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Abstract

A conventional wisdom in economics is that a model dealing frictionless markets with a large number of agents always yields a Walrasian outcome. In this paper we assess the above argument in a dynamic framework by modeling centralized auction markets, and show that in such markets the outcomes are not necessarily Walrasian; the set of stationary equilibria in our model is a continuum which includes the Walrasian equilibrium. Moreover, we also build a model on decentralized auction markets and obtain similar results.

Keywords: Dynamic General Equilibrium, Auction, Walrasian Market, Real Indeterminacy of Stationary Equilibria

Journal of Economic Literature Classification Number: C72, C78, D44, D51, D83, E40

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To an economic theorist, all competitive markets are the same. Regardless of their institutional differences, we use the Walrasian equilibrium. (...) The conventional wisdom can then be summed up in the conclusion that frictionless markets are Walrasian. Since the Walrasian theory itself has nothing to say on this subject, it remains an interesting and open question whether all frictionless markets are indeed Walrasian. (Gale [5])

1 Introduction

A conventional wisdom in economics is that a model dealing a large number of agents in frictionless markets always yields a Walrasian outcome. A frictionless market is defined as a market in which there is no transaction cost, no informational asymmetries, and so forth. It seems that many economists have shared this view, and only a few have challenged to provide a theoretical evidence against it. Rubinstein and Wolinsky [23], among others, were the first to refute this conventional wisdom by offering a random matching model, where the matched agents bargain over terms of trade. They argue that in such markets, the outcome is not necessarily Walrasian. (See also Rubinstein and Wolinsky [24].) On the other hand, Gale [5] [6] [7], and Gale and Sabourian [9] argue against them and confirm the conventional wisdom. They offer other frictionless and decentralized market models in which the equilibrium outcome is Walrasian.

As for dynamic centralized market games, there are few literature on the nature of the equilibrium. Thus, in this paper we attempt to present a theoretical evidence that in some markets the outcome is not necessarily Walrasian. We build a dynamic general equilibrium model on centralized auction markets, and yield the set of stationary equilibria which is a continuum. Our model is essentially dynamic; agents live forever and are involved in either production or consumption in each period. This stands in contrast with the above random matching models. (For the discussion, see Section 6.) Moreover, we also build a model on decentralized auction markets and obtain similar results.

Recently, real indeterminacy of stationary equilibria has been found in both specific and general search models with divisible money. (See, for example, Green and Zhou [11] [12], Kamiya and Shimizu [16], Matsui and Shimizu [20], and Zhou [28].) Although the nature of indeterminate equilibria is very different between search models and Walrasian market models; for example, overlapping generations models have a continuum of equilibria in some cases, but the stationary equilibria are generically determinate. We consider a counterpart of money search models in Walrasian market models is a cash-in-advance model with infinitely lived consumers. In the model, it is known that the stationary equilibrium is determinate. (See, for example, Lucas [18].) Given these arguments, the stationary equilibria in money search models have a quite different feature from those in Walrasian market models. Our question is whether some centralized market models other than Walrasian models have real indeterminacy of stationary equilibria. Our results provide theoretical evidence that real indeterminacy can occur even in a centralized auction market model with divisible money.

Now we turn to describe our model. We build a dynamic general equilibrium model with fiat money and a finite number of perishable goods. For each good, there is a centralized market, where goods are traded by the uniform price auction using fiat money. Each agent can produce a good which she cannot consume but is consumed by other agents. The utility and cost of production are the same for all agents, and therefore there is no informational asymmetry. In each period, she can visit just one centralized market. For example, at period t, if an agent wishes to obtain money, she visits the market of her production good and then in period $t+1$ she buys her consumption good using the money she obtained at t.[1] The conditions for a stationary equilibrium are: (i) each agent maximizes the expected value of utility-streams, (ii) the money holdings distribution of the economy is stationary, i.e., time-invariant, and (iii) the total amount of money the agents have is equal to exogenously given amount of money. We compare the equilibrium with that in the Walrasian market with cash-in-advance constraints, i.e., the case that equilibria are determined at the price that demand is equal to supply and the expenditure of each consumer is constrained by the amount of her money holding. We show that the set of equilibrium allocations with auction markets is a continuum, i.e., indeterminate, while that of the Walrasian market model is a singleton. Therefore the sets of equilibria do not coincide.

The general consensus of indeterminacy is either (i) due to the absence of some important equation, or (ii) equilibria in the real world economy are intrinsically fragile.[2] In this paper, we confine ourselves neither to (i) nor to (ii). That is, in the real world economy, auction markets might have some important feature, which is missing in our model, and it might lead the economy to determinate equilibria. In this paper, we do

[1] Even without this limited participation constraint, we can obtain almost the results. See Section B in Appendix.

[2] The ways are closely related to the Green and Zhou [12]'s statement that assumptions or features of specification that make the difference between a model economy having determinate or indeterminate equilibrium should be regarded as economically crucial.

not investigate the problem and it is an important topic for future research.

The plan of this paper is as follows. In Section 2, we present the common environment in this paper. In Section 3, we investigate a dynamic general equilibrium model with auction markets and show that there is a continuum of stationary equilibria. Then in Section 4, we show that if the trading institution is the Walrasian market with cash-in-advance constraints, the stationary equilibrium is unique. We keep the other environments the same besides the trading institution we introduced in Section 2. In Section 5, we investigate the logic behind the indeterminacy by classifying monetary trades into two types. Section 6 contains discussion of some related literature. Finally, in Appendix we present some related models including a decentralized auction market model.

2 The Environment

In this section, we present the environment common in this paper. Time is discrete denoted by $t = 1, 2, \ldots$. There is a continuum of agents of which measure is one. There are $k \geq 3$ types of agents with equal fractions and the same number of types of goods. Only one unit of indivisible and perishable good i can be produced by a type $i-1$ (mod k) agent with production cost $c > 0$. A type i agent obtains utility $u > 0$ only when she consumes one unit of good i. Let $\theta = u/c$ and assume $\theta > 1$. Let $\gamma > 0$ be the discount factor. There is completely divisible and durable fiat money of which nominal stock is $M > 0$. Finally, the above exogenously given parameters are common knowledge among the agents, and therefore there is no informational asymmetry.

3 Centralized Auction Markets

In each time period, a centralized market is open for each good. At the jth market, good j is traded by the auction we specify below. In each period, each agent has a choice of joining one market or not joining any market. If a type i agent chooses to join the $(i+1)$th market, then she becomes a seller of her product, i.e., good $i+1$. If a type i agent chooses to join the ith market, then she becomes a buyer and bids a price. The feature of this trade is that an agent cannot involve in two different transactions in the same period. However, even if we relax this constraint, we can obtain the similar results. (See Section B in Appendix.)

We consider the uniform price sealed bid auction. (For the auction, see Krishna [17].)

If we consider the case that money holdings distribution is discrete, it is sufficient to analyze just two following cases:

(i) each realized bid is made by a positive measure of agents, and

(ii) only one bid is made by just one agent and the other bids are made by a positive measure of agents.

Note that the strategy is an equilibrium no matter what the specifications in the other cases are.

