.30 and 0.49 for the marginal and joint distributions with multiple replicating tools, while that for the existing method is 0.17. In sum, all of the statistics were dramatically improved by the extension to multiple replicating tools with long and short positions. Although this example replicated the distributional properties of CS/Tremont managed futures index (i.e. Gaussian mixtures and Gaussian copula), the internet appendix shows that our method can generate various distributions and dependencies by simulation analyses.

4.2 Out-of-Sample Analysis

In this subsection, let us consider the replication in more realistic situation. In the previous subsection, it is assumed that the all of the models and parameters are known and that there are no transaction costs. However, in the real world, we do not know the true model and parameters of the target index return and trading asset prices, and there exist transaction costs. Therefore, we estimate parameters and select models using the available data at each trading date. The transaction costs are assumed to be 1 basis point for the sale and purchase of all assets, and assumed our management fee is 2% per year. The replication procedures are performed in the same manner as in the previous subsection except parameter estimations and model selections.

The model parameters are estimated using available data at each trading date. For example, consider the parameter estimation for the replication in January 2003. At that time, the data from January 2003 to December 2009 is not available. Therefore, the parameters are estimated using the data before December 2002 in practice. This subsection performs the parameter estimation and model selection using the past data at that time.

In the estimation of the target monthly log return distribution and copula between the monthly log return of the target index and the investor's existing portfolio, the monthly data from January 1995 are used. The estimation procedure is same as the previous subsection. The parameter estimations and model selections are performed for every month.

In the inference of the trading assets' price processes, the daily data from 1999 are utilized. First let us consider the same estimation procedure as the previous subsection. In other words, parameters are estimated by maximum likelihood. The parameters are assumed to be constant during one month, and they are updated at month start.

Then, the replicating strategies are performed. Figure 4 and Table 4 show the replication results. The clones by multiple replicating tools were incurred drawdown under Lehman shock. This failure is because the maximum likelihood estimation for trading asset price processes could not capture the structural break of the market. Therefore, the replicating strategy could not control market exposures efficiently. However, managed futures got profit by capturing the structural break quickly and taking short positions.

Next, consider the replication by another estimation method with capturing market trends. In the inference of the trading assets' price processes, take latest data into account more heavily. Here, let us use exponentially weighted moving average method. Let $\{r_n^i\}_{n=1}^N$ be the time-series data of daily log-return of asset i. Suppose r_1^i is the last sample data and r_N^i is the oldest sample. The variance of asset i is estimated by

$$Var(r^i) = \sum_{k=1}^{N} \frac{\lambda^{k-1}}{\sum_{n=1}^{N} \lambda^{n-1}} {r_k^i}^2.$$

The covariances are estimated in the same way. The drift coefficient μ^i is estimated by

$$\mu^i = \sum_{k=1}^{N} \frac{\lambda^{k-1}}{\sum_{n=1}^{N} \lambda^{n-1}} r_k^i + \frac{1}{2}\sqrt{Var(r^i)}.$$

Here, the decay factor λ is assumed to be 0.99. Then, the half value period is 68 days. The parameters are assumed to be constant during one month, and they are updated at month start.

Figure 5 shows the performances of the target and the replicating strategies. Both of the clones by multiple replicating tools performed very well even under the Lehman shock. The estimation method succeeded in capturing the turning point of the market trend, and the replicating strategies earned profits by taking short position. Figure 6 illustrates that the clones by multiple replicating tools succeeded in the replication of the joint distribution with the investor's existing portfolio. Especially, the returns in October 2008 are very close to the target. On the other hand, the replica by single replicating tool incurred large loss in the month. This is because the replicating strategy cannot take short position even if the expected return of the replicating tool is negative.

The clone for the marginal distribution outperforms that for the joint distribution as in the in-sample analysis. The reason is also same. The replication of the dependency on the investor's portfolio requires additional cost. Let us see more in detail. The returns are different especially in 2005. The most significant difference of the trading strategy is the second term in equation (8). By this term, the dependency on the investor's portfolio is created. Figure 7 shows the replicating portfolios for the marginal and joint distributions, and Figure 8 shows the performance of the investor's portfolio. The replicating strategy for the marginal distribution heavily weighed on the investor's portfolio in 2005 to enjoy the Japanese bull market trend. On the other hand, that for the joint distribution takes position on the investor's portfolio not so much to realize low correlation with the investor's portfolio. Therefore, the clone for the joint distribution could not earn profits from the Japanese bull market.

Table 5 exhibits the summary statistics of the target and replicated monthly log returns. The mean returns of the clones by multiple replicating tools are higher than the target. This means that the replication products succeeded in delivering the returns of managed futures at lower cost. If the performance fee is assumed to be 20%, the clones have close return level to the managed futures index. Standard deviations are very close to the target. Skews and kurtosis are close to zero and three, respectively, like the target, while the replication by single replicating tool end up with much higher kurtosis. The dependency on the investor's existing portfolio is also replicated successfully. The target correlation is -0.10. The clone for joint distribution has the correlation with the investor's portfolio -0.06, while that by the single replicating tool is 0.48. Since we replicate the target with reflecting the trading strategy of the target (i.e. trend following strategy), the clones exhibit correlated returns to the target to some extent. The correlations of replicas for marginal and joint distributions with the target are 0.56 and 0.58 respectively. The correlated returns are also confirmed in Figure 9. On the other hand, the clone by single replicating tool has low correlation to the target.

In sum, on out-of-sample basis, the parameter estimation for trading asset price processes is crucial for the replication performance. The replication with maximum likelihood estimation did not work. However, the estimation method with reflecting trend following strategy, that is exponentially weighted moving average method, resulted in very good replication performance. The empirical example showed that the extension to multiple replicating tools with long and short positions improved the replication performance substantially in practice.

5 Conclusion

This article developed a new hedge fund replication method with the dynamic optimal portfolio by extending Kat and Palaro (2005) and Papageorgiou *et al.* (2008) to multiple trading assets with both long and short positions. It generates a target payoff distribution by the cheapest dynamic portfolio. The cheapest payoff is theoretically supported by continuous-time version of the payoff distribution pricing model developed by Dybvig (1988). It can be shown that the cost minimization for some distribution is equivalent to the maximization of a von Neumann-Morgenstern utility. The dynamic replicating strategy is obtained by martingale method with Malliavin calculus.

The method was applied to the replication of CS/Tremont managed futures index. The replication performances were examined on in-sample and out-of-sample basis. Empirical results showed that the extension to multiple replicating tools with long and short positions dramatically improved the replication performance in practice. Especially, the new replication method got high returns in Lehman shock as the replication target while the replication based on the existing method incurred a large loss during this period. Moreover, the replication with an estimation method that reflects trend following strategy as the target brings correlated returns to the target to some extent. The extension to multiple assets with long and short positions and the estimation procedure with reflecting the investment strategy of the target overcome the two shortcomings of the existing distribution replication method that were pointed out by Amenc *et al.* (2008). (They are quoted in pp. 5.)

The implementation for the Markovian coefficients case including a stochastic volatility model as well as a stochastic interest rate model is a challenging task. Also, the application of our method to creating new attractive trading strategies is a next research topic.

References

Agarwal, V. and N.Y. Naik (2004). Risks and Portfolio Decisions Involving Hedge Funds. *The Review of Financial Studies*, Vol. 17(1), pp. 63-98.

Amenc, N., W. Géhin, L. Martellini, and J.C. Meyfredi (2008). Passive Hedge Fund Replication: A Critical Assessment of Existing Techniques. *The Journal of Alternative Investments*, Vol. 11(2), pp. 69-83.

Amin, G. and H.M. Kat (2003). Hedge Fund Performance 1990-2000: Do the Money Machines Really Add Value? *Journal of Financial and Quantitative Analysis*, Vol. 38(2), pp. 251-274.

Detemple, J., R. Garcia, and M. Rindisbacher (2003). A Monte Carlo Method for Optimal Portfolios. *Journal of Finance*, Vol. 58(1), pp. 401-446.

Detemple, J., R. Garcia, and M. Rindisbacher (2008). Simulation Methods for Optimal Portfolios. *Handbooks in OR & MS,* Vol. 15, pp. 867-923.

Duarte, J., F. Longstaff, and F. Yu (2007). Risk and Return in Fixed Income Arbitrage: Nickels in front of a Steamroller? *The Review of Financial Studies*, Vol. 20(3), pp. 769-811.

Dybvig, P. (1988). Distributional Analysis of Portfolio Choice. *Journal of Business*, Vol. 61, pp. 369-393.

Fung, W. and D.A. Hsieh (1997). Empirical Characteristics of Dynamic Trading Strategies: The Case of Hedge Funds. *The Review of Financial Studies*, Vol. 10(2), pp. 275-302.

Fung, W. and D.A. Hsieh (2000). Performance Characteristics of Hedge Funds and Commodity Funds: Natural vs. Spurious Biases. *The Journal of Financial and Quantitative Analysis*, Vol. 35(3), pp. 291-307.

Fung, W. and D.A. Hsieh (2001). The Risk in Hedge Fund Strategies: Theory and Evidence from Trend Followers. *The Review of Financial Studies*, Vol. 14(2), pp. 313-341.

Fung, W. and D.A. Hsieh (2007a). Will Hedge Funds Regress towards Index-like Products? *Journal of Investment Management*, Vol. 5(2), pp. 46-55.

Fung, W. and D.A. Hsieh (2007b). Hedge Fund Replication Strategies: Implications for Investors and Regulators. *Banque de France, Financial Stability Review, Special issue on hedge funds*, Vol. 10 April, pp. 55-66.

Géhin, W. (2007). Hedge Fund Clones: Key Issues in Distinguishing between Them. http://www.edhec-risk.com/latest_news/RISKArticle.2007-10-11.3936?newsletter=yes

Hakamada, T., A. Takahashi, and K. Yamamoto (2007). Selection and Performance Analysis of Asia-Pacific Hedge Funds. *The Journal of Alternative Investments*, Vol. 10(3), pp. 7-29.

Joe, H. (1997). *Multivariate Models and Dependence Concepts.* Chapman and Hall.

Kaplan, P.D. and J.A. Knowles (2004). Kappa: A Generalized Downside Risk-Adjusted Performance Measure. *Journal of Performance Measurement*, Vol. 8, pp. 42-54.

Karatzas, I. and S. Shreve (1998). *Methods of Mathematical Finance.* Springer.

Kat, H.M. and H.P. Palaro (2005). Hedge Fund Returns: You Can Make Them Yourself! *Journal of Wealth Management*, Vol. 8(2), pp. 62-68.

Lo, A. and J. Hasanhodzic (2007). Can Hedge-Fund Returns Be Replicated?: The Linear Case. *Journal of Investment Management*, Vol. 5(2), pp. 5-45.

Mitchell, M. and T. Pulvino (2001). Characteristics of Risk in Risk Arbitrage. *Journal of Finance*, Vol. 56, pp. 2067-2109.

Nelsen, R.B. (1999). *An introduction to copulas: Vol. 139 of Lecture Notes in Statistics*. Springer-Verlag, New York.

Nualart, D. (2006). *The Malliavin Calculus and Related Topics 2nd Edition*, Springer.

Ocone, D., and I. Karatzas (1991). A Generalized Clark Representation Formula, with Applications to Optimal Portfolios. *Stochastics*, Vol. 37, pp. 187-220.

Papageorgiou, N., B. Remillard, and A. Hocquard (2008). Replicating the Properties of Hedge Fund Returns. *The Journal of Alternative Investments*, Vol. 11(2), pp. 8-38.

Passow, A. (2005). Omega Portfolio Construction with Johnson Distributions. *Risk*, Vol. 18(4), pp. 85-90.

Pérez, P. G. (2004). An Approach to the Non-normal Behavior of Hedge Fund Indices Using Johnson Distributions. Working Paper.

Takahashi, A., and K. Yamamoto (2010). Hedge Fund Replication. Chapter 2 in *The Recent Trend of Hedge Fund Strategies*, Nova Science Publishers, Forthcoming in 3rd Quarter in 2010.

Takahashi, A., and N. Yoshida (2004). An Asymptotic Expansion Scheme for Optimal Investment Problems. *Statistical Inference for Stochastic Process*, Vol. 7, pp. 153-188.

