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Fair allocation of indivisible goods among two agents

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DISCUSSION PAPER

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Abstract

One must allocate a finite set of indivisible goods among two agents without monetary compensation. We impose Pareto-efficiency, anonymity, a weak notion of no-envy, a welfare lower bound based on each agent's ranking of the sets of goods, and a monotonicity property relative to changes in agents' preferences. We prove that there is a rule satisfying these axioms. If there are three goods, it is the only rule, with one of its subcorrespondences, satisfying each fairness axiom and not discriminating between goods. Further, we confirm the clear gap between these economies and those with more than two agents.

Keywords: indivisible goods, no monetary compensation, no-envy, lower bound, preference-monotonicity.

JEL Classification: D61, D63

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1 Introduction

We study allocation problems of indivisible goods among two agents when monetary compensation is impossible or not customary. Such problems are frequent. Think of family members who must allocate goods inherited from relatives (as handkerchiefs, chairs, tools...) among themselves, managers who must assign tasks or responsibilities among the direction board of their firm, or city councils that must share time blocks between users of a facility. Agents may get more than one good. Preferences over subsets of goods are strict and additively separable. Goods are desirable.

Our approach is axiomatic. The objective is to identify allocation *rules*, i.e. mappings that systematically give one allocations, satisfying (Pareto-)*efficiency* and axioms embodying fairness fundamentals. Agents' names should not matter. No agent should prefer another's bundle to her own. One should secure a minimal welfare level to each agent. As the consumption of each good is private, differences in preferences may generate welfare surplus. The less similar another's preferences are, the weakly better off an agent should be, hence the more similar another's preferences are, the weakly worse off she should be.

We prove that there is such a rule. If there are three goods, it is the only rule, together with one of its subcorrespondences, that is desirable according to each fairness property and that does not discriminate between goods.

To reach our objective, we use *anonymity* embodying the first fairness property and we introduce three axioms embodying the other three fairness properties resp. First is *conditional no-envy*, i.e. if possible and not contradicting *efficiency*, a rule should select *envy-free* allocations. Second is the *unanimity bound*, i.e. each agent should find her bundle at least as good as to the worst bundle of the allocations *efficiency* and minimization of inequality among equals recommend when each other agent has her preferences. Third is *preference-monotonicity*, i.e. if an agent's preferences change such that she now disagrees with another on at least one pair of subsets in addition to those they previously disagreed on, the latter should not find herself worse off on average.

We identify this rule. For each problem, the *maximin rule* maximizes, across allocations, the minimal rank, across agents, of an allocation, where the rank of an agent's bundle is its position in her preferences, from worst to best of all subsets. This rule embodies the fairness fundamental according to which one should first care for the least fortunates. As it uses ordinal information on preferences over subsets, it does not assume specific real-valued functions representing preferences nor welfare comparability, and it takes each good it allocates into account. Further, as there is a procedure such that an allocation is a solution of it if and only if it is an allocation the maximin rule selects (Herreiner and Puppe, 2002), one easily applies it.[1]

The literature on fair allocation problems of possibly several indivisible goods per agents without monetary compensation studies two of these fairness fundamentals, namely avoiding envy and caring for the least fortunates (Brams and Fishburn, 2000; Edelman and Fishburn, 2001; Brams *et al.*, 2003; Brams and King, 2005). It gives an in-depth study of necessary and sufficient for *envy-free*, and *efficient* and *envy-free* allocations to be, and an estimation of the likelihood of such allocations. If e.g. agents have the same most preferred good and prefer it to the subset containing all other goods, no allocation is *envy-free*. Thus, it is best to study other fairness fundamentals. The literature studies the existence of allocations in which one cares for the least fortunates, and of *efficient* and/or *envy-free* such allocations. Further, it gives procedures, in particular yielding *envy-free* allocations if they exist (also Herreiner and Puppe, 2002; Brams *et al.*, 2009). Existence and compatibility depend on the number of agents, from two to more, number of goods, and preferences. We detail results and relate them to ours in the Concluding Comments.[2] Besides, the literature studies solidarity properties w.r.t. changes in the set of goods, agents, or in preferences (Klaus and Miyagawa, 2001; Elhers and Klaus, 2003). They may impose selecting allocations that are not *efficient* nor *envy-free* even if these exist.

The literature is mute regarding fairness fundamentals we impose, namely those the *unanimity bound* and *preference-monotonicity* embody. The former is frequent in classical problems and those with both perfectly divisible and indivisible goods (Steinhaus, 1948; Moulin, 1990a/b, 1991, 1992; Beviá, 1996). The latter appears in problems with public goods (Sprumont, 1993). Yet, their formulation crucially depends on the problem one studies. Further, though avoiding envy or caring for the least fortunates appear as the properties one first and foremost associates with fairness in such problems, there is no axiomatic study of these properties nor, obvious from *supra*, of lower bounds or monotonicity w.r.t changes in preferences.

The main lesson we draw from our study is as follows. The possible non-existence of *envy-free* allocations urges to study other fairness fundamentals. The literature focuses on caring for the least fortunates. Other such fundamentals are as essential. In particular, agents' names should not matter, one should secure each agent with a minimal welfare level, and appropriately share welfare surplus due to differences in preferences. These fundamentals, together with avoiding envy, induce one to care for the least fortunates.

We explicitly study two-agent problems. Indeed, there is a clear gap between these and those with more than two. In the former case, not in the latter, if *envy-free* allocations exist, at least one is *efficient*. In the former case, one may explicitly identify the worst bundle of the allocations *efficiency* and minimization of inequality among equals recommend when each other agent has her preferences, hence precisely determine the minimal welfare level the *unanimity bound* secures to each agent. In the latter case, the maximin rule violates *efficiency* and as each of its subcorrespondences, *conditional no-envy*. This clear gap leads one to regard two-agent problems and those with more than two as different, with different problems calling for different solutions. We leave the question of identify desirable rules for problems with more than two agents, for further research. Yet, we give hints and conjectures.

We end this section with an example. Two siblings inherit from a relative. They have to share an aquarium, a book, and a coat. Both prefer the aquarium to the book, and the book to the coat. Fairness considerations intuitively lead to the following conclusion. Each should get at least one good. If one gets the aquarium, the other should get the book. Possibly, in addition to the book, she should get the coat. If both prefer the aquarium to the book and the coat together or vice versa, one could allocate these bundles either way. Yet, if one prefers the aquarium to the book and the coat together and the other prefers the book and the coat together to the aquarium, *efficiency* requires to allocate these bundles accordingly. To design rules satisfying *efficiency* and axioms embodying fairness fundamentals, one must consider preferences over subsets of goods rather than only over goods.[3]

In Sect. 2, we formally introduce the model. In Sect. 3, we define the axioms we impose on rules. In Sect. 4, we prove the maximin rule is desirable, if not the only one. In Sect. 5, we study the gap between two-agent problems and those with more than two. We formulate the definitions of Sect. 1 and 2 for each set of agents, possibly with more than two elements. Doing so, the axioms we introduce, extend to these problems.

2 Model

There is a non-empty and finite set of indivisible goods A to allocate among a set of agents N with $\#A > \#N = 2$.[4] Each agent $i \in N$ has a complete and transitive preference relation R_i over the set of all subsets of goods $\mathcal{S}$. Let P_i and I_i be the strict preference and indifference relation associated with R_i resp. We assume *desirability*, i.e. for each $\alpha \in A$, we have $\{\alpha\}\ R_i\ \emptyset$; *additive separability*, i.e. there is a real-valued function $u : A \cup \emptyset \to \mathbb{R}$ fitting R_i, in the sense, $u(\emptyset) = 0$ and for each $S, T \in \mathcal{S}$, we have $\sum_{\alpha \in S} u(\alpha) \geq \sum_{\alpha \in T} u(\alpha)$ if and only if $S\ R_i\ T$; and *strictness*, i.e. for each $S, T \in \mathcal{S}$ with $S \neq T$, we have $S\ P_i\ T$ or $T\ P_i\ S$.

Let $\mathcal{R}$ be the set of all preferences and $\mathcal{R}^N \equiv \times_{i \in N} \mathcal{R}$ the set of all preference profiles. We do not study effects of changes in the set of goods nor agents. For simplicity, an *economy* is a preference profile $\mathbf{R} \equiv (R_i)_{i \in N} \in \mathcal{R}^N$. For each $\mathbf{R} \in \mathcal{R}^N$ and each $i \in N$, let $\mathbf{R_{-i}} \equiv (R_j)_{j \in N \setminus \boldsymbol{i}} \in \mathcal{R}^{N \setminus \boldsymbol{i}}$ be the preference profile of all agents but i w.r.t. $\mathbf{R}$ and $\mathbf{R_i} \equiv (R_i, ..., R_i) \in \mathcal{R}^N$ the unanimity preference profile of i w.r.t. $\mathbf{R}$.

An *allocation* $\mathbf{x} \equiv (x_i)_{i \in N}$ is a list of bundles with $\cup_{i \in N} x_i \subset A$, $\cup_{i \in N} x_i \neq \emptyset$, and for each $i, j \in N$, if $i \neq j$, then $x_i \cap x_j = \emptyset$. Let X be the set of all allocations. An (allocation) *rule* Φ is a correspondence associating with each economy $\mathbf{R} \in \mathcal{R}^N$, a non-empty set of allocations $\Phi(\mathbf{R}) \subset X$.

For each $i \in N$ and each $R_i \in \mathcal{R}$, we formalize the *rank* she associates to each subset, from worst to best, w.r.t. R_i, with the bijection $r(.; R_i) : \mathcal{S} \leftrightarrow \{1, ..., 2^{\#A}\}$ such that for each $S, T \in \mathcal{S}$, we have $r(S; R_i) > r(T; R_i)$ if and only if $S\ P_i\ T$. We gather and illustrate properties of such a bijection in the next lemma and example resp.[5]

Before, we introduce notational shortcuts. Let $\#A \equiv a$, $\#N \equiv n$, for each $S \in \mathcal{S}$, let $\#S \equiv s$ and $S^c \equiv A \setminus S$, and for each $Y \subset X$, let $\#Y \equiv y$. For each $\alpha, ..., \beta \in A$, let $\boldsymbol{\alpha...\beta} \equiv \{\alpha, ..., \beta\}$ and for each

Figure 1:

	R_1	R_2
8	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$
7	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha\beta}$
6	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\gamma}$
5	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\alpha}$
4	$\boldsymbol{\alpha}$	$\boldsymbol{\beta\gamma}$
3	$\boldsymbol{\beta}$	$\boldsymbol{\beta}$
2	$\boldsymbol{\gamma}$	$\boldsymbol{\gamma}$
1	$\emptyset$	$\emptyset$

$i, j \in N$, let $\boldsymbol{ij} \equiv \{i, j\}$. For each $i \in N$ and each $S \in \mathcal{S}$, for $R_i \in \mathcal{R}$, let $r_i(S) \equiv r(S; R_i)$; for $R'_i \in \mathcal{R}$, let $r'_i(S) \equiv r(S; R'_i)$; and so on.

Lemma 1 *For each $i \in N$ and each $R_i \in \mathcal{R}$,*

- $r_i(A) = 2^a$;
- $r_i(\emptyset) = 1$;
- *For each $S \in \mathcal{S}$, we have $r_i(S) + r_i(S^c) = 2^a + 1$;*
- *For each $S, T \in \mathcal{S}$, we have $r_i(S) > r_i(T)$ if and only if $r_i(S^c) < r_i(T^c)$.*

Ex. 1 Let 1, 2, α, β, γ denote the siblings, aquarium, book, and coat of the Introduction resp. Then, $N = \mathbf{12}$, $A = \boldsymbol{\alpha\beta\gamma}$, and $R_1, R_2 \in \mathcal{R}$ are as in Fig. 1. There is one row for each subset, one column for each sibling. The 1st column gives the rank each sibling associates to each subset. One's way to rank subsets does not depend on the other's. E.g. 1 ranks $\boldsymbol{\beta\gamma}$ as her 5th least preferred subset and 2 ranks it 4th, i.e. $r_1(\boldsymbol{\beta\gamma}) = 5$ and $r_2(\boldsymbol{\beta\gamma}) = 4$. By Lem. 1, $r_1(\boldsymbol{\alpha\beta}) > r_2(\boldsymbol{\beta\gamma})$ if and only if $r_1(\boldsymbol{\gamma}) < r_2(\boldsymbol{\alpha})$.□

3 Axioms

We now define the axioms we impose on rules. Let Φ be a rule.

Efficiency is standard. There should be no allocation that each agent finds at least as good as a selected allocation and at least one agent prefers. Allocation $\mathbf{x} \in X$ is (Pareto-)*efficient* for $\mathbf{R} \in \mathcal{R}^N$ if there is no $\mathbf{y} \in X$ with for each $i \in N$, we have $y_i \; R_i \; x_i$ and for at least one $i \in N$, we have $y_i \; P_i \; x_i$. By desirability, for each $\mathbf{R} \in \mathcal{R}^N$ and each $\mathbf{x} \in X$, we have $\mathbf{x}$ is *efficient* for $\mathbf{R}$ if and only if there is no *free disposal*, i.e. $\cup_{i \in N} x_i = A$. For each $\mathbf{R} \in \mathcal{R}^N$, let $P(\mathbf{R})$ be the set of all *efficient* allocations for $\mathbf{R}$.

(Pareto-)efficiency: *For each $\mathbf{R} \in \mathcal{R}^N$, we have $\Phi(\mathbf{R}) \subset P(\mathbf{R})$.*

Fairness is as follows. First, agents' names should not matter. If one permute agents' preferences, one should permute the selected bundles accordingly. Let G be the set of all permutations on N.

Anonymity: *For each $\mathbf{R} \in \mathcal{R}^N$, each $\mathbf{x} \in \Phi(\mathbf{R})$, and each $g \in G$, we have $(x_{g(i)})_{i \in N} \in \Phi((R_{g(i)})_{i \in N})$.*

Second, no agent should prefer another's bundle to her own. Allocation $\mathbf{x} \in X$ is *envy-free* for $\mathbf{R} \in \mathcal{R}^N$ if there is no $i \in N$ with for $j \in N$, if $j \neq i$, then $x_j \; P_i \; x_i$. For each $\mathbf{R} \in \mathcal{R}^N$, let $F(\mathbf{R})$ be the set of all *envy-free* allocations for $\mathbf{R}$ and $PF(\mathbf{R}) \equiv P(\mathbf{R}) \cap F(\mathbf{R})$ the set of all *efficient* allocations for $\mathbf{R}$ of this set. If e.g. agents have the same most preferred good and prefer it to the subset containing all other goods, no allocation is *envy-free*.[6] If such an allocation exists, it need not be *efficient*. We introduce a weaker notion of *no-envy*. If possible and not contradicting *efficiency*, a rule should select *envy-free* allocations.

Conditional no-envy: *For each* $\mathbf{R} \in \mathcal{R}^N$ *with* $PF(\mathbf{R}) \neq \emptyset$*, we have* $\Phi(\mathbf{R}) \subset F(\mathbf{R})$.

Third, one should secure a minimal welfare level to each agent. This level should be as high as possible. Also, it should be decentralized, in the sense, it should only depend on the feasibility constraints of the economy and on the agent's own characteristics. To formalize this idea, consider what follows (Moulin, 1990a/b, 1991, 1992).

