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The complementarity foundations of industrial organization

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Abstract

In this paper we review the state of the art of Games with Strategic Complementarities (GSC), which are fundamental tools in modern Industrial Organization. GSC are a beautiful merge of comparative statics results and fixpoint results. We organize a large amount of material in a unified and self-contained frame, and we clarify the basic mathematical modeling so that the reader may rapidly develop his own ability to deal with more applied research. We will concentrate on why the assumptions are made, and why that type of assumptions and not others. Furthermore, we will shed light on the intuitions and conceptual points that lie in the background of the theory; and on how all the various pieces blend together to set up a consistent methodological framework.

Keywords: strategic complementarity, oligopoly theory, supermodularity, quasisupermodularity, Nash equilibria, lattices.

JEL Classification: C60, C70, C72

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1. Introduction

Industrial Organization (IO) lies at the centerpiece of economic theory. The "oligopoly problem" - the question of whether prices are determinate or not in presence of only a few competitors - remains central in economic literature and, indeed, it has proved to be one of the more resilient problems in economic theory[1].

Since the seminal contributions of Bertrand and Cournot, IO ideas and models have helped extensively in advancing our understanding of price formation in presence of a limited number of agents. Furthermore IO findings have furnished fundamental insights into firms' strategic behavior in oligopoly contexts, and as such those findings are nowadays widely used to assist decision makers and practitioners in the regulation of industries and in concrete, court-based regulatory cases.

Modern IO is firmly grounded into game theory, and recently new and powerful tools have been added to those traditionally available to IO scholars. These tools are usually denoted, in a concise way, as games with strategic complementarities (GSC), also known as supermodular or quasisupermodular games.

GSC are built up and studied on the basis of a fixpoint theory and of a theory of comparative statics that do not require neither continuity nor convexity assumptions, while working for example with discrete choice sets. In this respect, GSC are pretty innovative even within the consolidated body of game-theoretical literature.

At the heart of GSC lies, however, an old and very familiar notion in economics: that of complementarity in the sense of Pareto and Edgeworth. In fact, once payoffs are shown to embody this kind of complementarity, the joint best reply of the game can be proved to be increasing in a certain sense. This increasingness notion, paired with topological assumptions, allows to apply (extensions of) Tarski's fixpoint theorem to the best reply of the game and to show existence and other important properties of Nash equilibria.

The theory of GSC is, as such, pretty complex; especially whenever full generality is at stake. Furthermore, the theory is scattered in a literature that spans a long time period and plenty of different research fields such as applied mathematics, economics and operations research.

[1] Vives (1999).

Only recently all the pieces have been put together, setting up a general and new framework for the analysis of IO problems.[2]

In this paper, we review the theory of GSC from scratch to the state of the art, with a strong emphasis on the notion of complementarity that underlines the entire construction and on how this notion is used to build up GSC. We organize a large amount of material in a unified and self-contained frame, and we clarify the basic mathematical modeling so that the reader may rapidly develop his own ability to deal with applied research not falling entirely in the realm of well-known examples. We will concentrate on why the assumptions are made, and why that type of assumptions and not others. Furthermore, and maybe even more importantly, we will shed light on the intuitions and conceptual points that lie in the background of the theory, and on how all the aforementioned pieces blend together to set up a consistent methodological framework. For new results in the field one may see Calciano (2007, 2009, 2010).

As an example of our approach, consider that complementarity and GSC are usually treated in the literature the context of lattices[3]. However, lattices do not have much to do with complementarity. They are just useful devices when the individual choice sets are not the product of totally ordered sets. However, it is exactly in the context of choice sets which *are* the product of totally ordered sets that complementarity can be better understood.

Accordingly, the treatment in this paper does not start with lattices, but introduce them when it is needed by the natural development of the analysis. We present complementarity for choice sets which are chains at first[4], then product of chains, and finally lattices. We furthermore distinguish between cardinal and ordinal complementarity, and show why the latter should be introduced and what role it plays in the theory of GSC: namely, that of making sure that properties obtained by using cardinal complementarity are indeed ordinal properties; in the sense of being retained under order-preserving transformations of payoffs.

We will prove every statement with the tools developed that far, even at the obvious cost of longer, more pedantic proofs and more

[2]Is is worth remarking, however, that the theory of GSC finds application also beyond IO.

[3]Lattices are partially ordered sets which have the infima and suprema of all their finite subsets. All the definitions will be given in the subsequent sections of the paper.

[4]Totally ordered sets are also called chains.

cumbersome notation. A cost due to our choice of not using lattice notions until strictly needed[5].

The IO literature is full of extremely relevant applications of GSC, and many papers are concerned with these applications, for example Amir (1996, 2005 a, 2005 b), Bulow et al. (1985), Topkis (1995), just to quote some of the better known works in the field. Furthermore, a very well-known advanced IO book, Vives (1999), is entirely devoted to applications of GSC. A classical exposition of the theory, with applications as well, can be found in Topkis (1998).

As a consequence, we will not fill our paper with applications. The interested reader is referred to the works quoted above, which contain themselves extended references. This paper is concerned with the formal and conceptual structure of the theory of GSC, and as such it complements the applied literature. Notwithstanding this, a final section contains selected examples from IO which are presented here for the sake of illustrating some aspects of the techniques that are less clear and not so used in the applied literature but that, we believe, can be useful and conductive of new applications.

The paper is organized as follows. Section 2 introduces and analyzes the cardinal and ordinal notion of complementarity. Section 3 compares and contrasts these two notions. Sections 4, 5, 6 examine the effects of complementarity on individual decision problems in the context of chains, products of chains, and lattices respectively. Section 7 introduces and studies games with strategic complementarities. Section 8 contains IO applications.

2. Cardinal and ordinal complementarity

Complementarity is an old notion in economics. Samuelson (1974) presents an authoritative historical perspective on it, and surveys the idea of Pareto-Edgeworth complementarity, giving it the meaning of an externality among activities. Samuelson's reconstruction lies at the basis of the current approach to complementarity.

Consider an agent having preferences whose cardinality allows him to add own utility indeces. Consider a consumption bundle where tea and lemon are present. According to Samuelson, tea and lemon are Pareto-Edgeworth complements whenever, keeping all other goods fixed, a joint increase of tea and lemon gives the agent a benefit exceeding the

[5]We will not prove purely topological result, because proving them would fall beside the scope of the paper. Full references will be given for those results.

sum of benefits that he would get by increasing them separately. This means that an increase of, say, tea makes an increase of lemon more desirable: the increase of tea exerts a positive externality on increasing lemon.

