Efficient Structure of Provision for Emergency Public Services

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1 Introduction

Emergency services (fire-fighting, ambulance...) are subject to spatial limitations. Consequently, the quality of the service is negatively affected by the distance between the users and the location of the service. In order to reach efficiency in the provision of emergency services, the competent Authority has to decide the *optimal* structure of provision taking into account the distance-sensitive utilities of the users.(1)

In France, the Departmental Authority has the competence to design the structure of provision for emergency services within the Department. A natural objective for the Departmental Authority is to maximize welfare. But, which structure of provision for emergency services should the Departmental Authority choose? The emergency service has to attend a group of geographically separated municipalities and this may require the construction of more than one emergency unit.(2) The

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(1) Up to now the literature has only focused on the demand-supply side of emergency services (see Ahlbrandt [1973], Brueckner [1981], Duncombe [1992], Duncombe and Yinger [1993]). The provision-location side of emergency services has been neglected.

(2) In France, an emergency unit serves usually more than one municipality. For example, the Department Haut-Rhin has 38 emergency units for 377 municipalities and 671319 inhabitants [sources: Service Départemental d'Incendie et de Secours du Haut Rhin, Colmar].

provision costs of each emergency unit have to be divided among the municipalities that use it. Also, the possibilities of communication between municipalities restrict the possible cooperation to connected municipalities, and each emergency unit serves (in priority) a connected subgroup of municipalities. Hence, the benefits derived by a municipality depend positively on the level of provision of the service and negatively on the aggregate distance between the subgroup of municipalities and the emergency unit. Indeed, when a municipality requires the service of the emergency unit, the firemen could be attending any other municipality that shares the emergency service. In such context, it is not excluded that a single emergency unit for the whole economy (joint provision of the service) is inefficient, because the benefits obtained by sharing the costs of provision can be offset by the negative effect of the distance on the quality of service.

To what extent does the impact of the distance on the quality of service prevent the joint provision of the emergency public service? Also, if the joint provision is inefficient, which is the efficient or stable structure of provision? These are the two questions that we give an answer to in this paper. The most natural approach to tackle these issues is the cooperative game-theoretic approach[(3)].

A well-known observation is that the location of municipalities in a region gives rise to a hierarchy between them (see Nystuen and Dacey [1961]). Then, the communication between municipalities can be represented by graphs: abstract networks of points and lines. The *radial geographical structure*, one of the simplest structures, emerges when some peripheral municipalities are connected by means of a central municipality. We answer our two questions in this geographical setting, called star-tree-graph. The choice of the star-tree-graph is motivated by its simplicity for presenting our results which are robust to any tree-graph (see the discussion in Section 4). Formally, we consider an economy formed by municipalities located at the nodes of a star-tree-graph. A central government, the Departmental Authority in the case of France, is involved in the costly provision of an emergency public service. The emergency service has to be located in the graph. Only connected municipalities are able to derive benefits from the joint provision. The municipalities are characterized by distance-sensitive utility functions: the quality of the service diminishes with the aggregate distance between the municipalities and the location of the service.

In order to maximize welfare, the central government chooses the location of the emergency service and the optimal provision. The optimal

[(3)] Gallastegui *et al.* [1997] have used a similar approach for a public good economy with congestion.

location is chosen so that the aggregate distance between municipalities and service is minimized. To sustain the optimal provision, the central government charges the Lindahl prices to the municipalities. Pricing the emergency public service according to Lindahl generates benefits that depend negatively on the degree in which the distance affects the quality. Therefore, if the quality is strongly affected by distance, the joint provision of the service may be inefficient in the sense that there exists another structure of provision generating greater benefits to the municipalities. But, whenever the joint provision is efficient, is there any distribution of benefits that guarantees the sustainability of the Lindahl equilibrium in the core of the economy? To answer this question, we identify the benefits that each coalition of municipalities obtains from the provision of the service with the characteristic function of a TU-game. Hence, the nonemptiness of the core of the game determines the sustainability of the Lindahl equilibrium in the core of the economy.