Ex. 1 (cont.) Consumption of each good is private. The more 2 differs from 1, the more welfare may one simultaneously secure for each of the two. Thus, 1 should find herself at least as good as when 2 has her preferences. Assume this is the case, i.e. consider $(R_1, R_1) \in \mathcal{R}^N$. One should select *efficient* allocations treating these agents with equal preferences equally. As each good is indivisible and each agent has strict preferences, it is impossible to allocate all goods giving these equal agents equal bundles or bundles they find indifferent. To treat them as equally as possible, one should allocate all goods minimizing how unequal bundles are, maximizing how well off the envier is. There are the allocations consisting of bundles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta\gamma}$. The worst bundle of such an allocation according to R_1 is $\boldsymbol{\alpha}$. Thus, in (R_1, R_2), one should select allocations such that 1 finds her bundle at least as good as $\boldsymbol{\alpha}$. By the same logic, in (R_1, R_2), agent 2 should find her bundle at least as good as $\boldsymbol{\beta\gamma}$. E.g. there is $(\boldsymbol{\beta\gamma}, \boldsymbol{\alpha}) \in X$ with $\boldsymbol{\beta\gamma}\ R_1\ \boldsymbol{\alpha}$ and $\boldsymbol{\alpha}\ R_2\ \boldsymbol{\beta\gamma}$.□

We require each agent to find her bundle at least as good as to the worst bundle of the allocations *efficiency* and minimization of inequality among equals recommend when each other agent has her preferences. For each $i \in N$, we formalize the maximal minimal rank across X and N resp. for a unanimity profile w.r.t. i with the function $\check{r}(.;i) : \mathcal{R}^N \to \{1, ..., 2^a\}$ such that for each $\mathbf{R} \in \mathcal{R}^N$, we have $\check{r}(\mathbf{R}; i) \equiv \max_{\mathbf{x} \in X} \min_{j \in N} r_i(x_j)$. This rank only depends on the feasibility constraints of the economy which a preference profile characterizes and on i's own preferences, not on others' preferences. To emphasize this, we use the following notational shortcut. For each $\mathbf{R} \in \mathcal{R}^N$ and each $i \in N$, let $\check{r}(\mathbf{R_i}) \equiv \check{r}(\mathbf{R}; i)$. (As in $\mathbf{R}$, feasibility constraints are implicit in $\mathbf{R_i}$.) Allocation $\mathbf{x} \in X$ *meets the unanimity bound* for $\mathbf{R} \in \mathcal{R}^N$ if for each $i \in N$, we have $r_i(x_i) \geq \check{r}(\mathbf{R_i})$. For each $\mathbf{R} \in \mathcal{R}^N$, let $B(\mathbf{R})$ be the set of all allocations meeting the *unanimity bound* for $\mathbf{R}$ and $PB(\mathbf{R}) \equiv P(\mathbf{R}) \cap B(\mathbf{R})$ the set of all *efficient* allocations for $\mathbf{R}$ of this set.

Unanimity bound: *For each* $\mathbf{R} \in \mathcal{R}^N$*, we have* $\Phi(\mathbf{R}) \subset B(\mathbf{R})$.

This axiom secures each agent with a minimal welfare level that is the highest and decentralized. As it sets this level in terms of welfare associated to a subset of goods, it applies to economies without compensating means. As it measures this level in terms of ranks, it only requires ordinal information on preferences. As it requires a minimal welfare level, it is compatible with *efficiency*.

Fourth, as consumption of each good is private, differences in preferences may generate welfare surplus. The less similar another's preferences are, the weakly better off an agent should be, hence the more similar another's preferences are, the weakly worse off she should be. To formalize this idea, consider what follows.

Ex. 1 (cont.) Let $R_1', R_2^{\wedge}, R_2^{+}, R_2^{*}, R_2^{\bullet}, R_2^{\circ} \in \mathcal{R}$ be as in Fig. 2. As 1 and 2 disagree in (R_1', R_2°) on pairs of subsets they agree on in $(R_1', R_2^{\wedge})$ (e.g. $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$), one should not say R_2° not more similar to R_1' than $R_2^{\wedge}$. As 1 and 2 disagree in $(R_1', R_2^{\wedge})$ on pairs of subsets they agree on in (R_1', R_2°) (e.g. $\boldsymbol{\beta}$ and $\boldsymbol{\alpha\gamma}$), one should say $R_2^{\wedge}$ not more similar to R_1' than R_2°. Thus, $R_2^{\wedge}$ and R_2° are incomparable w.r.t. R_1'. Conversely, as 1 and 2 agree in (R_1', R_2^{+}) on two pairs of subsets in addition to those they agree on in $(R_1', R_2^{\wedge})$ ($\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$, $\boldsymbol{\alpha\beta}$ and $\boldsymbol{\beta\gamma}$), one should say R_2^{+} more similar to R_1' than $R_2^{\wedge}$.□

For each $i, j \in N$ and each $R_i, R_j', R_j^* \in \mathcal{R}$, say R_j^* *closer* to R_i than R_j' if i and j agree in (R_i, R_j^*) on at least one pair of subsets in addition to those they agree on in (R_i, R_j'), i.e. $\{S, T \in \mathcal{S} : S\ R_i\ T$ and $S\ R_j^*\ T\} \supsetneq \{S, T \in \mathcal{S} : S\ R_i\ T$ and $S\ R_j'\ T\}$. In Ex. 1, R_2^{+} is closer to R_1' than $R_2^{\wedge}$. Also, $R_2^{\bullet}$ is closer to R_1' than R_2°, R_2^{*} is closer to R_1' than $R_2^{\bullet}$, hence R_2^{*} is closer to R_1' than R_2°.

Figure 2:

R_1'	$R_2^\wedge$	R_1'	R_2^*	R_1'	$R_2^\bullet$	R_1'	R_2°	R_1'	R_2^+
$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$
$\boldsymbol{\alpha\beta}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha\beta}$
$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\beta\gamma}$
$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$
$\boldsymbol{\beta\gamma}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\gamma}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\gamma}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\alpha\gamma}$
$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\beta}$	$\boldsymbol{\beta}$	$\boldsymbol{\beta}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$
$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\gamma}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\gamma}$	$\boldsymbol{\gamma}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Ex. 1 (cont.) Assume $\Phi(R_1', R_2) = \{(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma}), (\boldsymbol{\beta\gamma}, \boldsymbol{\alpha})\}$ and $\Phi(R_1', R_2^*) = \{(\boldsymbol{\alpha\beta}, \boldsymbol{\gamma}), (\boldsymbol{\alpha}, \boldsymbol{\beta\gamma}), (\boldsymbol{\beta}, \boldsymbol{\alpha\gamma})\}$. Then, R_2 closer to R_1 than R_2^*. To ascertain if the closer 2's preferences are to 1's, the weakly worse off 1 is, one compares non-empty sets of allocations. Preferences are defined on the set of subsets of goods. It is not immediate how preferences on the set of non-empty sets of subsets of goods, hence bundles and *a fortiori* allocations, relate to them. E.g. 1 prefers $\boldsymbol{\alpha\beta}$ to $\boldsymbol{\alpha}$ or $\boldsymbol{\beta\gamma}$, and $\boldsymbol{\alpha}$ or $\boldsymbol{\beta\gamma}$ to $\boldsymbol{\beta}$.$\square$

Selecting a multi-valued set of allocations is a first step. By definition, the allocations it contains are mutually exclusive. *In fine*, only one materializes. The outcome of such a comparison depends on the agent's predictions regarding how, if the rule produces such a tie, it is broken, and on her feeling toward the uncertainty such a context involves. We assume, given the fairness considerations guiding our study, the tiebreaker is an even-chance random device, each agent knows it, and each agent is an expected utility maximizer. Let U be the set of all $u : A \cup \emptyset \to \mathbb{R}$ with for some $i \in N$ and $R_i \in \mathcal{R}$, we have u fitting R_i. For each $i \in N$, each $u \in U$, and each non-empty $Y \subset X$, let $E(Y, u) \equiv \sum_{\mathbf{x} \in Y} y^{-1} u(x_i)$ be i's *expected utility* of Y w.r.t. u. Then, for each $i \in N$, each $R_i \in \mathcal{R}$, each $u \in U$ fitting R_i, and each non-empty $Y, Z \subset X$, agent i finds Y at least as good as Z if and only if $E(Y, u) \geq E(Z, u)$.

One knows each agent's preferences on the set of subsets of goods, not her expected utility of each set of subsets of goods w.r.t. a real-valued function fitting her preferences. We require each agent's expected utility of the selected set of allocations to always be at least equal to the one of the selected set of allocations for each profile consisting of her preferences and another's preferences closer to hers. For each $\mathbf{R} \in \mathcal{R}^N$, each $i, j \in N$ with $i \neq j$, each $R_j' \in \mathcal{R}$ closer to R_i than R_j, and each $u \in U$ fitting R_i, we have $E(\Phi(R_j, \mathbf{R_{-j}}), u) \geq E(\Phi(R_j', \mathbf{R_{-j}}), u)$. This holds if and only if according to her preferences, the selected set of allocations first-stochastically dominates the set of allocations selected for each such profile (Fishburn, 1964).

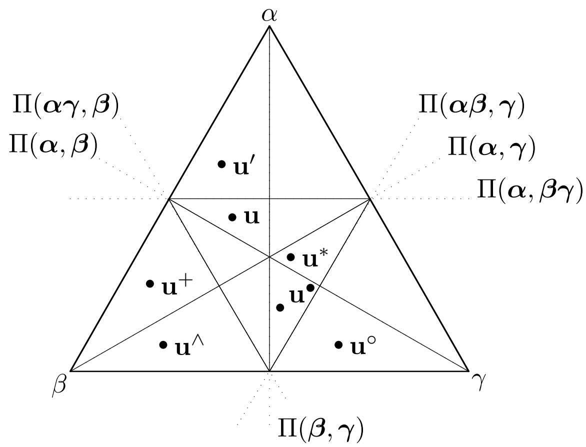
Preference-monotonicity: *For each $\mathbf{R} \in \mathcal{R}^N$, each $i, j \in N$ with $i \neq j$, each $R_j' \in \mathcal{R}$ closer to R_i than R_j, and each $\mathbf{x} \in \Phi(R_j, \mathbf{R_{-j}}) \cup \Phi(R_j', \mathbf{R_{-j}})$, we have*

$$\frac{\#\{\mathbf{y} \in \Phi(R_j, \mathbf{R_{-j}}) : x_i R_i y_i\}}{\#\Phi(R_j, \mathbf{R_{-j}})} \leq \frac{\#\{\mathbf{y} \in \Phi(R_j', \mathbf{R_{-j}}) : x_i R_i y_i\}}{\#\Phi(R_j', \mathbf{R_{-j}})}.$$

Let us come back on the definition of closer. Let $\Delta \equiv \{\mathbf{u} \in \mathbb{R}_+^A : \sum_{\alpha \in A} u_\alpha = 1\}$ be the $(a-1)$-dimensional simplex. Identifying each vertex as a good, each point in Δ gives a ranking of the subsets of goods according to how it is sited w.r.t. each vertex. We may represent each preferences as such a point. (Not conversely: Points in separating hyperplanes gives relations admitting indifferences.) Separating hyperplanes define polyhedrons with all points in their interior fitting the same preferences. For each $S, T \in \mathcal{S}$ with $S \cap T = \emptyset$, let $\Pi(S, T) \equiv \{\mathbf{u} \in \mathbb{R}_+^A : \sum_{\alpha \in S} u_\alpha = \sum_{\alpha \in T} u_\alpha\}$ be the separating hyperplane between S and T.

E.g. we may depict each three-good economy, in particular those of Ex. 1, in an equilateral triangle as in Fig. 3. We identify the top, left, right vertex as α, β, and γ resp. Point $\mathbf{u} \in \Delta$ is such that $u_\alpha > u_\beta$, $u_\alpha > u_\gamma$, $u_\beta > u_\gamma$, $u_\alpha < u_\beta + u_\gamma$, $u_\beta < u_\alpha + u_\gamma$, and $u_\gamma < u_\alpha + u_\beta$. The associated real-valued function $u \in U$,

Figure 3:



i.e. $u(\alpha) \equiv u_\alpha$, $u(\beta) \equiv u_\beta$, and $u(\gamma) \equiv u_\gamma$, fits R_1. As each point in the interior of the smallest triangle including $\mathbf{u}$ is sited as $\mathbf{u}$ w.r.t. each separating hyperplane, the real-valued function associated with each such point, also fits R_1. The real-valued function associated with each point in the interior of the smallest triangle including $\mathbf{u}', \mathbf{u}^\wedge, \mathbf{u}^+, \mathbf{u}^*, \mathbf{u}^\bullet, \mathbf{u}^\circ \in \mathbb{R}_+^A$, fits R_1', $R_2^\wedge$, R_2^+, R_2^*, $R_2^\bullet$, and R_2° resp.

This geometric representation leads to the following reformulation. For each $i, j \in N$ and each $R_i, R_j', R_j^* \in \mathcal{R}$, we have R_j^* closer to R_i than R_j' if and only if there are $\mathbf{u}, \mathbf{u}', \mathbf{u}^* \in \Delta$ with the associated real-valued function fitting R_i, R_j', and R_j^* resp. and the set of hyperplanes between $\mathbf{u}^*$ and $\mathbf{u}$ properly included in the one between $\mathbf{u}'$ and $\mathbf{u}$. The difference between R_j' and R_j^* w.r.t. R_i is a list of consecutive hyperplanes, i.e. a list of consecutive switches between adjacent bundles. Thus, there is a list of preferences starting with R_j', ending with R_j^*, and with each closer to R_i than the preceding one due to switches between adjacent bundles. In Ex. 1, for R_2^* closer to R_1' than R_2°, there is $(R_2^\circ, R_2^\bullet, R_2^*)$.

Lemma 2 *For each $i, j \in N$, each $R_i, R_j', R_j^* \in \mathcal{R}$ with R_j^* closer to R_i than R_j', there are $\overline{q} \in \mathbb{N}$ and $(R_j^q)^{q \in \{1,...,\overline{q}\}} \in \mathcal{R}^{\{1,...,\overline{q}\}}$ with $R_j^1 = R_j'$, $R_j^{\overline{q}} = R_j^*$, and for each $q \in \{2, ..., \overline{q}\}$,*

- *R_j^q closer to R_i than R_j^{q-1};*
- *for each $S, T \in \mathcal{S}$, we have $r_j^q(S) > r_j^q(T)$ and $r_j^{q-1}(S) < r_j^{q-1}(T)$ if and only if $r_j^{q-1}(S) = r_j^q(T)$, $r_j^{q-1}(T) = r_j^q(S)$, and $r_j^q(S) - r_j^q(T) = r_j^{q-1}(T) - r_j^{q-1}(S) = 1$.*

4 Results

We now identify a rule satisfying the axioms we impose. Further, we prove that if there are three goods, it is the only rule, with one of its subcorrespondences, satisfying the fairness axioms we impose and not discriminating between goods.

This rule embodies the fairness fundamental according to which one should first care for the least fortunates. For each preference profile, it selects the allocations maximizing, across allocations, the minimal rank, across agents, of an allocation.