Let $S \in [0,1]$ be the measure of sellers, and $b_1, b_2, \ldots, b_L$ be the bids made by positive measures of agents, and $B_1, B_2, \ldots, B_L \in [0,1]$ be the corresponding measures. Without loss of generality, we can assume $b_1 > b_2 > \cdots > b_L \geq 0$. Denote $\tilde{B}_\ell = \sum_{i=1}^{\ell} B_i$ and let $\tilde{B}_0 = 0$.

First, we consider the case (i). If $S < \tilde{B}_L$, there exists ℓ such that $S \geq \tilde{B}_{\ell-1}$ and $S < \tilde{B}_\ell$. Then the buyers with b_i, $i < \ell$, obtain the good with probability one. The buyers with b_ℓ obtain the good with probability $\frac{S-\tilde{B}_{\ell-1}}{B_\ell}$. Of course, any seller can sell her good. The uniform price is b_ℓ. If $S \geq \tilde{B}_L$, then all buyers obtain one unit of goods with price b_L. Any seller can sell her good with probability $\frac{\tilde{B}_L}{S}$.

Next, we consider the case (ii). Let $\hat{b}$ be the bid by the single agent, say Buyer 0. Consider that the distribution of the other bids is the same as in the case of (i). Then for the buyers other than Buyer 0, the uniform price and the buyers' possibility of winning is defined as in the case of (i). As for Buyer 0, if $S < \tilde{B}_L$ and $\hat{b} > b_\ell$, she can obtain the good with probability one. If $S < \tilde{B}_L$ and $\hat{b} = b_\ell$, she can obtain the good with probability $\frac{S-\tilde{B}_{\ell-1}}{B_\ell}$. If $S < \tilde{B}_L$ and $\hat{b} < b_\ell$, she cannot obtain the good. Finally, if $S \geq \tilde{B}_L$, she can obtain the good with probability one.

In this paper, we focus on a stationary equilibrium which has the following features. First, money holdings distribution has a support $\{0, p, \ldots, Np\}$, where $p > 0$ is an equilibrium price. Thus the money holdings distribution is expressed as $(h_0, h_1, \ldots, h_N)$, where h_n is the measure of agents with money holding np. Second, equilibrium strategies are Markovian, i.e., we seek for an equilibrium strategy depending only on money holdings. Finally, the actions taken by the agents with identical characteristics are symmetric. Thus a candidate for an equilibrium strategy can be defined as a function of a money holding $\eta \in R_+$,

$$\xi : R_+ \to \{\sigma\} \cup (\{\beta\} \times R_+) \cup \{\nu\},$$

where $\xi(\eta) = \sigma$ implies that the agent chooses to be a seller, $\xi(\eta) = (\beta, b)$ implies that she chooses to be a buyer and her bid price is b, and $\xi(\eta) = \nu$ implies that she does nothing. Thus if there is a finite number of bid prices $(b_1, b_2, \ldots, b_L)$ such that $\sum_{\{n|b_i=\xi(np)\}} h_n > 0$, $i = 1, \ldots L$, then the price of each good is determined by the above rule of the uniform price auction. The price is expressed as a function of (h, ξ), denoted by $\pi(h, \xi)$. Moreover, as for sellers, the probability of selling good is expressed as a function of (h, ξ), denoted by $\nu_S(h, \xi)$. Similarly, for each bid $b \in R_+$, the probability of obtaining good is expressed as a function of (b, h, ξ), denoted by $\nu_B(b, h, \xi)$.

The stationary equilibrium is defined as follows.

Definition 1 $\langle p, h, \xi, V \rangle$, where $V : R_+ \to R$ is a value function, is said to be a *stationary equilibrium with a discrete money holdings distribution* if

- h is stationary under the strategy ξ,
- $p = \pi(h, \xi)$,
- $\sum_{n=0}^{N} pnh_n = M$,
- there is a finite number of bid prices $(b_1, b_2, \ldots, b_L)$ such that $\sum_{\{n|b_i=\xi(np)\}} h_n > 0$, $i = 1, \ldots L$, and
- given h and ξ, the value function V, together with ξ, solves the Bellman equation, i.e., for $\eta \in R_+$,

$$\begin{aligned} V(\eta) = \max \big\{ & \nu_S(h, \xi) \left(-c + \gamma V(\eta + \pi(h, \xi))\right) + (1 - \nu_S(h, \xi))\gamma V(\eta), \\ & \max_{b \in R_+} [\nu_B(b, h, \xi) \left(u + \gamma V(\eta - \pi(h, \xi))\right) + (1 - \nu_B(b, h, \xi))\gamma V(\eta)], \gamma V(\eta) \big\}. \end{aligned}$$

In this section, we focus on stationary equilibria with $N = 1$. We consider the following strategy as a candidate for equilibrium:

- an agent with $\eta \in [0, p)$ chooses to be a seller, and
- an agent with $\eta \in [p, \infty)$ chooses to be a buyer and bids η.

We consider the case of $h_0 \leq \frac{1}{2}$. This implies that the measure of buyers is larger than or equal to that of sellers, and therefore an agent with $\eta \in (0, p)$ could not win even if she had chosen to be a buyer.

The stationarity of money holdings distributions is expressed as follows. Since $\nu_B(p, h, \xi) = \frac{h_0}{1-h_0}$ holds, the measure of agents who can buy is $(1 - h_0)\frac{h_0}{1-h_0}$ and their money holdings become 0. On the other hand, since agents without money can always sell, the measure of agents who can sell is h_0 and their money holdings become p. Thus the stationarity of money holdings at 0 is

$$(1-h_0)\frac{h_0}{1-h_0} = h_0.$$

Similarly, the stationarity of money holdings at p is

$$h_0 = (1-h_0)\frac{h_0}{1-h_0}.$$

Both of them are (the same) identities and thus any (h_0, h_1) satisfying $h_0 + h_1 = 1, h_0 \geq 0$, and $h_1 \geq 0$, is a stationary distribution.

Under the strategy, the value function is expressed as follows:

$$V(\eta) = \begin{cases} -c + \gamma V(\eta + p), & \text{if } \eta < p, \\ r\left(u + \gamma V(0)\right) + (1-r)\gamma V(p), & \text{if } \eta = p, \\ u + \gamma V(\eta - p), & \text{if } \eta > p, \end{cases}$$

where

$$r = \frac{h_0}{1-h_0}.$$

Let n be the integer part of η and ι be the residual, then η is uniquely expressed as $\eta = np + \iota$. Thus the value function is expressed as follows:

$$V(np+\iota) = \begin{cases} \frac{r\gamma u - (1-\gamma+r\gamma)c}{(1-\gamma)(1+r\gamma)}, & \text{if } \iota = 0,\ n = 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^{n-1}}{1+r\gamma}\left[(1-r+r\gamma)u + r\gamma c\right]\right\}, & \text{if } \iota = 0,\ n \neq 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^n}{1+\gamma}(u+c)\right\}, & \text{if } \iota \neq 0. \end{cases} \tag{1}$$

We need to check the following incentive conditions:

(i) incentive for an agent with $\eta \in [0, p)$ to be a seller instead of doing nothing,

(ii) incentive for an agent with $\eta \in [0, p)$ to be a seller instead of being a buyer,

(iii) incentive for an agent with $\eta \in [p, \infty)$ to be a buyer instead of doing nothing,

(iv) incentive for an agent with $\eta \in [p, \infty)$ to be a buyer instead of being a seller, and

(v) incentive for a buyer with $\eta \in [p, \infty)$ to bids η instead of bidding another price.