Our main results are the following ones. First, if the quality of service is not affected by distance, the joint provision of the emergency service is efficient. Moreover, the Lindahl equilibrium belongs to the core of the economy. However, for public services affected by distance, the joint provision is not always efficient. In this case, there exists a *minimum quality of service* compatible with its joint provision. This minimum or critical quality of service can be interpreted as the limit value below which the positive external effect of the joint provision of the service is dominated by the negative external effect: the distance. Finally, for any quality of service below the critical one, the joint provision of the service is inefficient. Nevertheless, we are able to identify the efficient and stable structure of provision for the emergency service. The paper is organized as follows. In Section 2 we develop our public good economy. Also, we determine the optimal location and provision, as well as the Lindahl prices, for the emergency service. Section 3 is devoted to determine the minimum quality level compatible with the joint provision of the service. We also identify the efficient and stable structure of provision for any quality level. In Section 4 we discuss some assumptions. Section 5 concludes.

2 The model

We develop a public good economy formed by some peripheral municipalities connected by means of a central municipality located in a star-tree-graph. A central government, whose objective is to maximize welfare, has to decide the location, provision and price of an emergency public service the quality of which diminishes with distance.

2.1 The location of the public service

Let us formalize the geographical setting. The central municipality and the peripheral municipalities are located in the star-tree-graph $T = \langle N, E \rangle$, where N is the finite set of nodes (or municipalities) and E is the finite set of edges joining each pair of nodes. We denote by $e_{ij} \in E$ the edge joining (in both directions) the nodes $i, j \in N$. A path between any two nodes of N is a sequence of distinct edges in E allowing both nodes to be joined. Each pair of different nodes of N is connected by exactly one path. The central node (central municipality) is directly connected to each of the remaining nodes (peripheral municipalities) The peripheral municipalities are connected by means of the central one. An example of our geographical setting is given in Figure 1.

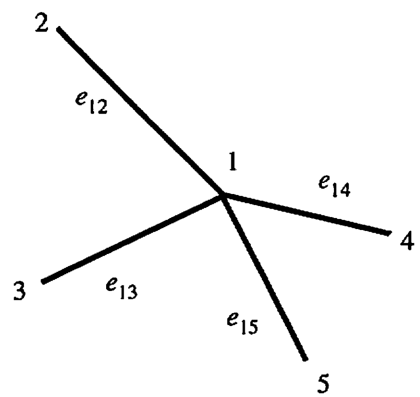

Figure 1: The geographical setting

Only connected municipalities have incentives to use the same provider. A coalition $S \subseteq N$ is connected if each pair of different nodes in the (sub)graph $\langle S, E(S) \rangle$ is connected by exactly one path, where $E(S) = \{e_{ij} \in E \mid i, j \in S\}$. We denote by $\mathcal{C}(N)$ the set of all connected coalitions. For each coalition $S \subseteq N$ we denote by $\mathcal{K}(S)$ the set of all components of S,

$$\mathcal{K}(S) = \{S^* \subseteq S \mid S^* \in \mathcal{C}(N) \text{ and } \nexists\, S' \supset S^*, S' \in \mathcal{C}(N), S' \subseteq S\}$$

In words $\mathcal{K}(S)$ is the set of largest connected subcoalitions in S.

Let $d(i, y)$ be the distance between municipality i and a point $y \in T$. The distance between coalition S and a point y is defined as the aggregate distance and denoted by $d(S, y) \equiv \sum_{i \in S} d(i, y)$. The quality of the public service provided to a municipality is only negatively affected by the aggregate distance between the set of municipalities and the location of the service. Therefore, for each connected coalition of municipalities, the optimal location chosen by the government, whose objective is to maximize welfare, is a point of the star-tree-graph that minimizes the aggregate distance between the municipalities and the service location. We denote by $|S|$ the cardinality of S. Thus, for each $S \in C(N)$, $|S| \geqslant 2$, the optimal location of the service is

$$y^*(S) = \min_{y \in T} \sum_{i \in S} d(i, y). \tag{1}$$

The distance between coalition S and the optimal location of the service is denoted by $d(S)$, i.e. $d(S) \equiv d(S, y^*(S))$.