Wallerstein, E., N.S. Tuchschmid, and S. Zaker (2010). How Do Hedge Fund Clones Manage the Real World? *The Journal of Alternative Investments*, Vol. 12(3), pp. 37-50.

	Investor's Portfolio	JPY/USD	Clude Oil	Gold	S&P 500	Single Replicating Tool
μ_i	4.06%	2.68%	16.07%	14.50%	0.56%	8.17%
Std. Dev.	13.33%	10.95%	42.24%	19.23%	22.05%	14.76%

Table 1: **Parameter Estimates of the Trading Assets.** This table shows the estimated parameters of the trading assets' price processes. They are assumed to be log-normal processes. The parameters are estimated by maximum likelihood using the data during January 2001-December 2009. The investor's portfolio is assumed to be equally weighted portfolio of TOPIX and Japanese government bond futures.

	Investor's Portfolio	JPY/USD	Clude Oil	Gold	S&P 500
Investor's Portfolio	1.00				
JPY/USD	0.27	1.00			
Clude Oil	0.13	0.06	1.00		
Gold	0.08	-0.13	0.24	1.00	
S&P 500	0.24	0.14	0.16	-0.04	1.00

Table 2: **Correlation Matrix of the Trading Assets.** This table shows the correlation matrix of the trading assets. They are estimated using the data during January 2001-December 2009. The investor's portfolio is assumed to be equally weighted portfolio of TOPIX and Japanese government bond futures.

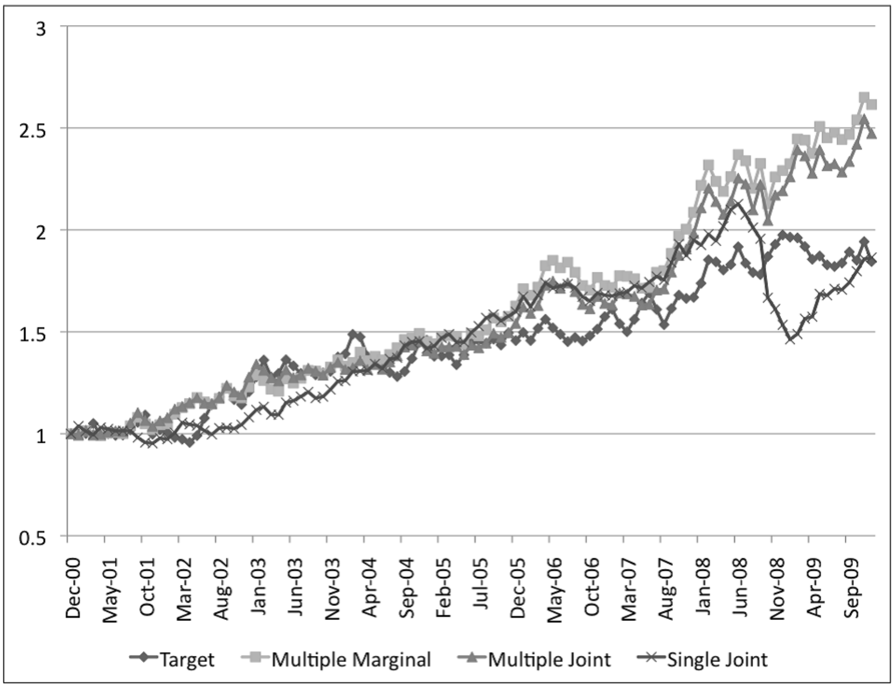

Figure 3: **Performances of the Target and Replicated Strategies Using Data during the Sample Period.** This figure shows the growth of the net asset values ($) of the target and its three clones from January 2001 to December 2009. The model parameters are estimated using data during the period. The target is CS/Tremont managed futures index. "Multiple Marginal" represents the clone for the marginal distribution by multiple replicating tools. "Multiple Joint" represents the clone for the joint distribution with the investor's existing portfolio by multiple replicating tools. The investor's portfolio is assumed to be equally weighted portfolio of TOPIX and Japanese government bond futures. Replicating tools are S&P 500 futures, NYMEX WTI crude oil futures, COMEX gold futures, and JPY against USD spot currency. "Single Joint" represents the clone for the joint distribution with the investor's existing portfolio by single replicating tool, which is the equally weighted portfolio of the four replicating tools. The initial net asset values for all the strategies are normalized to one at the end of December 2000.

	Target	Multiple Marginal	Multiple Joint	Single Joint
Mean	0.57%	0.89%	0.84%	0.58%
Std. Dev.	3.48%	2.89%	3.01%	2.94%
Mean/Std. Dev.	0.16	0.31	0.28	0.20
Skew	-0.17	-0.33	-0.17	-1.63
Kurtosis	2.50	2.95	2.56	11.22
Correlation with investor's portfolio	-0.10	0.02	-0.11	0.47
Correlation with the target		0.27	0.29	0.10
K-S test p-value		0.74	0.85	0.19

Table 3: **Summary Statistics of the Target and Replicated Log Returns Using Data during the Sample Period.** This table shows the summary monthly statistics of the target and replicated log returns from January 2001 to December 2009. The model parameters are estimated using data during the period. The target is CS/Tremont managed futures index. "Multiple Marginal" represents the clone for the marginal distribution by multiple replicating tools. "Multiple Joint" represents the clone for the joint distribution with the investor's existing portfolio by multiple replicating tools. The investor's portfolio is assumed to be equally weighted portfolio of TOPIX and Japanese government bond futures. Replicating tools are S&P 500 futures, NYMEX WTI crude oil futures, COMEX gold futures, and JPY against USD spot currency. "Single Joint" represents the clone for the joint distribution with the investor's existing portfolio by single replicating tool, which is the equally weighted portfolio of the four replicating tools. K-S test p-value represents p-value of two-sample Kolmogorov-Smirnov test with the target. In the test, demeaned time-series data are used.

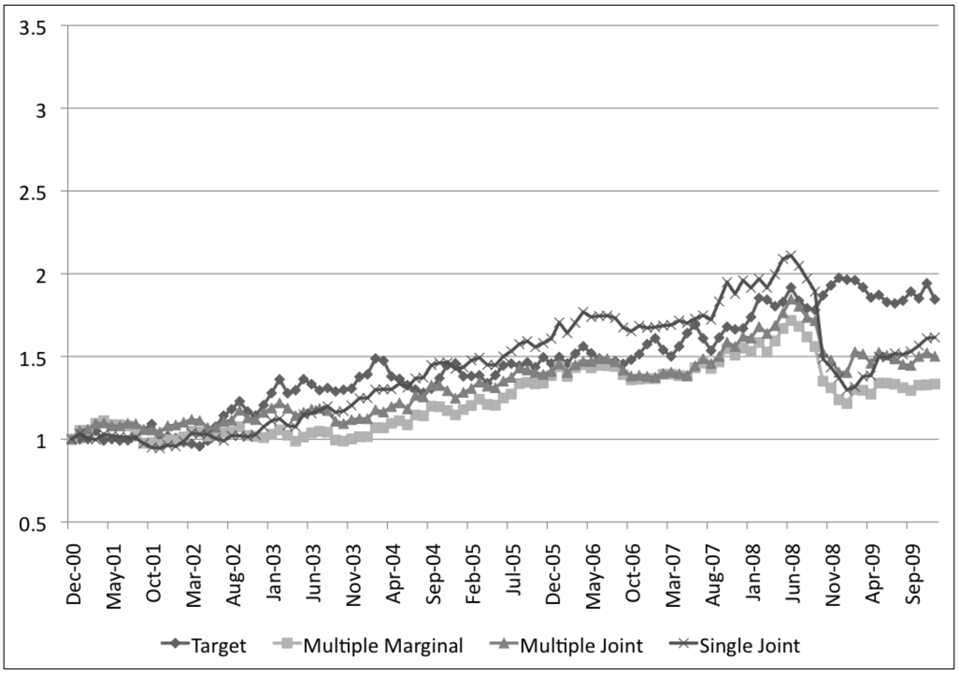

Figure 4: **Performances of the Target and Replicated Strategies Using Past Data with Maximum Likelihood.** This figure shows the growth of the net asset values ($) of the target and its three clones from January 2001 to December 2009. The model parameters for trading assets are estimated using the available data at each trading date with maximum likelihood. The target is CS/Tremont managed futures index. "Multiple Marginal" represents the clone for the marginal distribution by multiple replicating tools. "Multiple Joint" represents the clone for the joint distribution with the investor's existing portfolio by multiple replicating tools. The investor's portfolio is assumed to be equally weighted portfolio of TOPIX and Japanese government bond futures. Replicating tools are S&P 500 futures, NYMEX WTI crude oil futures, COMEX gold futures, and JPY against USD spot currency. "Single Joint" represents the clone for the joint distribution with the investor's existing portfolio by single replicating tool, which is the equally weighted portfolio of the four replicating tools. The initial net asset values for all the strategies are normalized to one at the end of December 2000.

	Target	Multiple Marginal	Multiple Joint	Single Joint
Mean	0.57%	0.27%	0.38%	0.44%
Std. Dev.	3.48%	3.09%	3.00%	3.64%
Mean/Std. Dev.	0.16	0.09	0.13	0.12
Skew	-0.17	-1.03	-0.36	-2.78
Kurtosis	2.50	6.83	4.68	21.65
Min	-9.01%	-14.38%	-11.86%	-24.19%
Max	8.28%	6.62%	8.52 %	7.86%
Correlation with investor's portfolio	-0.10	0.44	0.14	0.52
Correlation with the target		0.10	0.17	0.09
K-S test p-value		0.32	0.52	0.19

Table 4: **Summary Statistics of the Target and Replicated Log Returns Using the Past Data with Maximum Likelihood.** This table shows the summary monthly statistics of the target and replicated log returns from January 2001 to December 2009. The model parameters for trading assets are estimated using the available data at each trading date with maximum likelihood. The target is CS/Tremont managed futures index. "Multiple Marginal" represents the clone for the marginal distribution by multiple replicating tools. "Multiple Joint" represents the clone for the joint distribution with the investor's existing portfolio by multiple replicating tools. The investor's portfolio is assumed to be equally weighted portfolio of TOPIX and Japanese government bond futures. Replicating tools are S&P 500 futures, NYMEX WTI crude oil futures, COMEX gold futures, and JPY against USD spot currency. "Single Joint" represents the clone for the joint distribution with the investor's existing portfolio by single replicating tool, which is the equally weighted portfolio of the four replicating tools. K-S test p-value represents p-value of two-sample Kolmogorov-Smirnov test with the target. In the test, demeaned time-series data are used.

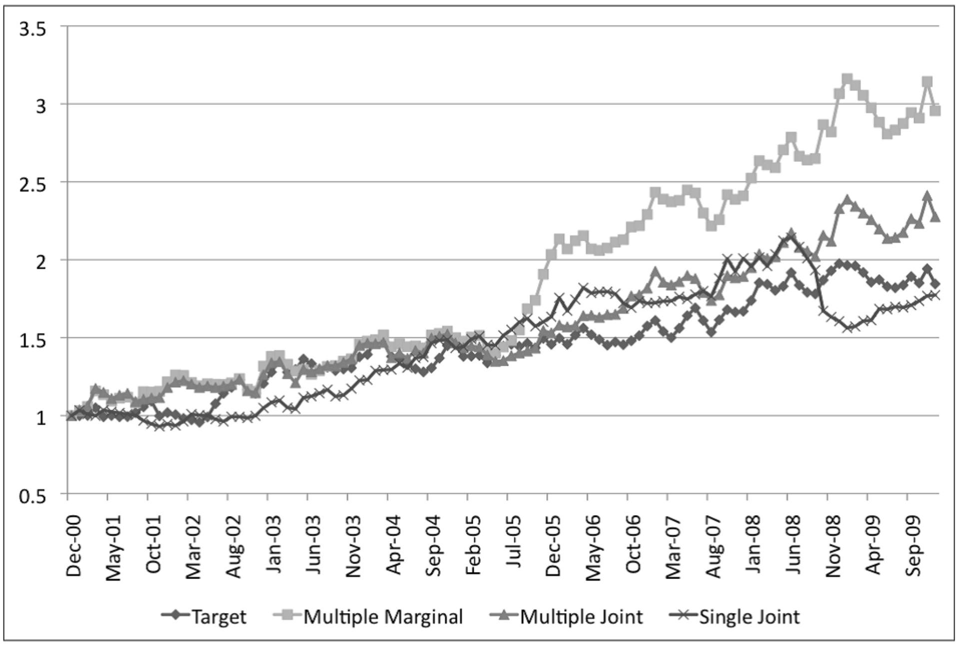

Figure 5: **Performances of the Target and Replicated Strategies Using Past Data with Exponentially Weighted Moving Average.** This figure shows the growth of the net asset values ($) of the target and its three clones from January 2001 to December 2009. The model parameters for trading assets are estimated using the available data at each trading date with exponentially weighted moving average. The target is CS/Tremont managed futures index. "Multiple Marginal" represents the clone for the marginal distribution by multiple replicating tools. "Multiple Joint" represents the clone for the joint distribution with the investor's existing portfolio by multiple replicating tools. The investor's portfolio is assumed to be equally weighted portfolio of TOPIX and Japanese government bond futures. Replicating tools are S&P 500 futures, NYMEX WTI crude oil futures, COMEX gold futures, and JPY against USD spot currency. "Single Joint" represents the clone for the joint distribution with the investor's existing portfolio by single replicating tool, which is the equally weighted portfolio of the four replicating tools. The initial net asset values for all the strategies are normalized to one at the end of December 2000.