It is easily verified that (ii) is reduced to (i), and (iii), (v) are automatically satisfied. (i) and (iv) are reduced to the following inequalities respectively:

(I) $V(0) \geq 0$,

(II) $V(p) \geq -c + \gamma V(2p)$.

By (1), the following inequality is equivalent to (I):

$$r \geq \frac{1-\gamma}{\gamma(\theta-1)}. \tag{2}$$

Again, by (1), the following inequality is equivalent to (II):

$$r \geq \frac{\gamma\theta - 1}{(1+\gamma-\gamma^2)\theta - \gamma^2}. \tag{3}$$

In this section, we restrict our attention to the case of $\gamma > \frac{1}{\theta}$. Then the RHSs of (2) and (3) lie in $(0, 1)$. Thus for any r such that

$$r \in \left[\max\left\{\frac{1-\gamma}{\gamma(\theta-1)}, \frac{\gamma\theta-1}{(1+\gamma-\gamma^2)\theta-\gamma^2}\right\}, 1\right], \tag{4}$$

there is the corresponding equilibrium, and moreover the above interval is non-empty. Since $r = \frac{h_0}{1-h_0}$, (4) is equivalent to

$$h_0 \in \left[\max\left\{\frac{1-\gamma}{\gamma(\theta-2)+1}, \frac{\gamma\theta-1}{(1+2\gamma-\gamma^2)\theta-1-\gamma^2}\right\}, \frac{1}{2}\right]. \tag{5}$$

Theorem 1 For any $\gamma \in (\frac{1}{\theta}, 1)$, there exists a stationary equilibrium with a discrete money holdings distribution with $N = 1$ for any h_0 satisfying (5). Moreover, the interval in (5) is non-empty.

4 Walrasian Markets

In this section, we define the concept of stationary Walrasian equilibrium, where a competitive market is open for each good. We hold the same environment as in the previous section. More precisely, (a) each agent maximizes the discounted sum of utility stream for given prices of goods under the budget constraint and the cash-in-advance constraint, (b) the markets of goods clear, i.e., for each good, the measure of sellers is equal to that of buyers, (c) the money demand is equal to supply, and (d) the money holdings distribution and the price of good are stationary, i.e., time-invariant.

Moreover, as in Section 3, we assume each agent can only join just one market in each period.

As in the previous section, we focus on stationary equilibria in which all agents with identical characteristics act similar and in which all of the k types are symmetric. Thus we seek for equilibria, where the prices of goods are the same. Let the price be $p \in R_+$. For a given p, the behavior of an agent with $\eta \in R_+$ is expressed in terms of a Bellman equation as follows:

$$V(\eta) = \max_{(\chi,\zeta)\in C} \chi u - \zeta c + \gamma V(\eta'), \tag{6}$$
$$\text{s.t. } \chi p + \eta' = \eta + \zeta p, \ \chi p \le \eta, \ \eta' \ge 0,$$

where $(\chi, \zeta) = (1, 0)$ when an agent is a buyer, $(\chi, \zeta) = (0, 1)$ when she is a seller, and $(\chi, \zeta) = (0, 0)$ when she does nothing. Since she cannot become a buyer and a seller at the same time, $C = \{(1, 0), (0, 1), (0, 0)\}$. Note that $\chi p \le \eta$ is the cash-in-advance constraint. The above Bellman equation is expressed by

$$V(\eta) = \begin{cases} \max\{u + \gamma V(\eta - p), -c + \gamma V(\eta + p), \gamma V(\eta)\}, & \text{if } \eta - p \ge 0, \\ \max\{-c + \gamma V(\eta + p), \gamma V(\eta)\}, & \text{if } \eta - p < 0. \end{cases}$$

For a given p, the unique value function $V : R_+ \to R$ and the optimal policy correspondence $\phi : R_+ \to \{\beta, \sigma, \nu\}$ are obtained, where β, σ, and ν stand for 'buy', 'sell', and 'do nothing', respectively. Let $\mathcal{B}$ be the Borel σ-algebra on $[0, \infty)$. A transition function $T : R_+ \times \mathcal{B} \to [0, 1]$ is said to be consistent with ϕ if, for any $A \in \mathcal{B}$, $T(\eta, A)$ is positive only if [3]

- $\sigma \in \phi(\eta)$ and $\eta + p \in A$, or
- $\beta \in \phi(\eta)$ and $\eta - p \in A$, or
- $\nu \in \phi(\eta)$ and $\eta \in A$.

Then a stationary Walrasian equilibrium is defined as follows.

Definition 2 $\langle p, F, V, \phi, T\rangle$, where F is a probability measure on $\mathcal{B}$, is said to be a *stationary Walrasian equilibrium* if

[3] A transition function $T : R_+ \times \mathcal{B} \to [0, 1]$ is a function such that

- for each $\eta \in R_+$, $T(\eta, \cdot)$ is a probability measure on $(R_+, \mathcal{B})$, and
- for each $A \in \mathcal{B}$, $T(\cdot, A)$ is a $\mathcal{B}$-measurable function.

For the details, see Stokey and Lucas [27].

(i) $\phi : R_+ \to \{\beta, \sigma, \nu\}$ is the optimal policy correspondence associated with V,

(ii) $T : R_+ \times \mathcal{B} \to [0, 1]$ is consistent with ϕ

(iii) F is stationary under T,

(iv) $\int_{n=0}^{\infty} \eta dF = M$,

(v) given p, V satisfies (6),

(vi) $T(\eta, \{\eta - p\})$ and $T(\eta, \{\eta + p\})$ are measurable functions of η, and

$$\int T(\eta, \{\eta - p\})dF = \int T(\eta, \{\eta + p\})dF$$

holds.

As for (i)-(v), no explanation is needed. (vi) is the market clearing condition for goods; namely, the measure of buyers is equal to that of sellers. Note that, by Walras' law, money demand is equal to money supply if (vi) is satisfied.