Let us illustrate the determination of the optimal location of the service with the following example. Take $N = \{1,2,3,4,5\}$ as the set of municipalities and $E = \{e_{12}, e_{13}, e_{14}, e_{15}\}$ as the set of edges (see Figure 1). Then, $C(N) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{1,2,3,4,5\}\}$. Notice that $\mathcal{K}(S) = S$ if $S \in C(N)$, but $\mathcal{K}(S) \neq S$ if $S \notin C(N)$. For example, $\mathcal{K}(\{2,3,4\}) = \{\{2\}, \{3\}, \{4\}\}$. Computing $\min_{y \in T} \sum_{i \in S} d(i, y)$, we obtain the optimal location for every connected coalition S in the star-tree-graph. For $\{N\}$ and all $S \in C(N)$ of cardinality 3 or 4, $y^*(S)$ is the central municipality. For all $S \in C(N)$ of cardinality 2, $y^*(S)$ is any point between the two municipalities. Obviously, for any coalition $\{i\}$, $y^*(\{i\})$ is municipality i, for all $i \in N$.

2.2 The Pareto optimal provision and the Lindahl equilibrium

In addition to the solving of the problem of location of the public service, the government has to decide the optimal provision of the public service f financed by a private good z, as well as to settle the Lindahl prices that municipalities have to pay for the provision of the service. Consider an economy formed by a finite set N of municipalities and two goods, one private and one public service. The public service is produced using the private good according to a technology, $c(f)$, which exhibits decreasing returns and is given by the following quadratic cost function:

$$c(f) = f^2/2. \tag{2}$$

Each municipality $i \in S$ is characterized by an endowment of the private good W_i and the following linear separable-additive utility function:

$$u_i(f, d(S), z_i) = \alpha_i \left(\frac{1}{[d(S)]^q} \right) f + z_i, \tag{3}$$

where f and z_i are the quantities of the public service and private good consumed by municipality i. The parameter α_i indicates the preferences of municipality i towards the public service and the private good. Remember that $d(S)$ is te distance between S and the optimal location, $y^*(S)$.[4] The parameter $q \geqslant 0$ measures the degree by which the quality of service is negatively affected by distance and $q = 0$ simply means that distance does not influence quality at all. The term between bracket in the utility function may be interpreted as the quality of provision of the emergency service. It is reasonable to assume that when a municipality is the only one to be attended by a provider located in its own municipality, its utility for using the service is not affected by distance. Without loss of generality, it reverts to assume that $d(\{i\}) = 1$ for all $i \in N$. Hence, for coalitions S, $|S| \geqslant 2$, we have that $d(S) > 1$. Note that in this model all municipalities are equally affected by distance.

The Pareto optimal allocations are obtained by maximizing the sum of utilities over all agents subject to the technological and feasibility restrictions.[5] First order conditions of the solution of this problem yield the Bowen-Lindahl-Samuelson condition: the sum of the marginal rates of substitution of the public good for the private one must be equal to the marginal rate of transformation of the public good for the private one (which is the marginal cost of output in terms of input):

$$\sum_{i \in S} \frac{\partial u_i / \partial f_S}{\partial u_i / \partial z_i} = \partial c / \partial f_S.$$

In our model, for an economy consisting of any connected coalition $S \in C(N)$ of municipalities, the optimal provision is given by

$$f_S^* = \frac{\alpha(S)}{[d(S)]^q}, \tag{4}$$

where $\alpha(S) = \sum_{i \in S} \alpha_i$.

It is well known that an optimal allocation can be attained as a Lindahl equilibrium. Let us normalize to one the price of the private good, and denote by $b_i(S, f_S^*)$ the share of the provider benefits assigned to municipality $i \in S$. Then, a Lindahl equilibrium is an allocation

(4) Our choice of this utility function, taking into account the aggregate distance, is well motivated to model the provision of emergency services. For example, when a municipality $i \in S$ requires the services of a fire station, firemen could be attending any other municipality belonging to the same group S. In France, an emergency unit as the *centre de secours* is able to do simultaneously two or three fire-interventions (sources: Service Départemental d'Incendie et de Secours du Haut Rhin, Colmar).