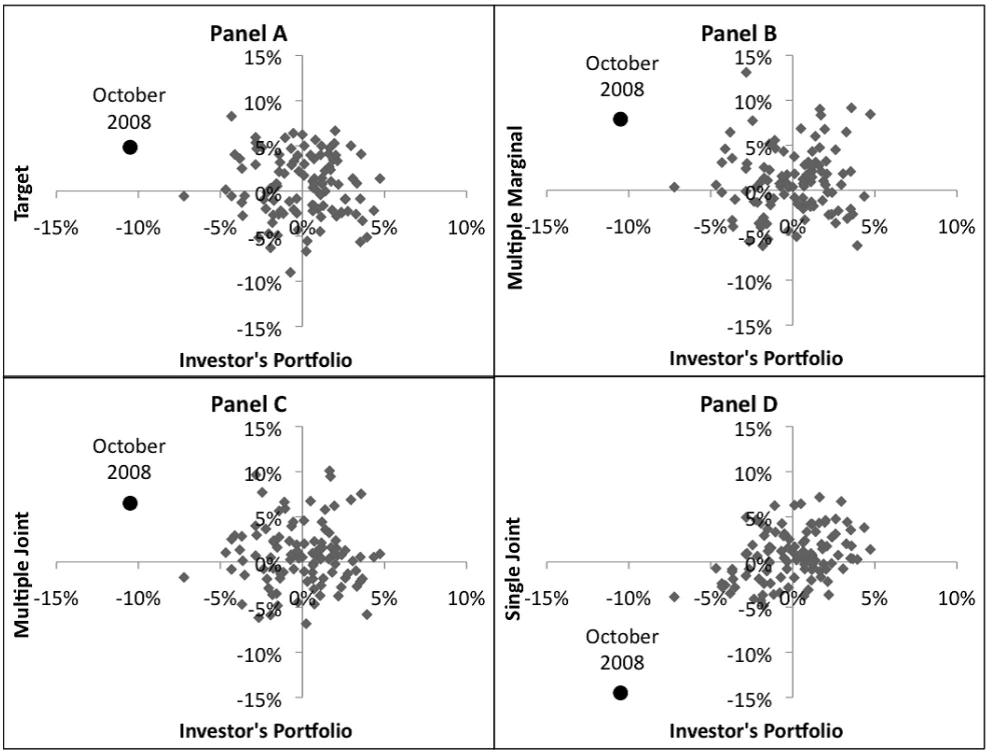

Figure 6: **Joint Distributions with the Investor's Existing Portfolio of the Target and its Clones.** This figure shows the joint distributions with the investor's portfolio of the target and its three clones. The sample period is from January 2001 to December 2009. The horizontal axis is the log return of investor's portfolio, and the vertical axes are those of the target and clones. The target is CS/Tremont managed futures index. The investor's portfolio is assumed to be equally weighted portfolio of TOPIX and Japanese government bond futures. Panel A represents the joint distribution of the target with the investor's portfolio. Panel B is that of the clone for the marginal distribution by multiple replicating tools. Panel C shows that of the clone for the joint distribution by multiple replicating tools. Replicating tools are S&P 500 futures, NYMEX WTI crude oil futures, COMEX gold futures, and JPY against USD spot currency. Panel D represents the joint distribution of the clone for the joint distribution with the investor's existing portfolio by single replicating tool, which is the equally weighted portfolio of the four replicating tools. The big black circles represent the samples in October 2008.

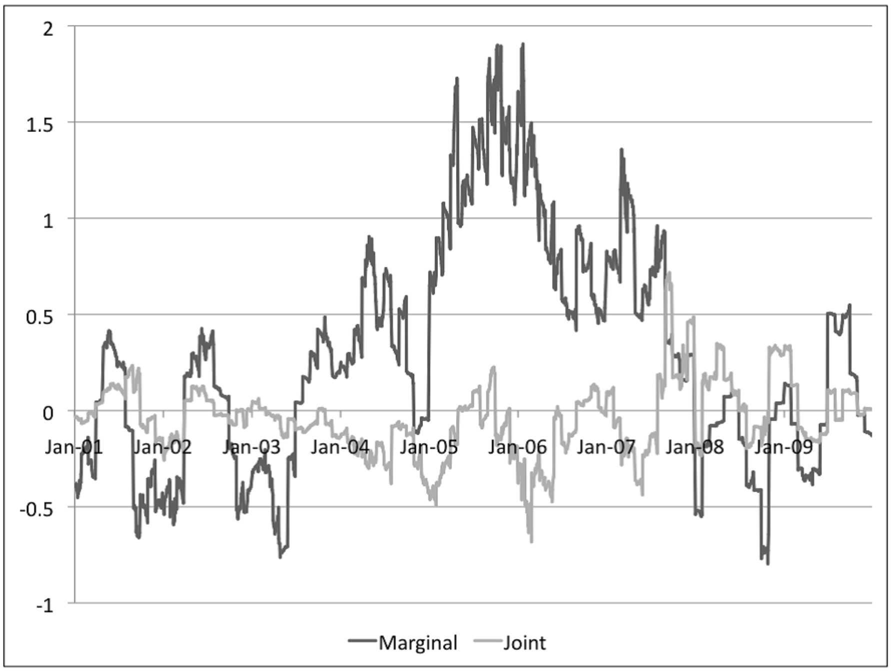

Figure 7: **Weights of the Replicating Strategies by Multiple Replicating Tools on the Investor's Portfolio.** This figure shows weights of the replicating strategies by multiple replicating tools on the investor's existing portfolio from January 2001 to December 2009. "Marginal" represents the clone for the marginal distribution and "Joint" represents the clone for the joint distribution with the investor's existing portfolio. The investor's portfolio is assumed to be equally weighted portfolio of TOPIX and Japanese government bond futures.

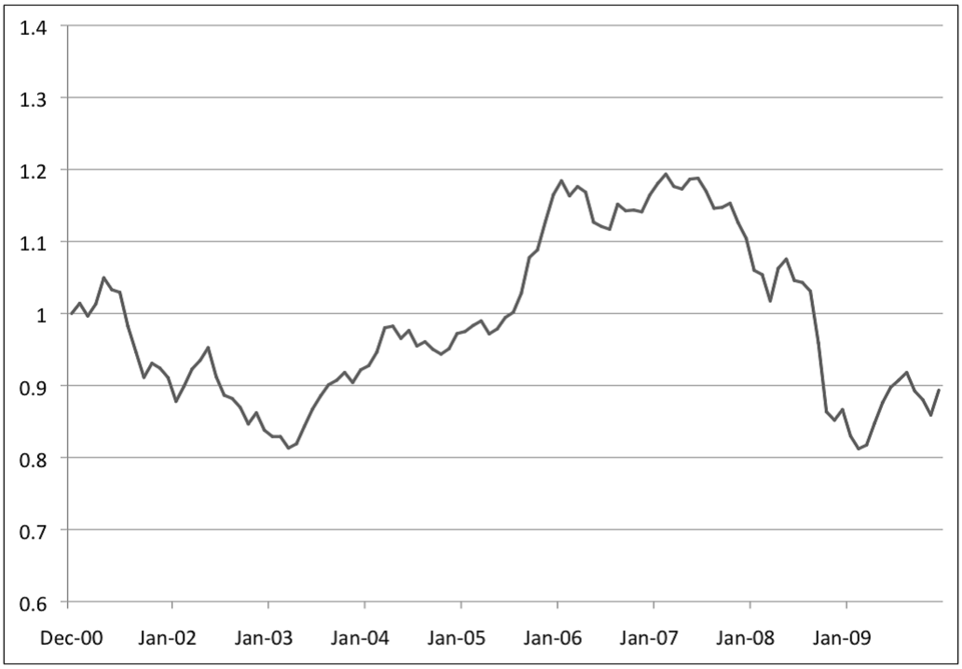

Figure 8: **Performance of the Investor's Portfolio.** This figure shows the growth of the net asset value ($) of the investor's portfolio from January 2001 to December 2009. The investor's portfolio is assumed to be equally weighted portfolio of TOPIX and Japanese government bond futures. The initial net asset value is normalized to one at the end of December 2000.

	Target	Multiple Marginal	Multiple Joint	Single Joint
Mean	0.57%	1.00%	0.76%	0.53%
Std. Dev.	3.48%	3.64%	3.45%	3.11%
Mean/Std. Dev.	0.16	0.28	0.22	0.17
Skew	-0.17	0.52	0.40	-0.83
Kurtosis	2.50	3.51	3.31	6.89
Min	-9.01%	-6.19%	-6.84%	-14.50%
Max	8.28%	13.11%	10.09%	7.18%
Correlation with investor's portfolio	-0.10	-0.02	-0.06	0.48
Correlation with the target		0.56	0.58	0.19
K-S test p-value		0.52	0.41	0.52

Table 5: **Summary Statistics of the Target and Replicated Log Returns Using the Past Data with Exponentially Weighted Moving Average.** This table shows the summary monthly statistics of the target and replicated log returns from January 2001 to December 2009. The model parameters for trading assets are estimated using the available data at each trading date with exponentially weighted moving average. The target is CS/Tremont managed futures index. "Multiple Marginal" represents the clone for the marginal distribution by multiple replicating tools. "Multiple Joint" represents the clone for the joint distribution with the investor's existing portfolio by multiple replicating tools. The investor's portfolio is assumed to be equally weighted portfolio of TOPIX and Japanese government bond futures. Replicating tools are S&P 500 futures, NYMEX WTI crude oil futures, COMEX gold futures, and JPY against USD spot currency. "Single Joint" represents the clone for the joint distribution with the investor's existing portfolio by single replicating tool, which is the equally weighted portfolio of the four replicating tools. K-S test p-value represents p-value of two-sample Kolmogorov-Smirnov test with the target. In the test, demeaned time-series data are used.