Clearly, $p = 0$ is not consistent with the incentive of sellers. Suppose $p > 0$ is an equilibrium price. An agent with $\eta \geq p$ clearly chooses to be a buyer. Thus $\eta \geq 2p$ is no longer a transient state. Similarly, an agent with $\eta < p$ cannot buy and chooses to be a seller if $\gamma V(\eta + p) - c > \gamma V(\eta)$. Thus, if the inequality holds, the market clearing condition implies that the measure of agents with $\eta \in [0, p)$ is equal to that with $\eta \in [p, 2p)$. Then the Bellman equation becomes as follows:

$$V(\eta) = \begin{cases} -c + \gamma V(\eta + p), & \text{if } \eta \in [0, p), \\ u + \gamma V(\eta - p), & \text{otherwise.} \end{cases}$$

The value function is obtained as follows:

$$V(\eta) = \begin{cases} \frac{\gamma u - c}{1 - \gamma^2}, & \text{if } \eta \in [0, p), \\ \frac{u - \gamma c}{1 - \gamma^2}, & \text{if } \eta \in [p, 2p), \\ \vdots & \\ \frac{1}{1-\gamma} \left\{ u - \frac{\gamma^n}{1+\gamma}(u + c) \right\}, & \text{if } \eta \in [np, (n+1)p), \\ \vdots & \end{cases} \tag{7}$$

Note that $\gamma V(\eta + p) - c > \gamma V(\eta)$ holds for $\eta \in [0, p)$ if

$$\gamma > \frac{1}{\theta},$$

is satisfied. It is straightforward to show that the stationary Walrasian equilibrium allocation is achieved when a half of agents have $\eta \in [0, p)$ and their value is $\frac{\gamma u - c}{1-\gamma^2}$, and the other half of agents have $\eta \in [p, 2p)$ and their value is $\frac{u - \gamma c}{1-\gamma^2}$. Note that the money holdings distribution is stationary if and only if $F([0, \eta]) = F([p, p + \eta])$ for any $\eta \in [0, p)$.

Theorem 2 For any $\gamma \in \left(\frac{1}{\theta}, 1\right)$, there exists a stationary Walrasian equilibrium $\langle p, F, V, \phi, T \rangle$. Moreover, any Walrasian equilibrium is characterized by

(I) V is given by (7),

(II)

$$\phi(\eta) = \begin{cases} \{\sigma\} & \text{if } \eta \in [0, p), \\ \{\beta\} & \text{if } \eta \in [p, \infty), \end{cases}$$

(III)

$$T(\eta, A) = 1 \text{ iff } \begin{cases} \eta + p \in A & \text{and} \quad \eta \in [0, p), \quad \text{or} \\ \eta - p \in A & \text{and} \quad \eta \in [p, \infty), \end{cases}$$

(IV) F satisfies $\int \eta dF = M$, $F([0, p)) = 1/2$, $F([p, 2p)) = 1/2$, and

$$F([0, \eta]) = F([p, p + \eta]), \quad \forall \eta \in [0, p).$$

Proof:
From the discussion in the above, for a given p, (V, ϕ, T) are characterized by (I), (II), and (III), and they clearly exist. Thus it is sufficient to show the existence of (F, p) satisfying (IV). Let $p^* = 2M$ and F^* be such that $F^*(\{0\}) = 1/2$ and $F^*(\{p\}) = 1/2$. They clearly satisfy (IV). ∎

In the proof of the above theorem, a probability measure satisfying (IV), F^*, is presented. Of course, there are other probability measures satisfying (IV). For example, $\tilde{p} = \frac{4M}{3}$ and $\tilde{F}$ such that $\tilde{F}(\{0\}) = \tilde{F}(\{\frac{1}{2}\tilde{p}\}) = \tilde{F}(\{\tilde{p}\}) = \tilde{F}(\{\frac{3}{2}\tilde{p}\}) = \frac{1}{4}$ satisfy (IV). In the equilibrium with $(\tilde{p}, \tilde{F})$, an agent with $\frac{1}{2}\tilde{p}$ deterministically alternates between acquiring $\tilde{p}$ as a seller and spending $\tilde{p}$ as a buyer, i.e., she alternates between states $\frac{1}{2}\tilde{p}$ and $\frac{3}{2}\tilde{p}$. Thus the behavior of an agent with $\frac{1}{2}\tilde{p}$ is exactly the same as that of an agent without money. Moreover the real allocation and transactions of goods on the equilibrium are completely the same as those on the equilibrium with (p^*, F^*). It is quite natural for one to think there is just one equilibrium.

Corollary 1 The distribution of value in stationary Walrasian equilibria is uniquely determined, i.e., the half of agents have $\frac{\gamma u-c}{1-\gamma^2}$, the case of $\eta \in [0,p)$, and the rest of agents have $\frac{u-\gamma c}{1-\gamma^2}$, the case of $\eta \in [p, 2p)$.

Thus in order to investigate real allocations, we can restrict our attention to the stationary Walrasian equilibrium with (p^*, F^*). [4]

Now we have described each market, we turn to compare the auction markets outcome derived in the previous section as versus stationary Walrasian equilibrium outcome derived in this section. Setting $h_0 = 1/2$ in the value function (1), we obtain $V(0) = \frac{\gamma u-c}{1-\gamma^2}$ and $V(p) = \frac{u-\gamma c}{1-\gamma^2}$. In this case, a half of agents have the former value and the rest of the agents have the latter value, i.e., exactly same as the case of the Walrasian market outcome in Corollary 1. As shown in Theorem 1, there are other stationary equilibria for $h_0 < \frac{1}{2}$ in the auction markets, where the values are different from those in Corollary 1.

Theorem 3 The set of outcomes in the auction market equilibrium does not coincide with that of the stationary Walrasian equilibrium.

5 Real Indeterminacy of Stationary Equilibria in Monetary Models

In this section, we explore the logic behind Theorem 3. As shown in the previous sections, the auction markets have a continuum of stationary equilibrium allocations, while the Walrasian markets have the unique equilibrium allocation, and thus the outcomes do not coincide. Below, we show that there are two types of fundamental natures of monetary trades; one has a continuum of stationary equilibrium and the other has locally unique equilibria. A typical example of the former case is the auction market, while that of the latter case is the Walrasian market.

In the Walrasian market, the price of goods is determined in the centralized markets. Thus, as shown in Section 4, it suffices to investigate money holdings distributions with a support expressed by $\{0, p\}$, where $p > 0$ is an equilibrium price. In auction markets, there may exist equilibrium money holdings distributions of which support are not $\{0, p\}$. However, in order to show that the outcomes in the latter case includes the outcomes in the former case, we only need to focus on distributions with a support

[4] If we assume that there is an infinitesimally small cost of holding money, then only the equilibrium with F^* survives.

expressed by $\{0, p\}$. Let $h = (h_0, h_1)$ be a probability distribution on the support, where h_n is a measure of agent with money holding np.