Maximin rule $\boldsymbol{\Gamma}$: *For each $\mathbf{R} \in R^N$, we have $\Gamma(\mathbf{R}) = \arg\max_{\mathbf{x} \in X} \min_{i \in N} r_i(x_i)$.*

It has subcorrespondences based on two distinct ideas. First, conditional on caring the least fortunates, if avoiding envy is impossible, the agents should have equal chance of being the enviee; if not, one should minimize inequality. For each preference profile, if no *envy-free* allocation exists, it selects the allocations

Figure 4:

	(R_1, R_2^*)	$(R_1, R_2^\wedge)$	(R_1', R_2^+)	$(R_1', R_2^\wedge)$	(R_1', R_2°)
Γ	$(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma}), (\boldsymbol{\alpha\beta}, \boldsymbol{\gamma})$	$(\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$	$(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma}), (\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$	$(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma}), (\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$	$(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma}), (\boldsymbol{\alpha\beta}, \boldsymbol{\gamma})$
Θ	$(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma}), (\boldsymbol{\alpha\beta}, \boldsymbol{\gamma})$	$(\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$	$(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma}), (\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$	$(\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$	$(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma}), (\boldsymbol{\alpha\beta}, \boldsymbol{\gamma})$
Λ	$(\boldsymbol{\alpha\beta}, \boldsymbol{\gamma})$	$(\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$	$(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma}), (\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$	$(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma})$	$(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma}), (\boldsymbol{\alpha\beta}, \boldsymbol{\gamma})$

the maximin rule selects; if not, it selects, among these allocations, those minimizing, across allocations, the maximal rank, across agents. Second, one should care for the least fortunates recursively. For each preference profile, it selects, among the allocations the maximin rule selects, those maximizing, across allocations, the minimal rank, across agents, whose rank is not the minimal rank of an allocation the maximin rule selects. The maximin rule and these subcorrespondences embody such ideas, using only ordinal information on preferences.

We define these subcorrespondences for two-agent economies. Doing so, we underline the dual aspect of these, specific to such economies. The next example illustrates it. We discuss their definitions for those with more than two agents in the next section.

Maximin-minimax rule $\boldsymbol{\Theta}$: *For each $\mathbf{R} \in R^N$, if $PF(\mathbf{R}) = \emptyset$, then $\Theta(\mathbf{R}) = \Gamma(\mathbf{R})$; if not, $\Theta(\mathbf{R}) = \arg\min_{\mathbf{x} \in \Gamma(\mathbf{R})} \max_{i \in N} r_i(R_i)$.*

Leximin rule $\boldsymbol{\Lambda}$: *For each $\mathbf{R} \in R^N$, we have $\Lambda(\mathbf{R}) = \arg\max_{x \in \Gamma(\mathbf{R})} \min_{i \in N \setminus \{j \in N:\ \text{there is } y \in \Gamma(\mathbf{R}) \text{ with } r_j(y_j) = \min_{k \in N} r_k(y_k)\}} r_i(x_i)$.*

Ex. 1 (cont.) The table in Fig. 4 gives the allocations that Γ, Θ, Λ select for (R_1, R_2^*), $(R_1, R_2^\wedge)$, (R_1', R_2^+), $(R_1', R_2^\wedge)$, and (R_1', R_2°) resp. There is one row for each rule, one column for each profile.□

To come to our main results, we distinguish rules not discriminating between goods. If all agents reverse their preferences over a pair of goods, one should permute the selected allocations accordingly. Let H be the set of all permutations on A. For each $h \in H$, each $i \in N$, and $R_i \in \mathcal{R}$, let $h(R_i) \in \mathcal{R}$ with for each $S, T \in \mathcal{S}$, we have $\cup_{\alpha \in S} h(\alpha)\ h(R_i) \cup_{\alpha \in T} h(\alpha)$ if and only if $S\ R_i\ T$. Let Φ be a rule.

Neutrality: *For each $\mathbf{R} \in \mathcal{R}^N$, each $\mathbf{x} \in \Phi(\mathbf{R})$, and each $h \in H$, we have $(h(x_i))_{i \in N} \in \Phi((h(R_i))_{i \in N})$.*

Further, we prove properties pertaining to our axioms.

Theorem 1

1. *If* envy-free *allocations exist,* efficient *and* envy-free *allocations exist.*
2. *Independently of an agent's preferences, the rank of the worst bundle of the allocations* efficiency *and* *minimization of inequality among equals recommend when each other agent has her preferences, equals* 2^{a-1}.

Proof. Let $N = \mathbf{12}$.

Stmt. 1 [For each $\mathbf{R} \in \mathcal{R}^N$, if $F(\mathbf{R}) \neq \emptyset$, then $PF(\mathbf{R}) \neq \emptyset$.] Let $\mathbf{R} \in \mathcal{R}^N$ and $\mathbf{x} \in F(\mathbf{R})$ with $\mathbf{x} \notin P(\mathbf{R})$. By strictness, there is $\mathbf{y} \in P(\mathbf{R})$ with (i) $y_1\ P_1\ x_1$ and $y_2\ P_2\ x_2$. As $\mathbf{x} \in F(\mathbf{R})$, we have $x_1\ R_1\ x_2$ and $x_2\ R_2\ x_1$. By (i) and Lem. 1, $x_1^c\ P_1\ y_1^c$ and $x_2^c\ P_2\ y_2^c$. As $n = 2$, we have $x_1^c = x_2$, $y_1^c = y_2$, $x_2^c = x_1$, and $y_2^c = y_1$. By (i), $y_1\ P_1\ y_2$ and $y_2\ P_2\ y_1$. Thus, $\mathbf{y} \in PF(\mathbf{R})$.

Stmt. 2 [For each $\mathbf{R} \in \mathcal{R}^N$ and each $i \in N$, we have $\check{r}(\mathbf{R_i}) = 2^{a-1}$.] Let $\mathbf{R} \in \mathcal{R}^N$, $i \in N$, and for $j \in N \setminus \boldsymbol{i}$, let $R_j' \in \mathcal{R}$ with $R_j' = R_i$. W.l.o.g. assume $i = 1$. By **1.** and **2.**, $\check{r}(\mathbf{R_1}) = 2^{a-1}$.

1. [There is $\mathbf{x} \in X$ with $r'_2(x_2) \geq 2^{a-1}$ and $r_1(x_1) \geq 2^{a-1}$.] Let $\mathbf{x} \in X$ with $r'_2(x_2) = 2^{a-1}$. By Lem. 1, $r'_2(x_2^c) = 2^{a-1} + 1$. As $n = 2$, we have $x_2^c = x_1$. As $R'_2 = R_1$, we have $r_1(x_1) > 2^{a-1}$.
2. [For each $\mathbf{x} \in X$, if $r'_2(x_2) \geq 2^{a-1}+1$, then $r_1(x_1) < 2^{a-1}+1$.] Let $\mathbf{x} \in X$ with $r'_2(x_2) \geq 2^{a-1}+1$. By Lem. 1, $r'_2(x_2^c) \leq 2^{a-1}$. As $n = 2$, we have $x_2^c = x_1$. As $R'_2 = R_1$, we have $r_1(x_1) < 2^{a-1}+1$.■

We are now ready to state and prove our main result. The maximin rule satisfies each axiom we impose. If there are three goods, it is the only rule, with the maximin-minimax rule, satisfying each fairness axiom and *neutrality*.

Theorem 2

1. *The maximin rule satisfies* efficiency*,* anonymity*,* conditional no-envy*, the* unanimity bound*, and* preference-monotonicity*.*
2. *In three-good economies, a rule satisfies* anonymity*,* conditional no-envy*, the* unanimity bound*,* preference-monotonicity*, and* neutrality *if and only if it is the maximin or maximin-minimax rule.*

Proof. Let $N = \mathbf{12}$. For each $\mathbf{x} \in X$ and each $\mathbf{R} \in \mathcal{R}^N$, let $\underline{r}(\mathbf{x}, \mathbf{R}) \equiv \min_{i \in N} r_i(x_i)$ be the minimal rank of $\mathbf{x}$ across N w.r.t. $\mathbf{R}$ and $\overline{r}(\mathbf{x}, \mathbf{R}) \equiv \max_{i \in N} r_i(x_i)$ the maximal rank of $\mathbf{x}$ across N w.r.t. $\mathbf{R}$. We use the following notational shortcut. Let Φ be a rule and $\mathbf{x} \in X$. For each $\mathbf{R} \in \mathcal{R}^N$, we have $\underline{r}(\Phi, \mathbf{R}) \equiv \underline{r}(\mathbf{x}, \mathbf{R})$ if and only if $\mathbf{x} \in \Phi(\mathbf{R})$.

Stmt. 1 • *Efficiency.* By contradiction, assume there are $\mathbf{R} \in \mathcal{R}^N$ and $\mathbf{x} \in \Gamma(\mathbf{R})$ with $\mathbf{x} \notin P(\mathbf{R})$. By strictness, as $n = 2$, there is $\mathbf{y} \in X$ with for each $i \in N$, we have $r_i(y_i) > r_i(x_i)$. Thus, $\underline{r}(\mathbf{y}, \mathbf{R}) > \underline{r}(\mathbf{x}, \mathbf{R})$, contradicting $\mathbf{x} \in \Gamma(R)$.

• *Anonymity.* As Γ never uses agents' names, it satisfies *anonymity.*

• *Conditional no-envy.* By contradiction, assume there are $\mathbf{R} \in \mathcal{R}^N$, $\mathbf{x} \in F(\mathbf{R})$, and $\mathbf{y} \in \Gamma(\mathbf{R})$ with $y \notin F(\mathbf{R})$. W.l.o.g. assume $\underline{r}(\mathbf{x}, \mathbf{R}) = r_1(x_1)$. For each $i, j \in N$ with $i \neq j$,

- By definition, $r_i(y_i) \geq \underline{r}(\mathbf{y}, \mathbf{R})$. As $\mathbf{y} \in \Gamma(\mathbf{R})$, we have $\underline{r}(\mathbf{y}, \mathbf{R}) \geq \underline{r}(\mathbf{x}, \mathbf{R})$. By assumption, $\underline{r}(\mathbf{x}, \mathbf{R}) = r_1(x_1)$. Thus, (i) $r_i(y_i) \geq r_1(x_1)$.
- As $\mathbf{x} \in F(\mathbf{R})$, we have $r_1(x_1) \geq r_1(x_2)$. By strictness (ii) $r_1(x_1) > r_1(x_2)$.
- By (i) and Lem. 1, $r_i(y_i^c) \leq r_1(x_1^c)$. As $n = 2$, we have $y_i^c = y_j$ and $x_1^c = x_2$. Thus, (iii) $r_1(x_2) \geq r_i(y_j)$.

By (i), (ii), and (iii), $r_1(y_1) > r_1(y_2)$ and $r_2(y_2) > r_2(y_1)$, contradicting $\mathbf{y} \notin F(\mathbf{R})$.

• *Unanimity bound.* By contradiction, assume there are $\mathbf{R} \in \mathcal{R}^N$, $\mathbf{x} \in \Gamma(\mathbf{R})$, and $i \in N$ with $r_i(x_i) < \check{r}(\mathbf{R_i})$. By Th. 1, $r_i(x_i) < 2^{a-1}$ and there is $\mathbf{y} \in X$ with for each $j \in N$, we have $r_j(y_j) \geq 2^{a-1}$. By definition, $\underline{r}(\mathbf{x}, \mathbf{R}) \leq r_i(x_i)$. Thus, $\underline{r}(\mathbf{x}, \mathbf{R}) < \underline{r}(\mathbf{y}, \mathbf{R})$, contradicting $\mathbf{x} \in \Gamma(\mathbf{R})$.

• *Preference-monotonicity.* By contradiction, assume there are $\mathbf{R} \in \mathcal{R}^N$, $R'_2 \in \mathcal{R}$ with R'_2 closer to R_1 than R_2, and $u \in U$ fitting R_1 with $E(\Gamma(\mathbf{R}), u) < E(\Gamma(R_1, R'_2), u)$. There are $\overline{q} \in \mathbb{N}$ and $(R_2^q)^{q \in \{1,\ldots,\overline{q}\}} \in \mathcal{R}^{\{1,\ldots,\overline{q}\}}$ with

- By Lem. 2, $R_2^1 = R_2$, $R_2^{\overline{q}} = R'_2$, and for each $q \in \{2, ..., \overline{q}\}$, we have R_2^q closer to R_1 than R_2^{q-1} and for each $S, T \in \mathcal{S}$, we have $r_2^q(S) > r_2^q(T)$ and $r_2^{q-1}(S) < r_2^{q-1}(T)$ if and only if $r_2^{q-1}(S) = r_2^q(T)$, $r_2^{q-1}(T) = r_2^q(S)$, and $r_2^q(S) - r_2^q(T) = r_2^{q-1}(T) - r_2^{q-1}(S) = 1$;
- There is $q \in \{2, ..., \overline{q}\}$ with $E(\Gamma(R_1, R_2^{q-1}), u) < E(\Gamma(R_1, R_2^q), u)$.

W.l.o.g. assume $\overline{q} = 2$. Let $\mathbf{R}' \in \mathcal{R}^N$ with $\mathbf{R}' = (R_1, R'_2)$. As $E(\Gamma(\mathbf{R}), u) < E(\Gamma(\mathbf{R}'), u)$, there are $\mathbf{x} \in \Gamma(\mathbf{R})$ and $\mathbf{y} \in \Gamma(\mathbf{R}')$ with $y_1 \; P_1 \; x_1$.

1. [$\underline{r}(\mathbf{y}, \mathbf{R}) = r_2(y_2)$.] By assumption, $r_1(y_1) > r_1(x_1)$. By definition, $r_1(x_1) \geq \underline{r}(\mathbf{x}, \mathbf{R})$. As $\mathbf{x} \in \Gamma(\mathbf{R})$, we have $\underline{r}(\mathbf{x}, \mathbf{R}) \geq \underline{r}(\mathbf{y}, \mathbf{R})$. Thus, $r_1(y_1) > \underline{r}(\mathbf{y}, \mathbf{R})$, implying $\underline{r}(\mathbf{y}, \mathbf{R}) = r_2(y_2)$.

2. [$\underline{r}(\mathbf{x}, \mathbf{R}') = r_1(x_1)$.] By contradiction, assume $\underline{r}(\mathbf{x}, \mathbf{R}') \neq r_1(x_1)$.

- By assumption, $r_1(y_1) > r_1(x_1)$. By Lem. 1, $r_1(y_1^c) < r_1(x_1^c)$. As $n = 2$, we have $y_1^c = y_2$ and $x_1^c = x_2$. Thus, (i) $r_1(x_2) > r_1(y_2)$.
- By definition, $r_2(x_2) \geq \underline{r}(\mathbf{x}, \mathbf{R}))$. As $\mathbf{x} \in \Gamma(\mathbf{R})$, we have $\underline{r}(\mathbf{x}, \mathbf{R}) \geq \underline{r}(\mathbf{y}, \mathbf{R})$. By **1.**, $\underline{r}(\mathbf{y}, \mathbf{R}) = r_2(y_2)$. By strictness, as $x \neq y$, (ii) $r_2(x_2) > r_2(y_2)$.

By assumption, $\underline{r}(\mathbf{x}, \mathbf{R}') = r_2'(x_2)$. As R_2' closer to R_1 than R_2, and (i) and (ii) hold, $r_2'(x_2) > r_2'(y_2)$. By definition, $r_2'(y_2) \geq \underline{r}(\mathbf{y}, \mathbf{R}')$. Thus, $\underline{r}(\mathbf{x}, \mathbf{R}') > \underline{r}(\mathbf{y}, \mathbf{R}')$, contradicting $\mathbf{y} \in \Gamma(\mathbf{R}')$.