(5) The welfare function weights equally all the municipalities. Indeed, an egalitarian power distribution among the municipalities is assumed.

$(z_i^*, f_s^*)_{i\in S} \in \mathbb{R}^{|S|} \times \mathbb{R}$ and a system of personalized prices for the public service $(t_i^*)_{i\in S} \in \mathbb{R}^{|S|}$ such that:

(i)

$$\text{For all } i \in S: \frac{\alpha_i}{[d(S)]^q} f_S^* + z_i^* \geqslant \frac{\alpha_i}{[d(S)]^q} f_S + z_i \qquad \forall (f_S, z_i) \in \mathbb{R}_+^2$$
$$\text{s.t.} : t_i^* f_S + z_i \leqslant W_i + b_i(S, f_S^*),$$

(ii)

$$\sum_{i\in S} t_i^* f_S^* - f_S^{*2}/2 \geqslant \sum_{i\in S} t_i^* f_S - f_S^2/2 \quad \forall f_S \in \mathbb{R}_+,$$

(iii)

$$\sum_{i\in S} z_i^* - f_S^{*2}/2 = \sum_{i\in S} W_i;$$

where condition (i) requires that municipalities maximize utility, condition (ii) requires that the provider of emergency service maximizes benefits and condition (iii) indicates that the market clearing condition is satisfied.

Having solved the system we obtain the Lindahl prices:

$$t_i^* = \frac{\alpha_i}{[d(S)]^q}, i \in S. \tag{5}$$

The Lindahl equilibrium benefits to be shared among the set of municipalities belonging to S are

$$B(S) = \sum_{i\in S} b_i(S, f_S^*) = \frac{1}{2}\left(\frac{\alpha(S)}{[d(S)]^q}\right)^2, \tag{6}$$

and the equilibrium utility derived by each municipality $i \in S$ is given by

$$u_i(f_S^*, d(S), z_i^*) = W_i + b_i(S, f_S^*). \tag{7}$$

The utility of each municipality $i \in S$ depends on the share of the total benefits it obtains and on its initial endowment. Therefore, whether or not the Lindahl equilibrium lies into the core of the economy formed by the municipalities belonging to S, will depend on the benefits distribution among them. We precisely investigate this issue in the next section.

3 The quality and the structure of provision

In order to know whether or not the Lindahl equilibrium is stable, we identify the benefits, $B(S)$, of each connected coalition of municipalities $S \in C(N)$ with the characteristic function of the star-tree game

(N, v), where the value of coalition S is given by:

$$v(S) = \sum_{S^* \in \mathcal{K}(S)} \frac{1}{2} \left(\frac{\alpha(S^*)}{[d(S^*)]^q} \right)^2 \text{ for all } S \subset N.$$

Our star-tree game (N, v) is a cooperative game[6] restricted to the star-tree-graph $T = \langle N, E \rangle$, such that the value that a coalition S can obtain, $v(S)$, is the sum of the values of its components. This star-tree game (N, v) depends on the parameter q which determines, given the aggregate distance, the quality of the service, $[d(S)]^{-q}$. This fact allows the relationship between q and the nonemptiness of the core of the game (N, v) (and hence the nonemptiness of the core of the economy) to be studied. Precisely, we analyze to what extent the degree in which the quality of service is affected by the aggregate distance prevents the optimal joint provision of service. The joint provision simply means a single provider for the whole economy.