Below, we compare the equilibrium conditions in the previous two sections. Limiting our analysis on the case that the measure of buyers is larger than or equal to that of sellers, the equilibrium condition for the auction markets is as follows:

$$\begin{aligned}
p &= \frac{M}{h_1}, \\
h_0 + h_1 &= 1, \\
h_1 \frac{h_0}{h_1} &= h_0, \\
h_0 &= h_1 \frac{h_0}{h_1}, \\
V(\eta) &= \max \big\{ (-c + \gamma V(\eta + \pi(h, \xi))) \\
&\quad \max_{b \in R_+} [\nu_B(b, h, \xi)\, (u + \gamma V(\eta - \pi(h, \xi))) + (1 - \nu_B(b, h, \xi)) \gamma \mathcal{V}(\eta)], \gamma V(\eta) \big\}.
\end{aligned}$$

On the other hand, the equilibrium condition for Walrasian markets is as follows:

$$\begin{aligned}
p &= \frac{M}{h_1}, \\
h_0 + h_1 &= 1, \\
h_1 T(p, \{0\}) &= h_0 T(0, \{p\}), \\
h_0 T(0, \{p\}) &= h_1 T(p, \{0\}), \\
V(\eta) &= \begin{cases} \gamma V(\eta + p) - c & \text{if } \eta \in [0, p), \\ \gamma V(\eta - p) + u & \text{otherwise,} \end{cases}
\end{aligned}$$

where the third and the forth equations are the conditions for stationarity of money holdings. In the both systems, for a given (h_0, h_1), p and V are uniquely determined by $p = \frac{M}{h_1}$ and the Bellman equations. As for (h_0, h_1), in the equilibrium condition for the auction markets, any (h_0, h_1) satisfying $h_0 + h_1 = 1$, $h_0 \geq 0$, and $h_1 \geq 0$, is a stationary distribution, since $h_1 \frac{h_0}{h_1} = h_0$ and $h_0 = h_1 \frac{h_0}{h_1}$ are identities. On the other hand, in the equilibrium condition for the Walrasian markets, (h_0, h_1) satisfying $h_0 + h_1 = 1$ is not necessarily a stationary distribution, since $h_1 T(p, \{0\}) = h_0 T(0, \{p\})$ is not an identity. (See the discussion below.) Thus there is one degree of freedom in the former system, while the solution is determinate in the latter system.

Below, we show that there are two types of fundamental natures of monetary trades; one has a continuum of stationary equilibrium and the other has only locally unique equilibria. They are:

- the amount of money the sellers obtain is always equal to that of buyers pay even out of equilibria, and
- the amount of money the sellers obtain is not necessarily equal to that of buyers pay.

It is clear that the auction market is classified as the former type and the Walrasian market is classified as the latter type. Indeed, in the auction markets, the amount of money the sellers obtain is ph_0 and that of buyers pay is $ph_1\frac{h_0}{h_1} = ph_0$. By this identity, any money holdings distribution is stationary, i.e., all (h_0, h_1) satisfying $h_0 + h_1 = 1$, $h_0 \geq 0$, and $h_1 \geq 0$ is stationarity, and the set of equilibria is a continuum. On the other hand, in the Walrasian markets, if p is large enough, then all agents without money choose to be sellers, i.e., $T(0, \{p\}) = 1$, and the amount of money the sellers obtain is ph_0 and the amount of money the buyers pay is at most $p(1 - h_0)$. Of course, they are not necessarily the same.

In order to understand the logic of indeterminacy, we investigate more general case, i.e., the case that a money holdings distribution is expressed as $h = (h_0, h_1, \ldots, h_N)$ for some positive integer N. That is, in the auction markets, we consider the case that some agents with np becomes sellers and the rest of them become buyers. Then the transition probability is represented by $Q_n^S \geq 0$ and $Q_n^B \geq 0$ such that $Q_n^S + Q_n^B = 1$, where Q_n^S is the measure of agents who have np and become sellers and Q_n^B the measure of agents who have np and become buyers. Then the stationarity of money holdings is written as:

$$
\begin{aligned}
D_0 &\equiv Q_1^B - Q_0^S = 0, \\
D_1 &\equiv Q_2^B + Q_0^S - Q_1^B - Q_1^S = 0, \\
\vdots & \qquad\quad \vdots \\
D_N &\equiv Q_{N-1}^S - Q_N^B = 0.
\end{aligned}
$$

In this case, the amount of money the sellers obtain is $\sum_{n=0}^{N-1} pQ_n^S$ and the amount of money the buyers pay is $\sum_{n=1}^{N} pQ_n^B$, and they are identically the same because of the rule of the uniform price auction. Thus

$$\begin{aligned}\sum_{n=0}^{N} npD_n &= 0pD_0 + 1pD_1 + 2pD_2 + \cdots + NpD_N \\ &= \sum_{n=0}^{N-1}((n+1)pQ_n^S - npQ_n^S) + \sum_{n=1}^{N}((n-1)pQ_n^B - npQ_n^B) \\ &= \sum_{n=0}^{N-1} pQ_n^S - \sum_{n=1}^{N} pQ_n^B \\ &= 0\end{aligned}$$

always holds and thus $\sum_{n=0}^{N} npD_n = 0$ is an identity. Moreover, Q_n^S (Q_n^B) appear twice in the above equations, and they are positive and negative, respectively. Thus another identity

$$\begin{aligned}\sum_{n=0}^{N} D_n &= D_0 + D_1 + D_2 + \cdots + D_N \\ &= 0\end{aligned}$$

holds. If $D_2 = D_3 = \cdots = D_N = 0$ holds, then by the above two identities, $D_1 = 0$ and $D_0 = 0$ follow. Hence among $D_n = 0, n = 0, 1, \ldots, N$, two equations are redundant. Since the number of variables $(h_0, h_1, \ldots, h_N)$ is $N + 1$ and the number of linearly independent equations including $\sum_{n=0}^{N} h_n = 1$ is N, there is at least one degree of freedom. Especially, in the case of $N = 1$ in Section 3, both $D_0 = 0$ and $D_1 = 0$ are redundant and any (h_0, h_1) satisfying $h_0 + h_1 = 1$ is a stationary distribution. Note that an example with $N = 2$ is given in Section A in Appendix.

In decentralized market models with divisible money, monetary trades typically classified as the first type. (See, for example, Green and Zhou [11], Kamiya and Shimizu [16], Matsui and Shimizu [20], and Zhou [28].) For example, suppose (a) a buyer randomly meets a seller, (b) then the buyer offers a price and an amount of good she wants to buy, and (c) finally the seller decides whether to accept or to reject the offer. In this case, in each matching the amount of money the buyer pays is always the same as that of the seller obtains even out of equilibria. Thus in the economy the total amount of money the buyers pay is always the same as that of the sellers obtain even out of equilibria. By the same logic as in the centralized economy, the equilibria are typically indeterminate. The only one difference between decentralized market models and centralized market models of the first type is that equilibrium price dispersion might occur in decentralized market models. (See Kamiya and Sato [15] and Matsui and Shimizu [20].)

6 Discussion

In this section, we classify the related literature into three categories and discuss them in turn.

The first category is cooperative game approach. Relevant literature analyze economies with a large population by making use of the concept of core. This approach is initiated by Edgeworth [4]'s classical work, and followed by Shubik [26], Debreu and Scarf [2], Aumann [1], and Hildenbrand [14], which show the core convergence result in considerably general frameworks. Particularly, Aumann [1] directly assumes that there is a continuum of agents and shows that the set of core allocations exactly coincides with that of Walrasian equilibrium. However, these papers do not give any specification of trading institution or mechanism, and neither lead to any institutional consideration about market.

Next we define the literature on non-cooperative game approach as the second category. This includes Cournot-type models as Novshek and Sonnenschein [21], and strategic market games as Shapley and Shubik [25],[5] and decentralized market models as Rubinstein and Wolinsky [23] and Gale [5], [7]. Although these models specify the details of trading process, they differ from the present paper in modeling the notion of 'time.'