Later literature (Bulow et al., 1985) independently rediscovers Samuelson's approach, calling "strategic complements" what Samuelson called Pareto-Edgeworth complements. Of course, in general there is no reason to qualify a complementarity relation as "strategic". But the terminology has spread out in economics at large. We point out, then, that strategic complementarity is exactly Pareto-Edgeworth complementarity, and call it simply "complementarity" from now on.

Samuelson heuristic description translates directly into a property of the utility function. Let $\mathbb{R}^2$ be the commodity space, with typical element (x,t), where x is the amount of lemon and t is the amount of tea. Start from a consumption bundle (x_1,t_1). Consider a consumption bundle (x_2,t_2), with $x_1 < x_2$ and $t_1 < t_2$. At (x_2,t_1) we would have increased only lemon. At (x_1,t_2) we would have increased only tea. At (x_2,t_2) we would have increased both.

Samuelson description of complementarity says that:

$$\begin{gathered} u(x_2,t_1) - u(x_1,t_1) + u(x_1,t_2) - u(x_1,t_1) \\ \leq \\ u(x_2,t_2) - u(x_1,t_1); \end{gathered}$$

that is,

$$u(x_2,t_1) - u(x_1,t_1) \leq u(x_2,t_2) - u(x_1,t_2). \tag{1}$$

This property has a modern name. Let X be a poset (a partially ordered set), T be a set, and $u : X \times T \to \mathbb{R}$. For fixed $x_1, x_2 \in X$, with $x_1 < x_2$, call the expression

$$u(x_2,t) - u(x_1,t)$$

a first difference of u in x. This is clearly a function of t.

Definition 1. (Increasing differences). Let $u(x,t) : X \times T \to \mathbb{R}$, where X and T are posets. u has increasing differences in (x,t) if every first difference of u in x is increasing in t; that is, if for every $x_1, x_2 \in X$ with $x_1 < x_2$, for every $t_1, t_2 \in T$ with $t_1 < t_2$, inequality (1) holds.

Definition 2. (Cardinal complementarity). Let $u(x,t) : X \times T \to \mathbb{R}$, where X and T are posets. We say that activities x and t are cardinal complements if u has increasing differences in (x,t).

It is clear from the definition that $u(x,t)$ has increasing differences in (x,t) if and only if it has also increasing differences in (t,x), by which we mean that any first difference of $u(x,t)$ in t, i.e. any

$$u(x,t_2) - u(x,t_1)$$

with $t_1 < t_2$, is increasing in x. Hence the cardinal complementarity relation is symmetric. We will come back to this important point in the sequel.

In applications, to check if two one-dimensional activities x,t are cardinal complements, one often uses the following immediate result.

Lemma 1. (Differential characterization of cardinal complementarity). *Let $u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be differentiable in x for every t, and let $u_x(x,t)$ be differentiable in t for every x.*

$u(x,t)$ has increasing differences in (x,t) if and only if for every $(x,t) \in \mathbb{R}^2$, $u_{xt}(x,t) \geq 0$.

Furthermore, if for every $(x,t) \in \mathbb{R}^2$, $u_{xt}(x,t) > 0$, then $u(x,t)$ has strict increasing differences in (x,t) (meaning that (1) holds with strict inequality).

Proof: If u has increasing differences in (x,t), then for every $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, and for every distinct $x, x_0 \in \mathbb{R}$, we have that

$$\frac{u(x,t_2) - u(x_0,t_2) - [u(x,t_1) - u(x_0,t_1)]}{x - x_0} \geq 0.$$

Taking limit as $x \to x_0$, we get that $u_x(x_0,t_2) \geq u_x(x_0,t_1)$. Hence $u_x(x,t)$ is increasing in t, and so for every x and every distinct t, t_0 in $\mathbb{R}$,

$$\frac{u_x(x,t) - u_x(x,t_0)}{t - t_0} \geq 0.$$

Taking limit as $t \to t_0$, we get the result.

Let now $u_{xt}(x,t) \geq (>)\, 0$ for every $(x,t) \in \mathbb{R}^2$. Since for every x $u_x(x,t)$ is differentiable in t, then by the intermediate value theorem, for every t_1, t_2 with $t_1 < t_2$, there exists some $\beta \in (t_1, t_2)$ such that

$$\frac{u_x(x,t_2) - u_x(x,t_1)}{t_2 - t_1} = u_{xt}(x,\beta) \geq (>)\, 0,$$

meaning that for every x, $u_x(x,t_2) - u_x(x,t_1) \geq (>) 0$. Hence for $u(x,t_2) - u(x,t_1)$, which is a differentiable function of x, by the intermediate value theorem we have that, for every $x_1 < x_2$, there is some $\alpha \in (x_1, x_2)$ such that

$$\frac{u(x_2,t_2) - u(x_2,t_1) - [u(x_1,t_2) - u(x_1,t_1)]}{x_2 - x_1} =$$

$$u_x(\alpha,t_2) - u_x(\alpha,t_1) \geq (>) 0.$$

Hence

$$u(x_2,t_2) - u(x_2,t_1) - [u(x_1,t_2) - u(x_1,t_1)] \geq (>) 0,$$

and u has (strictly) increasing differences in (x,t). ■

Remark: conventional complementarity. Bulow etc. (1985) define "conventional" complementarity as a positive effects of increasing one activity, say t, on total profits. In these terms, x is a conventional complement of t if $u_x(x,t) \geq 0$, i.e. if payoff is increasing in t for every x. On the other hand, strategic complementarity means that increasing activity t has a positive effects on the *marginal profits* associated to x, that is, $u_{xt}(x,t) \geq 0$. While this approach justifies the distinction, the term "strategic" still sounds arbitrary. It is referred to the fact that activity t is controlled by some opponent of the player at stake in a game setting.

Lemma 2. (Multidimensional increasing differences). *Let $u(x,t) : X \times T \to \mathbb{R}$, with $X = X_1 \times \cdots \times X_m$ and $T = T_1 \times \cdots \times T_n$, each factor in the products being a poset.*

$u(x,t)$ has increasing differences in $((x_1,\ldots,x_m),(t_1,\ldots,t_n))$ on $X \times T$ if and only if, for every $(x'_1,\ldots,x'_m,t'_1,\ldots,t'_n)$ in $X \times T$, for every $i = 1,\ldots,m$ and every $j = 1,\ldots,n$, the function

$$u(x'_1,\ldots,x_i,\ldots,x'_m,t'_1,\ldots,t_j,\ldots,t'_n) : X_i \times T_j \to \mathbb{R}$$

has increasing differences in (x_i,t_j) on $X_i \times T_j$.