3. [$r_2(y_2) < r_2'(y_2)$.] By contradiction, assume $r_2(y_2) \geq r_2'(y_2)$. As $\mathbf{y} \in \Gamma(\mathbf{R}')$, we have $\underline{r}(\mathbf{y}, \mathbf{R}') \geq \underline{r}(\mathbf{x}, \mathbf{R}')$. By **2.**, $\underline{r}(\mathbf{x}, \mathbf{R}') = r_1(x_1)$. By definition, $r_1(x_1) \geq \underline{r}(\mathbf{x}, \mathbf{R})$. As $\mathbf{x} \in \Gamma(\mathbf{R})$, we have $\underline{r}(\mathbf{x}, \mathbf{R}) \geq \underline{r}(\mathbf{y}, \mathbf{R})$. By **1.**, $\underline{r}(\mathbf{y}, \mathbf{R}) = r_2(y_2)$. By assumption, $r_2(y_2) \geq r_2'(y_2)$. By definition, $r_2'(y_2) \geq \underline{r}(y, \mathbf{R}')$. Thus, $\underline{r}(\mathbf{y}, \mathbf{R}') = \underline{r}(\mathbf{x}, \mathbf{R}') = \underline{r}(\mathbf{x}, \mathbf{R}) = \underline{r}(\mathbf{y}, \mathbf{R})$. By definition, $\#\Gamma(\mathbf{R}) \in \{1, ..., n\}$. As $n = 2$, $\mathbf{x} \neq \mathbf{y}$, $\mathbf{x} \in \Gamma(\mathbf{R})$ and $\mathbf{y} \in \Gamma(\mathbf{R}')$, we have $\Gamma(\mathbf{R}) = \Gamma(\mathbf{R}') = \{\mathbf{x}, \mathbf{y}\}$, contradicting $E(\Gamma(\mathbf{R}), u) < E(\Gamma(\mathbf{R}'), u)$.

4. [Contradiction.]

- By **3.**, $r_2(y_2) < r_2'(y_2)$. As $\overline{q} = 2$, (i) there is $\mathbf{z} \in X$ with $r_2(z_2) = r_2'(y_2)$, $r_2(y_2) = r_2'(z_2)$, and $r_2(z_2) - r_2(y_2) = r_2'(y_2) - r_2'(z_2) = 1$.
- As R_2' is closer to R_1 than R_2, we have $r_1(z_2) < r_1(y_2)$. By Lem. 1, $r_1(z_2^c) > r_1(y_2^c)$. As $n = 2$, we have $z_2^c = z_1$ and $y_2^c = y_1$. Thus, (ii) $r_1(z_1) > r_1(y_1)$.
- By (ii), $r_1(z_1) > r_1(y_1)$. By assumption, $r_1(y_1) > r_1(x_1)$. By definition, $r_1(x_1) \geq \underline{r}(\mathbf{x}, \mathbf{R})$. As $\mathbf{x} \in \Gamma(\mathbf{R})$, we have $\underline{r}(\mathbf{x}, \mathbf{R}) \geq \underline{r}(\mathbf{z}, \mathbf{R})$. Thus, $r_1(z_1) > \underline{r}(\mathbf{z}, \mathbf{R})$, implying (iii) $\underline{r}(\mathbf{z}, \mathbf{R}) = r_2(z_2)$.

By **2.**, $\underline{r}(\mathbf{x}, \mathbf{R}') = r_1(x_1)$. By definition, $r_1(x_1) \geq \underline{r}(\mathbf{x}, \mathbf{R})$. As $\mathbf{x} \in \Gamma(\mathbf{R})$, we have $\underline{r}(\mathbf{x}, \mathbf{R}) \geq \underline{r}(\mathbf{z}, \mathbf{R})$. By (iii), $\underline{r}(\mathbf{z}, \mathbf{R}) = r_2(z_2)$. By (i), $r_2(z_2) = r_2'(y_2)$. By definition, $r_2'(y_2) \geq \underline{r}(y, \mathbf{R}')$. As $\mathbf{y} \in \Gamma(\mathbf{R}')$, we have $\underline{r}(\mathbf{y}, \mathbf{R}') \geq \underline{r}(\mathbf{x}, \mathbf{R}')$. Thus, $\underline{r}(\mathbf{x}, \mathbf{R}') = \underline{r}(\mathbf{x}, \mathbf{R}) = \underline{r}(\mathbf{z}, \mathbf{R}) = \underline{r}(\mathbf{y}, \mathbf{R}')$. By definition, $\#\Gamma(\mathbf{R}) \in \{1, ..., n\}$. As $n = 2$, $\mathbf{x} \neq \mathbf{y} \neq \mathbf{z}$, $\mathbf{x} \in \Gamma(\mathbf{R})$ and $\mathbf{y} \in \Gamma(\mathbf{R}')$, we have $\Gamma(\mathbf{R}) = \{\mathbf{x}, \mathbf{z}\}$ and $\Gamma(\mathbf{R}') = \{\mathbf{x}, \mathbf{y}\}$. By (ii), $E(\Gamma(\mathbf{R}), u) > E(\Gamma(\mathbf{R}'), u)$.

Stmt. 2 • *The maximin and maximin-minimax rules satisfy the axioms of Th. 2.2.* By Th. 2.1, Γ satisfies the fairness axioms. As it never uses names of goods, it satisfies *neutrality*. As Θ never uses names of agents nor of goods, it satisfies *anonymity* and *neutrality*. As it is a subcorrespondence of Γ, it satisfies *conditional no-envy* and the *unanimity bound*. Let $a = 3$. Then, it satisfies *preference-monotonicity*: The proof of Th. 2.1 holds for Θ, considering the cases where Θ may differ from Γ.

3. [$r_2(y_2) < r_2'(y_2)$.] (cont.) We have $\underline{r}(\mathbf{x}, \mathbf{R}) = \underline{r}(\mathbf{y}, \mathbf{R}) = \underline{r}(\mathbf{x}, \mathbf{R}') = \underline{r}(\mathbf{y}, \mathbf{R}')$, hence (...), $\Gamma(\mathbf{R}) = \Gamma(\mathbf{R}') = \{\mathbf{x}, \mathbf{y}\}$. W.l.o.g. assume $\Theta(\mathbf{R}) \neq \Gamma(\mathbf{R})$, $\Theta(\mathbf{R}') \neq \Gamma(\mathbf{R}')$, or both.

- By **1.**, $\underline{r}(\mathbf{y}, \mathbf{R}) = r_2(y_2)$. By **2.**, $\underline{r}(\mathbf{x}, \mathbf{R}') = r_1(x_1)$. As $\underline{r}(\mathbf{y}, \mathbf{R}) = \underline{r}(\mathbf{x}, \mathbf{R})$, $\underline{r}(\mathbf{x}, \mathbf{R}') = \underline{r}(\mathbf{y}, \mathbf{R}')$, $n = 2$, and $\mathbf{x} \neq \mathbf{y}$, (i) $\underline{r}(\mathbf{x}, \mathbf{R}) = r_1(x_1)$ and $\underline{r}(\mathbf{y}, \mathbf{R}') = r_2'(y_2)$.
- Assume $\underline{r}(\Gamma, \mathbf{R}) = 2^{a-1}$. As $\underline{r}(\Gamma, \mathbf{R}) = \underline{r}(\Gamma, \mathbf{R}')$, we have $\underline{r}(\Gamma, \mathbf{R}') = 2^{a-1}$. As $\mathbf{x} \in \Gamma(\mathbf{R})$, $\mathbf{y} \in \Gamma(\mathbf{R}')$, and (i) holds, $r_1(x_1) = r_2'(y_2) = 2^{a-1}$. By Lem. 1, $r_1(x_1^c) = r_2'(y_2) = 2^{a-1} + 1$. As $n = 2$, we have $x_1^c = x_2$ and $y_2^c = y_1$. Thus, $r_1(x_2) = r_2'(y_1) = 2^{a-1} + 1$. Thus, $r_1(x_1) < r_1(x_2)$ and $r_2'(y_2) < r_2'(y_1)$. As $\mathbf{x} \in \Gamma(\mathbf{R})$ and $\mathbf{y} \in \Gamma(\mathbf{R}')$, and Γ satisfies *efficiency* and *conditional no-envy*, $PF(\mathbf{R}) = PF(\mathbf{R}') = \emptyset$, contradicting $\Theta(\mathbf{R}) \neq \Gamma(\mathbf{R})$, $\Theta(\mathbf{R}') \neq \Gamma(\mathbf{R}')$, or both. Thus, $\underline{r}(\Gamma, \mathbf{R}) \neq 2^{a-1}$. By Th. 1.2 and 2.1, $\underline{r}(\Gamma, \mathbf{R}) \geq 2^{a-1} + 1$. As $\underline{r}(\Gamma, \mathbf{R}) = \underline{r}(\Gamma, \mathbf{R}')$, we have $\underline{r}(\Gamma, \mathbf{R}') \geq 2^{a-1} + 1$. As $x \in \Gamma(\mathbf{R})$, $y \in \Gamma(\mathbf{R}')$, and (i) holds, $r_1(x_1) = r_2'(y_2) \geq 2^{a-1} + 1$. By Lem. 1, $r_1(x_1^c) = r_2'(y_2) \leq 2^{a-1}$. As $n = 2$, we have $x_1^c = x_2$ and $y_2^c = y_1$. Thus, $r_1(x_2) = r_2'(y_1) \leq 2^{a-1}$. Thus, $r_1(x_1) > r_1(x_2)$ and $r_2'(y_2) > r_2'(y_1)$. Thus, (ii) $PF(\mathbf{R}) = PF(\mathbf{R}') \neq \emptyset$.
- By (i) and **1.**, $\underline{r}(\mathbf{x}, \mathbf{R}) = r_1(x_1)$ and $\underline{r}(\mathbf{y}, \mathbf{R}) = r_2(y_2)$. By (i) and **2.**, $\underline{r}(\mathbf{x}, \mathbf{R}') = r_1(x_1)$ and $\underline{r}(\mathbf{y}, \mathbf{R}') = r_2'(y_2)$. As $n = 2$ and $\mathbf{x} \neq \mathbf{y}$, we have $\overline{r}(\mathbf{x}, \mathbf{R}) = r_2(x_2)$ and $\overline{r}(\mathbf{y}, \mathbf{R}) = r_1(y_1)$, and $\overline{r}(\mathbf{x}, \mathbf{R}') = r_2'(x_2)$ and $\overline{r}(\mathbf{y}, \mathbf{R}') = r_1(y_1)$. As $\mathbf{x} \in \Theta(\mathbf{R})$, $\mathbf{y} \in \Theta(\mathbf{R}')$, $\underline{r}(\mathbf{x}, \mathbf{R}) = \underline{r}(\mathbf{y}, \mathbf{R})$, $\underline{r}(\mathbf{x}, \mathbf{R}') = \underline{r}(\mathbf{y}, \mathbf{R}')$, and $\Theta(\mathbf{R}) \neq \Gamma(\mathbf{R})$, $\Theta(\mathbf{R}') \neq \Gamma(\mathbf{R}')$, or both, and (ii) holds, $\overline{r}(\mathbf{x}, \mathbf{R}) \leq$

$\overline{r}(\mathbf{y},\mathbf{R})$ and $\overline{r}(\mathbf{x},\mathbf{R}') \geq \underline{r}(\mathbf{y},\mathbf{R}')$ with strict inequality for at least one. Thus, (iii) $r_2(x_2) < r'_2(x_2)$.

As $s = 2$ and (iii) holds, there is $\mathbf{z} \in X$ with $r_2(z_2) > r_2(x_2)$, $r'_2(z_2) < r'_2(x_2)$, $r_2(z_2) = r_2(x_2)+1$, $r'_2(z_2) = r_2(x_2)$, and $r'_2(x_2) = r_2(z_2)$. As $\mathbf{x} \in \Theta(\mathbf{R})$, we have $r_1(z_1) < r_1(x_1)$. By Lem. 1, $r_1(z_1^c) > r_1(x_1^c)$. As $n = 2$, we have $z_1^c = z_2$ and $x_1^c = x_2$. Thus, $r_1(z_2) > r_1(x_2)$. By assumption, $r_2(z_2) > r_2(x_2)$ and $r'_2(z_2) < r'_2(x_2)$, contradicting R'_2 closer to R_1 than R_2.

4. [Contradiction.] (cont.) We have $\underline{r}(\mathbf{x},\mathbf{R}) = \underline{r}(\mathbf{z},\mathbf{R}) = \underline{r}(\mathbf{x},\mathbf{R}') = \underline{r}(\mathbf{y},\mathbf{R}')$, hence (...), $\Gamma(\mathbf{R}) = \{\mathbf{x},\mathbf{z}\}$ and $\Gamma(\mathbf{R}') = \{\mathbf{x},\mathbf{y}\}$. W.l.o.g. assume $\Theta(\mathbf{R}) \neq \Gamma(\mathbf{R})$, $\Theta(\mathbf{R}') \neq \Gamma(\mathbf{R}')$, or both.

- By (iii), $\underline{r}(\mathbf{z},\mathbf{R}) = r_2(z_2)$. As $\underline{r}(\mathbf{z},\mathbf{R}) = \underline{r}(\mathbf{x},\mathbf{R})$ and $n = 2$, $\underline{r}(\mathbf{x},\mathbf{R}) = r_1(x_1)$. Thus, (iv) $r_1(x_1) = r_2(z_2)$.
- Assume $r_2(z_2) = 2^{a-1}+1$. By Lem. 1, $r_2(z_2^c) = 2^{a-1}$. As $n = 2$, we have $z_2^c = z_1$. Thus, (v) $r_2(z_1) = 2^{a-1}$. By (i), $r_2(z_2) - r_2(y_2) = 1$. By assumption, $r_2(z_2) = 2^{a-1}+1$. Thus, $r_2(y_2) = 2^{a-1}$. By (v), $r_2(y_2) = r_2(z_1)$. By strictness, (vi) $y_2 = z_1$. By assumption, $r_1(y_1) > r_1(x_1)$. By (iv), $r_1(x_1) = r_2(z_2)$. By assumption, $r_2(z_2) = 2^{a-1}+1$. Thus, $r_1(y_1) > 2^{a-1}+1$. By Lem. 1, $r_1(y_1^c) < 2^{a-1}$. As $n = 2$, we have $y_1^c = y_2$. Thus, $r_1(y_2) < 2^{a-1}$. By (vi), $r_1(z_1) < 2^{a-1}$. By assumption, $r_2(z_2) = 2^{a-1}+1$. Thus, $r_1(z_1) < r_2(z_2)$, contradicting (iii). Thus, (vii) $r_2(z_2) \neq 2^{a-1}+1$.

As $\mathbf{z} \in \Gamma(\mathbf{R})$, by Th. 1 and 2.1, $\underline{r}(\mathbf{z},\mathbf{R}) \geq 2^{a-1}$. As $a = 3$, we have $\underline{r}(\mathbf{z},\mathbf{R}) \leq 2^{a-1}+1$. By (iii), (iv), and (vii), $r_1(x_1) = 2^{a-1}$. By Lem. 1, $r_1(x_1^c) = 2^{a-1}+1$. As $n = 2$, we have $x_1^c = x_2$. Thus, $r_1(x_2) = 2^{a-1}+1$. Thus, $r_1(x_1) < r_1(x_2)$. As $\mathbf{x} \in \Gamma(\mathbf{R}) \cap \Gamma(\mathbf{R}')$ and Γ satisfies *efficiency* and *conditional no-envy*, $PF(\mathbf{R}) = PF(\mathbf{R}') = \emptyset$, contradicting $\Theta(\mathbf{R}) \neq \Gamma(\mathbf{R})$, $\Theta(\mathbf{R}') \neq \Gamma(\mathbf{R}')$, or both.

• *A rule satisfying the axioms of Th. 2.2 is the maximin or maximin-minimax rule.* Let Φ be a rule satisfying these axioms. Let $A = \boldsymbol{\gamma\delta\epsilon}$ and $\mathbf{R} \in \mathcal{R}^N$.

1. [$\Phi(\mathbf{R}) \subset \Gamma(\mathbf{R})$.] By contradiction, assume there is $\mathbf{x} \in \Phi(\mathbf{R})$ with $\mathbf{x} \notin \Gamma(\mathbf{R})$. Let $\mathbf{y} \in \Gamma(\mathbf{R})$.