Marshall [19] defines the tripartite division of time: a day, a short period, and a long period. (See also Hicks [13].) The production decision cannot be changed in a day, but the amount of production good can be changed for a given capital if given a short period, and in a long period the amount of capital can be changed as well. Following Marshall's definition, we label the models within a day as static models and those within a short period as dynamic models. Then any model referred in the previous paragraph is static, since they all describe an economy in which the production is done. On the other hand, the environment of the present paper is dynamic, since agents are involved in either production or consumption in each period.

The third category, which is most closely related to the present paper, is consisted from search theoretic models with divisible money. This category includes Green and

[5]Dubey, Mas-Colell, and Shubik [3] present axiomatic approach to this issue. For a survey on strategic market games, see Giraud [10]

Zhou [11] [12], Kamiya and Shimizu [16], Matsui and Shimizu [20], and Zhou [28]. In this category, real indeterminacy of stationary equilibria has been found in both specific and general models, and now there are theories that explain the stationary equilibria have a quite different feature from those in Walrasian market models.

Our model belongs to this category. We consider the centralized auction markets, and show that the real indeterminacy of stationary equilibria occurs in such markets. We emphasize that any other model offered in this category assume an agent is randomly matched with other agents and transaction is made in a decentralized way. Therefore, our paper has significance in showing that decentralized feature of money search models is not crucial for the real indeterminacy result.

Finally, we would like to note that at the end of his book, Gale [8] claims that convincing models of general equilibrium are the ones in which markets of goods are distinct, and agents do not trade most goods and do not therefore participate in most markets, although his formal model in the book does not have the feature. In our model, there are $k \geq 3$ types of agents and the same number of types of goods; a type i agent produces type $i+1$ good and she consumes type i good. Moreover, each agent can transact in just one market in each time period.

Appendix

A Mixed Strategy Equilibria in Centralized Auction Markets

In this section, we prove the existence of mixed strategy stationary equilibria in the centralized auction market model investigated in Section 3. First, we define a mixed strategy stationary equilibrium. A candidate for an equilibrium mixed strategy can be defined as a function of a money holding $\eta \in R_+$,

$$\tilde{\xi} : R_+ \to \Omega\left(\{\sigma\} \cup (\{\beta\} \times R_+) \cup \{\nu\}\right),$$

where $\Omega(A)$ is the set of probability distributions on a set A. In parallel with Definition 1 in Section 3, a mixed strategy stationary equilibrium is defined in the case of a finite number of bid prices.

We consider the following mixed (behavioral) strategy:

- an agent with $\eta \in [0, p)$ chooses to be a seller,

- an agent with p chooses to be a buyer and bids p with probability $\delta \in (0,1)$, and chooses to be a seller with probability $1-\delta$, and
- an agent with $\eta \in (p,\infty)$ chooses to be a buyer and bids η.

Under the above strategy, money holding $2p$ is not a transient state. Let $h = (h_0, h_1, h_2)$ and

$$r = \frac{h_0 + (1-\delta)h_1 - h_2}{\delta h_1}. \tag{8}$$

An agent with p can buy a good with probability r. It is shown that $r \in [0,1]$ holds in equilibria. Under the above strategy, the stationarity of money holding distribution can be written as follows:

$$\begin{aligned}
&r\delta h_1 = h_0,\\
&h_2 + h_0 = r\delta h_1 + (1-\delta)h_1,\\
&h_2 = (1-\delta)h_1,\\
&h_0 + h_1 + h_2 = 1.
\end{aligned}$$

As we have shown in Section 5, two of the first three equations are redundant. The stationary distribution is obtained as follows:

$$h_0 = \frac{\delta r}{2-\delta+\delta r},\ h_1 = \frac{1}{2-\delta+\delta r},\ h_2 = \frac{1-\delta}{2-\delta+\delta r}. \tag{9}$$

Clearly, $r \geq 0$ implies $h_n \in [0,1]$ for $n = 0,1,2$.

If $\delta > 0$, the uniform price is p. Thus we obtain the Bellman equation as (1) in Section 3, where r is defined by (8). The incentive conditions for choosing the above strategy are as follows:

(i) $V(0) \geq 0$,

(ii) $V(p) = -c + \gamma V(2p)$.

The first inequality is the incentive to be a seller for an agent who possesses no money. Since $V(p)$ is the value when an agent becomes a buyer, the second equality implies that the agent is indifferent between being a buyer or being a seller. As in Section 3, the other incentive conditions can be easily checked. (i) is equivalent to (2) in Section 3, where r is defined by (8), and (ii) is equivalent to

$$r = \frac{\gamma\theta - 1}{(1+\gamma-\gamma^2)\theta - \gamma^2}. \tag{10}$$

Note that $\gamma > \frac{1}{\theta}$ implies $r \in (0,1)$. It is also worthwhile noting that, substituting (10) into (9) , (h_0, h_1, h_2) can be parametrized by δ. Then substituting (10) into (2), we obtain the following inequality:

$$-(\theta+1)\gamma^3 + (\theta^2+\theta+1)\gamma^2 - (\theta-1)\gamma - \theta \geq 0.$$

It is verified that there exists $\underline{\gamma} \in \left(\frac{1}{\theta}, 1\right)$ such that the above inequality holds for any $\gamma \in \left(\underline{\gamma}, 1\right)$.

Theorem 4 There exists $\underline{\gamma} \in \left(\frac{1}{\theta}, 1\right)$ such that, for any given $\gamma \in \left(\underline{\gamma}, 1\right)$, the above mixed strategy is a stationary equilibrium with a discrete money holdings distribution for any $\delta \in (0,1]$.

Note that since (h_0, h_1, h_2) and (V_0, V_1, V_2) depend on δ, there is real indeterminacy in equilibria.

B Relaxing the Limited Participation Constraint

In this section we relax the limited participation constraint and show that the similar results can be obtained as in Sections 3 and 4. We do so by assuming that agents in this section can simultaneously can involve in both transactions: sell and buy in the same period.

B.1 Centralized Auction Markets

First, we analyze centralized auction markets. We focus on stationary equilibria in which money holdings distribution has support $\{0, p\}$. We investigate the following strategy:

- an agent with $\eta \in [0, p)$ chooses only to be a seller,
- an agent with p chooses only to be a buyer and bids p,
- an agent with $\eta \in (p, 2p)$ chooses to be a buyer and seller, and as a buyer, bids η, and
- an agent with $\eta \in [2p, \infty)$ chooses only to be a buyer and bids η.