Proof: Necessity is trivial. For sufficiency, we need two steps.

STEP 1. We first prove that $u(x,t)$ has increasing differences in

$$(x_i,(t_1,\ldots,t_n))$$

for any $i = 1,\ldots,m$ and any fixed x_{-i}, where given any x in X, x_{-i} is the projection of x onto all of its coordinates except for the ith one. Take $i = 1$. Fix any $(x_2,\ldots,x_n)$. Pick any $(x'_i,t'_1,\ldots,t'_n) \leq (x''_i,t''_1,\ldots,t''_n)$.

Set $(t_2, \ldots, t_n) = (t'_2, \ldots, t'_n)$. By assumption u has increasing differences in (x_1, t_1) for fixed $(x_2, \ldots, x_n)$ and $(t'_2, \ldots, t'_n)$; that is,

$$u(x''_1, x_2, \ldots, x_m, t'_1, t'_2, \ldots, t'_n) - u(x'_1, x_2, \ldots, x_m, t'_1, t'_2, \ldots, t'_n)$$
$$\leq$$
$$u(x''_1, x_2, \ldots, x_m, t''_1, t'_2, \ldots, t'_n) - u(x'_1, x_2, \ldots, x_m, t''_1, t'_2, \ldots, t'_n)$$

Set now $(t_1, t_3, \ldots, t_n) = (t''_1, t'_3, \ldots, t'_n)$. By assumption, u has increasing differences in (x_1, t_2) for fixed $(x_2, \ldots, x_n)$ and $(t''_1, t'_3, \ldots, t'_n)$. Hence the inequality above can be continued into:

$$\leq$$
$$u(x''_1, x_2, \ldots, x_m, t''_1, t''_2, \ldots, t'_n) - u(x'_1, x_2, \ldots, x_m, t''_1, t''_2, \ldots, t'_n)$$

Set now $(t_1, t_2, t_4 \ldots, t_n) = (t''_1, t''_2, t'_4 \ldots, t'_n)$. By assumption, u has increasing differences in (x_1, t_3) for fixed $(x_2, \ldots, x_n)$ and $(t''_1, t''_2, t'_4 \ldots, t'_n)$. Hece continue the inequalities above into:

$$\leq$$
$$u(x''_1, x_2, \ldots, x_m, t''_1, t''_2, t''_3, t'_4 \ldots, t'_n) - u(x'_1, x_2, \ldots, x_m, t''_1, t''_2, t''_3, t'_4 \ldots, t'_n)$$

Proceeding this way, we get a string of inequalities ending as:

$$\cdots$$
$$\leq$$
$$u(x''_1, x_2, \ldots, x_m, t''_1, t''_2, \ldots, t''_n) - u(x'_1, x_2, \ldots, x_m, t''_1, t''_2, \ldots, t''_n).$$

Hence, for fixed $(x_2, \ldots, x_n)$,

$$u(x''_1, x_2, \ldots, x_m, t'_1, t'_2, \ldots, t'_n) - u(x'_1, x_2, \ldots, x_m, t'_1, t'_2, \ldots, t'_n)$$
$$\leq$$
$$u(x''_1, x_2, \ldots, x_m, t''_1, t''_2, \ldots, t''_n) - u(x'_1, x_2, \ldots, x_m, t''_1, t''_2, \ldots, t''_n).$$

So u has increasing differences in $(x_i, (t_1, \ldots, t_n))$ for any fixed $(x_2, \ldots, x_n)$. Redo the argument for $i = 2, \ldots, m$.

STEP 2. We now prove the result. Pick any

$$(x'_1, \ldots, x'_m, t'_1, \ldots, t'_n) \leq (x''_1, \ldots, x''_m, t''_1, \ldots, t''_n).$$

By the previous step, fixing $(x_2, x_3, \ldots, x_m) = (x'_2, x'_3, \ldots, x'_m)$, u has increasing differences in $(x_1, (t_1, \ldots, t_n))$, and hence in $((t_1, \ldots, t_n), x_1)$ (here we use that increasing differences does not distinguish between the first and second variable). That is,

$$u(x'_1, x'_2, \ldots, x'_m, t''_1, \ldots, t''_n) - u(x'_1, x'_2, \ldots, x'_m, t'_1, \ldots, t'_n)$$
$$\leq$$
$$u(x''_1, x'_2, \ldots, x'_m, t''_1, \ldots, t''_n) - u(x''_1, x'_2, \ldots, x'_m, t'_1, \ldots, t'_n).$$

Fix now $(x_1, x_3, \ldots, x_m) = (x_1'', x_3', \ldots, x_m')$. By Step 1, u has increasing differences in $(x_2, (t_1, \ldots, t_n))$, and so u has increasing differences in $((t_1, \ldots, t_n), x_2)$. Thus, we can continue the inequality above into:

$$\begin{gathered} \leq \\ u(x_1'', x_2'', \ldots, x_m', t_1'', \ldots, t_n'') - u(x_1'', x_2'', \ldots, x_m', t_1', \ldots, t_n'). \end{gathered}$$

Proceeding this way, we get a string of inequalities ending as:

$$\begin{gathered} \ldots \\ \leq \\ u(x_1'', x_2'', \ldots, x_m'', t_1'', \ldots, t_n'') - u(x_1'', x_2'', \ldots, x_m'', t_1', \ldots, t_n'). \end{gathered}$$

Hence,

$$\begin{gathered} u(x_1', x_2', \ldots, x_m', t_1'', \ldots, t_n'') - u(x_1', x_2', \ldots, x_m', t_1', \ldots, t_n') \\ \leq \\ u(x_1'', x_2'', \ldots, x_m'', t_1'', \ldots, t_n'') - u(x_1'', x_2'', \ldots, x_m'', t_1', \ldots, t_n'). \end{gathered}$$

This prove that u has increasing differences in $((x_1, \ldots, x_m), (t_1, \ldots, t_n))$ on $X \times T$. ■

Corollary 1. (Differential characterization of multidimensional increasing differences). *Let $u(x,t) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be twice differentiable on $\mathbb{R}^m \times \mathbb{R}^n$ (but we need less, see the assumptions in Lemma 1).*

$u(x,t)$ has increasing differences in (x,t) on $\mathbb{R}^m \times \mathbb{R}^n$ if and only if, for every $i = 1, \ldots, m$; every $j = 1, \ldots, n$; and every (x,t) in $\mathbb{R}^m \times \mathbb{R}^n$, $u_{x_i t_j}(x,t) \geq 0$.