3.1 The minimum quality of service

We determine the minimum quality of service that guarantees the nonemptiness of the core of the game (N, v) and the stability of the joint provision of the service. This minimum quality is a critical value indicating when the external positive effect of the joint provision of the service is offset by distance. In the game (N, v), the allocations $x \in \mathbb{R}^{|N|}$ of the value $v(N)$ belonging to the core of the game are given by:

$$C(N, v) = \{x \in \mathbb{R}^{|N|} \mid x(N) = v(N) \text{ and } x(S) \geqslant v(S) \ \forall S \in \mathcal{C}(N)\},$$

where $x(S) = \sum_{i \in S} x_i$ and x_i is the payoff of municipality i according to $x \in \mathbb{R}^{|N|}$. To verify whether or not an allocation belongs to the core of the game (N, v), we have to check if the connected coalitions $S \in \mathcal{C}(N)$ receive at least their values, $v(S)$. First, we consider the limit case where the quality of the service is not affected by distance, and hence $q = 0$. In this case, we show that the efficient and stable structure of provision of the service is a single provider located at the central municipality for the whole economy.

[6] Myerson [1977] introduced the graph-restricted games by assuming that only players who can communicate with each other are able to cooperate. Our star-tree game is a class of graph-restricted games.

Proposition 1 *If $q = 0$, then the core of the game (N, v) is nonempty.*

Proof

For $q = 0$, the characteristic function of the game (N, v) is

$$v(S) = \sum_{S^* \in \mathcal{K}(S)} \frac{[\alpha(S^*)]^2}{2} \text{ for all } S \subset N.$$

Function $v(.)$ is convex and $\alpha(.)$ is a non-negative additive function. Therefore, following Shapley [1971], the game (N, v) is convex and it has a nonempty core. □

However, we can easily find values of q large enough for which the core of the game (N, v) is empty. The following numerical example illustrates this eventuality.

Example 1 (core is empty) Let $N = \{1, 2, 3, 4\}$ be the set of municipalities located in the nodes of a star-tree-graph: municipality 1 is the central node connected with the remaining nodes. Let $\alpha_i = 2$ for all $i \in N$. The distances among the municipalities are: $d(1, 2) = 2.25$, $d(1, 3) = 1.5$, $d(1, 4) = 2.75$ and the impact of the distance on the quality of the service is $q = 0.5$. The characteristic function v is: $v(1) = v(2) = v(3) = v(4) = 2$, $v(12) = 3.55$, $v(13) = 5.33$, $v(14) = 2.91$, $v(123) = 4.8$, $v(124) = 3.6$, $v(134) = 4.23$, $v(1234) = 4.92$, and the sum of the values of its components for non-connected coalitions. Since $v(123) + v(4) = 6.8 > v(1234) = 4.92$, the core of the game (N, v) is empty.

Nevertheless, there exist values of $q > 0$ such that the core of the game (N, v) is nonempty. Setting q equal to 0.2 in our previous Example 1, illustrates this fact.

Example 2 (core is nonempty) Considering Example 1 and setting now $q = 0.2$, the characteristic function of the game (N, v) reverts to: $v(1) = v(2) = v(3) = v(4) = 2$, $v(12) = 5.78$, $v(13) = 6.8$, $v(14) = 5.34$, $v(123) = 10.61$, $v(124) = 9.45$, $v(134) = 10.09$, $v(1234) = 15.13$, and the sum of the values of its components for non-connected coalitions. It is easy to verify that the allocation: $x_1 = 4.51$, $x_2 = 3.52$, $x_3 = 3.84$, $x_4 = 3.26$, is in the core.

Which is the minimum quality of service that guarantees the efficiency or stability of the joint provision? The next proposition answers this question. Let us denote by $\mathcal{P}(N)$ the set of partitions of N formed by connected coalitions. An element of this set is denoted by P.

Proposition 2 *The core of the game (N, v) is nonempty if and only if $q \leqslant q^*$ where*

$$q^* = \min_{\{q\}} \left\{ q \left| \sum_{S \in P} [\alpha(S)]^2 \left[\frac{d(N)}{d(S)}\right]^{2q} = [\alpha(N)]^2 \text{ for all } P \in \mathcal{P}(N) \right. \right\}$$

Proof

Proposition 2 is proved in four steps. But first we recall a few well-known definitions. Let N be a set of players and let B be a collection of nonempty sets. Define $B_i = \{S \in B \mid i \in S\}$ for all $i \in N$. B is said to be a balanced collection of sets if there exist positive numbers $(\lambda_S)_{S \in B}$ such that $\sum_{S \in B_i} \lambda_S = 1$ for all $i \in N$. A balanced collection not containing a balanced subcollection is called minimal.