Under the strategy, the value function is expressed as follows:

$$V(\eta) = \begin{cases} -c + \gamma V(\eta + p), & \text{if } \eta \in [0, p), \\ r\left(u + \gamma V(0)\right) + (1 - r)\gamma V(p), & \text{if } \eta = p, \\ u - c + \gamma V(\eta), & \text{if } \eta \in (p, 2p), \\ u + \gamma V(\eta - p), & \text{if } \eta \in [2p, \infty), \end{cases}$$

where

$$r = \frac{h_0}{1 - h_0}.$$

Let n be the integer part of η and ι be the residual, then η is uniquely expressed as $\eta = np + \iota$. Thus the value function becomes as follows:

$$V(np + \iota) = \begin{cases} \frac{r\gamma u - (1-\gamma+r\gamma)c}{(1-\gamma)(1+r\gamma)}, & \text{if } \iota = 0,\ n = 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^{n-1}}{1+r\gamma}\left[(1 - r + r\gamma)u + r\gamma c\right]\right\}, & \text{if } \iota = 0,\ n \neq 0, \\ \frac{\gamma u - c}{1-\gamma}, & \text{if } \iota \neq 0,\ n = 0, \\ \frac{u - \gamma^{n-1}c}{1-\gamma}, & \text{if } \iota \neq 0,\ n \neq 0. \end{cases}$$

We consider the case of $\gamma > \frac{1}{\theta}$. Then the incentive conditions are as follows:

(i) incentive for an agent without money only to be a seller instead of doing nothing,

(ii) incentive for an agent with p only to be a buyer instead of being only a seller,

(iii) incentive for an agent with p only to be a buyer instead of being a buyer and seller, and

(iv) incentive for an agent with $2p$ only to be a buyer instead of being a buyer and seller.

Since the value of $\eta = np$ is the same as the one in Section 3, (i) and (ii) are equivalent to (2) and (3) respectively.

(iii) is expressed as

$$V(p) \geq r\left[u - c + \gamma V(p)\right] + (1 - r)\left[-c + \gamma V(2p)\right],$$

and it is equivalent to

$$-\left[\gamma(2-\gamma)\theta - \gamma^2\right] r^2 + \left[\gamma(2-\gamma)\theta - \gamma^2\right] r - (\gamma\theta - 1) \geq 0. \tag{11}$$

Substituting (2) with equality into the RHS of (11), we obtain

$$\frac{\gamma\theta-1}{\gamma(\theta-1)^2}\left\{(\theta+1)\gamma^2-(\theta^2+\theta+2)\gamma+2\theta\right\}.$$

Denote the expression in the braces by $F(\gamma)$, then we have

$$\begin{aligned}&F\left(\frac{1}{\theta}\right)>0,\\&F(1)<0.\end{aligned}$$

Therefore there exists $\overline{\gamma}_1\in\left(\frac{1}{\theta},1\right)$ such that for any $\gamma\in\left(\frac{1}{\theta},\overline{\gamma}_1\right)$, there exists $\overline{r}\in\left(\frac{1-\gamma}{\gamma(\theta-1)},1\right]$ such that (2) and (11) hold for any $r\in[\underline{r},\overline{r}]$, where $\underline{r}=\frac{1-\gamma}{\gamma(\theta-1)}$. Moreover, we can choose $\overline{\gamma}_1$ sufficiently close to $\frac{1}{\theta}$ such that (3) holds for any $r\in[\underline{r},\overline{r}]$.

Next, (iv) is expressed by

$$V(2p)\geq u-c+\gamma V(2p),$$

and it is equivalent to

$$-r(\theta+1)\gamma^2+\left[r(\theta+1)-\theta\right]\gamma+1\geq 0. \tag{12}$$

Denote the LHS by $G(\gamma)$. Then it is verified that

$$\begin{aligned}&G\left(\frac{1}{\theta}\right)>0,\quad \text{if } r>0,\\&G(1)<0.\end{aligned}$$

Then there exists $\overline{\gamma}_2\in\left(\frac{1}{\theta},1\right)$ such that for any $\gamma\in\left(\frac{1}{\theta},\overline{\gamma}_2\right)$, (12) holds for any $r>0$.

Finally, let $\overline{\gamma}=\min\{\overline{\gamma}_1,\overline{\gamma}_2\}$. Then for any $\gamma\in\left(\frac{1}{\theta},\overline{\gamma}\right)$, there exist $\underline{r}$ and $\overline{r}$ satisfying $0<\underline{r}<\overline{r}\leq 1$ such that there exists a stationary equilibrium for any $r\in[\underline{r},\overline{r}]$. Since $r=\frac{h_0}{1-h_0}$, we obtain the following result:

Theorem 5 There exists $\overline{\gamma}\in\left(\frac{1}{\theta},1\right)$ such that for any $\gamma\in\left(\frac{1}{\theta},\overline{\gamma}\right)$, there exist $\overline{h}_0$ and $\underline{h}_0$ satisfying $0<\underline{h}_0<\overline{h}_0\leq\frac{1}{2}$ such that there exists a stationary equilibrium for any $h_0\in\left[\underline{h}_0,\overline{h}_0\right]$.

B.2 Walrasian Markets

Next, we consider Walrasian markets. Holding fix the other conditions, we redefine C and ϕ. Here, we define C as $C=\{(1,1),(1,0),(0,1),(0,0)\}$ and ϕ as $\phi: R_+\rightarrow$

$\{\omega, \beta, \sigma, \nu\}$, where ω stands for 'sell and buy'. Note that the Bellman equation is also expressed by (6).

Below, we consider the case of $\gamma > 1/\theta = c/u$. As in the discussion in Section 4, an equilibrium price, if it exists, is $p > 0$. Since an agent with $\eta \in [0, p)$ cannot buy,

$$V(\eta) = \max\{0, -c + \gamma V(\eta + p)\}$$

holds. Since an agent with $\eta + p$ can buy and sell,

$$V(\eta + p) \geq \frac{u - c}{1 - \gamma},$$

holds. Thus by

$$-c + \gamma \frac{u - c}{1 - \gamma} = \frac{\gamma u - c}{1 - \gamma} > 0,$$

we obtain

$$V(\eta) = -c + \gamma V(\eta + p), \tag{13}$$

i.e., $\phi(\eta) = \{\sigma\}$. Thus by (13),

$$u - c + \gamma V(\eta) = u - c + V(\eta - p) + c > u + \gamma V(\eta - p).$$

holds for any $\eta \in [p, 2p)$. Since it is easily verified that $\sigma \notin \phi(\eta)$, we obtain $\phi(\eta) = \{\omega\}$. Moreover, it is also easily verified that $\phi(\eta) = \{\beta\}$ for any $\eta \in [2p, \infty)$. Then we obtain the following result.

Theorem 6 For any $\gamma \in \left(\frac{1}{\theta}, 1\right)$, there exists a stationary Walrasian equilibrium $\langle p, F, V, \phi, T \rangle$. Moreover, any Walrasian equilibrium is characterized by

(i)

$$\phi(\eta) = \begin{cases} \{\sigma\} & \text{if } \eta \in [0, p), \\ \{\omega\} & \text{if } \eta \in [p, 2p), \\ \{\beta\} & \text{if } \eta \in [2p, \infty), \end{cases}$$

(ii) V is defined as

$$V(\eta) = \begin{cases} \frac{\gamma u - c}{1 - \gamma}, & \text{if } \eta \in [0, p), \\ \frac{u - c}{1 - \gamma}, & \text{if } \eta \in [p, 2p), \\ \quad \vdots & \\ \frac{1}{1 - \gamma}\left\{u - \gamma^{n-1} c\right\}, & \text{if } \eta \in [np, (n+1)p), \\ \quad \vdots & \end{cases}$$

(iii)

$$T(\eta, A) = 1, \quad \text{iff} \begin{cases} \eta + p \in A & \text{and} \quad \eta \in [0, p), \quad \text{or} \\ \eta \in A & \text{and} \quad \eta \in [p, 2p), \quad \text{or} \\ \eta - p \in A & \text{and} \quad \eta \in [2p, \infty), \end{cases}$$

(iv) F satisfies

$$F([p, 2p)) = 1,$$
$$\int_p^{2p} \eta dF = M.$$

Corollary 2 The distribution of value in stationary Walrasian equilibria is uniquely determined, i.e., the value of any agent is $\frac{u-c}{1-\gamma}$.