Proof: Immediate by applying first Lemma 2 and then Lemma 1 .■

A weaker, ordinal (preserved under increasing transformations of payoffs) notion of complementarity has been introduced by Milgrom and Shannon (1994) by observing that, in the inequality defining increasing differences, if the left-hand-side difference is (strictly) greater than some real k, then the right-hand-side difference must also be (strictly) greater than k. Normalizing $k = 0$, this means that whenever an increase of lemon is desirable at a fixed level of tea, this increase remains desirable if tea increases too. This property represents a weak notion of Pareto-Edgeworth complementarity, and is formalized in the definition of single crossing property.

Definition 3. (Single crossing property). Let X and T be posets. Function $u(x,t) : X \times T \to \mathbb{R}$ has the single crossing property in (x,t) if any first difference of $u(x,t)$ in x which is (strictly) positive at some t, remains (strictly) positive as t increases. That is, if for every $x_1, x_2 \in X$ with $x_1 < x_2$, for every $t_1, t_2 \in T$ with $t_1 < t_2$,

$$\begin{aligned} u(x_1,t_1) \leq u(x_2,t_1) &\Rightarrow u(x_1,t_2) \leq u(x_2,t_2); \\ u(x_1,t_1) < u(x_2,t_1) &\Rightarrow u(x_1,t_2) < u(x_2,t_2). \end{aligned}$$

Clearly, if function $u(x,t)$ has increasing differences in (x,t), it also satisfies the single crossing property in (x,t), while the contrary does not hold necessarily, as can be shown easily.

Definition 4. (Ordinal complementarity). Let $u(x,t) : X \times T \to \mathbb{R}$, where X and T are posets. We say that activities x and t are ordinal complements if $u(x,t)$ has the single crossing property in (x,t).

The fact that $u(x,t)$ has the single crossing property in (x,t) does not imply that it has the single crossing property in (t,x), i.e. it does not imply that whenever any first difference of $u(x,t)$ in t is (strictly) positive at some x, it remains so as x increases. Hence, contrary to cardinal complementarity, ordinal complementarity does not generate a symmetric relation. We elaborate on this fact in the next Section.

3. Cardinal versus ordinal complementarity

3.1. On the lack of symmetry of ordinal complementarity.

We have seen that a function $u(x,t)$ has increasing differences in (x,t) if and only if it has increasing differences in (t,x). Hence, if increasing differences in (x,t) is meant to define the binary relation: "x is a cardinal complement of t", this relation is symmetric.

On contrary, the single crossing condition depends on the variable with respect to whom the first difference is taken. Single crossing in (x,t) does not imply single crossing in (t,x), as the following example shows.

Consider the function:

$$u(x,t) = \frac{x}{t} + t$$

with $t \neq 0$. For any $x_1 < x_2$, the corresponding first difference of u in x is:

$$\frac{1}{t}(x_2 - x_1).$$

If any such first difference is strictly positive[6], then t must be strictly positive as well, and so this first difference stays strictly positive as t increases, albeit being decreasing in t. Thus u has the single crossing property in (x, t).

On the other hand, $u(x, t)$ fails to have the single crossing property in (t, x), since for any $t_1 < t_2$ the corresponding first difference of u in t takes the form of

$$x\left(\frac{1}{t_2} - \frac{1}{t_1}\right) + t_2 - t_1$$

which for $t_1 = 1$ and $t_2 = 2$ becomes $-\frac{x}{2} + 1$, which in turn is strictly positive at $x = 1$ but nonpositive at any $x \geq 2$.

If the single crossing property in (x, t) is take as a definition of the binary relation "x is an ordinal complement of t", this relation is not symmetric.

This lack of symmetry is much more than a curiosity. It will require to add extra assumptions on payoffs when studying the comparative statics of the set of maximizers of $u(x, t)$ in a purely ordinal context. We will turn to this issue in subsection 5.2.

3.2. **On some apparent inconsistency between cardinal and ordinal complementarity.** We now exhibit a payoff function where the two activities x and t are both ordinal complements and cardinal substitutes.

Consider again the function:

$$u(x, t) = \frac{x}{t} + t$$

with $t \neq 0$. We have seen that it satisfies the single crossing property in (x, t). On the restricted domain $\{(x, t) : x \leq 1 \leq t\}$, the function satisfies also the single crossing property in (t, x). Indeed, for every $t_1 < t_2$, the expression

$$\left(\frac{1}{t_2} - \frac{1}{t_1}\right) + t_2 - t_1$$

is (strictly) positive iff $t_1 t_2 \geq 1$ (iff $t_1 t_2 > 1$). Hence, when we rescale the negative term $\left(\frac{1}{t_2} - \frac{1}{t_1}\right)$ multiplying it by any $x \leq 1$, we see that this first difference remain positive or, respectively, strictly positive. Hence $u(x, t)$ satisfies the single crossing property in (t, x) on the stated domain.

[6] It can never be 0

However, our function has *decreasing differences* in (x,t), since

$$u_{xt}(x,t) = -\frac{1}{t^2} < 0$$

for all x and all $t \neq 0$. Hence for this function, on the stated domain, the two activities x and t are both ordinal complements and cardinal substitutes.

This fact is due to the increased generality of the notion of ordinal complementarity relative to that of cardinal complementarity, but however does not represent an instance of logical inconsistency of the two notions. Indeed, as we will see in the sequel, any GSC-result driven by cardinal complementarity of payoffs holds for any order-preserving transformation of these payoffs, and so what really drives the result are not the cardinal properties, but the ordinal ones, which are preserved by definition under these transformations. Thus we can well transform cardinal complementarity into cardinal substitutability, for example getting the function of the example, without loosing any GSC-conclusion.

In terms of the complementarity relations, in our example the activities being cardinal substitutes are also ordinal substitutes. Hence they are indeed ordinally “neutral” with each other.

4. Monotone comparative statics on chains

Let B_t be the set of maximizers over X of function $u(x,t) : X \times T \to \mathbb{R}$. We keep the assumption, in the comparative statics results below, that B_t is nonempty for every t in T. Conditions to assure that this is indeed the case in the various contexts will be introduced separately. In view of these conditions, however, we introduce here an appropriate intrinsic topology for posets.

Definition 5. (Interval topology). Let X be a poset. The interval topology of X is the topology generated by taking the closed intervals

$$[y,z] = \{x \in X : y \leq x \leq z\},$$

with $y, z \in X$, as a subbasis for closed sets.

For the sake of applications, we recall that in $\mathbb{R}^n$ the interval topology is equivalent to the standard topology (Frink, see Birkhoff, 1967, Ch. X).

