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A Note on The Intensity of Downside Risk Aversion

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Abstract

In this note we show that the measure of intensity of downside risk aversion proposed recently by Crainich and Eeckhoudt (2007) cannot be guaranteed to exist. We do this by means of an example in which the existence of the measure depends upon the values of the parameters in the problem.

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1 Introduction

The issue of downside risk aversion has been subject to a fair amount of study lately. One particular issue that has proven to me of importance is the measure of the intensity of downside risk aversion, that is, the study of the utility characteristics that differentiate between individuals according to who is the more downside risk averse. To that end, in a recent paper Crainich and Eeckhoudt (2007), from now on CE, have proposed that the ratio of the third derivative of utility to the first derivative captures the intensity of downside risk aversion. While this particular mesaure is not new to the literature (see Menezes et al. (1980) for the seminal paper in which this measure is mentioned, and also Modica and Scarsini (2005)), the manner in which CE derive the measure is novel. However, as we shall show in this note, there is no guarantee that the method proposed by CE actually works generally.

1.1 A quick overview of downside risk aversion

Consider two lotteries: the first one (the primary risk) gives a one-half chance of the loss of $k > 0$ (the "downside") and a one-half chance of a gain of 0 (the "upside"); the second lottery is defined by a random variable $\widetilde{\varepsilon}$ with zero mean $E\widetilde{\varepsilon} = 0$ (the secondary risk), and it can be placed either on the "upside" or the "downside" of the first risk. Then:

1. If $\widetilde{\varepsilon}$ is on the upside, it gives a expected utility of

$$E_U = \frac{1}{2}Eu(x + \widetilde{\varepsilon}) + \frac{1}{2}u(x - k).$$

2. If $\widetilde{\varepsilon}$ is on the downside, it gives a expected utility of

$$E_D = \frac{1}{2}u(x) + \frac{1}{2}Eu(x - k + \widetilde{\varepsilon}).$$

where x is the initial wealth.

We assume that marginal utility is convex (i.e. $u''' > 0$). From Jensen's inequality for any strictly convex function v and any random variable $\widetilde{y}$ we know that $v(E\widetilde{y}) < Ev(\widetilde{y})$. Using $v = u'$ and the random variable $\widetilde{y} = w + \widetilde{\varepsilon}$ we get:

$$u'(w) = u'(w + E\widetilde{\varepsilon}) = u'(E(w + \widetilde{\varepsilon})) \leq Eu'(w + \widetilde{\varepsilon}) \Longrightarrow Eu'(w + \widetilde{\varepsilon}) - u'(w) \geq 0$$

which implies that $J(w) = Eu(w + \widetilde{\varepsilon}) - u(w)$ is an increasing function in w. Then, we conclude that:

$$E_U - E_D = \frac{1}{2}[Eu(x+\widetilde{\varepsilon}) - u(x)] + \frac{1}{2}[u(x-k) - Eu(x-k+\widetilde{\varepsilon})] = \frac{1}{2}[J(x) - J(x-k)] > 0$$

that is, E_U (the expected utility of having $\widetilde{\varepsilon}$ on the upside) is always greater than E_D (the expected utility of having it on the downside), situation called downside risk aversion.

In order to measure the intensity of downside risk aversion in terms of the shape of the utility function u, CE place the zero mean lottery on the less preferred downside of the primary lottery, but then introduce a compensation given by an amount of money (m) received in the upside of the primary lottery, such that the decision maker is indifferent to having the zero mean risk placed directly on the upside. That is, m must satisfy:

$$\frac{1}{2}Eu(x + \widetilde{\varepsilon}) + \frac{1}{2}u(x - k) = \frac{1}{2}u(x + m) + \frac{1}{2}Eu(x - k + \widetilde{\varepsilon})$$

Then using a second-order Taylor expansion, CE show that m can be expressed as a function of $\frac{u'''}{u'}$, and therefore $\frac{u'''}{u'}$ can be taken as being a measure of the intensity of downside risk aversion, since the greater is $\frac{u'''}{u'}$, the greater would have to be the upside compensation, m, for having the risk located on the downside.