Step 1. A necessary and sufficient condition for any game (N, v) to have a nonempty core is that

$$v(N) \geqslant \sum_{S \in B} \lambda_S \cdot v(S) \tag{8}$$

for every minimal balanced collection B (see Theorem X.5.1 in Owen [1995]).

Step 2. Demange [1994] has shown that *any balanced family of connected coalitions in a tree-graph contains a partition* (Lemma 2 in Demange [1994]).

Step 3. Step 1 and Step 2 allow us to rewrite the conditions (8) of Theorem X.5.1 of Owen [1995] for tree-games as:

$$v(N) \geqslant \sum_{S \in P} \lambda_S \cdot v(S) \text{ for all } P \in \mathcal{P}(N) \text{ with } \lambda_S = 1 \text{ for all } S \in P. \tag{9}$$

Step 4. Therefore, the conditions (9) for our star-tree game (N, v) are:

$$\frac{1}{2}[\alpha(N)]^2 [d(N)]^{-2q} \geqslant \sum_{S \in P} \frac{1}{2}[\alpha(S)]^2 [d(S)]^{-2q} \text{ for all } P \in \mathcal{P}(N). \tag{10}$$

Reordering, we obtain:

$$[\alpha(N)]^2 \geqslant \sum_{S \in P} [\alpha(S)]^2 \left[\frac{d(N)}{d(S)}\right]^{2q} \text{ for all } P \in \mathcal{P}(N). \tag{11}$$

Note that $[d(N)/d(S)]$ increases with q since $d(N) > d(S)$ for all $S \subset N$. This fact implies that if for a given partition P there exists a value $\overline{q}$ such that

$$[\alpha(N)]^2 \geqslant \sum_{S\in P} [\alpha(S)]^2 \left[\frac{d(N)}{d(S)}\right]^{2\overline{q}},$$

then, this inequality holds for any value $q < \overline{q}$. Thus, the value

$$q^* = \min_{\{q\}} \left\{ q \,\middle|\, \sum_{S\in P} [\alpha(S)]^2 \left[\frac{d(N)}{d(S)}\right]^{2q} = [\alpha(N)]^2 \;\forall P \in \mathcal{P}(N) \right\}$$

guarantees that the core of the game (N, v) is nonempty $\forall q \leqslant q^*$. Consider now that $q > q^*$. In this case, we have that

$$\sum_{S\in P} [\alpha(S)]^2 \left[\frac{d(N)}{d(S)}\right]^{2q} > [\alpha(N)]^2 \text{ for some } P \in \mathcal{P}(N),$$

and hence, for such partition $P \in \mathcal{P}(N)$,

$$v(N) < \sum_{S\in P} v(S).$$

Then, the game (N, v) has an empty core. Therefore, any star-tree game (N, v) has a nonempty core if and only if $q \leqslant q^*$. □

As the quality of service $[d(S)]^{-q}$ is smaller the bigger is parameter q, for values of $q > q^*$, the external positive effect of the joint provision of the service is offset by distance. Let us illustrate Proposition 2 with a numerical example.

Example 3 (minimum quality) Consider the game in Example 1 . Computing the critical values of q given by conditions (11) for each partition of $\mathcal{P}(N)$ we obtain:

Partition	q
$\{\{123\},\{4\}\}$	0.31
$\{\{124\},\{3\}\}$	0.42
$\{\{134\},\{2\}$	0.36
$\{\{12\},\{3\},\{4\}\}$	0.35
$\{\{13\},\{2\},\{4\}\}$	0.30
$\{\{14\},\{2\},\{3\}\}$	0.38
$\{\{1\},\{2\},\{3\},\{4\}\}$	0.37

Therefore, $q^* = 0.30$. The game (N, v) we obtain for $q^* = 0.30$ is: $v(1) = v(2) = v(3) = v(4) = 2$, $v(12) = 4.92$, $v(13) = 6.27$, $v(14) = 4.36$, $v(123) = 8.14$, $v(124) = 6.85$, $v(134) = 7.55$, $v(1234) = 10.41$, and the sum of the values of their components for non-connected coalitions. Note that the allocation: $x_1 = 3.595$, $x_2 = 2$, $x_3 = 2.675$, $x_4 = 2$, verifies the core conditions.