The simplest case is that in which the choice set X is a chain (a totally ordered set, for example a subset of $\mathbb{R}$). We have the following comparative statics result.

Theorem 1. (Comparative statics on chains, cardinal case). *Let X be a chain, T be a poset and let $u(x,t) : X \times T \to \mathbb{R}$ have increasing differences in (x,t). For every $t_1 < t_2$, for every $a \in B_{t_1}$ and every $b \in B_{t_2}$, $\min\{a,b\} \in B_{t_1}$ and $\max\{a,b\} \in B_{t_2}$.*

Proof: If $a \leq b$ we are done. Let then $b < a$. We have that

$$0 \leq u(a,t_1) - u(b,t_1) \leq u(a,t_2) - u(b,t_2) \leq 0,$$

where the first and last inequality follow from optimality, and the middle-one follows from increasing differences. Hence $b \in B_{t_1}$, and $a \in B_{t_2}$. ■

The generalization to the ordinal case is immediate, and the proof is a prototype of how the inequalities of the single crossing property work in monotone comparative statics theorems. Hence it is instructive to fill in all the details.

Theorem 2. (Comparative statics on chains, ordinal case). *Let X be a chain, T be a poset and let $u(x,t) : X \times T \to \mathbb{R}$ satisfy the single crossing property in (x,t). For every $t_1 < t_2$, for every $a \in B_{t_1}$ and every $b \in B_{t_2}$, $\min\{a,b\} \in B_{t_1}$ and $\max\{a,b\} \in B_{t_2}$.*

Proof: If $a \leq b$ we are done. Let then $b < a$. By the optimality of a at t_1

$$u(a,t_1) - u(b,t_1) \geq 0,$$

and hence by the single crossing property

$$u(a,t_2) - u(b,t_2) \geq 0,$$

which implies, by the optimality of b at t_2, that $a \in B_{t_2}$. On the other hand, by optimality of b at t_2, the following inequality fails:

$$u(a,t_2) - u(b,t_2) > 0.$$

Hence, by the single crossing property,

$$u(a,t_1) - u(b,t_1) \leq 0.$$

Thus, by the optimality of a at t_1 we have that $b \in B_{t_1}$, and we are done. ■

Theorem 3. (Increasing extremal selections). *Let X be a chain, T be a poset and let $u : X \times T \to \mathbb{R}$ satisfy the single crossing* *property in (x, t).*

If B_t has either a least element $a_* (t)$ or a greatest element $a^* (t)$ for every $t \in T$ (or both), then this element is an increasing functions.

Proof: Take any $t_1 < t_2$. Consider first $a_* (t)$. Take any $a \in B_{t_1}$. By the previous theorem, $c := \min \{a, a_* (t_2)\} \in B_{t_1}$. Hence, $a_* (t_1) \leq c \leq a_* (t_2)$. Analogously, for any $b \in B_{t_2}$, by the previous theorem $d := \max \{b, a^* (t_1)\} \in B_{t_2}$, and so $a^* (t_1) \leq d \leq a^* (t_2)$. ∎

Conditions assuring that the argmax B_t has a least element and a greatest element of every t in T are here the same as the standard ones making it nonempty. Assume X compact in its interval topology and $u (x, t)$ upper semicontinuous in x for every t. Hence every B_t is nonempty and compact. Compactness in the interval topology implies that B_t has indeed a least and a greatest element. To understand why this is so we will need an important topological result, that will be presented later on in the paper as Theorem 7. See also the discussion after Corollary 2 as well.

5. Monotone comparative statics on finite products of chains

If $X = Y \times Z$, where Y and Z are chains (for example, X is a box in $\mathbb{R}^2$), then a natural way to extend to this context the definition of monotonicity of the argmax B_t of $u (y, z, t)$ over $Y \times Z$ is to use coordinate-wise minima and maxima.

Pick any (y_1, z_1),$(y_2, z_2) \in Y \times Z$. Define infima and suprema as

$$(y_1, z_1) \wedge (y_2, z_2) = (\min \{y_1, y_2\} , \min \{z_1, z_2\}) ,$$

$$(y_1, z_1) \vee (y_2, z_2) = (\max \{y_1, y_2\} , \max \{z_1, z_2\}) .$$

Note that such infima and suprema are well defined exactly because Y and Z are chains.

We say that B_t is increasing in t on T if for every $t_1 \leq t_2$ in T, for every $(y_1, z_1) \in B_{t_1}$ and every $(y_2, z_2) \in B_{t_2}$,

$$(y_1, z_1) \wedge (y_2, z_2) \in B_{t_1},$$

$$(y_1, z_1) \vee (y_2, z_2) \in B_{t_2}.$$

In this context, increasing differences of $u (x, t)$ in (x, t), with $x = (y, z)$, and hence the single crossing property $u (x, t)$ in (x, t), do no longer suffice to guarantee that B_t is increasing. The reason is examined in the following subsections.

5.1. **Using cardinal complementarity for comparative statics over a product of chains.** Let:

$$X = \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2.$$

X is the product of chain $\{0,1\}$ by itself. Let $T = \{t_1, t_2\}$, with $t_1 < t_2$. Let $u : X \times T \to \mathbb{R}$ be defined as:

$$\begin{array}{ll} u((0,0), t_1) = 0 & u((0,0), t_2) = 0 \\ u((0,1), t_1) = 8 & u((0,1), t_2) = 16 \\ u((1,0), t_1) = 10 & u((1,0), t_2) = 15 \\ u((1,1), t_1) = 1 & u((1,1), t_2) = 10 \end{array}$$

This function has increasing differences (hence satisfies the single crossing property) in (x,t). Indeed, check for all $x_1 \leq x_2$, with

$$x_1, x_2 \in \{(0,0), (0,1), (1,0), (1,1)\}.$$

For $x_1 = (0,0)$:

$$\begin{array}{c} u((1,0), t_1) - u((0,0), t_1) = 10 < 15 = u((1,0), t_2) - u((0,0), t_2) \\ u((0,1), t_1) - u((0,0), t_1) = 8 < 16 = u((0,1), t_2) - u((0,0), t_2) \\ u((1,1), t_1) - u((0,0), t_1) = 1 < 10 = u((1,1), t_2) - u((0,0), t_2) \end{array}$$

For $x_1 = (0,1)$:

$$u((1,1), t_1) - u((0,1), t_1) = -7 < -6 = u((1,1), t_2) - u((0,1), t_2)$$

For $x_1 = (1,0)$:

$$u((1,1), t_1) - u((1,0), t_1) = -9 < -5 = u((1,1), t_2) - u((1,0), t_2)$$

For this function, $B_{t_1} = \{(1,0)\}$ and $B_{t_2} = \{(0,1)\}$. Hence increasingness of the argmax fails. What has happened here?