- As $\mathbf{x} \notin \Gamma(\mathbf{R})$ and $\mathbf{y} \in \Gamma(\mathbf{R})$, we have $\underline{r}(\mathbf{x},\mathbf{R}) < \underline{r}(\mathbf{y},\mathbf{R})$. As $\mathbf{x} \in \Phi(\mathbf{R})$ and Φ satisfies the *unanimity bound*, by Th. 1, $\underline{r}(\mathbf{x},\mathbf{R}) \geq 2^{a-1}$. As $\mathbf{y} \in \Gamma(\mathbf{R})$, by Th. 1 and 2.1, $\underline{r}(\mathbf{y},\mathbf{R}) \geq 2^{a-1}$. As $a = 3$, we have $\underline{r}(\mathbf{y},\mathbf{R}) \leq 2^{a-1}+1$. Thus, (i) $\underline{r}(\mathbf{x},\mathbf{R}) = 2^{a-1}$ and $\underline{r}(\mathbf{y},\mathbf{R}) = 2^{a-1}+1$.
- By definition, $r_1(y_1) \geq \underline{r}(\mathbf{y},\mathbf{R})$ and $r_2(y_2) \geq \underline{r}(\mathbf{y},\mathbf{R})$. By (i), $\underline{r}(\mathbf{y},\mathbf{R}) = 2^{a-1}+1$. By Lem. 1, $r_1(y_1^c) \leq 2^{a-1}$ and $r_2(y_2^c) \leq 2^{a-1}$. As $n = 2$, we have $y_1^c = x_2$ and $y_2^c = x_1$. Thus, $r_1(y_2) > r_1(y_2)$ and $r_2(y_2) > r_2(y_1)$. As $\mathbf{y} \in \Gamma(\mathbf{R})$ and Γ satisfies *efficiency* and *conditional no-envy*, (ii) $PF(\mathbf{R}) \neq \emptyset$.

W.l.o.g. assume $\underline{r}(\mathbf{x},\mathbf{R}) = r_1(x_1)$. By (i), $r_1(x_1) = 2^{a-1}$. By Lem. 1, $r_1(x_1^c) = 2^{a-1}+1$. As $n = 2$, we have $x_1^c = x_2$. Thus, $x_1(x_2) > r_1(x_1)$. By (ii), $PF(\mathbf{R}) \neq \emptyset$, contradicting *conditional no-envy*.

2. [$\Phi(\mathbf{R}) = \Gamma(\mathbf{R})$ or $\Phi(\mathbf{R}) = \Theta(\mathbf{R})$.] By contradiction, assume $\Phi(\mathbf{R}) \neq \Gamma(\mathbf{R})$ and $\Phi(\mathbf{R}) \neq \Theta(\mathbf{R})$. By definition, $\#\Gamma(\mathbf{R}) \in \{1,...,n\}$. As $n = 2$ and **1.** holds, there are $\mathbf{x},\mathbf{y} \in X$ with $\mathbf{x} \neq \mathbf{y}$, $\Phi(\mathbf{R}) = \{\mathbf{x}\}$, and $\Gamma(\mathbf{R}) = \{\mathbf{x},\mathbf{y}\}$. As $\mathbf{x} \in \Gamma(\mathbf{R})$, by Th. 1 and 2.1, $\underline{r}(\mathbf{x},\mathbf{R}) \geq 2^{a-1}$. As $a = 3$, we have $\underline{r}(\mathbf{x},\mathbf{R}) \leq 2^{a-1}+1$. W.l.o.g. assume $\underline{r}(\mathbf{x},\mathbf{R}) = r_1(x_1)$. Distinguish 2 cases.

***a* :** $r_1(x_1) = 2^{a-1}$. Let $R'_2 \in \mathcal{R}$ with $R'_2 = R_1$. Then, R'_2 is closer to R_1 than R_2, there is $\mathbf{z} \in X$ with $r_1(z_1) = 2^{a-1}+1$ and $r'_2(z_2) = 2^{a-1}$, and $B(R_1,R'_2) = \{\mathbf{x},\mathbf{z}\}$. As Φ satisfies *anonymity* and the *unanimity bound*, $\Phi(R_1,R'_2) = \{\mathbf{x},\mathbf{z}\}$. By assumption, $r_1(x_1) = 2^{a-1}$ and $r_1(z_1) = 2^{a-1}+1$. For each $u \in U$ fitting R_1, we have $E(\Phi(\mathbf{R}),u) < E(\Phi(R_1,R'_2),u)$, contradicting *preference-monotonicity*.

***b* :** $r_1(x_1) = 2^{a-1}+1$. As $n = 2$, $a = 3$, and $\#\Gamma(\mathbf{R}) = 2$, there are $\alpha,\beta \in A$ with $\mathbf{x} = (\boldsymbol{\alpha}, A/\boldsymbol{\alpha})$ and $\mathbf{y} = (A/\boldsymbol{\beta},\boldsymbol{\beta})$. W.l.o.g. assume $\mathbf{x} = (\boldsymbol{\delta},\boldsymbol{\gamma\epsilon})$ and $\mathbf{y} = (\boldsymbol{\gamma\epsilon},\boldsymbol{\epsilon})$. As $\underline{r}(\mathbf{x},\mathbf{R}) = r_1(x_1)$, $\underline{r}(\mathbf{x},\mathbf{R}) = \underline{r}(\mathbf{y},\mathbf{R})$, and $n = 2$, we have $\underline{r}(\mathbf{y},\mathbf{R}) = r_2(y_2)$. As $\mathbf{x} \neq \mathbf{y}$, we have $r_1(x_1) = 2^{a-1}+1$,

$r_2(x_2) > 2^{a-1}+1$, $r_1(y_1) > 2^{a-1}+1$, and $r_2(y_2) = 2^{a-1}+1$. As $\Phi(\mathbf{R}) \neq \Theta(\mathbf{R})$, we have $r_1(y_1) \leq r_2(x_2)$. Distinguish 2 cases.

b.1 $r_1(y_1) = r_2(x_2)$. Let $h \in H$ with $h(\delta) = \epsilon$ and $h(\epsilon) = \delta$. Let $\mathbf{R}' \in \mathcal{R}^N$ with $\mathbf{R}' = h(\mathbf{R})$. As $\Phi(\mathbf{R}) = (\boldsymbol{\delta}, \boldsymbol{\gamma\epsilon})$ and Φ satisfies *neutrality*, $\Phi(R'_1, R'_2) = \{(\boldsymbol{\epsilon}, \boldsymbol{\gamma\delta})\}$. As $r_1(y_1) = r_2(x_2)$ and $a = 3$, we have $\mathbf{R}' = (R_2, R_1)$. Thus, $\Phi(R_2, R_1) = \{(\boldsymbol{\epsilon}, \boldsymbol{\gamma\delta})\}$, contradicting *anonymity*.

b.2 $r_1(y_1) < r_2(x_2)$. Let $z_1 \in \mathcal{S}$ with $r_1(z_1) = r_2(x_2)$. As $a = 3$, we have $r_1(y_1) = r_1(z_1) - 1$. Let $R'_1 \in \mathcal{R}$ with $r'_1(y_1) = r_1(z_1)$, $r'_1(z_1) = r_1(y_1)$ and for each $S \in \mathcal{S}\backslash\{x_1, z_1\}$, we have $r'_1(S) = r_1(S)$. Then, R_1 is closer to R_2 than R'_1 and $B(R'_1, R_2) = \{\mathbf{x}, \mathbf{y}\}$. As $r'_1(y_1) = r_2(x_2)$, $a = 3$, and Φ satisfies *anonymity*, the *unanimity bound*, and *neutrality*, by the logic of the previous paragraph, $\Phi(R'_1, R_2) = \{\mathbf{x}, \mathbf{y}\}$. By assumption, $r_2(x_2) > 2^{a-1}+1$ and $r_2(y_2) = 2^{a-1}+1$. For each $u \in U$ fitting R_2, we have $E(\Phi(\mathbf{R}), u) > E(\Phi(R'_1, R_2), u)$, contradicting *preference-monotonicity*.

By **2.**, $\Phi(\mathbf{R}) = \Gamma(\mathbf{R})$ or $\Phi(\mathbf{R}) = \Theta(\mathbf{R})$. This holds for each $\mathbf{R} \in \mathcal{R}^N$. As Φ satisfies *neutrality*, if there is $\mathbf{R} \in \mathcal{R}^N$ with $PF(\mathbf{R}) \neq \emptyset$ and $\Phi(\mathbf{R}) = \Theta(\mathbf{R})$, then for each $\mathbf{R} \in \mathcal{R}^N$ with $PF(\mathbf{R}) \neq \emptyset$, we have $\Phi(\mathbf{R}) = \Theta(\mathbf{R})$. Thus, for each $\mathbf{R} \in \mathcal{R}^N$, we have $\Phi(\mathbf{R}) = \Gamma(\mathbf{R})$, or for each $\mathbf{R} \in \mathcal{R}^N$, we have $\Phi(\mathbf{R}) = \Theta(\mathbf{R})$.■

One cannot drop an axiom in Th. 2.2.[7] Let Φ be a rule. Assume there is an ordered list $(\alpha_1, ..., \alpha_a)$ of A with for each $\mathbf{R} \in \mathcal{R}^N$, if $\underline{r}(\Gamma, \mathbf{R}) \neq 2^{a-1}+1$, then $\Phi(\mathbf{R}) = \Gamma(\mathbf{R})$; if not, $\Phi(\mathbf{R}) = \{\mathbf{x} \in \Gamma(\mathbf{R})$: there are $i \in N$ with $r(x_i) = \underline{r}(\Gamma, \mathbf{R})$ and $q \in \{1, ..., a\}$ with $\alpha_q \in y_i$, and for each $j \in N$ with $r(x_j) = \underline{r}(\Gamma, \mathbf{R})$ and each $p \in \{1, ..., a\}$ with $p < q$, we have $\alpha_p \notin x_j\}$. Then, Φ satisfies all axioms but *neutrality*. Assume for each $\mathbf{R} \in \mathcal{R}^N$, we have $\Phi(\mathbf{R}) = \Lambda(\mathbf{R})$. Then, Φ satisfies all axioms but *preference-monotonicity*. Assume for each $\mathbf{R} \in \mathcal{R}^N$, if $PF(\mathbf{R}) \neq \emptyset$, then $\Phi(\mathbf{R}) = \Gamma(\mathbf{R})$; if not, $\Phi(\mathbf{R}) = \{(\emptyset, A), (A, \emptyset)\}$. Then, Φ satisfies all axioms but the *unanimity bound*. Assume for each $\mathbf{R} \in \mathcal{R}^N$, we have $\Phi(\mathbf{R}) = PB(\mathbf{R})$. Then, Φ satisfies all axioms but *conditional no-envy*. Assume there is $i \in N$ with for each $\mathbf{R} \in \mathcal{R}^N$, we have $\Phi(\mathbf{R}) = \{\mathbf{x} \in \Gamma(\mathbf{R})$: for each $\mathbf{y} \in \Gamma(\mathbf{R})$, we have $r_i(y_i) \leq r_i(x_i)\}$. Then, Φ satisfies all axioms but *anonymity*.

The intuition for why the maximin rule satisfies *preference-monotonicity*, not the leximin rule, is simple. Consider Ex. 1. In $(R'_1, R^{\wedge}_2)$, the maximin rule selects $(\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma})$. The maximal rank of $(\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$ is the rank of $\boldsymbol{\alpha\gamma}$ w.r.t. R'_1. The one of $(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma})$ is the rank of $\boldsymbol{\beta\gamma}$ w.r.t. $R^{\wedge}_2$. As the latter is one rank higher than the former, the leximin rule only selects $(\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$. In (R'_1, R^{+}_2), agents 1 and 2 further agree on $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$, hence on $\boldsymbol{\beta\gamma}$ and $\boldsymbol{\alpha\beta}$. To be so, $\boldsymbol{\beta\gamma}$ is one rank lower for R^{+}_2 than $R^{\wedge}_2$. The maximal ranks of $(\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma})$ equalize. As their minimal ranks are unchanged, both rules select $(\boldsymbol{\alpha\gamma}, \boldsymbol{\beta})$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta\gamma})$. The selections of the maximin rule are the same in both economies. The one of the leximin rule includes one more allocation in the latter, with this allocation being precisely the one, among the two with minimal ranks, 1 prefers. Her expected utility is unchanged with the maximin rule and increases with the leximin rule. It is exactly the difference between the two rules that makes the former satisfy *preference-monotonicity* and the latter not.

If rules different from the maximin rule satisfy the axioms we impose in economies with more than three goods, is an open question. Natural candidates do not.

Ex. 2 Let $N = \mathbf{12}$, $A = \boldsymbol{\alpha\beta\gamma\delta}$, and $R_1, R_2, R'_2, R^{*}_2, R^{\wedge}_2 \in \mathcal{R}$ be as in Fig. 5.

- Consider Θ. As it is a subcorrespondence of Γ, by Th. 2.1, it satisfies *efficiency*, *conditional no-envy*, and the *unanimity bound*. As it never uses agents' names, it satisfies *anonymity*. Yet, it violates *preference-monotonicity*: $R^{\wedge}_2$ is closer to R'_1 than R^{*}_2, $\Theta(R'_1, R^{*}_2) = \{(\boldsymbol{\alpha\beta}, \boldsymbol{\gamma\delta})\}$, and $\Theta(R'_1, R^{\wedge}_2) = \{(\boldsymbol{\alpha\beta}, \boldsymbol{\gamma\delta}), (\boldsymbol{\beta\gamma}, \boldsymbol{\alpha\delta})\}$. For each $u \in U$ fitting R_1, we have $E(\Theta(R'_1, R^{*}_2), u) < E(\Theta(R'_1, R^{\wedge}_2), u)$.
- Let Φ be the rule with for each $\mathbf{R} \in \mathcal{R}^N$, if $PF(\mathbf{R}) = \emptyset$, then $\Phi(\mathbf{R}) = PB(\mathbf{R})$; if not, $\Phi(\mathbf{R}) = PF(\mathbf{R})$. (Note that Φ is a supercorrespondence of Γ and if $a = 3$, for each $\mathbf{R} \in \mathcal{R}^N$, we have $\Phi(\mathbf{R}) = \Gamma(\mathbf{R})$.) By definition, it satisfies *efficiency* and *conditional no-envy*. As it never uses agents' names, it satisfies *anonymity*. As $n = 2$, if an allocation is *envy-free*, each agent has at

Figure 5:

R_1	R_2	R_1	R_2'	R_1'	R_2^*	R_1'	$R_2^{\wedge}$
$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$
$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\gamma\delta}$
$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\beta\gamma\delta}$
$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\beta\delta}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\beta\delta}$	$\boldsymbol{\alpha\beta\delta}$
$\boldsymbol{\alpha\delta}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\delta}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{deg}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{def}$
$\boldsymbol{\alpha\beta}$	$\boldsymbol{\gamma\delta}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\beta\delta}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\gamma\delta}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\gamma\delta}$
$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\beta\delta}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\gamma\delta}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha\delta}$
$\boldsymbol{\alpha}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha\delta}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha\gamma}$
$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\delta}$	$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\delta}$	$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\beta\delta}$
$\boldsymbol{\beta\delta}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\beta\delta}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\gamma\delta}$	$\boldsymbol{\beta\delta}$	$\boldsymbol{\gamma\delta}$	$\boldsymbol{\beta\gamma}$
$\boldsymbol{\gamma\delta}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\gamma\delta}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\delta}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha\delta}$	$\boldsymbol{\alpha\beta}$
$\boldsymbol{\beta\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\delta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\delta}$
$\boldsymbol{\delta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\gamma}$
$\boldsymbol{\beta}$	$\boldsymbol{\beta}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$
$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

least her $2^{a-1}+1th$ least preferred good, hence it satisfies the *unanimity bound*. Yet, it violates *preference-monotonicity*: R_2' is closer to R_1 than R_2, $\Phi(R_1,R_2)=\{(\boldsymbol{\alpha},\boldsymbol{\beta\gamma\delta}),(\boldsymbol{\alpha\beta},\boldsymbol{\gamma\delta}),(\boldsymbol{\alpha\delta},\boldsymbol{\beta\gamma})\}$, and $\Phi(R_1,R_2')=\{(\boldsymbol{\alpha},\boldsymbol{\beta\gamma\delta}),(\boldsymbol{\alpha\gamma},\boldsymbol{\beta\delta}),(\boldsymbol{\alpha\beta},\boldsymbol{\gamma\delta}),(\boldsymbol{\alpha\delta},\boldsymbol{\beta\gamma})\}$. For $u\in U$ fitting R_1 with $u(\boldsymbol{\alpha})=3.4$, $u(\boldsymbol{\beta})=1.1$, $u(\boldsymbol{\gamma})=1$, and $u(\boldsymbol{\delta})=1.2$, we have $E(\Phi(R_1,R_2),u)<E(\Phi(R_1,R_2'),u)$.$\square$

Let us end this section with a remark on *preference-monotonicity*. Defining it, we have introduced a notion of (dis-)similarity w.r.t. preferences. We have deemed incomparable those such that a change from one to the other implies opposite movements, i.e. agreements and disagreements on pairs of subsets of goods. Assume one instead only focuses on agreements, disregarding the shaky interpretation. For each $i,j\in N$ and each $R_i,R_j',R_j^*\in\mathcal{R}$, say R_j^* *weakly closer* to R_i than R_j' if i and j agree in (R_i,R_j^*) on at least one more pair of subsets than in (R_i,R_j'), i.e. $\#\{S,T\in\mathcal{S}: S\ R_i\ T \text{ and } S\ R_j^*\ T\}>\#\{S,T\in\mathcal{S}: S\ R_i\ T \text{ and } S\ R_j'\ T\}$.[8] This notion is weaker, in the sense, if preferences are closer, they are weakly closer, not conversely. In Ex. 1, $R_2^{\wedge}$ is weakly closer to R_1' than R_2°. As 1 and 2 disagree in $(R_1',R_2^{\wedge})$ on pairs of subsets they agree on in (R_1',R_2°), it is not closer. This notion implies a stronger monotonicity axiom that the maximin and maximin-minimax rules violate. Thus, no rule satisfies this axiom, the other fairness axioms, and *neutrality*.

Ex. 1 (cont.) Consider $(R_1,R_2^{\wedge})$ and (R_1,R_2^*). Then, R_2^* weakly closer to R_1 than $R_2^{\wedge}$, $\Gamma(R_1,R_2^{\wedge})=\{(\boldsymbol{\gamma\delta},\boldsymbol{\epsilon})\}$, and $\Gamma(R_1,R_2^*)=\{(\boldsymbol{\delta\epsilon},\boldsymbol{\gamma}),(\boldsymbol{\delta},\boldsymbol{\gamma\epsilon})\}$. For $u\in U$ fitting R_1 with $u(\boldsymbol{\delta})=3.5$, $u(\boldsymbol{\epsilon})=3$, and $u(\boldsymbol{\gamma})=1$, we have $E(\Gamma(R_1,R_2^{\wedge}),u)<E(\Gamma(R_1,R_2^*),u)$. Consider (R_1',R_2°) and $(R_1',R_2^{\wedge})$. Then, $R_2^{\wedge}$ weakly closer to R_1' than R_2°, $\Theta(R_1',R_2^{\circ})=\{(\boldsymbol{\delta\epsilon},\boldsymbol{\gamma}),(\boldsymbol{\delta},\boldsymbol{\gamma\epsilon})\}$, and $\Theta(R_1',R_2^{\wedge})=\{(\boldsymbol{\gamma\delta},\boldsymbol{\epsilon})\}$. For $u\in U$ fitting R_1' with $u(\boldsymbol{\delta})=6$, $u(\boldsymbol{\epsilon})=3$, and $u(\boldsymbol{\gamma})=2$, we have $E(\Theta(R_1',R_2^{\circ}),u)<E(\Theta(R_1',R_2^{\wedge}),u)$.$\square$

5 Two-agent vs. More-than-two-agent economies

We now study the gap between two-agent economies and those with more than two.

In two-agent economies, if *envy-free* allocations exist, at least one is *efficient* (Th. 1.1). In economies with one more good than agents, each *envy-free* allocation is *efficient*. Yet, in other economies, even if there are *envy-free* allocations, it may be that none is *efficient*. Further, in two-agent economies, independently of an agent's preferences, the rank of the worst bundle of the allocations *efficiency* and minimization of inequality among equals recommend when each other agent has her preferences, equals 2^{a-1} (Th. 1.2). In economies with one more good than agents, independently of her preferences, it equals 4. Yet, in other economies, it may vary along with preferences.

Theorem 3

1. *If $a = n + 1$, each* envy-free *allocation is* efficient*. If $a > n + 1$ and $n > 2$, there are economies with* envy-free *allocations, and no* efficient *and* envy-free *allocation.*
2. *If $a = n + 1$, independently of an agent's preferences, the rank of the worst bundle of the allocations* efficiency *and minimization of inequality among equals recommend when each other agent has her preferences, equals 4. If $a > n + 1$ and $n > 2$, it varies along preferences.*

Proof.

Stmt. 1

- [If $a = n + 1$, for each $\mathbf{R} \in \mathcal{R}^N$, we have $F(\mathbf{R}) \subset P(\mathbf{R})$.] Assume $a = n + 1$ and by contradiction, there are $\mathbf{R} \in \mathcal{R}^N$ and $\mathbf{x} \in F(\mathbf{R})$ with $\mathbf{x} \notin P(\mathbf{R})$. There is $\mathbf{y} \in P(\mathbf{R})$ with for each $i \in N$, we have $y_i \; R_i \; x_i$ and there is $i \in N$ with $y_i \; P_i \; x_i$. By strictness, for each $i \in N$ with $y_i \neq x_i$, we have $y_i \; P_i \; x_i$. As $a = n + 1$ and by desirability, there is $j \in N$ with $\#x_j = 2$ and for each $l \in N \backslash \boldsymbol{j}$, we have $\#x_l = 1$, and there is $k \in N$ with $\#y_k = 2$ and for each $l \in N \backslash \boldsymbol{k}$, we have $\#y_l = 1$. Distinguish 2 cases.
  - ***a* :** $i = j$ and $i \neq k$, or $i \neq j$ and $i = k$. Then, $\#x_j = 2$, $\#y_j = 1$, and $y_j \; P_j \; x_j$. By desirability, $x_j \not\supset y_j$, hence there is $l \in N \backslash \boldsymbol{j}$ with $x_l = y_i$. Thus, $x_l \; P_i \; x_i$, contradicting $\mathbf{x} \in F(\mathbf{R})$.
  - ***b* :** $i = j$ and $i = k$, or $i \neq j$ and $i \neq k$. There is $l \in N$ with $\#x_l = 1$, $\#y_l = 1$, and $y_l \; P_l \; x_l$, hence there is $m \in N \backslash \boldsymbol{l}$ with $x_m \supset y_l$. By desirability, $x_m \; R_l \; y_l$. Thus, $x_m \; P_l \; x_l$, contradicting $\mathbf{x} \in F(\mathbf{R})$.
- [If $a > n + 1$ and $n > 2$, there is $\mathbf{R} \in \mathcal{R}^N$ with $F(\mathbf{R}) \neq \emptyset$ and $PF(\mathbf{R}) = \emptyset$.] Let $N = \mathbf{123}$, $A = \boldsymbol{\alpha\beta\gamma\delta\epsilon}$, and $\mathbf{R} \in \mathcal{R}^N$ be as in Fig. 6. Let $\mathbf{x}, \mathbf{y} \in X$ with $x = (\boldsymbol{\alpha\epsilon}, \boldsymbol{\beta\delta}, \boldsymbol{\gamma})$ and $y = (\boldsymbol{\alpha\beta}, \boldsymbol{\delta\epsilon}, \boldsymbol{\gamma})$. Then, $F(\mathbf{R}) = \{\mathbf{x}\}$. As $y_1 \; P_1 \; x_1$, $y_2 \; P_2 \; x_2$, and $y_3 \; I_3 \; x_3$, we have $x \notin P(\mathbf{R})$. Thus, $PF(\mathbf{R}) = \emptyset$.

Stmt. 2

- [If $a = n + 1$, for each $\mathbf{R} \in \mathcal{R}^N$ and each $i \in N$, we have $\check{r}(\mathbf{R_i}) = 4$.] Assume $a = n + 1$. Let $\mathbf{R} \in \mathcal{R}^N$, $i \in N$, and for each $j \in N \backslash \boldsymbol{i}$, let $R'_j \in \mathcal{R}$ with $R'_j = R_i$. Let $\alpha, \beta, \gamma \in A$ be i's 1st, 2nd, and 3rd least preferred good resp., i.e. $r_i(\boldsymbol{\alpha}) = 2$, $r_i(\boldsymbol{\beta}) = 3$, and $r_i(\boldsymbol{\gamma}) = 4$ if and only if $r_i(\boldsymbol{\alpha\beta}) = 5$. W.l.o.g. assume $r_i(\boldsymbol{\gamma}) = 4$. By **1.** and **2.**, $\check{r}(\mathbf{R_i}) = 4$.
  1. [There is $\mathbf{x} \in X$ with for each $j \in N \backslash \boldsymbol{i}$, we have $r'_j(x_j) \geq 4$ and $r_i(x_i) \geq 4$.] Let $\mathbf{x} \in X$ with for each $j \in N \backslash \boldsymbol{i}$, we have $\#x_j = 1$ and $x_j \notin \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$, and $x_i = \boldsymbol{\alpha\beta}$. As for each $j \in N \backslash \boldsymbol{i}$, we have $R'_j = R_i$, for each $j \in N \backslash \boldsymbol{i}$, we have $\boldsymbol{r'_j(x_j) \geq 4}$, and $r_i(x_i) \geq 4$.
  2. [For each $\mathbf{x} \in X$, if for each $j \in N \backslash \boldsymbol{i}$, we have $r'_j(x_j) \geq 5$, then $r_i(x_i) < 5$.] Let $\mathbf{x} \in X$ with for each $j \in N \backslash \boldsymbol{i}$, we have $r'_j(x_j) \geq 5$. As for each $j \in N \backslash \boldsymbol{i}$, we have $R'_j = R_i$, for each $j \in N \backslash \boldsymbol{i}$, we have $x_j \notin \{\emptyset, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$. As $a = n + 1$, there are $j, k \in N \backslash \boldsymbol{i}$ with $\#x_j = \#x_k = 2$ and for each $l \in N \backslash \boldsymbol{ijk}$, we have $\#x_l = 1$. If $j \neq k$, then $\sum_{l \in N \backslash \boldsymbol{i}} \#x_l = a$. Thus, $x_i = \emptyset$. If $j = k$, then $\sum_{l \in N \backslash \boldsymbol{i}} \#x_l = a - 1$. As for each $l \in N \backslash \boldsymbol{ij}$, we have $\#x_l = 1$ and $x_l \notin \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$, we have $\cup_{l \in N \backslash \boldsymbol{ij}} x_l = A \backslash \boldsymbol{\alpha\beta\gamma}$. As $x_j \notin \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$, we have $x_j = \boldsymbol{\alpha\beta}$. Thus, $x_i = \boldsymbol{\gamma}$. In both cases, $r_i(x_i) < 5$.
- [If $a > n + 1$ and $n > 2$, there are $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$ and $i \in N$ with $\check{r}(\mathbf{R_i}) \neq \check{r}(\mathbf{R'_i})$.] Let $N = \mathbf{123}$, $A = \boldsymbol{\alpha\beta\gamma\delta\epsilon}$, and $\mathbf{R} \in \mathcal{R}^N$ be as in Fig. 6. Then, $\check{r}(\mathbf{R_1}) = 7$, $\check{r}(\mathbf{R_2}) = 8$, and $\check{r}(\mathbf{R_3}) = 8$.■

By Th. 1.2 and 3.2, if $n = 2$, $a = n + 1$, or both, the rank of the worst bundle of the allocations *efficiency* and minimization of inequality among equals recommend when each other agent has an agent's preferences, has a closed-form expression. One knows exactly what the *unanimity bound* requires, i.e. each agent should

Figure 6:

R_1	R_2	R_3
$\boldsymbol{\alpha\beta\gamma\delta\epsilon}$	$\boldsymbol{\alpha\beta\gamma\delta\epsilon}$	$\boldsymbol{\alpha\beta\gamma\delta\epsilon}$
$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\beta\gamma\delta\epsilon}$	$\boldsymbol{\alpha\beta\gamma\delta}$
$\boldsymbol{\alpha\gamma\delta\epsilon}$	$\boldsymbol{\alpha\gamma\delta\epsilon}$	$\boldsymbol{\alpha\gamma\delta\epsilon}$
$\boldsymbol{\alpha\beta\gamma\epsilon}$	$\boldsymbol{\gamma\delta\epsilon}$	$\boldsymbol{\alpha\gamma\delta}$
$\boldsymbol{\alpha\beta\delta\epsilon}$	$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\epsilon}$
$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma}$
$\boldsymbol{\beta\gamma\delta\epsilon}$	$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\alpha\gamma\epsilon}$
$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\gamma\delta}$	$\boldsymbol{\alpha\gamma}$
$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\alpha\beta\delta\epsilon}$	$\boldsymbol{\beta\gamma\delta\epsilon}$
$\boldsymbol{\alpha\gamma\epsilon}$	$\boldsymbol{\beta\delta\epsilon}$	$\boldsymbol{\beta\gamma\delta}$
$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\epsilon}$	$\boldsymbol{\alpha\beta\delta\epsilon}$
$\boldsymbol{\alpha\delta\epsilon}$	$\boldsymbol{\alpha\delta\epsilon}$	$\boldsymbol{\alpha\beta\delta}$
$\boldsymbol{\alpha\beta\epsilon}$	$\boldsymbol{\beta\gamma\epsilon}$	$\boldsymbol{\gamma\delta\epsilon}$
$\boldsymbol{\gamma\delta\epsilon}$	$\boldsymbol{\delta\epsilon}$	$\boldsymbol{\gamma\delta}$
$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\gamma\epsilon}$	$\boldsymbol{\beta\gamma\epsilon}$
$\boldsymbol{\beta\gamma\epsilon}$	$\boldsymbol{\gamma\epsilon}$	$\boldsymbol{\beta\gamma}$
$\boldsymbol{\alpha\delta}$	$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\alpha\delta\epsilon}$
$\boldsymbol{\beta\delta\epsilon}$	$\boldsymbol{\beta\delta}$	$\boldsymbol{\alpha\delta}$
$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\epsilon}$
$\boldsymbol{\gamma\delta}$	$\boldsymbol{\alpha\delta}$	$\boldsymbol{\alpha\beta}$
$\boldsymbol{\beta\gamma}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\gamma\epsilon}$
$\boldsymbol{\alpha\epsilon}$	$\boldsymbol{\delta}$	$\boldsymbol{\gamma}$
$\boldsymbol{\beta\delta}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\epsilon}$
$\boldsymbol{\gamma\epsilon}$	$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$
$\boldsymbol{\delta\epsilon}$	$\boldsymbol{\alpha\beta\epsilon}$	$\boldsymbol{\beta\delta\epsilon}$
$\boldsymbol{\alpha}$	$\boldsymbol{\beta\epsilon}$	$\boldsymbol{\beta\delta}$
$\boldsymbol{\beta\epsilon}$	$\boldsymbol{\alpha\epsilon}$	$\boldsymbol{\delta\epsilon}$
$\boldsymbol{\gamma}$	$\boldsymbol{\epsilon}$	$\boldsymbol{\delta}$
$\boldsymbol{\delta}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\beta\epsilon}$
$\boldsymbol{\beta}$	$\boldsymbol{\beta}$	$\boldsymbol{\beta}$
$\boldsymbol{\epsilon}$	$\boldsymbol{\alpha}$	$\boldsymbol{\epsilon}$
$\emptyset$	$\emptyset$	$\emptyset$

find her bundle at least as good as the subset she ranks: (i) if $n=2$, then 2^{a-1}th; (ii) if $a=n+1$, then 4th. If $n>2$ and $a>n+1$, we fear it is not the case. Computational complexity hardens the study of an axiom.[9] Yet, in what follows, we prove that the minimum, across preferences, of this rank has a closed-form expression. This result gives a necessary condition for *unanimity bound*, in the sense, the *unanimity bound* secures each agent finds her bundle at least as good as the subset whose rank equals this minimal value, that avoids such possible complexity.