2 A problem with this approach

The CE approach has been criticised in the literature. For example, Keenan and Snow (2009) note that an increase in the intensity measure proposed by CE is "neither necessary nor sufficient for greater downside risk aversion, whether for small or large changes in risk preference." Here we shall concentrate on a separate issue, related to the very existence of the compensation proposed by CE.

Although E_U is greater than E_D and introducing the compensation m we can always increase the value of $E_D(m) = \frac{1}{2}u(x+m) + \frac{1}{2}Eu(x-k+\widetilde{\varepsilon})$, it is not clear that the curve $E_D(m)$ can always reach the value E_U. The curve $E_D(m)$ starts from an initial value $E_D(0) < E_U$, and $E_D(m)$ is an increasing, concave function, but it is possible (as we will see) that $E_D(m)$ never reaches the value E_U, even for infinite compensation levels.

We only need a counter-example to prove this result. To that end, let us consider the typical CARA utility function:

$$u(w) = C - \alpha e^{-\rho w}$$

where $\rho > 0$ represents the absolute risk aversion; then:

$$\begin{aligned}
u'(w) &= \alpha\rho e^{-\rho w} > 0 \\
u''(w) &= -\alpha\rho^2 e^{-\rho w} < 0 \\
u'''(w) &= \alpha\rho^3 e^{-\rho w} > 0
\end{aligned}$$

Let us also consider the uniform random variable $\widetilde{\varepsilon}$ defined by the density function $f(\varepsilon) = \frac{1}{2}$ over the interval $[-1, 1]$. Clearly, this is a zero mean random variable.

The expected utility of wealth $w + \widetilde{\varepsilon}$ is given by

$$\begin{aligned}
Eu(w+\widetilde{\varepsilon}) &= \int_{-1}^{1} u(w+\varepsilon)f(\varepsilon)d\varepsilon \\
&= \frac{1}{2}\int_{-1}^{1}[C-\alpha e^{-\rho(w+\varepsilon)}]d\varepsilon \\
&= \frac{1}{2}[2C-\alpha e^{-\rho w}\int_{-1}^{1} e^{-\rho\varepsilon}d\varepsilon] \\
&= C+\frac{1}{2\rho}\alpha e^{-\rho w}\ e^{-\rho\varepsilon}\Big|_{\varepsilon=-1}^{\varepsilon=1} \\
&= C+\frac{1}{2\rho}\alpha e^{-\rho w}(e^{-\rho}-e^{\rho})
\end{aligned}$$

The upside of the primary lottery is the case in which $w = x$, while the downside is $w = x - k$. Thus, placing the zero mean lottery on the upside gives

$$\begin{aligned}
E_U &= \frac{1}{2}Eu(x+\widetilde{\varepsilon})+\frac{1}{2}u(x-k) \\
&= \frac{1}{2}[C+\frac{1}{2\rho}\alpha e^{-\rho x}(e^{-\rho}-e^{\rho})+C-\alpha e^{-\rho(x-k)}] \\
&= C+\frac{\alpha}{2}e^{-\rho x}[\frac{1}{2\rho}(e^{-\rho}-e^{\rho})-e^{\rho k}]
\end{aligned}$$

On the other hand, placing the zero mean lottery on the downside of the primary lottery, with compensation of m on the upside, gives

$$\begin{aligned}
E_D(m) &= \frac{1}{2}u(x+m)+\frac{1}{2}Eu(x-k+\widetilde{\varepsilon}) \\
&= \frac{1}{2}[C-\alpha e^{-\rho(x+m)}+C+\frac{1}{2\rho}\alpha e^{-\rho(x-k)}(e^{-\rho}-e^{\rho})] \\
&= C+\frac{\alpha}{2}e^{-\rho x}[-e^{-\rho m}+\frac{1}{2\rho}e^{\rho k}(e^{-\rho}-e^{\rho})]
\end{aligned}$$