The problem is of course not in the notion of increasingness that we adopted, which is the most natural one. The problem lies in the definition of complementarity that we used.

Starting at the optimal bundle $(1,0)$, a shift of the parameter from t_1 to t_2 makes indeed desirable to increase the amount of the second good, as increasing differences shows by taking $x_1 = (1,0) < (1,1) = x_2$; that is, the marginal utility of the second good increases with t:

$$u((1,1), t_1) - u((1,0), t_1) = -9 < -5 = u((1,1), t_2) - u((1,0), t_2).$$

Furthermore, a shift of the parameter from t_1 to t_2 makes desirable to increase the amount of the first good too (i.e. to keep it to 1, which is the maximum amount allowed by our feasible set), as the last inequality above also shows. So why $(1,1)$ is not an optimal bundle at t_2?

The point is that we have not considered so far the effect that an increase in one of the two goods can have on the utility of increasing the other good, everything else kept fixed.

Indeed, for our utility function, as we will see immediately below the marginal utility of the first good *decreases* as the amount of the second good increases, and vice versa, at any fixed level of t. Hence, the two goods are *substitutes* to each other, and the net effect of increasing the parameter t - as this effect is determined by the function of the example - is to increase the optimal consumption of the second good but to decrease that of the first good, ending up the re-optimization process at bundle $(0, 1)$.

In other words, the problem in this example is that $u(y, z, t)$, albeit having increasing differences in the pair $((y, z), t)$, has *decreasing differences* in (y, z) for every t.

Take in fact $y_1 = 0$ and $y_2 = 1$. For fixed t, the corresponding first difference of $u(y, z, t)$ in y is decreasing in z, i.e. as z shifts from 0 to 1. Let's check this.

For $t = t_1$:

$$u((1,0), t_1) - u((0,0), t_1) = 10 > -7 = u((1,1), t_1) - u((0,1), t_1).$$

For $t = t_2$:

$$u((1,0), t_2) - u((0,0), t_2) = 15 > -6 = u((1,1), t_2) - u((0,1), t_2).$$

Remark. When X is a chain, even a multidimensional chain, the fact illustrated above can not happen. Indeed, in a chain, any reallocation following a parameter's shift needs to take the form of either an increase in the level of all goods, or a decrease. Hence the level of all goods move in the same direction. Then, complementarity between the bundle in X and the parameter suffices for monotone comparative statics to hold. In some sense, on a chain all goods can be seen as behaving as complements to each others.

Summing up. In this example monotone comparative statics has failed because notwithstanding that each good is a complement to the parameter, i.e. that $u(y, z, t)$ has increasing difference in $((y, z), t)$, the two goods are substitutes to each other, i.e. $u(y, z, t)$ has decreasing differences in (y, z) for every t.

In order to obtain the desired monotone comparative statics, we need to assure not only that each one of the activities y and z is a complement to the parameter t, but also that the activities y and z are complements to each other for each level of t. Hence we need to assume both increasing differences of $u(y,z,t)$ in $((y,z),t)$, and increasing differences of $u(y,z,t)$ in (y,z) for any fixed t. Complementarities must be pervasive. The comparative statics theorem in this context is then the following:

Theorem 4. (Comp. statics on a product of chains, cardinal case). *Let Y and Z be a chains, T be a poset and $u: Y \times Z \times T \to \mathbb{R}$ have increasing differences in $((y,z),t)$, and in (y,z) for every t in T.*

For every $t_1 < t_2$, for every $(y_1,z_1) \in B_{t_1}$ and every $(y_2,z_2) \in B_{t_2}$, $(y_1,z_1) \wedge (y_2,z_2) \in B_{t_1}$ and $(y_1,z_1) \vee (y_2,z_2) \in B_{t_2}$.

Proof: Apply the proof of theorem 5 below. ■

5.2. **Using ordinal complementarity for comparative statics over a product of chains.** If the choice space X is the product of chains Y and Z, and if we use the ordinal notion of complementarity, we need to take care of the non-symmetry of the induced complementarity relation. For $u(y,z,t)$, assuming the single crossing property of u in (y,z) for any fixed t, and of u in $((y,z),t)$, is not enough.

Indeed, we are assuming that y is an ordinal complement to z at any fixed t, and that both y and z are ordinal complements to t. However, we are not assuming that z is an ordinal complement to y for fixed t. As a result, we could well get that increasing t makes in the first place the level of both y and z increase, then the increase in z makes y increase as well, but then we could get that the increase in y makes z decrease. Hence there would be no definite result, ex-ante, on the comparative statics of the maximizers' set.

To avoid this, we need to assume also that z is an ordinal complement to y. Hence, we need to assume for $u(y,z,t)$ that the single crossing property is satisfied in both (y,z) and (z,y), for any fixed t.

This is done in the next theorem. After reading Section 6, it will be clear that the proof of the theorem could have been made much shorter and less pedantic. In the spirit of this paper, however, we have chosen to explain in details the role and working of ordinal complementarity, at the cost of a much longer and involved proof. A role that becomes less clear when one works in a more general context such as that of Section 6.

Theorem 5. (COMP. STATICS ON A PRODUCT OF CHAINS, ORDINAL CASE). *Let Y and Z be a chains and T be a poset. Let $u : Y \times Z \times T \to \mathbb{R}$ satisfy the single crossing property in $((y,z),t)$, and in both (y,z) and (z,y) for every $t \in T$.*

For every $t_1 < t_2$, for every $(y_1, z_1) \in B_{t_1}$ and every $(y_2, z_2) \in B_{t_2}$, $(y_1, z_1) \wedge (y_2, z_2) \in B_{t_1}$ and $(y_1, z_1) \vee (y_2, z_2) \in B_{t_2}$.

Proof: Pick any $(y_1, z_1) \in B_{t_1}$ and any $(y_2, z_2) \in B_{t_2}$. If $(y_1, z_1) \leq (y_2, z_2)$, we are done.

Case (A): If $(y_2, z_2) \leq (y_1, z_1)$ then, by the theorem on comparative statics on chains, the single crossing property in $((y,z),t)$ suffices for the result.