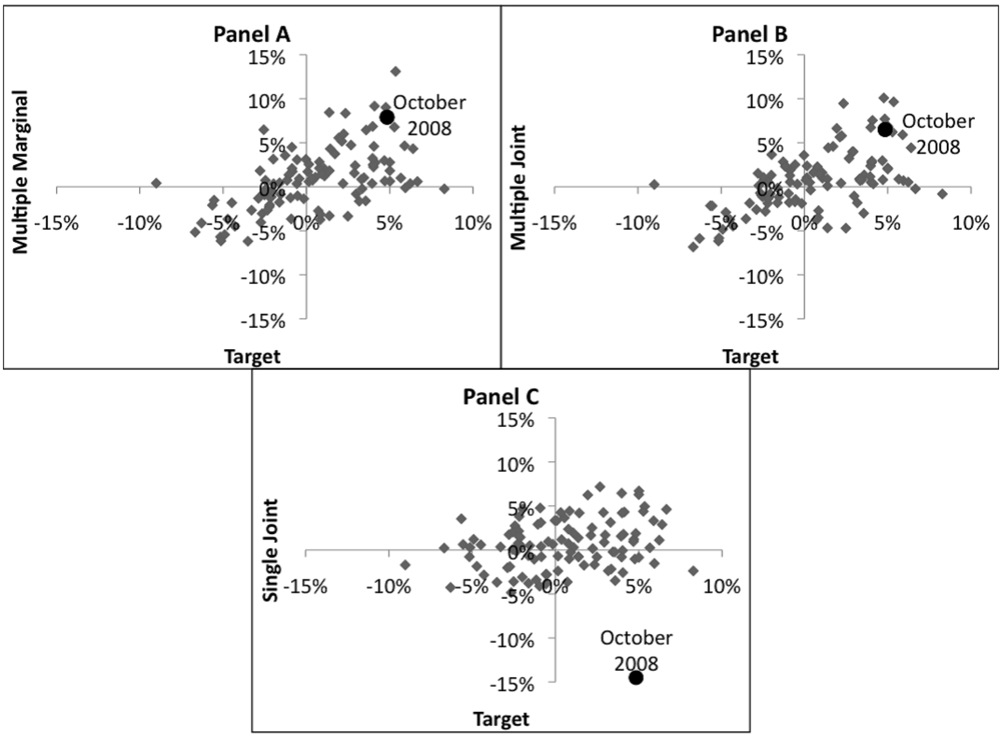

Figure 9: **Joint Distributions of the Clones with the Target.** This figure shows the joint distributions the clones with the target. The sample period is from January 2001 to December 2009. The horizontal axis is the log return of the target, and the vertical axes are those of the clones. The target is CS/Tremont managed futures index. Panel A represents the joint distribution of the target and the clone for the marginal distribution by multiple replicating tools. Panel B is that of the clone for the joint distribution by multiple replicating tools. Replicating tools are S&P 500 futures, NYMEX WTI crude oil futures, COMEX gold futures, and JPY against USD spot currency. Panel C shows the joint distribution of the clone for the joint distribution with the investor's existing portfolio by single replicating tool, which is the equally weighted portfolio of the four replicating tools. The big black circles represent the samples in October 2008.

Internet Appendix
for
Generating a Target Payoff Distribution with the Cheapest Dynamic Portfolio: An Application to Hedge Fund Replication

Akihiko Takahashi and Kyo Yamamoto

A Mathematical Setting of the Financial Market

This section describes the financial market with mathematically technical conditions. Let us begin with a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is given a n-dimensional standard Brownian motion $W_t = (W_t^1, \cdots, W_t^n)'$, $0 \le t \le T$, where $'$ represents transposition so that W_t is a column vector. $W_0 = 0$ is satisfied almost surely. Let $\{\mathcal{F}_t^W\}_{0 \le t \le T}$ be the filtration generated by W_t. This paper uses the augmented filtration

$$\mathcal{F}_t = \sigma(\mathcal{F}_t^W \cup \mathcal{N}) \quad \text{for any } t < T,$$

where $\mathcal{N}$ denotes $\mathbb{P}$-null subsets of $\mathcal{F}_T^W$.

Suppose that the price processes of the financial assets S^i $(i = 0, \cdots, n)$, $\{S_t^i\}_{t=0}^T$, follow stochastic differential equations (SDEs)

$$dS_t^0 = r_t S_t^0 dt,$$

$$dS_t^i = \mu_t^i S_t^i dt + \sum_{j=1}^{i} \sigma_t^{ij} S_t^i dW_t^j \quad (i = 1, \cdots, n),$$

where r_t, μ_t^i, and σ_t^{ij} are progressively measurable and satisfy

$$\int_0^T |r_t| dt < \infty \text{ a.s.},$$
$$\int_0^T |\mu_t^i| dt < \infty \text{ a.s.},$$
$$\int_0^T (\sigma_t^{ij})^2 dt < \infty \text{ a.s.}$$

for any $1 \leq j \leq i \leq n$. Suppose that σ_t is invertible almost surely. Then, there exists the unique market price of risk

$$\theta_t = \sigma_t^{-1}(\mu_t - r_t \vec{1}).$$

In other words, the financial market is complete. The financial market is denoted by $\mathcal{M} = (r, \mu, \sigma)$.

In complete market $\mathcal{M}$, the unique state price density process is given by

$$H_t = \exp\left\{ - \int_0^t r_u du - \frac{1}{2}\int_0^t ||\theta_u||^2 du - \int_0^t \theta_u' dW_u \right\}.$$

The no-arbitrage price of any $\mathcal{F}_T$-measurable payoff X is given by $x = E[H_T X]$. X can be replicated by a dynamic trading of the financial assets with initial cost x. (See, for example, Karatzas and Shreve (1998).)

B The Payoff for a Target Marginal Distribution

The payoff that attains a target probability distribution with the minimum cost is presented here. Let ξ be $\mathcal{F}_T$-measurable random variable that has the target payoff distribution. In this paper, it is assumed that ξ is positive, and has a continuous strictly increasing distribution function. Theorem 1 in Dybvig (1988) shows that the cheapest way to obtain a given distribution is by allocating terminal wealth in the reverse order of the state price density in an equally probable finite state setting. For convenience, the minus logarithm state price density process L_t is introduced:

$$L_t = -\log H_t = \int_0^t r_u du + \frac{1}{2}\int_0^t ||\theta_u||^2 du + \int_0^t \theta_u' dW_u. \tag{B.1}$$

Let F_ξ and F_{L_T} denote the distribution functions of ξ and L_T, respectively. Assume that F_{L_T} is also continuous and strictly increasing. If X is defined as follows, X has the same distribution with ξ:

$$X = f(L_T), \tag{B.2}$$

where

$$f(l) = F_\xi^{-1}\big(F_{L_T}(l)\big).$$

The next theorem asserts that X is the unique cheapest payoff among the random variables that has the same distribution with ξ.

Theorem B.1 *Assume ξ is a positive $\mathcal{F}_T$-measurable random variable, and F_ξ and F_{L_T} are continuous and strictly increasing. In a complete market $\mathcal{M}$, the unique cheapest payoff X that has the same distribution with ξ is given by equation (B.2).*

Proof. First, let us show that the cheapest way to obtain a target distribution is by allocating terminal wealth in the reverse order of the state price density. Reverse order of the state price density is mathematically defined as follows. Let Z be a random variable. For $\omega^* \in \Omega$, define $U_Z(\omega^*)$ as

$$U_Z(\omega^*) = \{\omega : H_T(\omega) < H_T(\omega^*)\} \cap \{\omega : Z(\omega) < Z(\omega^*)\},$$

and let

$$V_Z = \cup_{\omega^* \in \Omega} U_Z(\omega^*).$$

Z is reverse order of the state price density if

$$\mathbb{P}(V_Z) = 0.$$

Assume that $\mathcal{F}_T$-measurable random variable Z has the same distribution with ξ, and is not the reverse order of the state price density. Then, there exist $a_1^- < a_1^+ \leq a_2^- < a_2^+$ such that

$$H_T(\omega_1) < H_T(\omega_2) \quad \text{for any } (\omega_1, \omega_2) \in A_1 \times A_2, \tag{B.3}$$

where $A_1 = \{a_1^- < Z < a_1^+\}$, $A_2 = \{a_2^- < Z < a_2^+\}$, and $\mathbb{P}(A_1) = \mathbb{P}(A_2) > 0$. Z is not reverse order of H_T on $A_1 \cup A_2$.

The following discussion shows that a payoff cheaper than Z can be created without changing the distribution by switching the values on A_1 and A_2. Let $p_i^\pm = \mathbb{P}(Z \leq a_i^\pm)$, $I_1 = (p_1^-, p_1^+]$ and $I_2 = (p_2^-, p_2^+]$. Define Z' as

$$Z' = G(F_\xi(Z)),$$

where function G is defined as

$$G(p) = \begin{cases} F_\xi^{-1}(p) & \text{on } (0,1) \setminus (I_1 \cup I_2), \\ F_\xi^{-1}(p + p_2^- - p_1^-) & \text{on } I_1, \\ F_\xi^{-1}(p + p_1^- - p_2^-) & \text{on } I_2. \end{cases}$$

Here, note that $p_1^+ - p_1^- = p_2^+ - p_2^-$. Then, Z' has the same distribution with ξ. This is because equation

$$\mathbb{P}(Z' \leq a) = F_\xi(a), \tag{B.4}$$

holds for any $a > 0$, as shown in the following. For $0 < a \leq a_1^-$,

$$\mathbb{P}(Z' \leq a) = F_\xi(a).$$

For $a \in (a_1^-, a_1^+]$,

$$\begin{aligned} \mathbb{P}(Z' \leq a) &= \mathbb{P}(Z' \leq a_1^-) + \mathbb{P}(a_1^- < Z' \leq a) \\ &= \mathbb{P}(Z \leq a_1^-) + \mathbb{P}(a_1^- < G(F_\xi(Z)) \leq a) \\ &= p_1^- + \mathbb{P}(F_\xi(a_1^-) < F_\xi(Z) + p_2^- - p_1^- \leq F_\xi(a)) \\ &= F_\xi(a). \end{aligned}$$

The same arguments prove that equation (B.4) holds for any a in other intervals.

The difference of cost is given by

$$E[H_T Z] - E[H_T Z'] = E[H_T(Z - Z')1_{A_2}] - E[H_T(Z' - Z)1_{A_1}].$$

Since Z' is created by switching the values of Z' on A_1 and A_2, $(Z - Z')1_{A_2}$ and $(Z' - Z)1_{A_1}$ have the same distribution. Noting inequality (B.3), it is obtained that

$$E[H_T Z] - E[H_T Z'] > 0.$$

From above the discussion, Z' has the same distribution with ξ with cheaper cost than Z. Therefore, to obtain the same distribution with ξ at time T, the terminal wealth should be in the reverse order of the state price density. Since X is reverse order of H_T, X is one of the cheapest payoffs that have the same distribution with ξ.

Next, let us prove the uniqueness. Let X' be the random variable that has the same distribution with ξ, and is reverse order of H_T. Then, $P(V_{X'}) = 0$. For any $\omega^* \in \Omega \setminus V_{X'}$,

$$\mathbb{P}\{\omega : L_T(\omega) \leq L_T(\omega^*)\} = \mathbb{P}\{\omega : X'(\omega) \leq X'(\omega^*)\}.$$

Therefore,

$$F_{L_T}(L_T(\omega^*)) = F_\xi(X'(\omega^*)).$$

By operating F_ξ^{-1} on the both sides of the equation,

$$F_\xi^{-1}(F_{L_T}(L_T(\omega^*))) = X'(\omega^*).$$

Hence, $X' = X$. □

C A Theoretical Support of the Cheapest Payoff

The dynamic trading strategy generating payoff X defined by (B.2) attains the given payoff distribution with the minimum cost. The next theorem asserts that the cost minimization is equivalent to an expected utility maximization. Theorem 2 in Dybvig (1988) is the equally probable finite state setting version of the theorem. To state the theorem in continuous-time framework, Inada condition is required for the utility function. Rogers (2009) derives a similar result independently.

Theorem C.1 *Assume ξ is a positive $\mathcal{F}_T$-measurable random variable, and F_ξ and F_{L_T} are continuous and strictly increasing. If X is the cheapest payoff that has the same distribution with ξ (i.e. X is defined by equation (B.2)), then, in a complete market $\mathcal{M}$, there exists a strictly increasing and strictly concave von Neumann-Morgenstern utility function $u(\cdot)$ such that (a) $\lim_{z\to+0} u'(z) = +\infty$, (b) $\lim_{z\to+\infty} u'(z) = 0$, and the dynamic trading strategy that attains payoff X is the optimal investment strategy for $u(\cdot)$.*

Conversely, if a dynamic trading strategy maximizes a strictly increasing and strictly concave von Neumann-Morgenstern utility function $u(\cdot)$ that satisfies conditions (a) and (b), it attains the cheapest payoff for some distribution.