Finally, then, the difference between these two is

$$\begin{aligned} E_U - E_D(m) &= \frac{\alpha}{2}e^{-\rho x}[\frac{1}{2\rho}(e^{-\rho} - e^{\rho}) - e^{\rho k} + e^{-\rho m} - \frac{1}{2\rho}e^{\rho k}(e^{-\rho} - e^{\rho})] \\ &= \frac{\alpha}{2}e^{-\rho x}[e^{-\rho m} - e^{\rho k} + \frac{1}{2\rho}(e^{-\rho} - e^{\rho})(1 - e^{\rho k})] \end{aligned}$$

In consequence, the sign of $E_U - E_D(m)$ is given by the sign of the function:

$$R(m) = e^{-\rho m} - e^{\rho k} + \frac{1}{2\rho}(e^{-\rho} - e^{\rho})(1 - e^{\rho k}).$$

We are looking for a level of compensation, say $\widehat{m}$, such that $E_U - E_D(\widehat{m}) = 0$, which is the same as saying that we are searching for $\widehat{m}$ such that $R(\widehat{m}) = 0$.

If no compensation is given, $m = 0$, then we should expect that $R > 0$, that is placing the zero mean risk on the upside is better than placing it on the downside. This can be confirmed by noting that:

$$\begin{aligned} R(0) &= 1 - e^{\rho k} + \frac{1}{2\rho}(e^{-\rho} - e^{\rho})(1 - e^{\rho k}) \\ &= (1 - e^{\rho k})[1 + \frac{e^{-\rho} - e^{\rho}}{2\rho}] \end{aligned}$$

But since $G(\rho) = e^{-\rho} - e^{\rho}$ is concave $(G''(\rho) = G(\rho) < 0)$ and decreasing, we have:

$$G(\rho) < G(0) + G'(0)\rho = -2\rho \Longrightarrow 1 + \frac{e^{-\rho} - e^{\rho}}{2\rho} < 1 - \frac{2\rho}{2\rho} = 1 - 1 = 0$$

Since for any positive number ρk we have $1 - e^{\rho k} < 0$, it turns out that $R(0)$ is the product of two negative numbers, and so as expected we have $R(0) > 0$.

It also happens that

$$\begin{aligned} R'(m) &= -\rho e^{-\rho m} < 0 \\ R''(m) &= \rho^2 e^{-\rho m} > 0 \end{aligned}$$

that is $R(m)$ is decreasing and convex in m. Thus, considering only non-negative values of m, we know that $R(m)$ starts off positive, and then decreases but at a diminishing rate. It is not clear that such a function will always reach a value of 0.

If the function were to reach a value of 0 for some m, then it would be negative as m goes infinite. We therefore consider the value of

$$\lim_{m \longrightarrow \infty} R(m) = -e^{\rho k} + \frac{1}{2\rho}(e^{-\rho} - e^{\rho})(1 - e^{\rho k}) \equiv H(\rho, k).$$

The existence of a value $\widehat{m}$ such that $R(\widehat{m}) = 0$ depends on the sign of $H(\rho, k)$.

But $H(\rho, k)$ can be positive or negative depending on the values of ρ and k. For example, taking $k = 1$ we get:

$$H(\rho, 1) = -e^{\rho} + \frac{1}{2\rho}(e^{-\rho} - e^{\rho})(1 - e^{\rho})$$

The graph of this function is shown in Figure 1 ($H(\rho, 1) = 0$ for $\rho = 1.36$):