Case (B): Let $y_2 < y_1$ and $z_1 < z_2$. Here we need the single crossing in $((y,z),t)$ and in (y,z). By optimality of (y_1, z_1) at $t = t_1$, we have that:

$$u(y_1, z_1, t_1) - u(y_2, z_1, t_1) \geq 0,$$

and so by the single crossing property in (y,z),

$$u(y_1, z_2, t_1) - u(y_2, z_2, t_1) \geq 0.$$

Since $(y_2, z_2) < (y_1, z_2)$ and $t_1 < t_2$, by the single crossing property in $((y,z),t)$ we have that:

$$u(y_1, z_2, t_2) - u(y_2, z_2, t_2) \geq 0,$$

and by optimality of (y_2, z_2) at $t = t_2$, $(y_1, z_2) = (y_1, z_1) \vee (y_2, z_2) \in B_{t_2}$. Analogously, by optimality of (y_2, z_2) at $t = t_2$,

$$u(y_2, z_2, t_2) - u(y_1, z_2, t_2) \geq 0,$$

and so by the single crossing in (y,z) and by the fact that u takes values in a chain,

$$u(y_2, z_1, t_2) - u(y_1, z_1, t_2) \geq 0.$$

Since $(y_2, z_2) < (y_1, z_2)$ and $t_1 < t_2$, then again by the single crossing in $((y,z),t)$,

$$u(y_2, z_1, t_1) - u(y_1, z_1, t_1) \geq 0.$$

Hence by optimality of (y_1, z_1) at $t = t_1$, we have that $(y_2, z_1) = (y_1, z_1) \wedge (y_2, z_2) \in B_{t_1}$, and we are done.

Case (C): Let now $y_1 < y_2$ and $z_2 < z_1$. Here we need the single crossing in $((y,z),t)$ and in (z,y). By optimality of (y_1, z_1) at $t = t_1$,

$$u(y_1, z_1, t_1) - u(y_1, z_2, t_1) \geq 0.$$

By the single crossing property in (z, y),

$$u(y_2, z_1, t_1) - u(y_2, z_2, t_1) \geq 0.$$

Since $(y_2, z_2) < (y_2, z_1)$ and $t_1 < t_2$, then by the single crossing in $((y, z), t)$ we have that:

$$u(y_2, z_1, t_2) - u(y_2, z_2, t_2) \geq 0.$$

Hence by optimality again, $(y_2, z_1) = (y_1, z_1) \vee (y_2, z_2) \in B_{t_2}$. In the same way, by optimality,

$$u(y_2, z_2, t_2) - u(y_2, z_1, t_2) \geq 0.$$

Hence by the single crossing in (z, y),

$$u(y_1, z_2, t_2) - u(y_1, z_1, t_2) \geq 0.$$

Again, because $(y_2, z_2) < (y_2, z_1)$ and $t_1 < t_2$, by the single crossing in $((y, z), t)$ we have that:

$$u(y_1, z_2, t_1) - u(y_1, z_1, t_1) \geq 0,$$

and so, by optimality, $(y_1, z_2) = (y_1, z_1) \wedge (y_2, z_2) \in B_{t_1}$. ■

Corollary 2. (Increasing extremal selections). *Let Y and Z be chains, T be a poset and $u : Y \times Z \times T \to \mathbb{R}$ satisfy the single crossing property in $((y, z) t)$ and in both (y, z) and (z, y) for every $t \in T$.*

If B_t has either a least element $a_* (t)$ or a greatest element $a^* (t)$ for every $t \in T$ (or both), then this element is an increasing function.

Proof: Take any $t_1 < t_2$. Consider $a_* (t)$. Take any $c \in B_{t_1}$. By the previous theorem, $c \wedge a_* (t_2) \in B_{t_1}$. Hence,

$$a_* (t_1) \leq c \wedge a_* (t_2) \leq a_* (t_2).$$

Analogously, for any $b \in B_{t_2}$, by the previous theorem $a^* (t_1) \vee b \in B_{t_2}$, and so

$$a^* (t_1) \leq a^* (t_1) \vee b \leq a^* (t_2).$$

■

The conditions assuring that the argmax B_t has indeed a least and a greatest element are the same conditions assuring that it is nonempty, namely that the choice set is compact in the interval topology and that the objective function is upper semicontinuous in the decision variables at each value of the parameter. To explain why, we need further definitions and an important topological result, presented as Theorem 7 below. See also the discussion after Corollary 3.

6. Monotone comparative statics on lattices

If the choice set X is a poset which is not a finite product of chains (for example a circle in the plane), then pointwise infima and suprema of pairs of elements of X do not need to exist (or to be in X). To extend to this context the notion of increasingness of B_t that we have used so far, we need to introduce lattices.

Definition 6. (Lattice) Let X be a nonempty poset. X is a lattice if for every $x_1, x_2 \in X$, $x_1 \wedge x_2 \in X$ and $x_1 \vee x_2 \in X$ (where the first expression denotes the infimum and the latter denotes the supremum of $\{x_1, x_2\}$ in X).

We now introduce, in the context of lattices, a notion of increasingness that is due to Veinott[7]. We call it Veinott-increasingness and point out that, if X is either a chain or a product of chains, it coincides with increasingness in the sense of the previous sections of this paper.

Definition 7. (Veinott-increasingness) Let X be a lattice and $F : x \in X \mapsto F_x \subseteq X$ be a correspondence. We say that F is Veinott-increasing if for every $x, y \in X$, with $x \leq y$, for every $v \in F_x$ and every $z \in F_y$, $v \wedge z \in F_x$ and $v \vee z \in F_y$.

There is an important difference with the previous sections. Albeit increasing differences and the single crossing property can still be defined on $X \times T$, these properties may be insufficient to investigate the behavior of the first differences of $u(x,t)$, hence to produce the desired comparative statics conclusions, when X is a general lattice.

Let us elaborate more on this. Let X be a lattice, T be a poset, and $u(x,t) : X \times T \to \mathbb{R}$ have the single crossing property in (x,t). Let as usual B_t denote the argmax of $u(x,t)$ over X. Take unordered $x_1 \in B_{t_1}$ and $x_2 \in B_{t_2}$. By optimality of x_1 at t_1, the first difference

$$u(x_1, t_1) - u(x_1 \wedge x_2, t_1)$$

is greater than or equal to zero. By the single crossing property, the same holds at $t = t_2$. And this is all we can say by using the single crossing property[8]. In particular, we do not reach any statement about the eventual non negativity of the first difference

$$u(x_1 \vee x_2, t_2) - u(x_2, t_2),$$

which is what we need to asses whether B_t is Veinott-increasing or not.

[7]See Topkis (1978).

[8]Analogously were we using increasing differences.