Proof. Let x be the initial cost for payoff X (i.e. $x = E[H_T X].$). Since X is defined by equation (B.2), $H_T = \exp[-f^{-1}(X)]$. $\exp[-f^{-1}(\cdot)]$ is a positive strictly decreasing function. Define $u(\cdot)$ as

$$u(z) = \lambda \int_0^z \exp[-f^{-1}(\zeta)]d\zeta,$$

where λ is a positive number. Then, $u(\cdot)$ is strictly increasing and strictly concave. It is also satisfied that $\lim_{z \to +0} u'(z) = +\infty$ and $\lim_{z \to +\infty} u'(z) = 0$. Moreover, $u'(X) = \lambda H_T$. This is the first order condition of the optimality for von Neumann-Morgenstern utility $u(\cdot)$. (See, for example, Karatzas and Shreve (1998).) Budget constraint $x = E[H_T u'^{-1}(\lambda H_T)]$ is also satisfied.

Conversely, assume a dynamic trading strategy with initial cost x' generating payoff X' maximizes a strictly increasing and strictly concave von Neumann-Morgenstern utility $u(\cdot)$ satisfying conditions (a) and (b). Then, it is satisfied that $u'(X') = \lambda' H_T$ for some $\lambda' > 0$. Therefore, X' is reverse order of H_T. By the argument in the proof of theorem B.1, the strategy attains the cheapest payoff among random variables whose distribution is same with X'. $\square$

D The Payoff for a Target Joint Distribution with the Investor's Portfolio

Many investors consider the payoff distribution of a fund together with its dependence structure on the returns of their existing portfolio, which is very important in practice because it can crucially affect the risk-return profile. Therefore, let us consider the cheapest payoff generating a target joint distribution with the investor's portfolio. Let ξ be a positive $\mathcal{F}_T$-measurable positive random variable. Denote the conditional distribution functions of ξ and L_T under condition $S_T^1 = s$ by $F_{\xi|s}$ and $F_{L_T|s}$ respectively. Assume that $F_{\xi|s}$ and $F_{L_T|s}$ are continuous and strictly increasing for any $s > 0$. If X is defined as follows, (S_T^1, X) has the same joint distribution with (S_T^1, ξ):

$$X = g(S_T^1, L_T), \tag{D.1}$$

where

$$g(s, l) = F_{\xi|s}^{-1}(F_{L_T|s}(l)).$$

The next theorem asserts that X is the unique cheapest payoff among the random variables whose joint distributions with S_T^1 are same as ξ.

Theorem D.1 *Assume ξ is a positive $\mathcal{F}_T$-measurable random variable, and $F_{\xi|s}$ and $F_{L_T|s}$ are continuous and strictly increasing for any $s > 0$. In a complete market $\mathcal{M}$, the unique cheapest payoff X among the random variables whose joint distributions with S_T^1 are same with ξ is given by equation (D.1).*

Proof. The basic idea is same with the proof of theorem B.1. Let Z be a random variable whose joint distributions with S_T^1 is same with (S_T^1, ξ). Define $\mathcal{S}_Z \subset \mathbb{R}_+$ and $B_Z^1 \subset \Omega$ as

$$\mathcal{S}_Z = \{s : Z \text{ is not reverse order of } H_T \text{ under the condition } S_T^1 = s\},$$

$$B_Z^1 = \{\omega : S_T^1(\omega) \in \mathcal{S}_Z\}.$$

Assume $\mathbb{P}(B_Z^1) > 0$. For any $s \in \mathcal{S}_Z$, there exist $a_1^{s-} < a_1^{s+} \leq a_2^{s-} < a_2^{s+}$ such that

$$H_T(\omega_1) < H_T(\omega_2) \quad \text{for any } (\omega_1, \omega_2) \in A_1^s \times A_2^s, \tag{D.2}$$

where $A_1^s = \{S_T^1 = s, a_1^{s-} < Z < a_1^{s+}\}$, $A_2^s = \{S_T^1 = s, a_2^{s-} < Z < a_2^{s+}\}$, and $\mathbb{P}(A_1^s | S_T^1 = s) = \mathbb{P}(A_2^s | S_T^1 = s) > 0$. Let $p_i^{s\pm} = \mathbb{P}(Z \leq a_i^{s\pm} | S_T^1 = s)$, $I_1^s = (p_1^{s-}, p_1^{s+}]$ and $I_2^s = (p_2^{s-}, p_2^{s+}]$. Introduce random variable Z' defined by

$$Z' = \begin{cases} G^1(S_T^1, F_{\xi|S_T^1}(Z)) & \text{on } B_Z^1, \\ Z \quad \text{on } \Omega \setminus B_Z^1, \end{cases}$$

where G^1 is defined as

$$G^1(s,p) = \begin{cases} F_{\xi|s}^{-1}(p) & \text{for } p \in (0,1) \setminus (I_1^s \cup I_2^s), \\ F_{\xi|s}^{-1}(p + p_2^{s-} - p_1^{s-}) & \text{for } p \in I_1^s, \\ F_{\xi|s}^{-1}(p + p_1^{s-} - p_2^{s-}) & \text{for } p \in I_2^s. \end{cases}$$

Here, note that $p_1^{s+} - p_1^{s-} = p_2^{s+} - p_2^{s-}$. Then, (S_T^1, Z') has the same distribution with (S_T^1, Z), and Z' is cheaper than Z, because of the same argument in the proof of theorem B.1. Therefore, in order to obtain the cheapest payoff whose joint distribution with S_T^1 is same with (S_T^1, ξ), the terminal wealth should be the reverse order of H_T under the condition that S_T^1 is known. Therefore, X defined by equation (D.1) is the cheapest payoff.

Next, let us prove the uniqueness. Suppose that X' is a random variable whose joint distribution with S_T^1 is same with (S_T^1, ξ), and is the reverse order of H_T under the condition that S_T^1 is known. Then, $\mathbb{P}(B_{X'}^1) = 0$. For $s \in \mathbb{R}_+ \setminus \mathcal{S}_Z$ and $\omega^* \in \{S_T^1 = s\}$, define $U_{X'}^s(\omega^*)$ as

$$U_{X'}^s(\omega^*) = \{\omega : S_T(\omega) = s, H_T(\omega) < H_T(\omega^*), \text{ and } X'(\omega) < X'(\omega^*)\},$$

and let

$$V_{X'}^s = \cup_{\omega^* \in \{S_T^1 = s\}} U_{X'}^s(\omega^*),$$

$$C_{X'} = (\Omega \setminus B^1_{X'}) \setminus (\cup_{s \in \mathcal{S}_{X'}} V^s_{X'}).$$

Then, $\mathbb{P}(V^s_{X'} | S^1_T = s) = 0$. Since it is satisfied that

$$\mathbb{P}(\cup_{s \in \mathcal{S}_{X'}} V^s_{X'}) = \int_{\mathcal{S}_{X'}} \mathbb{P}(V^s_{X'} | S^1_T = s) F_{S^1_T}(ds) = 0,$$

$\mathbb{P}(C_{X'}) = 1$. For any $s \in \mathbb{R}_+ \setminus \mathcal{S}_Z$ and $\omega^* \in C_{X'}$,

$$\mathbb{P}\{\omega : L_T(\omega) \leq L_T(\omega^*) | S^1_T = s\} = \mathbb{P}\{\omega : X'(\omega) \leq X'(\omega^*) | S^1_T = s\}.$$

Therefore,

$$F_{L_T|s}(L_T(\omega^*)) = F_{\xi|s}(X'(\omega^*)).$$

By operating $F^{-1}_{\xi|s}$ on both of the sides of the equation,

$$F^{-1}_{\xi|s}(F_{L_T|s}(L_T(\omega^*))) = X'(\omega^*).$$

Hence, $X' = X$. $\square$

E Derivation of the Dynamic Replicating Portfolios

For the case of the Markovian coefficients, the concrete expression for the dynamic replicating portfolios can be obtained. Suppose that a k-dimensional state variable Y_t follows SDE

$$dY_t = \mu^Y(Y_t)dt + \sigma^Y(Y_t)dW_t \tag{E.1}$$

and r_t, μ_t, and σ_t can be described by differentiable functions of state variable Y_t: $r_t = r(Y_t)$, $\mu_t = \mu(Y_t)$, and $\sigma_t = \sigma(Y_t)$. The following proposition gives the dynamic portfolios generating the payoffs.

Proposition E.1 *Assume that r, μ and σ are functions of Y_t following SDE (E.1). Then, in a complete market $\mathcal{M}$, the dynamic portfolio generating payoff $f(L_T)$ is given by*

$$\pi^M_t = \sigma'(t)^{-1}\phi^M_t, \tag{E.2}$$

where

$$\phi^M_t = X_t\theta_t + \frac{1}{H_t}E[H_T\{f'(L_T) - X_T\}D_{L_T}|\mathcal{F}_t]. \tag{E.3}$$

D_{L_T} is given by

$$
\begin{aligned}
D_{L_T} = \quad & \int_t^T \partial r(Y_s) A_{t,s} \sigma^Y(Y_t) ds + \int_t^T \sum_{j=1}^n \theta^j(Y_s) \partial\theta^j(Y_s) A_{t,s} \sigma^Y(Y_t) ds \\
& + \int_t^T \sum_{j=1}^n \partial\theta(Y_s) A_{t,s} \sigma^Y(Y_t) dW_s^j + \theta(Y_t)',
\end{aligned}
$$

where $A_{t,s}$ is a $k \times k$-valued unique solution of the SDE

$$
dA_{t,s} = \sum_{i=1}^n \partial\sigma_i^Y(Y_s) A_{t,s} dW_s^i
$$

with an initial condition $A_{t,t} = I$. $\sigma_i^Y(\cdot)$ and I denotes i-th row of matrix $\sigma^Y(\cdot)$ and the $k \times k$ identity matrix, respectively.

The portfolio that attains payoff $g(S_T^1, L_T)$ is given by

$$
\pi_t^J = \sigma'(t)^{-1} \phi_t^J, \tag{E.4}
$$

where

$$
\phi_t^J = X_t\theta_t + \frac{1}{H_t} E[H_T g_1(S_T^1, L_T) D_{S_T^1} + H_T\{g_2(S_T^1, L_T) - X_T\} D_{L_T} | \mathcal{F}_t]. \tag{E.5}
$$

g_i ($i = 1, 2$) represents the partial derivative of g with respect to the i-th argument. $D_{S_T^1}$ is given by

$$
\begin{aligned}
D_{S_T^1} = \quad & S_T^1 \int_t^T \{\partial\mu^1(Y_s) - \sigma^{11}(Y_s)\partial\sigma^{11}(Y_u)\} A_{t,s} \sigma^Y(Y_t) ds \\
+ \quad & S_T^1 \int_t^T \partial\sigma^{11}(Y_s) A_{t,s} \sigma^Y(Y_t) dW_s^1 \\
+ \quad & (S_T^1 \sigma^{11}(Y_t), 0, \cdots, 0).
\end{aligned}
$$

Proof. By the argument in Karatzas and Shreve (1998), π_t is described as

$$
\pi_t = \sigma_t'^{-1} \left(X_t\theta_t + \frac{\psi_t}{H_t} \right), \tag{E.6}
$$

where ψ_t is given by the martingale representation:

$$
M_t = E[H_T X_T | \mathcal{F}_t] = x + \int_0^t \psi_u' dW_u.
$$

Applying Clark-Ocone formula (See, for example, pp. 46 in Nualart (2006)), ψ_t is given by

$$\psi_t = E[\mathcal{D}_t \eta | \mathcal{F}_t], \tag{E.7}$$

where $\eta = H_T X_T$ and $\mathcal{D}_t$ denotes Malliavin derivative: $\mathcal{D}_t \eta = (\mathcal{D}_{1t}\eta, \cdots, \mathcal{D}_{nt}\eta)$.

To generate the marginal payoff distribution, $X_T = f(L_T)$. Then, $\mathcal{D}_t \eta$ can be calculated as

$$\mathcal{D}_t \eta = (\mathcal{D}_t H_T) X_T + H_T f'(L_T) \mathcal{D}_t L_T. \tag{E.8}$$

For generating the joint distribution with the investor's portfolio, $X_T = g(S_T^1, L_T)$. Then, $\mathcal{D}_t \eta$ can be calculated as

$$\mathcal{D}_t \eta = (\mathcal{D}_t H_T) X_T + H_T g_1(S_T^1, L_T) \mathcal{D}_t S_T^1 + H_T g_2(S_T^1, L_T) \mathcal{D}_t L_T, \tag{E.9}$$

where g_i $(i = 1, 2)$ represents the partial derivative of g with respect to i-th argument.