3.2 The stable geographical structure

Whether or not the joint provision is efficient, we are able to determine the structure of provision compatible with any degree of quality, even for $q > q^*$. Next proposition establishes that, for any game (N, v), there always exists at least a stable partition $P \in \mathcal{P}(N)$, which represents the efficient and stable geographical structure of provision. Aumann and Drèze [1974] analyze games with coalition structure. A coalition structure P on N, is a partition of N. A game with coalition structure P is a triple (N, v, P). The core of a game with coalition structure (N, v, P) is defined as $C(N, v, P) = \{x \in \mathbb{R}^{|N|} \mid x(S) \geqslant v(S)$ $\forall S \subseteq N$, $x(S_k) = v(S_k)$ $\forall S_k \in P\}$. A coalition structure P is stable whenever $C(N, v, P) \neq \varnothing$. Let $P^* = \max_{P \in \mathcal{P}(N)} \sum_{S \in P} v(S)$ be the partition of maximum value.

Proposition 3 *Let q be any positive number. Then, there exists at least a partition $P \in \mathcal{P}(N)$ such that $C(N, v, P) \neq \varnothing$.*

Proof

We distinguish two cases.

(i) case 1: $0 < q \leqslant q^*$. Since $C(N, v, \{N\}) = C(N, v)$, by Proposition 2 we have that $C(N, v, \{N\}) \neq \varnothing$.

(ii) case 2: $q > q^*$. To prove that there exists a stable coalition structure in this case, we select an allocation and we show that it lies into the core of the partition of maximum value P^*. Note that, in a star-tree game, any coalition $S \in C(N)$ of cardinality $|S| \geqslant 2$ contains the central node. (Hereafter player 1 denotes the central node). Hence, any partition $P \in \mathcal{P}(N)$ takes the form $P = \{\{S\}, \{i\}, ..., \{j\}\}$, $1 \in S$ and $i \neq j \neq 1$. Now, consider the following allocation $x \in \mathbb{R}^{|N|}$: $x_i = v(\{i\})$ for $i \in N \backslash \{1\}$ and $x_1 = \sum_{S \in P^*} v(S) - \sum_{i \in N \backslash \{1\}} v(\{i\})$. Note that allocation x satisfies $x(S) = v(S)$ for all $S \in P^*$. Furthermore, suppose that, for some $T \neq S$, $S \in P^*$, $T \subset N$ with $|T| \geqslant 2$ we have $x(T) = \sum_{S \in P^*} v(S) - \sum_{i \in N \backslash \{1\}} v(\{i\}) + \sum_{i \in T \backslash \{1\}} v(\{i\}) < v(T)$. Since $v(T) + \sum_{i \in N \backslash T} v(\{i\}) < \sum_{S \in P^*} v(S)$, we have $x(T) > v(T)$. Therefore $v(T) \leqslant x(T)$ for all $T \subseteq N$. □

The next numerical example illustrates Proposition 3.

Example 4 Let $N = \{1,2,3,4,5\}$ be the set of municipalities located at the nodes of a star-tree-graph, with $\alpha_i = 2$ for all $i \in N$; the distances are $d(1,2) = 1.2$, $d(1,3) = 1.2$, $d(1,4) = 1.8$, $d(1,5) = 2.2$; and $q = 0.4$. The characteristic function v is $v(1) = v(2) = v(3) = v(4) = v(5) = 2$, $v(12) = 6.91$, $v(13) = 6.91$, $v(14) = 5$, $v(15) = 4.26$, $v(123) = 8.93$, $v(124) = 7.47$, $v(125) = 6.76$, $v(134) = 7.47$, $v(135) = 6.76$, $v(145) = 5.94$, $v(1234) = 10.15$, $v(1235) = 9.44$, $v(1245) = 8.56$, $v(1345) = 8.56$, $v(12345) = 11.32$, and for the value of non-connected coalitions the sum of the values of its components. In this example the partition of maximum value is $P^* = \{\{123\},\{4\},\{5\}\}$ and $\sum_{S\in P^*} v(S) = 12.93$. The allocation $x_i = 2$ for all $i \in N\backslash\{1\}$ and $x_1 = 4.93$ is in the core of the game (N, v, P^*). In other words, the stable geographical structure for the provision of the service requires three points of provision, one for each of the following coalitions: $\{123\}$, $\{4\}$, and $\{5\}$. The service for coalition $\{123\}$ will be located in the central node or municipality.