The above corollary implies that the indeterminacy is not a real one but a nominal one as in Section 4.

C Decentralized Auctions

In the same environment as in Section 2, we consider an economy, where trades take place in decentralized second price auction markets, and show that there is also a continuum of stationary equilibria.

In each period, each agent is given a choice of becoming either a seller or a buyer, or not doing anything. Each seller posts a minimum bid of her second-price auction.[6] After observing the distribution of posted minimum bids, each buyer simultaneously chooses which auction he participates in. After observing the number of the other participants in the auction he participates in and bids a price. In this environment, we consider the following strategy:

- an agent with $\eta \in [0, p)$ chooses to be a seller and post a minimum bid p,
- an agent with p chooses to be a buyer and always bids p,
- an agent with $\eta \in [p, \infty)$ chooses to be a buyer, and
  - bids p if there is no participant in the auction,
  - bids η if there are other participants in the auction.

[6]Nothing would change if we assume the sellers also choose her auction format.

Moreover, we consider the stationary equilibria in which

(i) the support of the money holdings distribution is $\{0, p\}$,

(ii) $h_0 \leq \frac{1}{2}$, and

(iii) every buyer randomizes with equal probabilities between auctions with the same minimum bids.

The equilibrium with (iii) is often investigated in many directed search model, e.g., Peters [22]. In such an equilibrium, although the measure of the sellers is no less than that of the buyers, a seller could be left out of the trade. To be more precise, let $\rho = \frac{1-h_0}{h_0}$ (then $\rho \geq 1$). Then the probability a seller succeeds to sell a good in each period is obtained as $1 - e^{-\rho}$, and the probability each buyer with p succeeds to buy a good in each period is obtained as $\frac{1-e^{-\rho}}{\rho}$.[7] Let $\alpha = 1 - e^{-\rho}$.

The value function is defined as follows:

$$V(\eta) = \begin{cases} \alpha\left(-c + \gamma V(\eta + p)\right) + (1-\alpha)\gamma V(\eta), & \text{if } \eta < p, \\ \frac{\alpha}{\rho}\left(u + \gamma V(0)\right) + \left(1 - \frac{\alpha}{\rho}\right)\gamma V(p), & \text{if } \eta = p, \\ u + \gamma V(\eta - p), & \text{if } \eta > p. \end{cases}$$

Let $\eta = np + \iota$, where n is the integer part of η and ι is the residual. Then we obtain

$$V(np+\iota) = \begin{cases} \frac{\gamma\alpha^2(u-c)-(1-\gamma)\alpha\rho c}{(1-\gamma)[(1-\gamma+\gamma\alpha)\rho+\gamma\alpha]}, & \text{if } \iota = 0,\ n = 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^{n-1}}{(1-\gamma+\gamma\alpha)\rho+\gamma\alpha}\left\{[(\rho-\alpha)(1-\gamma+\gamma\alpha)+\gamma\alpha]\,u + \gamma\alpha^2 c\right\}\right\}, & \text{if } \iota = 0,\ n \neq 0, \\ \frac{1}{1-\gamma}\left\{u - \frac{\gamma^n}{1+\gamma\alpha}(u+\alpha c)\right\}, & \text{if } \iota \neq 0. \end{cases}$$

The incentive conditions are the same as in the centralized auction model, i.e.,

(I) $V(0) \geq 0$.

(II) $V(p) \geq -c + \gamma V(2p)$.

(I) is equivalent to

$$\gamma \geq \underline{\gamma}(\rho) = \frac{\rho}{(\theta-1)\alpha+\rho}.$$

Note that $\underline{\gamma}(\rho) \in (1/\theta, 1)$. Similarly, (II) is equivalent to

$$\begin{aligned} H(\gamma;\rho) = {} & \left[\rho(1-\alpha)\theta - \alpha(2-\alpha)\theta - \alpha^2\right]\gamma^2 \\ & + \left[-\rho(1-\alpha+\theta) + \alpha^2\theta + \alpha(1-\alpha)\right]\gamma + \rho + \alpha\theta \geq 0. \end{aligned}$$

[7] We obtain these probabilities in the limit of finite economies. See Peters [22] for the details.

Since

$$
\begin{aligned}
&H(0;\rho) > 0,\\
&H(1/\theta;\rho) > 0,\\
&H(1;\rho) < 0,\\
&\frac{\partial H(\gamma;\rho)}{\partial \gamma} < 0, \quad \forall \gamma \in [0,1],
\end{aligned}
$$

there exists $\overline{\gamma}(\rho) \in \left(\frac{1}{\theta}, 1\right)$ such that, for $\gamma \in (0,1)$,

$$\gamma \leq \overline{\gamma}(\rho) \Leftrightarrow H(\gamma;\rho) \geq 0.$$

By tedious calculations, we obtain

$$H\left(\underline{\gamma}(1);1\right) = \frac{\alpha^*(\theta^2-1)}{[\alpha^*\theta + (1-\alpha^*)]^2}\left\{(\alpha^*)^2(\theta-1) + (3\alpha^*-1)\right\},$$

where $\alpha^* = 1 - e^{-1}$. Since $\alpha^* > \frac{1}{3}$, it is verified $H\left(\underline{\gamma}(1);1\right) > 0$, and therefore $\overline{\gamma}(1) > \underline{\gamma}(1)$. The continuity of $\overline{\gamma}(\rho)$ and $\underline{\gamma}(\rho)$ at $\rho = 1$ implies there exist $\underline{\gamma}$, $\overline{\gamma}$ and $\overline{\rho}$ such that

$$
\begin{aligned}
&\frac{1}{\theta} < \underline{\gamma} < \overline{\gamma} < 1,\\
&\overline{\rho} > 1,
\end{aligned}
$$

and for any $\gamma \in (\underline{\gamma}, \overline{\gamma})$, there exists a stationary equilibrium with the specified strategy with any $\rho \in [1, \overline{\rho})$. Let $\overline{h}_0 = \frac{1}{\overline{\rho}+1}$, then we obtain the following result:

Theorem 7 There exist $\underline{\gamma}$, $\overline{\gamma}$, and $\overline{h}_0$ satisfying $\frac{1}{\theta} < \underline{\gamma} < \overline{\gamma} < 1$ and $\overline{h}_0 \in (0, \frac{1}{2})$ such that for any $\gamma \in (\underline{\gamma}, \overline{\gamma})$, there exists a stationary equilibrium for any $h_0 \in \left(\overline{h}_0, \frac{1}{2}\right]$.

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