For generating the marginal distribution, to obtain ψ_t, it is necessary to calculate $\mathcal{D}_t H_T$ and $\mathcal{D}_t L_T$. To obtain the joint distribution with the investor 's portfolio, the calculation of $\mathcal{D}_t S_T^1$ is needed. $\mathcal{D}_t H_T$, $\mathcal{D}_t L_T$ and $\mathcal{D}_t S_T^1$ are calculated as

$$\mathcal{D}_t H_T = -H_T \mathcal{D}_t L_T,$$

$$\begin{aligned}
\mathcal{D}_t L_T = \quad & \int_t^T \partial r(Y_s) \mathcal{D}_t Y_s ds + \int_t^T \sum_{j=1}^n \theta^j(Y_s) \partial \theta^j(Y_s) \mathcal{D}_t Y_s ds \\
& + \int_t^T \sum_{j=1}^n \partial \theta(Y_s) \mathcal{D}_t Y_s dW_s^j + \theta(Y_t)',
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_t S_T^1 = \quad & S_T^1 \int_t^T \{\partial \mu^1(Y_s) - \sigma^{11}(Y_s) \partial \sigma^{11}(Y_u)\} \mathcal{D}_t Y_s ds \\
+ \quad & S_T^1 \int_t^T \partial \sigma^{11}(Y_s) \mathcal{D}_t Y_s dW_s^1 \\
+ \quad & (S_T^1 \sigma^{11}(Y_t), 0, \cdots, 0).
\end{aligned}$$

In this case, $\mathcal{D}_t Y_s$ can be calculated concretely. Let $A_{t,s}$ be a $k \times k$-valued unique solution of the SDE

$$dA_{t,s} = \sum_{i=1}^n \partial \sigma_i^Y(Y_s) A_{t,s} dW_s^i$$

with initial condition $A_{t,t} = I$, where $\sigma_i^Y(\cdot)$ denotes i-th row of matrix $\sigma^Y(\cdot)$, and I is the $k \times k$ identity matrix. $\mathcal{D}_t Y_s$ is represented as

$$\mathcal{D}_t Y_s = A_{t,s}\sigma^Y(Y_t). \tag{E.10}$$

(See, for example, pp. 126 in Nualart (2006).) Therefore, the proposition is obtained. □

Proposition 1 in the main text can be directly derived from this proposition.

F Simulation Analysis

This section investigates the dynamics of portfolios and their realized return distributions through simulation analyses. The target joint distributions with the investor's existing portfolio are generated by dynamic trading of an investor's existing portfolio S^1 and four assets $S^2, \cdots, S^5$. In this section, the coefficients for the stochastic process of the investor's portfolio and the replicating tools are assumed as constants. Let us set the parameters to the estimated values for the assets used in the main text. $r = 0.02$, $\mu = (0.0668, 0.0304, 0.1644, 0.1405, -0.0424)'$, and

$$\sigma = \begin{pmatrix} 0.1221 & & & & \\ 0.0134 & 0.1011 & & O & \\ 0.0506 & 0.0004 & 0.3778 & & \\ 0.0189 & -0.04864 & 0.0444 & 0.1727 & \\ 0.0592 & 0.0373 & 0.0108 & -0.0154 & 0.1976 \end{pmatrix}.$$

Then, $\theta = (0.3836, 0.0524, 0.3309, 0.5856, -0.4132)'$. Note that the market price of risk for the investor's portfolio is positive and relatively high. Suppose that the initial wealth of the investor is 1. If the initial cost for the trading strategy is less (more) than 1, then the target payoff can be realized by a lower (higher) cost and so the remaining (shortage of) money is invested (funded) in the risk-free asset.

The target joint distributions are generated by the simulated path of $S^1, \cdots, S^5$ for 300 months. The portfolio is rebalanced on a daily basis. In the following, let us consider all the combinations of the two marginal distributions and three dependence structures with the investor's existing portfolio. The benchmark distribution is Gaussian. The Johnson distribution can create skewness and fat-tails. As for dependencies, the benchmark is the Gaussian copula. The Clayton and Gumbel copulas can generate asymmetric dependencies. Moreover, different parameter sets are implemented in a combination of a distribution and a copula.

First, we generate the payoff, X, such that $(\log S_T^1, \log X)$ are bivariate Gaussian distributions, as benchmark cases. The target mean and standard deviation of log return are 1.5% and 5.77%, respectively. Then, the annual log return is normally distributed with a mean of 18% and standard deviation of 20%. As for the dependence structure with the investor's portfolio, consider the following three correlations: $\rho = -0.75$, 0, and 0.75.

Table 1 and Figures 1-3 confirm that the bivariate Gaussian distributions are successfully realized. Note that the statistics for the realized returns in table 1 are not those for the realized sample of $\log X$, but those for realized log return with an initial cost of 1. Therefore, the realized mean returns are affected by the initial costs. The mean return increases in correlation because the initial cost decreases in correlation. The weights on W_t^1 (which represents the investor's portfolio risk factor) are 0 and negative for the cases of $\rho = 0$ and -0.75, respectively, which means the zero and short positions for the risk factor that brings a positive expected return. Thus, the initial cost becomes high. The right-hand side figures that exhibit weights on the assets show that the strategy takes the short, zero, and long positions of S^1 for the case of $\rho = -0.75$, 0, and 0.75, respectively. Since the dynamic portfolio generates the log-normal return with the log-normal process, the portfolio weights do not change with the passage of time.

Secondly, generate the log returns that have skewness and fat-tails. Consider the Johnson unbounded distribution to model higher moments. The distribution function is given by

$$F(a) = N\left(\alpha + \beta h\Big(\frac{a-\gamma}{\delta}\Big)\right),$$

where α and β are shape parameters, γ and δ are location and scale parameters respectively, $N(\cdot)$ is the distribution function of the standard normal distribution, and $h(\cdot)$ is the following function:

$$h(z) = \log\left(z + \sqrt{z^2+1}\right).$$

Consider the following two parameter sets for $(\alpha,\ \beta,\ \gamma,\ \delta)$: p_1=(1.2, 1.475, 0.0686, 0.047) and p_2=(-1.2, 1.475, -0.0386, 0.047). For both cases, the annual mean logarithmic return and volatility are 18% and 20%, and the excess kurtosis is 10. For the cases of p_1 and p_2, the skews are -2 and 2, respectively. The dependence structure with the investor's portfolio is a Gaussian copula with the correlation $\rho = 0$, to enable comparison with the benchmark case.

From Figures 4 and 5, the joint distributions seem to be generally realized by the dynamic trading strategies. However, tail events heavily depend on the sample. Some negative tail events for Johnson(p_1) were successfully generated, but positive tail sample for Johnson(p_2) did not realize. Then, the skewness and kurtosis for Johnson(p_2) were smaller than the target, which can be confirmed by table 1. Note that the mean and standard deviation are also crucially affected by the tail sample. If even one positive tail event occurred, the statistics came closer to the target. Let us examine the dynamic optimal portfolio. For the case of Johnson(p_1), the portfolio is leveraged to realize a negative tail return if the process L_t starts to decline. Conversely, the portfolio is scaled down, when L_t increases. On the other hand, for Johnson(p_2) case, the opposite strategy is taken. The portfolio is scaled up if the process L_t starts to go up, and it is scaled down when L_t goes down. Through these operations, skewness and fat-tails are generated. It is also seen that the weights in the risky asset portfolio do not change with the passage of time.

Next, we generate asymmetric dependences with investor's portfolio. To this end, let us consider the Clayton and Gumbel copulas. The copula functions for Clayton and Gumbel are given by

$$C_{Clayton}(u_1, u_2) = (u_1^{-\alpha} + u_2^{-\alpha})^{-1/\alpha},$$

$$C_{Gumbel}(u_1, u_2) = \exp[-\{(-\log u_1)^\beta + (-\log u_2)^\beta\}^{1/\beta}],$$

respectively. The parameters are set to $\alpha = 6$ and $\beta = 4$. The Clayton copula has more dependence in the lower tail, and the Gumbel copula has more dependence in the upper tail. The target marginal distribution is $N(0.015, 0.0577^2)$.

Figures 6 and 7 confirm that the two dependence structures are successfully achieved. Let us observe the dynamics of the portfolio for the Clayton copula. The figures on the right show that the weight on S^1 continues to be high in the lower tail event to realize strong dependence, whereas it is reduced in the upper tail event to decrease the dependency. The dynamics of the portfolio for the Gumbel copula have the opposite character to those for the Clayton copula case. The weight on S^1 increases in the upper tail event to exhibit a strong dependence. It decreases in the lower tail event to show independence. The mean returns for these dependences are lower than the Gaussian copula with $\rho = 0$ and 0.75 cases. This is because the strategies for the Clayton and Gumbel copulas realize the lower and the upper tails, respectively, by heavily weighing on W_t^1 in order to generate the asymmetric dependences. This restricts the exposure to the other risk factors. On the other tails, the strategies reduce the weights on W_t^1. Because of the restrictions on the accessibility to the risk factors, the initial costs are relatively high.

Finally, Figures 8-11 show that the combinations of the Johnson distribution and the Clayton or Gumbel copula can also be generated. Then, both of the marginal distributions of the log returns and dependence structures are asymmetric. For the case of Johnson(p_1) and Clayton copula, the target marginal distribution has a negative skewness, and the dependence is stronger in the lower tail. To realize this character, when the investor's portfolio starts to decline, the weight on S^1 increases with high leverage. Conversely, for the Johnson(p_2) and Gumbel copula case, the target distribution has a positive skewness, and the dependence is stronger in the upper tail. Therefore, if the investor's portfolio goes up, the weight on S^1 increases with high leverage. For the other cases, the fatter tails of the target marginal distributions are lowly dependent on the investor's portfolio. Hence, the weights on S^1 are not so high.

In summary, it was confirmed that this method could generate various distributions and dependencies. To generate skewness or fat-tails, the total volume of the risky asset portfolio is adjusted accordingly. The weight on S^1 affects the dependence structure with the investor's existing portfolio. This method gives us a recipe to attain a target distribution and dependency.

References

Dybvig, P. (1988). Distributional Analysis of Portfolio Choice. *Journal of Business*, Vol. 61, pp. 369-393.

Karatzas, I. and S. Shreve (1998). *Methods of Mathematical Finance*, Springer.

Nualart, D. (2006). *The Malliavin Calculus and Related Topics 2nd Edition*, Springer.

Rogers, L.C.G. (2009). Optimal and Robust Contracts for a Risk-constrained Principal. *Mathematics and Financial Economics*, Vol. 2(3), pp. 151-188.