Theorem 4 *Independently of an agent's preferences, the rank of the worst bundle of the allocations* efficiency *and minimization of inequality among equals recommend when each other agent has her preferences, is at least equal to* $r_0 \in \{1,...,2^a\}$ *with*

$$r_0 = \sum_{p\in\{0,...,\lfloor \frac{a}{n}\rfloor\}} \binom{a}{p} - [\min\{1, n\lceil \tfrac{a}{n}\rceil - a - 1\} \sum_{q\in\{1,...,n\lceil \frac{a}{n}\rceil - a - 1\}} \binom{a-q}{\lfloor \frac{a}{n}\rfloor - 1}].$$

Proof. [For each $\mathbf{R}\in\mathcal{R}^N$ and each $i\in N$, we have $\check{r}(\mathbf{R_i}) \geq r_0$.] Let $\mathbf{R}\in\mathcal{R}^N$, $i\in N$, and for each $S\in\mathcal{S}$ and each $q\in\{1,...,s\}$, let $\hat{r}(S,R_i,1)\equiv \max_{\alpha\in S}\{r_i(\boldsymbol{\alpha})\}$, ..., and $\hat{r}(S,R_i,q)\equiv \max_{\alpha\in S\setminus\{\beta\in S:\text{ there is } p\in\{1,...,q-1\} \text{ with } r_i(\boldsymbol{\beta})=\hat{r}(S,R_i,p)\}}\{r_i(\boldsymbol{\alpha})\}$.
Let $R_i^*\in\mathcal{R}$ be lexicographic w.r.t. R_i, in the sense, for each $S,T\in\mathcal{S}$, if $s\neq t$, then $S\ P_i^*\ T$ if and only if $s>t$; if not, $S\ P_j^*\ T$ if and only if there is $q\in\{1,...,s\}$ with for each $p\in\{1,...,q-1\}$, we have $\hat{r}(S,R_i,p)\geq\hat{r}(T,R_i,p)$ with strict inequality for q.
Let $S\in\mathcal{S}$ be non-empty and $t_0,...,t_s\in\{0,...,a\}$ with $t_0=0$, $\hat{r}(A,R_i^*,t_1)=\hat{r}(S,R_i^*,1)$, ..., and $\hat{r}(A,R_i^*,t_s)=\hat{r}(S,R_i^*,s)$. Then,

$$r_i^*(S) = \sum_{p\in\{0,...,s\}} \binom{a}{p} - \sum_{p\in\{1,...,s\}} [\min\{1, t_p - t_{p-1} - 1\} \sum_{q\in\{t_{p-1}+1,...,t_p-1\}} \binom{a-q}{s-p}].$$

W.l.o.g. let $A=\boldsymbol{\alpha_1...\alpha_a}$ with $r_i(\boldsymbol{\alpha_1})>...>r_i(\boldsymbol{\alpha_a})$. The 1st term counts all subsets of size 0 to s. To this, I subtract the 2nd term, i.e. (**i**) all subsets of size s in which the 1st preferred good in the subset is α_1, ..., α_{t_1-1}; (**ii**) all subsets of size s in which the 1st preferred good in the subset is α_{t_1} and the 2nd preferred good in the subset is the α_{t_1+1}, ..., α_{t_2-1}; ...
Assume for each $p\in\{2,...,s\}$, we have $(t_p-t_{p-1})=1$. Then,

$$r_i^*(S) = \sum_{p\in\{0,...,s\}} \binom{a}{p} - \min\{1, t_1-1\} \sum_{q\in\{1,...,t_1-1\}} \binom{a-q}{s-1}.$$

Let $\Sigma(s,t_1)$ denote this formula. For each $j\in N\backslash\boldsymbol{i}$, let $R_j^*\in\mathcal{R}$ with $R_j^*=R_i^*$. By **1.** and **2.**, $\check{r}(\mathbf{R_i^*})=\Sigma(\lfloor\frac{a}{n}\rfloor, n\lceil\frac{a}{n}\rceil-a)$. Thus, $\check{r}(\mathbf{R_i})=\Sigma(\lfloor\frac{a}{n}\rfloor, n\lceil\frac{a}{n}\rceil-a)$.

1. [There is $\mathbf{x}\in X$ with for each $j\in N$, we have $r_j^*(x_j)\geq\Sigma(\lfloor\frac{a}{n}\rfloor, n\lceil\frac{a}{n}\rceil-a$.] Let $(i_1,...,i_n)$ be an ordered list of N, $\overline{q}\in\{1,...,n\}$, and $\mathbf{x}\in X$ with

   - $\overline{q}\lfloor\frac{a}{n}\rfloor+(n-\overline{q})\lceil\frac{a}{n}\rceil=a$,
   - $x_{i_{\overline{q}}}=\{\alpha_{\overline{q}},...,\alpha_{\overline{q}+\lfloor\frac{a}{n}\rfloor-1}\}$,
   - For each $q\in\{1,...,\overline{q}-1\}$, we have $x_{i_q}=\{\alpha_q,\alpha_{\overline{q}+(\overline{q}-q)\lfloor\frac{a}{n}\rfloor-(\overline{q}-q-1)},...,\alpha_{\overline{q}+(\overline{q}-q+1)\lfloor\frac{a}{n}\rfloor-(\overline{q}-q+1)}\}$,
   - For each $q\in\{\overline{q}-1,...,n\}$, we have $\#x_{i_q}=\lfloor\frac{a}{n}\rfloor+1$.

   W.l.o.g. assume $i_{\overline{q}}=i$. Then, $\#x_i=\lfloor\frac{a}{n}\rfloor$. Also, $t_1=\overline{q}$, $t_2=\overline{q}+1$, ..., $t_{s_1}=\overline{q}+\lfloor\frac{a}{n}\rfloor-2$, and $t_s=\overline{q}+\lfloor\frac{a}{n}\rfloor-1$ implying for each $p\in\{2,...,s\}$, we have $(t_p-t_{p-1})=1$. Thus, $r_i^*(x_i)=\Sigma(\lfloor\frac{a}{n}\rfloor,\overline{q})$. As $\overline{q}\lfloor\frac{a}{n}\rfloor+(n-\overline{q})\lceil\frac{a}{n}\rceil=a$, we have $\overline{q}=n\lceil\frac{a}{n}\rceil-a$. Thus, $r_i^*(x_i)=\Sigma(\lfloor\frac{a}{n}\rfloor, n\lceil\frac{a}{n}\rceil-a)$.
   Let $q\in\{1,...,n\}$. If $q<\overline{q}$, then $\#x_{i_q}=\lfloor\frac{a}{n}\rfloor=\#x_i$ and $\hat{r}(x_{i_q},R_{i_q}^*,1)=\hat{r}(A,R_i^*,q)>\hat{r}(A,R_i^*,\overline{q})=\hat{r}(x_i,R_i^*,1)$. If $q>\overline{q}$, then $\#x_{i_q}=\lfloor\frac{a}{n}\rfloor+1$. Thus, $r_{i_q}^*(x_{i_q})>\sum_{p\in\{0,...,\lfloor\frac{a}{n}\rfloor\}}\binom{a}{p}$. In both cases, $r_{i_q}^*(x_{i_q})>r_i^*(x_i)$.

2. [Let $\mathbf{x} \in X$ be as in **1.** There is no $\mathbf{y} \in X$ with for each $j \in N$, we have $r_j^*(y_j) > r_i^*(x_i)$.] By contradiction, assume there is $\mathbf{y} \in X$ with for each $j \in N$, we have $r_j^*(y_j) > r_i^*(x_i)$. For each $j \in N$, we have $\#y_j = \lfloor \frac{a}{n} \rfloor$ or $\#y_j = \lfloor \frac{a}{n} \rfloor + 1$. As $r_i^*(y_i) > r_i^*(x_i)$, we have $\#x_i < \#y_i$ or $\#x_i = \#y_i$. Distinguish 2 cases.

***a* :** $\#x_i < \#y_i$. There is $j \in N \backslash \boldsymbol{i}$ with $\#x_j = \lfloor \frac{a}{n} \rfloor + 1$ and $\#y_j = \lfloor \frac{a}{n} \rfloor$. If $\hat{r}(y_j, R_i, 1) \leq \hat{r}(A, R_i, \overline{q})$, then $r_j^*(y_j) \leq \Sigma(\lfloor \frac{a}{n} \rfloor, \overline{q})$, contradicting $r_j^*(y_j) > r_i^*(x_i)$. Thus, $\hat{r}(y_j, R_i, 1) > \hat{r}(A, R_i, \overline{q})$, hence there is $k \in N \backslash \boldsymbol{i}$ with $\#x_k = \lfloor \frac{a}{n} \rfloor$ and $\hat{r}(y_k, R_i, 1) < \hat{r}(A, R_i, \overline{q})$. If $\#y_k = \lfloor \frac{a}{n} \rfloor$, then $r_k^*(y_k) < \Sigma(\lfloor \frac{a}{n} \rfloor, \overline{q})$, contradicting $r_k^*(y_k) > r_i^*(x_i)$. Thus, $\#y_k = \lfloor \frac{a}{n} \rfloor + 1$. Repeating this logic, there is $g \in G$ with $(x_{g(l)})_{l \in N} = y$. By **1.**, for each $l \in N$, we have $r_l^*(y_l) \geq \Sigma(\lfloor \frac{a}{n} \rfloor, n\lceil \frac{a}{n} \rceil - a)$ with equality for $l \in N \backslash \boldsymbol{i}$, contradicting $r_l^*(y_l) > r_i^*(x_i)$.

***b* :** $\#x_i = \#y_i$. Thus, $\hat{r}(y_i, R_i, 1) > \hat{r}(A, R_i, \overline{q})$, hence there is $j \in N \backslash \boldsymbol{i}$ with $\#x_j = \lfloor \frac{a}{n} \rfloor$ and $\hat{r}(y_j, R_i, 1) < \hat{r}(A, R_i, \overline{q})$. If $\#y_j = \lfloor \frac{a}{n} \rfloor$, then $r_k^*(y_j) < \Sigma(\lfloor \frac{a}{n} \rfloor, \overline{q})$, contradicting $r_j^*(y_j) > r_i^*(x_i)$. Thus, $\#y_j = \lfloor \frac{a}{n} \rfloor + 1$. By the logic of Case ***a*.**, there is $g \in G$ with $(x_{g(k)})_{k \in N} = y$. By **1.**, for each $k \in N$, we have $r_k^*(y_k) \geq \Sigma(\lfloor \frac{a}{n} \rfloor, n\lceil \frac{a}{n} \rceil - a)$ with equality for $k \in N \backslash \boldsymbol{i}$, contradicting $r_k^*(y_k) > r_i^*(x_i)$.■

The clear gap between two-agent economies and those with more than two leads one to regard them as different. Different problems call for different solutions. Indeed, by Th. 2, in two-agent economies, the maximin rule satisfies the axioms we impose. In economies with three goods, it is the only rule, with one of its subcorrespondences, satisfying the fairness axioms we impose and not discriminating between goods. Yet, in economies with more than two agents, it violates *efficiency* and as each of its subcorrespondences, *conditional no-envy*. We prove this in Ex. 3. As there seem to be no other desirable rule in two-agent economies, an open question is whether this gap implies an incompatibility between *efficiency*, *anonymity*, *conditional no-envy*, the *unanimity bound*, and *preference-monotonicity* in those with more than two.

Before, we comment on the definitions of the maximin-minimax and leximin rules. The extension of the former to economies with more than two agents is not immediate, in particular if avoiding envy is possible. Minimizing inequality, e.g. recursively from the agent with the maximal rank and onwards, forces a violation of *efficiency*. The latter lexicographically applies the maximin rule, in the sense, it first selects the allocations maximizing the minimal rank across agents; among these allocations, it selects the allocations maximizing the second minimal rank across agents; ...; This is done until no further distinction is possible (e.g. Sen, 1970; d'Aspremont and Gevers, 1977).[10] The dual aspects of these rules illustrate the conflict between *efficiency* and fairness fundamentals that only appears in economies with more than two agents.

Ex. 3 Let $N = \mathbf{123}$, $A = \boldsymbol{\alpha\beta\gamma\delta}$, and $R_1, R_2, R_3, R_3' \in \mathcal{R}$ be as in Fig. 7, and $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ with $\mathbf{w} = (\boldsymbol{\beta\gamma}, \boldsymbol{\delta}, \boldsymbol{\alpha})$, $\mathbf{x} = (\boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\beta\gamma})$, $\mathbf{y} = (\boldsymbol{\alpha\beta}, \boldsymbol{\delta}, \boldsymbol{\gamma})$, and $\mathbf{z} = (\boldsymbol{\alpha}, \boldsymbol{\beta\delta}, \boldsymbol{\gamma})$. First, $\Gamma(R_1, R_2, R_3) = \{\mathbf{w}, \mathbf{x}\}$. As $x_1 \; P_1 \; w_1$, $x_2 \; I_2 \; w_2$, and $x_3 \; P_3 \; w_3$, we have $\mathbf{w} \notin P(R_1, R_2, R_3)$. Thus, $\Gamma(R_1, R_2, R_3) \not\subset P(R_1, R_2, R_3)$. Second, $\Gamma(R_1, R_2, R_3') = \{\mathbf{z}\}$. As $z_2 \; P_2 \; z_1$, we have $\mathbf{z} \notin F(R_1, R_2, R_3')$. As $y_1 \; P_1 \; y_2$, $y_1 \; P_1 \; y_3$, $y_2 \; P_2 \; y_1$, $y_2 \; P_2 \; y_3$, $y_3 \; P_3' \; y_1$, and $y_3 \; P_3' \; y_2$, we have $\mathbf{y} \in F(R_1, R_2, R_3')$. As $a = n + 1$, by Th. 3.1, $\mathbf{y} \in PF(R_1, R_2, R_3')$. Thus, $\Gamma(R_1, R_2, R_3') \not\subset F(R_1, R_2, R_3')$ and $PF(R_1, R_2, R_3') \neq \emptyset$.□

6 Concluding comments

Our objective was to identify mappings that systematically give one *efficient* and fair allocations of indivisible goods among two agents when monetary compensation is impossible or not customary. We have assumed strict and additively separable preferences over subsets, and desirable goods.