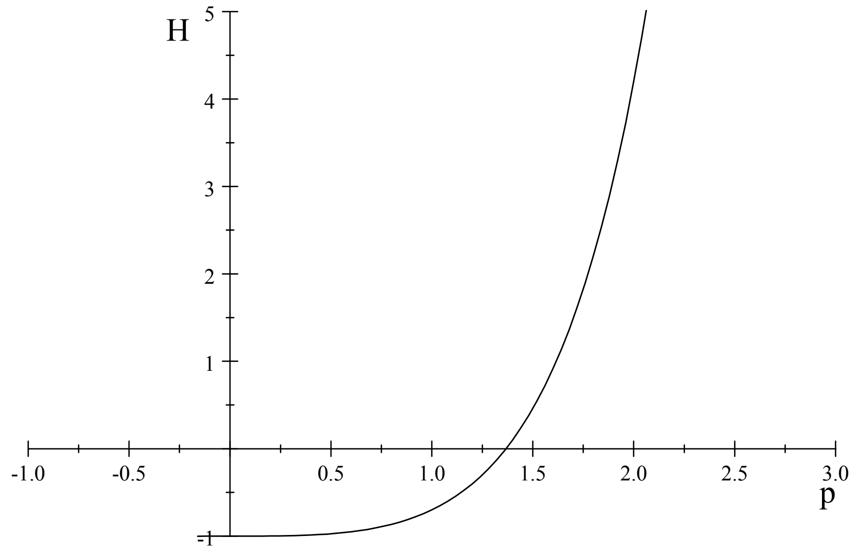

Figure 1

In consequence, if $k = 1$ and $\rho > 1.36$ (recall that ρ is the measure of constant absolute risk aversion) there is no $\widehat{m}$ such that $R(\widehat{m}) = 0$, that is, such that $E_U - E_D(\widehat{m}) = 0$. It is an empirical matter whether or not absolute risk aversion of 1.36 is reasonable, but in any case whatever is the level of absolute risk aversion required, one can always find a value of k such that $R(m) > 0$ for all m.

Figure 2 shows two graphs of $E_U - E_D(m)$: the first one (plotted towards the bottom of the figure) for $k = 1$ and $\rho = 1$, and the other one for $k = 1$ and $\rho = 2$:

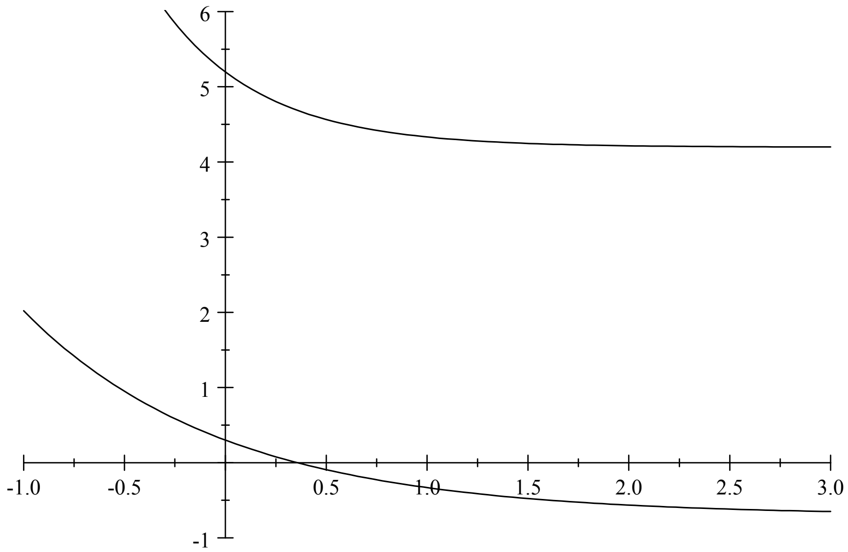

Figure 2

As can be seen, in the first case $E_U - E_D(m)$ is positive for small values of m and negative for larger ones, and so for $k = \rho = 1$ there does exist an $\widehat{m}$. On the other hand, for $k = 1$ and $\rho = 2$ (the top graph), we can see that $E_U - E_D(m) > 0$ for all values of compensation m. In this case, there is no compensation that ever works to equate the two utility levels.

3 Conclusion

In this note we have considered the validity of the Crainich and Eeckhoudt measure for the intensity of downside risk aversion. We find that the method used by Crainich and Eeckhoudt for developing their measure cannot be guaranteed to work generally. We have found a concrete example, using constant absolute risk aversion, for which there is no possible compensation that makes the decision maker indifferent between having the zero-mean risk on the downside and monetary compensation on the upside, and having the zero-mean risk on the upside. Of course, if the required compensation were not to exist, it is not possible to study how it is affected by the shape of the utility function.

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