Target Distribution		Normal	Normal	Normal	Johnson(p1)	Johnson(p2)	Normal
Target Dependence		Gaussian(-0.75)	Gaussian(0)	Gaussian(0.75)	Gaussian(0)	Gaussian(0)	Clayton(6)
Mean	Target	1.50%	1.50%	1.50%	1.50%	1.50%	1.50%
	Realized	0.85%	1.44%	1.91%	1.40%	1.24%	0.93%
Std. Dev.	Target	5.77%	5.77%	5.77%	5.77%	5.77%	5.77%
	Realized	5.61%	5.67%	5.89%	5.83%	5.45%	6.01%
Skew	Target	0	0	0	-2	2	-0.02
	Realized	-0.04	-0.09	-0.07	-2.09	1.47	-0.02
Excess Kurtosis	Target	0	0	0	10	10	0.45
	Realized	-0.11	0.25	-0.30	8.00	4.51	0.45
Correlation with Investor's Portfolio	Target	-0.75	0	0.75	0	0	
	Realized	-0.74	0.05	0.79	-0.03	-0.03	0.90

Target Distribution		Normal	Johnson(p1)	Johnson(p2)	Johnson(p1)	Johnson(p2)
Target Dependence		Gumbel(4)	Clayton(6)	Clayton(6)	Gumbel(4)	Gumbel(4)
Mean	Target	1.50%	1.50%	1.50%	1.50%	1.50%
	Realized	1.19%	0.76%	1.12%	1.29%	1.42%
Std. Dev.	Target	5.77%	5.77%	5.77%	5.77%	5.77%
	Realized	5.70%	6.34%	6.29%	5.35%	6.23%
Skew	Target	0	-2	2	-2	2
	Realized	-0.02	-1.99	1.70	-1.48	1.88
Excess Kurtosis	Target	0	10	10	10	10
	Realized	-0.22	7.41	5.79	4.12	5.63
Correlation with Investor's Portfolio	Target					
	Realized	0.93	0.87	0.83	0.88	0.88

Table 1: **Statistics of the target and generated log returns.**

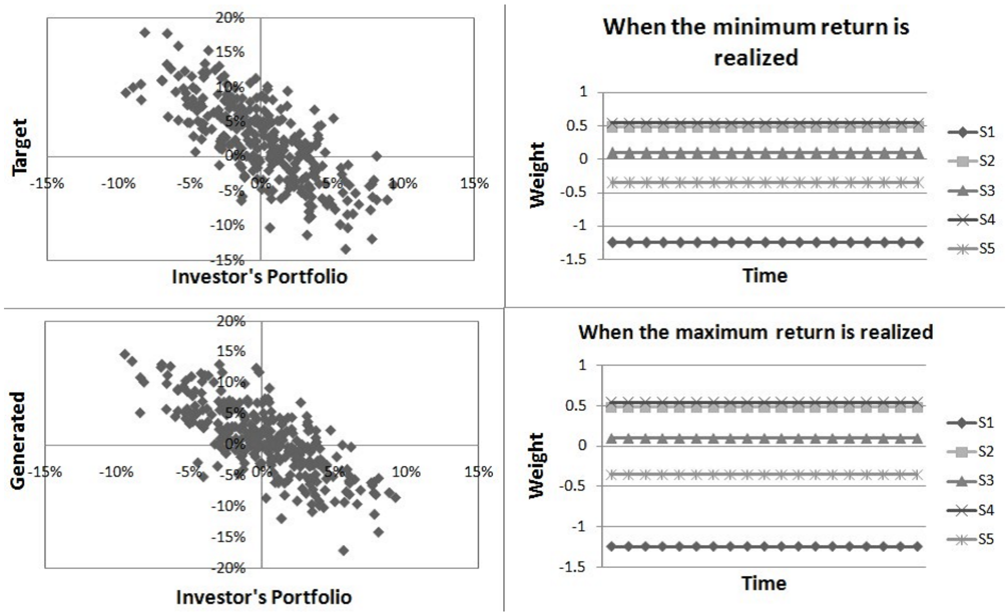

Figure 1: $\mathbf{N(0.015, 0.0577^2)}$ **and Gaussian copula with** $\rho = \mathbf{-0.75}$. Left: Scatter plot of the target and generated log returns against the investor's portfolio. The upper figure represents the sample points extracted from the joint distribution of the target and investor's portfolio log returns. The lower figure plots the generated log returns by the optimal dynamic trading strategy against log return of the investor's portfolio. Right: Weights of the dynamic optimal portfolio for each of the situation described on the top of the figure. The composition is same for Figures 2-11.

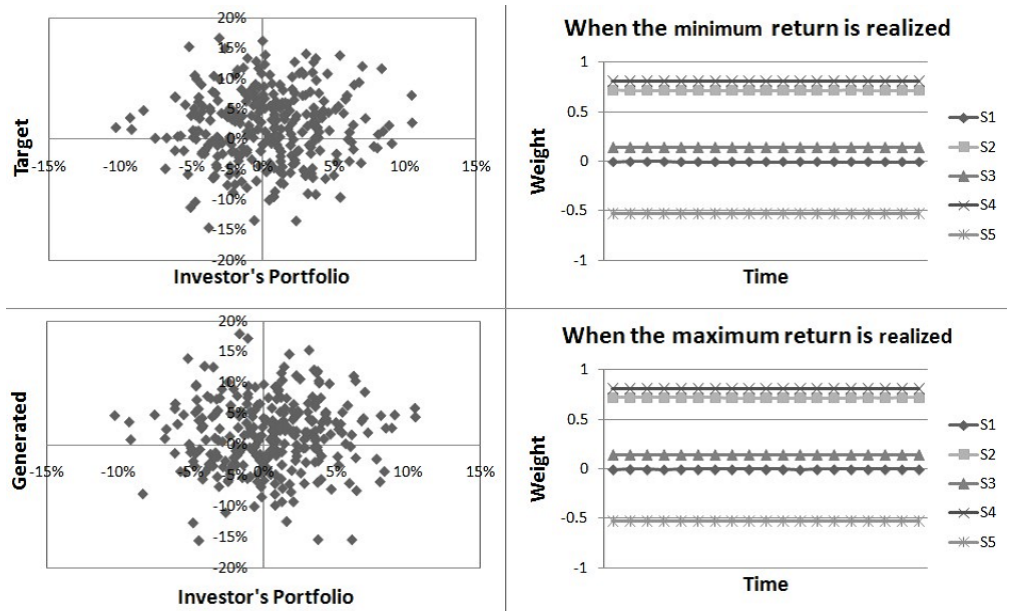

Figure 2: $\mathbf{N(0.015, 0.0577^2)}$ **and Gaussian copula with** $\rho = \mathbf{0}$.

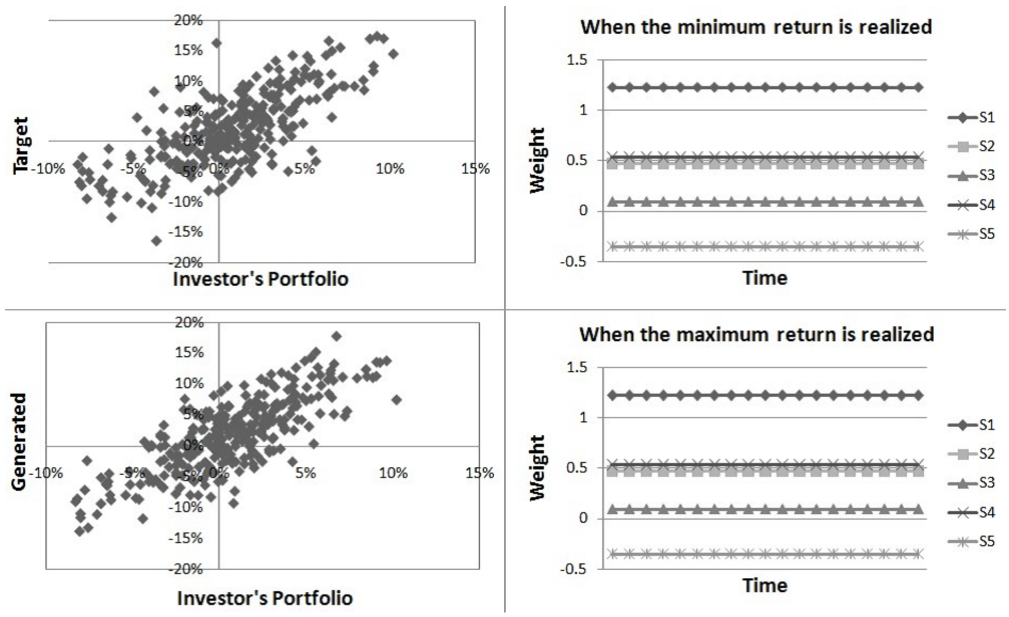

Figure 3: $\mathbf{N(0.015, 0.0577^2)}$ **and Gaussian copula with** $\rho = \mathbf{0.75}$.

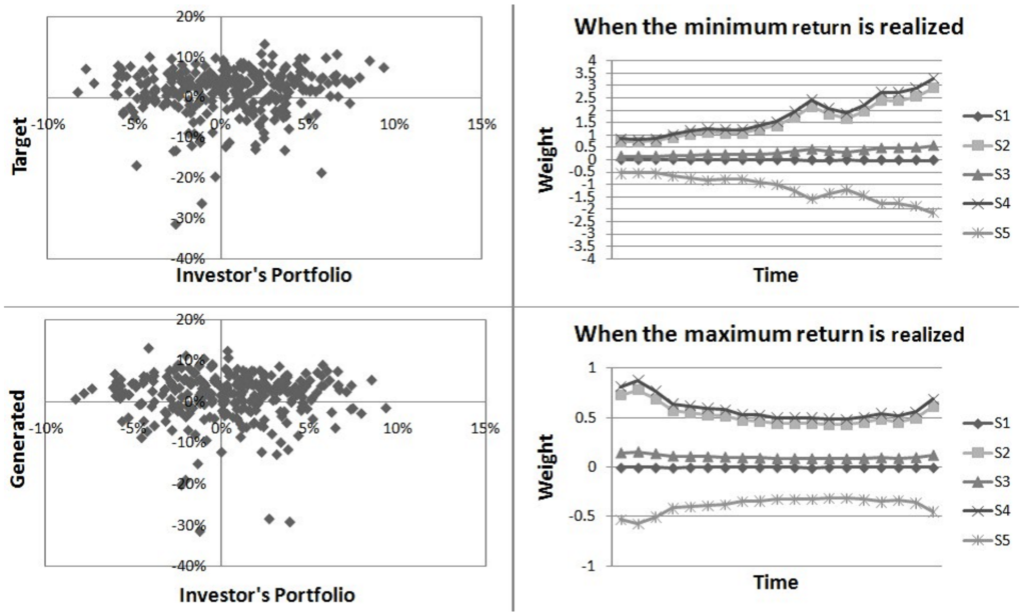

Figure 4: **Johnson($\mathbf{p_1}$) and Gaussian copula with $\rho = \mathbf{0}$.**

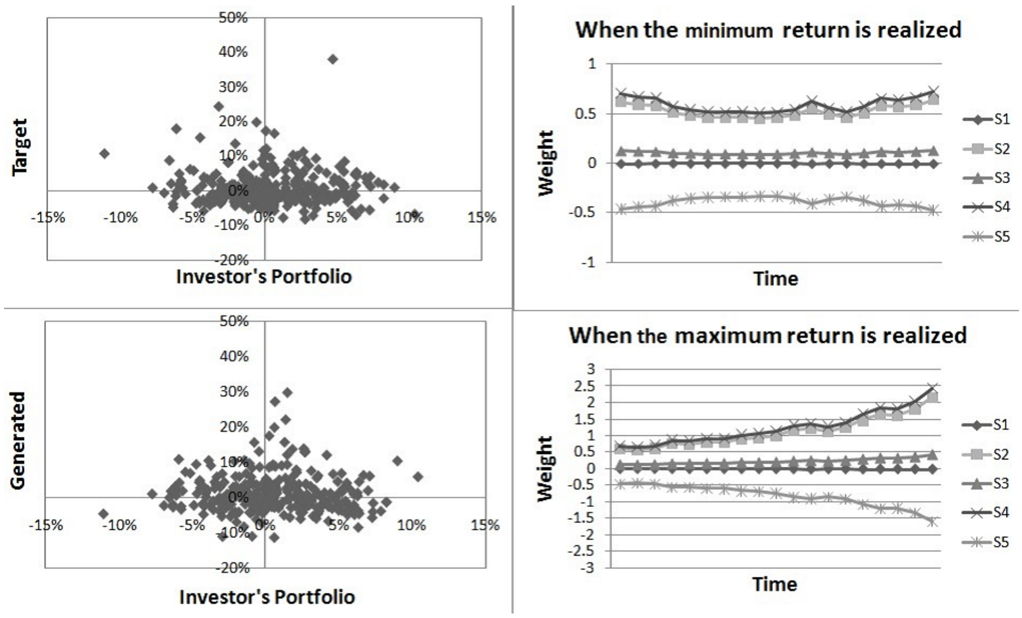

Figure 5: **Johnson($\mathbf{p_2}$) and Gaussian copula with $\rho = \mathbf{0}$.**

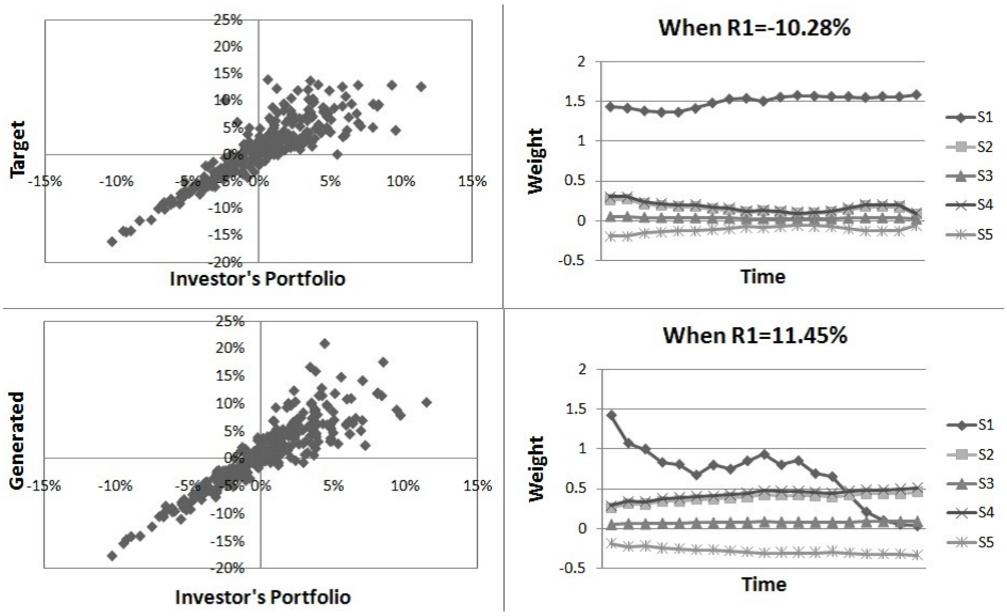

Figure 6: $\mathbf{N(0.015, 0.0577^2)}$ **and Clayton copula with** $\alpha = \mathbf{6}$.