As no large number of goods ever replaces money as a compensating means, the search for *efficient* and fair allocations is even harder than when money is available. We have focused on several fairness fundamentals. *Anonymity* is classical, in the sense, one studies it much in the literature and its adaptation is straightforward. As avoiding envy is not always possible, yet sometimes, we have introduced *conditional no-envy*. As properties of welfare lower bounds and monotonicity w.r.t. changes in preferences are crucial to judge allocations on the

Figure 7:

R_1	R_2	R_3	R_1	R_2	R'_3
$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma\delta}$
$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\beta\gamma\delta}$
$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\gamma\delta}$
$\boldsymbol{\alpha\beta}$	$\boldsymbol{\gamma\delta}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\gamma\delta}$	$\boldsymbol{\gamma\delta}$
$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\alpha\gamma\delta}$	$\boldsymbol{\alpha\beta\delta}$	$\boldsymbol{\alpha\beta\gamma}$
$\boldsymbol{\alpha\delta}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\delta}$	$\boldsymbol{\alpha\beta\gamma}$	$\boldsymbol{\beta\gamma}$
$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\delta}$	$\boldsymbol{\alpha\delta}$	$\boldsymbol{\beta\gamma\delta}$	$\boldsymbol{\alpha\delta}$	$\boldsymbol{\alpha\gamma}$
$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\alpha\gamma}$	$\boldsymbol{\gamma}$
$\boldsymbol{\beta\delta}$	$\boldsymbol{\beta\delta}$	$\boldsymbol{\beta\delta}$	$\boldsymbol{\beta\delta}$	$\boldsymbol{\beta\delta}$	$\boldsymbol{\alpha\beta\delta}$
$\boldsymbol{\alpha}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\beta\delta}$
$\boldsymbol{\beta\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\alpha\delta}$
$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\beta}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$
$\boldsymbol{\gamma\delta}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\gamma\delta}$	$\boldsymbol{\gamma\delta}$	$\boldsymbol{\alpha\beta}$	$\boldsymbol{\alpha\beta}$
$\boldsymbol{\delta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\delta}$	$\boldsymbol{\delta}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$
$\boldsymbol{\gamma}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\gamma}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

basis of fairness in each possible economy and as their adaptation is not straightforward, we have introduced the *unanimity bound* and *preference-monotonicity.* The axioms we have introduced, apply to each set of agents, possibly with more than two elements.

We have reached our objective identifying the maximin rule as a desirable rule. First, it satisfies each axiom we imposed. If there are three goods, it is the only rule, with one of its subcorrespondences, satisfying each fairness axiom and not discriminating between goods. If there are more than three goods, it seems to be among the very few rules satisfying these axioms. Natural subcorrespondences and supercorrespondences, as the maximin-minimax rule, the leximin rule, the rule selecting for each $\mathbf{R} \in \mathcal{R}^N$, each $\mathbf{x} \in PB(\mathbf{R})$, or the rule selecting for each $\mathbf{R} \in \mathcal{R}^N$, if $PF(\mathbf{R}) = \emptyset$, each $\mathbf{x} \in PB(\mathbf{R})$; if not, each $\mathbf{x} \in PF(\mathbf{R})$, violate one of these properties. Finally, one easily applies it: There is a procedure such that an allocation is a solution of it if and only if it is an allocation the maximin rule selects (Herreiner and Puppe, 2002).[1]

The maximin rule embodies the fairness fundamental according to which one should first care for the least fortunates. Other formulations are possible. In particular, *balanced divisions* maximize, across allocations, the minimal utility, across agents, of an allocation, equating the agents' utilities of the subset containing all goods (Brams and Fishburn, 2000; Edelman and Fishburn, 2001); *maxmin divisions* or *Borda maximin allocations* maximize, across allocations, the minimal sum of points, across agents, of an allocation, where the point of a good in an agent's bundle is a, ..., 1 if her most, ..., least preferred of all goods resp. (Brams *et al.*, 2003; Brams and King, 2005); *maximin allocations* maximize, across allocations with each agent's bundle counting (a/n) only if possible, the minimal rank, across agents, of the list consisting of each agent's least preferred good of her bundle in an allocation (Brams and King, 2005).[11] As the maximin rule uses ordinal information on preferences over subsets, it does not assume specific real-valued functions representing preferences nor welfare comparability, and it takes each good it allocates into account.

We have explicitly studied two-agent economies. Indeed, there is a clear gap between these and those with more than two. If there are two agents and both have the same preferences over singletons, the existence of *envy-free* allocations implies the one of *efficient* and *envy-free* allocations (Brams and Fishburn, 2000). If there are more than two agents and all have the same preferences over singletons, it is not true. There are economies with $n = 3$ and $a \geq 9$ with *envy-free* allocations, none of which is *efficient* (Edelman and Fishburn, 2001). Intuitively, the less similar the agents' preferences are, the more *envy-free* allocations and the fewer *efficient* allocations there should be. It is not obvious how the set of *envy-free* and *efficient* allocations reacts.[12] We have proved that for $n = 2$, if *envy-free* allocations exist, at least one is *efficient*, but there are economies with $n = 3$ and $a = 5$ with *envy-free* allocations, none of which is *efficient*. Thus, the implication for $n = 2$ versus the incompatibility for $n > 2$ holding under similarity of preferences over singletons, generalize. We have proved that 5 is the smallest a for there to be such an incompatibility.

For $n = 2$, if there are allocations with for each preference profile in which the agents keep these preferences over singletons, these are *envy-free* allocations, at least one is *efficient* for at least one such profile; for $n > 2$, there are preference profiles and allocations with for each preference profile in which the agents keep these preferences over singletons, these are *envy-free* allocations and for no such profile, it is *efficient* (Brams *et al.*, 2003). This understanding of avoiding envy is stronger, and possible only if there is $q \in \mathbb{N}$ with $qn = a$. Though the incompatibility for $n > 2$ follows from it, this statement says nothing on the implication for $n = 2$, and the number of goods in the example proving it, need not be be the smallest for there to be such an incompatibility. Finding this number, we have revealed a class of economies for which the sets of *envy-free*, and *efficient* and *envy-free* allocations coincide.

Further, we have proved that in two-agent economies, independently of an agent's preferences, the rank of the worst bundle of the allocations *efficiency* and minimization of inequality among equals recommend when each other agent has her preferences, equals 2^{a-1}, hence it has a closed-form expression. In economies with more than two agents, it may vary along with preferences and we fear computational complexity. We have proved that 5 is the smallest a for there to be such a situation. We leave for further research the study of the *unanimity bound* in such economies and possibly, the computational complexity pertaining to it, hence the one pertaining to determining if a rule, as the maximin rule, satisfies it. We have given a necessary condition for *unanimity bound* avoiding such possible complexity.

Our results strengthen the known gap between two-agent economies and those with more than two. We fear it implies an incompatibility between *efficiency* and fairness axioms we imposed, in the latter economies. The maximin rule does not satisfy *efficiency* in such economies. By opposition, the leximin rule and the rule selecting for each $\mathbf{R} \in \mathcal{R}^N$, each $\mathbf{x} \in P(\mathbf{R}) \cap \Gamma(\mathbf{R})$,[13] satisfy *efficiency*. They satisfy *anonymity* and *neutrality*. They may satisfy the *unanimity bound*, and the latter *preference-monotonicity*. As they coincide with the maximin rule in $\mathbf{R}'$ of Ex. 3, they violate *conditional no-envy*.

Incompatibility between avoiding envy and caring for the least fortunates seems inexorable in economies with more than two agents, irrespective of how one formulates the latter. There are economies with $n = 3$ and $a = 6$, preference profiles, and allocations with for each preference profile in which the agents keep these preferences over singletons, these are *envy-free* allocations, but each Borda maximin allocation implies envy for at least one such profile (Brams *et al.*, 2003). If $n > 2$ and there is $q \in \mathbb{N}$ with $qn = a$, there are preferences profiles with a Borda maximin allocation that is *efficient* for at least one preference profile in which the agents keep these preferences over singletons, ensuring envy, i.e. for each such profile (Brams and King, 2005).

We have proved that one may avoid this incompatibility in two-agents economies. Yet, one must be careful formulating the latter. Assume that there is $q \in \mathbb{N}$ with $qn = a$. There are preferences profiles with a maximin allocation that is *efficient* for at least one preference profile in which the agents keep these preferences over singletons, ensuring envy. This holds for Borda maximin allocations only if $q = 1$. (Brams and King, 2005). Yet, it does not mean that even if possible, such allocations are *envy-free*, as opposed to the allocations the maximin rule selects. Further, we have proved that there is more behind compatibility, in the sense, avoiding envy, together with other fairness fundamentals, induce one to care for the least fortunates.

If indeed *efficiency* and fairness axioms we imposed, are incompatible in economies with more than two agents, one should differentiate classes, and perhaps, focus on rules that fairly allocate goods in each economy rather than on those that avoid envy in a limited number of them. One might then determine if these fundamentals, and which of them, also induce one to care for the least fortunates in such economies.

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Notes

[1] *Descending Demand Procedure*: Order the set of agents. Each agent, one after the other w.r.t. the order, gives her most preferred subset. If we get an allocation accordingly, stop; if not, continue. Each agent, one after the other w.r.t. the order, gives her second most preferred subset. (...) Stop when for the first time, we get an allocation from the bundles given up to here. Each *efficient* allocation among the allocations we get accordingly, is a solution of the procedure. In each economy, possibly with more than two agents, each allocation it yields is one the maximin rule selects. One considers refinements to explicitly find *efficient* allocations in more-than-two-agent economies or if possible, secure avoiding envy in two-agent economies, in which preferences reveal complementarities (Herreiner and Puppe, 2002).
[2] For a general review, see Brams (2006).
[3] Procedures one relies on to get the allocations rules select, use such preferences as informational inputs (Brams and Fishburn, 2000; Herreiner and Puppe, 2002). To obtain these, one may face practical issues. The number of subsets of goods may be large, over a million if $n = 20$ (Brams *et al.*, 2003; Brams and King, 2005). Yet, if preferences are additively separable, one may ask for cardinal information over goods to extract the needed ordinal information over subsets. The use of point assignments, which need not perfectly mirror an agent's utility, as *in fine* one only uses ordinal information, solves the tractability issue surely for n up to 30 or so (Brams and Fishburn, 2000). Besides, agents may behave strategically giving the information, and possibly falsify their preferences in view of welfare gains. Yet, it would require remarkable knowledge of the others' preferences and computational ability, hence procedures could be immune from such strategic manipulation (Brams and Fishburn, 2000; Herreiner and Puppe, 2002; Brams *et al.*, 2009).
[4] Solutions to problems with one or two goods are obvious. In the former case, *efficiency* requires to give the good, and *anonymity* requires to give the agents equal chance of being the "lucky" one. In the latter case, other fairness axioms that are vacuously satisfied in the former case, become compelling. These problems amounts to three cases. One easily extends our results to these. To avoid unnecessary notational difficulty, we do not study them explicitly.
[5] Pt. 4 is a notable property of complements due to the order, desirability, and independence assumptions. Indeed, if (*i*) R_i on $\mathcal{S}$ is a weak order; (*ii*) for each $S \in \mathcal{S}$, if $S \neq \emptyset$, then $S\ P_i\ \emptyset$; and (*iii*) for each $S, T, T' \in \mathcal{S}$, if $S \cap T' = T \cap T' = \emptyset$, then $S \cup T'\ P_i\ T \cup T'$ if and only if $S\ P_i\ T$, then (*iv*) for each $S, T \in \mathcal{S}$, we have $S\ R_i\ T$ if and only if $T^c\ R_i\ S^c$ (Fishburn, 1970). Unless $a \leq 4$, (*i*), (*ii*), (*iii*), and (v) R_i on $\mathcal{S}$ is a linear order w.r.t singletons, are not sufficient for additive separability (Kraft *et al.*, 1959).
[6] Each agent getting an empty bundle is always *envy-free*. Regarding "no allocation" as a potential solution to allocation problems seems absurd. In any case, by desirability, *efficiency* precludes it. To avoid unnecessary notational difficulty, we explicitly do not include it in X.
[7] Proof available upon request.
[8] Then, R_j^* weakly closer to R_i than R_j' if and only if the Kemeny distance (Kemeny and Snell, 1962) between R_j^* and R_i is less than the one between on R_j' and R_i. Geometrically, there are $\mathbf{u}, \mathbf{u}', \mathbf{u}^* \in \Delta$ with the associated real-valued function fitting R_i, R_j', and R_j^* resp. and the number of hyperplanes between $\mathbf{u}^*$ and $\mathbf{u}$ less than the one between $\mathbf{u}'$ and $\mathbf{u}$.
[9] Bouveret and Lang (2008) study the problem of determining the existence of *efficient* and *envy-free* allocations from compact representation and computational complexity viewpoints. Demko and Hill (1998) prove the computational complexity of the problem of determining if allocations maximizing, across allocations, the minimal utility, across agents, of an allocation, equating the agents' utilities of the subset containing all goods to 1, are such that each agent's utility of her bundle is at least $(1/n)$, i.e. at least NP-complete. Such a constraint does not define a lower bound, in the sense, one may only secure it in a limited number of economies. Further, it requires more than ordinal information on preferences.
[10] For each $\mathbf{x} \in X$, each $\mathbf{R} \in \mathcal{R}^N$, and each $q \in \{1, ..., n\}$, let $\underline{r}(\mathbf{x}, \mathbf{R}, 1) \equiv \min_{i \in N} r_i(x_i)$, ..., $\underline{r}(\mathbf{x}, \mathbf{R}, q) \equiv \min_{i \in N \setminus \{j \in N: \text{ there is } p \in \{1,...,q-1\} \text{ with } r_j(x_j) = \underline{r}(\mathbf{x}, \mathbf{R}, q)\}} r_i(x_i)$. For each $\mathbf{R} \in R^N$, we have $\Lambda(\mathbf{R}) = \{\mathbf{x} \in X$: there is $\overline{q} \in \{1, ..., n-1\}$ with $\mathbf{x} \in \cap_{q \in \{1,...\overline{q}\}} \arg\max_{\mathbf{y} \in X} \underline{r}(\mathbf{y}, \mathbf{R}, q)$ and $\arg\max_{\mathbf{y} \in X} \underline{r}(\mathbf{y}, \mathbf{R}, \overline{q}) = \arg\max_{\mathbf{y} \in X} \underline{r}(\mathbf{y}, \mathbf{R}, \overline{q} + 1)\}$.
[11] The *deterministic-distribution value solution* corresponds to rules selecting balanced divisions (Demko and Hill, 1998). *Equimax divisions* are the lexicographic extension of maxmin divisions (Brams *et al.*, 2003; Brams and King, 2005).
[12] Brams and Fishburn (2000) prove the implication under the assumptions of Footn. 3. Edelman and Fishburn (2001) leave open the question of how small a may be for the incompatibility to be, but conjecture 9. If $n = 2$, w.l.o.g. $N = \{1, 2\}$, for each $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$ with $R_1' = R_1$ and R_2' closer to R_1 than R_2, we have $F(\mathbf{R}) \supset F(\mathbf{R}')$, $P(\mathbf{R}) \subset P(\mathbf{R}')$, $PF(\mathbf{R}) \supset PF(\mathbf{R}')$, $PF(\mathbf{R}) \subset PF(\mathbf{R}')$, or both, if $a = 3$, then $PF(\mathbf{R}) \supset PF(\mathbf{R}')$. Proof available upon request.
[13] This rule selects the allocations Herreiner and Puppe (2002) define as *balanced*.

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