4 Discussion

Before concluding we comment upon some of the main assumptions of the model. Section 4.1 contains a brief discussion about the assumption that the municipalities are only differentiated through their locations. In Section 4.2 we point out that our results are robust to a more general geographical setting. Finally, Section 4.3 is devoted to the class of public services that fits our model.

4.1 The size of the municipalities

We have analyzed a situation in which a central government has to decide on the provision and location of an emergency public service in order to maximize welfare. This emergency service has to attend a group of identical municipalities, which are only differentiated through their locations. In other words, it is assumed that the municipalities have the same size, the same power of decision in sharing the benefits,... Nevertheless, we argue that our results do not change qualitatively once the size or power of the municipality is taken into account. One way to differentiate the municipalities relatively to their size or power is to weight the distance separating them. In this case, the aggregate distance between a coalition of municipalities and a location of the service is defined as a weighted sum of the individual distances in which the weight of each municipality represents its relative size or power. This case has been studied in an extended version. By comparing our egalitarian distribution of power with a distribution in which more weight is given to the central municipality, we found that the minimum quality of service necessary for the efficiency of the joint provision is smaller the more powerful (or the bigger) is the central municipality.

4.2 The geographical setting

For sake of simplicity we have presented our results using, as geographical setting, a star-tree-graph. Indeed, our results obtained for this radial geographical structure are robust to any other tree structure. For example, to compute the minimum quality of service compatible with the joint provision of service (Proposition 2) we have only to take into account the partitions of N formed by the different connected coalitions of municipalities. Moreover, Grafe *et al.* [1998] have shown that, for the class of tree games, the efficient coalition structure is always stable (in the sense that the core of the game associated to that efficient coalition structure is nonempty) (Proposition 3).

4.3 The emergency public services

We have restricted our analysis to the class of emergency public services. Therefore, we have assumed that the utility function of any municipality depends negatively on the aggregate distance between the coalition it belongs to and the service location. Nevertheless, our model can be easily adapted to analyze the provision of other classes of public services (hospitals, schools...). In these cases, the utility function of each municipality also depends on the distance between the municipality and the location of the service. But then, the optimal location would not be invariant to the specification of the government welfare function.

5 Concluding remarks

Up until now, the existing models on public services in which distance has a role, have developed powerful algorithms in order to solve locational-allocation aspects. In this type of literature a-priori levels of provision are assumed, putting aside decision problems such as the determination of the optimal provision and pricing (see e.g. Granot [1987]). Lea [1983] presented an evaluation of these models and pointed out the convenience of studying the locational-allocation aspects within the scope of the theory of public goods. This is precisely what we have done in this paper.

We have considered an economy in which a central government has to decide on the location and provision of an emergency public service the quality of which diminishes with distance. We have derived the optimal level of the service and the Lindahl prices that sustain it. Since we have assumed convex technology of production the optimal provision of the service generates benefits, the distribution of which remains to be solved. To this respect, a question we have answered is whether there exists a distribution of benefits sustaining the optimal level of service so that joint provision is stable. A distribution of benefits capable of sustaining the joint provision of services may not exist when the distance affects quality too strongly. Also, we have determined the minimum quality that allows the sharing of benefits to sustain the optimal provision.

But, if the quality of service does not permit the joint provision of the emergency service, alternative structures of provision are required. We have shown that for any quality of service there always exists a stable structure determining the points of provision for the emergency public service.

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