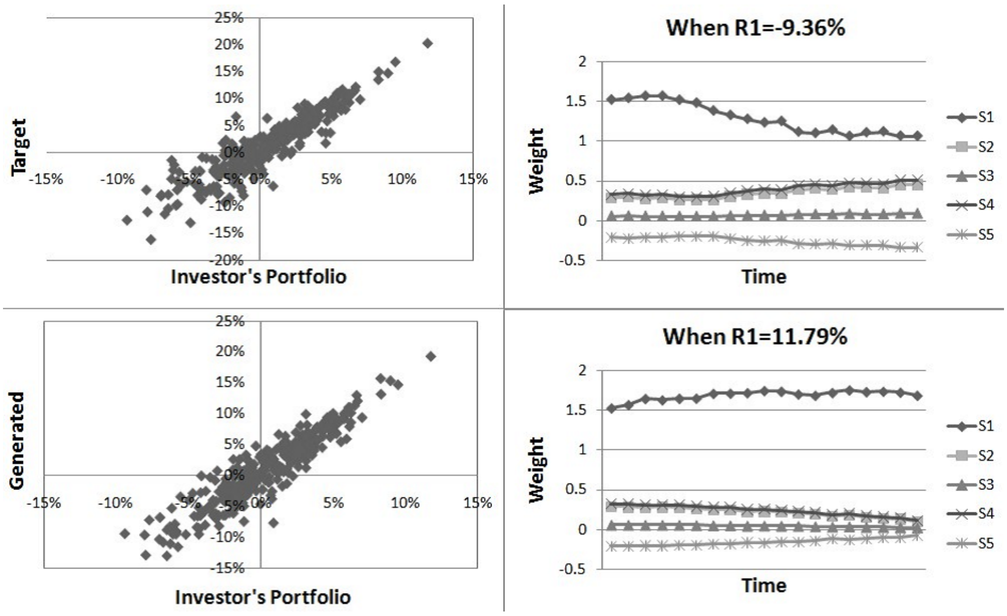

Figure 7: $\mathbf{N(0.015, 0.0577^2)}$ **and Gumbel copula with** $\beta = \mathbf{4}$.

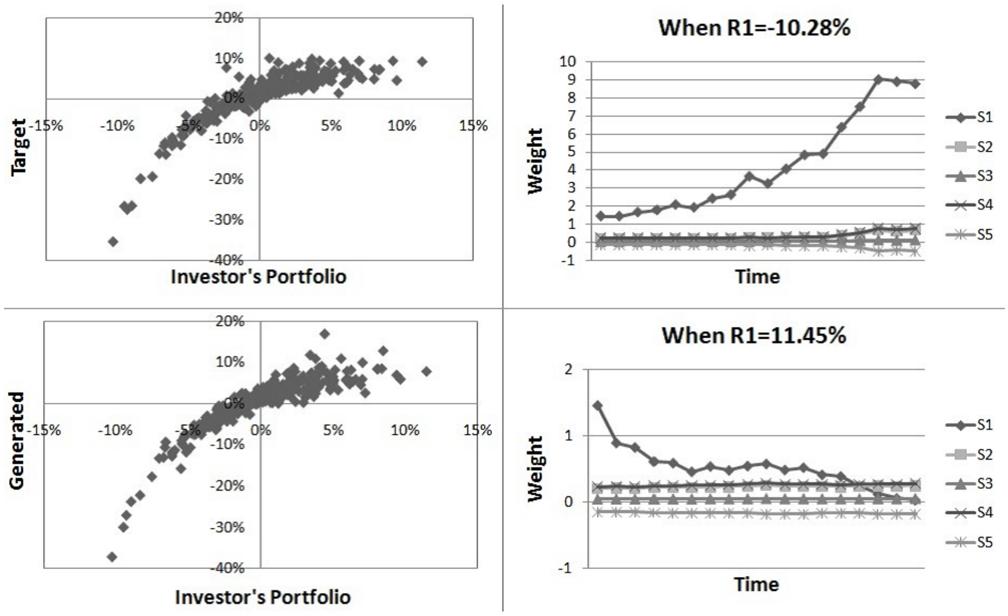

Figure 8: **Johnson($\mathbf{p_1}$) and Clayton copula with $\alpha = \mathbf{6}$.**

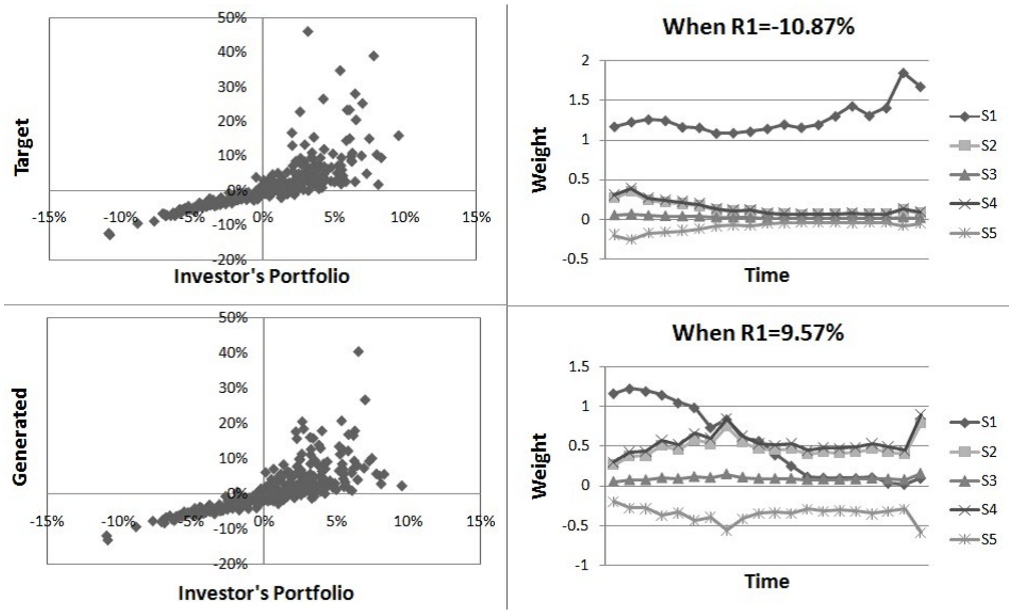

Figure 9: **Johnson($\mathbf{p_2}$) and Clayton copula with $\alpha = \mathbf{6}$.**

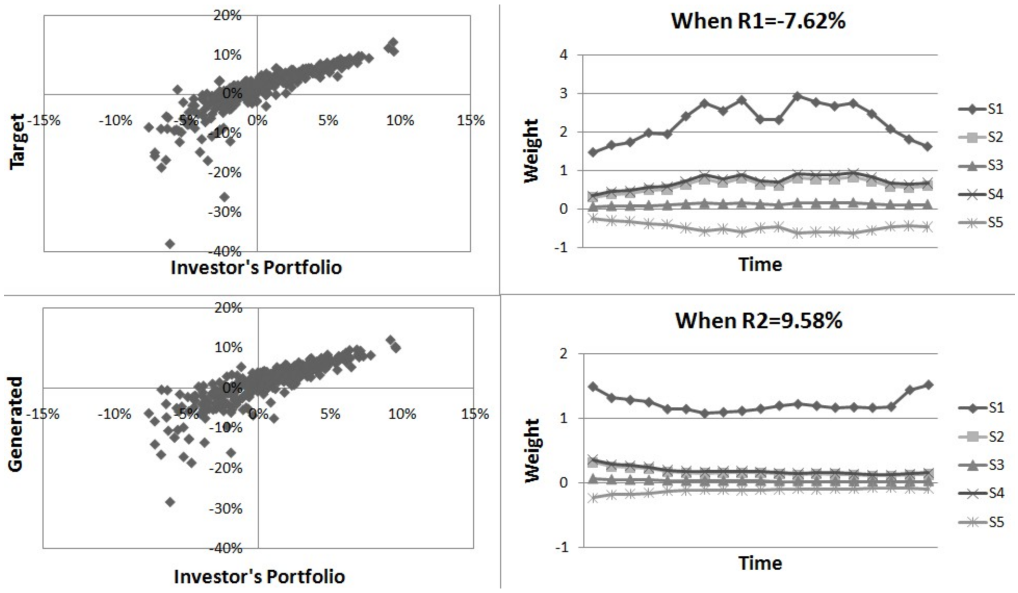

Figure 10: **Johnson($\mathbf{p_1}$) and Gumbel copula with $\beta = 4$.**

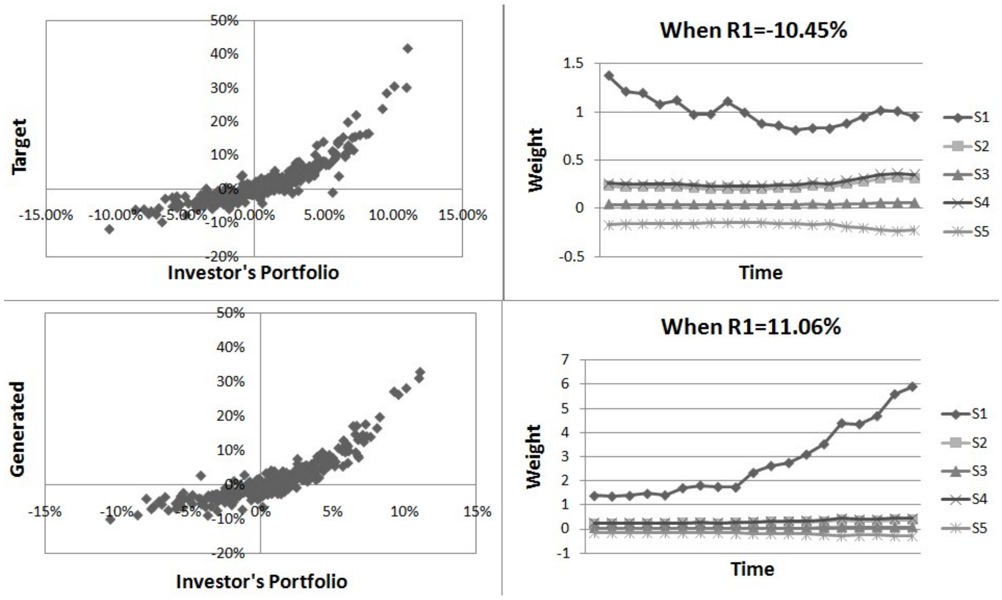

Figure 11: **Johnson($\mathbf{p_1}$) and Gumbel copula with $\beta = 4$.**