Note that this problem is due exactly to the fact that lattice X is no longer assumed to be the product of chains. A solution to the problem consists in having a property of $u(x,t)$ that relates in the right way, for fixed t in T and for any x_1, x_2 in X, the two first differences above. This property is called supermodularity. Its ordinal version, quasisupermodularity, which remains preserved under ordinal transformations of payoffs, has been introduced by Milgrom and Shannon (1994).

Definition 8. ((QUASI)SUPERMODULARITY). Let X be a lattice. A function $u(x): X \to \mathbb{R}$. is supermodular on X if for every $x_1, x_2 \in X$,

$$u(x_1) + u(x_2) \leq u(x_1 \wedge x_2) + u(x_1 \vee x_2).$$

Function $u(x)$ is quasisupermodular on X if for every $x_1, x_2 \in X$,

$$\begin{array}{l} u(x_1 \wedge x_2) \leq u(x_1) \Rightarrow u(x_2) \leq u(x_1 \vee x_2); \\ u(x_1 \wedge x_2) < u(x_1) \Rightarrow u(x_2) < u(x_1 \vee x_2). \end{array}$$

In general lattices X, (quasi)supermodularity has not a direct interpretation in terms of complementarity. It only make the choice variables in X behave "consistently" with complementarity, but it is not complementarity itself. It is essentially a useful mathematical device. But, as Lemma 3 and Lemma 6 show, the interpretation in terms of complementarity is completely restored as soon as X takes the more familiar forms that we considered in the previous sections.

Lemma 3. (CHARACTERIZATION OF SUPERMODULARITY IN TERMS OF INCREASING DIFFERENCES). *Let $X = Y \times Z$ and $u(y,z): X \to \mathbb{R}$.*

(i) If Y and Z are chains and $u(y,z)$ has increasing differences in (y,z) on $Y \times Z$, then it is supermodular in (y,z) on $Y \times Z$.

(ii) If Y and Z are lattices, and $u(y,z)$ is supermodular in (y,z) on $Y \times Z$, then it has increasing differences in (y,z) on $Y \times Z$.

Proof: (i) Take any unordered $x' = (y', z'), x'' = (y'', z'')$ in $Y \times Z$ (if they are ordered supermodularity holds trivially). Let, without loss of generality, $x' \wedge x'' = (y', z'')$ and $x' \vee x'' = (y'', z')$ (here we are using the assumption that Y and Z are chains). By increasing differences,

$$u(y'', z'') - u(y', z'') \leq u(y'', z') - u(y', z'),$$

which is supermodularity.

(ii) Pick any $(y', z') \leq (y'', z'')$ in the lattice $Y \times Z$. For $x' := (y', z'')$ and $x'' := (y'', z')$, we have that $x' \wedge x'' = (y', z')$ and $x' \vee x'' = (y'', z'')$.

By supermodularity,

$$u(x'') - u(x' \wedge x'') \leq u(x' \vee x'') - u(x'),$$

which is increasing differences. ■

Lemma 4. (VECTOR SUPERMODULARITY). *Let $u(x) : X \to \mathbb{R}$, where $X = X_1 \times \cdots \times X_n$ and each factor is a chain. Function $u(x)$ is supermodular in $(x_1, \ldots, x_n)$ on X if and only if for every $(x'_1, \ldots, x'_n)$ in X, for every $h, i = 1, \ldots, n$, with $h < i$, the function*

$$u\left(x'_1, \ldots, x_h, x'_{h+1} \ldots, x_i, \ldots, x'_n\right) : X_h \times X_i \to \mathbb{R}$$

is supermodular in (x_h, x_i) on $X_h \times X_i$.

Proof: Necessity is trivial. As for sufficiency, since $u(x)$ is supermodular in each pair of variables (x_h, x_i), then by Lemma 3 $u(x)$ has increasing differences in each (x_h, x_i) (fix the value of all the other entries x_k, $k \neq h, i$, and apply the proof of point (ii) of Lemma 3). To get the statement, apply then Lemma 2. ■

Lemma 5. (DIFFERENTIAL CHARACTERIZATION OF SUPERMODULARITY). *Let $u(x) : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable on $\mathbb{R}^n$. Function $u(x)$ is supermodular on $\mathbb{R}^n$ if and only, for every x in $\mathbb{R}^n$ and every $h, i = 1, \ldots, n$ with $h \neq i$, $u_{x_h x_i}(x) \geq 0$.*

Proof: By Lemma 4, $u(x)$ has increasing difference in each pair of variables. Apply then Lemma 1, the differential characterization of increasing differences. ■

The difference between Lemma 6 below and Lemma 3 - in which we have characterized supermodularity in terms of increasing differences - lies in the fact that, in the definition of quasisupermodularity, when we interchange the roles of the two initial points x_1, x_2 of lattice X we obtain different statements. This is not true for supermodularity.

Lemma 6. (CHARACTERIZATION OF QUASISUPERMODULARITY IN TERMS OF THE SINGLE CROSSING PROPERTY) *Let $X := Y \times Z$ and $u(y, z) : X \to \mathbb{R}$.*

(i) If Y and Z are chains and $u(y, z)$ has the single crossing property in both (y, z) and (z, y) on $Y \times Z$, then it is quasisupermodular in (y, z) on $Y \times Z$.

(ii) If Y and Z are lattices and $u(y, z)$ is quasisupermodular in (y, z) on $Y \times Z$, then it has the single crossing property in both (y, z) and (z, y) on $Y \times Z$.

Proof: (i) Take any unordered $(y', z'), (y'', z'')$ in $Y \times Z$ (if they are ordered quasisupermodularity holds trivially). Let, without loss of generality, $x' \wedge x'' = (y', z'')$ and $x' \vee x'' = (y'', z')$ (here we are using the assumption that Y and Z are chains). Let

$$u(y', z') - u(y', z'') \geq (>)\, 0.$$

Hence this first difference of $u(y, z)$ in z is (strictly) positive at y', and by the single crossing property in (z, y) it stays (strictly) positive at y'', meaning that:

$$u(y'', z') - u(y'', z'') \geq (>)\, 0.$$

Let now

$$u(y'', z'') - u(y', z'') \geq (>)\, 0.$$

Hence this first difference of $u(y, z)$ in y is (strictly) positive at z'', and by the single crossing property in (y, z) it stays (strictly) positive at z', meaning that

$$u(y'', z') - u(y', z') \geq (>)\, 0.$$

Hence quasisupermodularity holds.

(ii) Pick any $(y', z') \leq (y'', z'')$ in the lattice $Y \times Z$. For $x' := (y', z'')$ and $x'' := (y'', z')$, $x' \wedge x'' = (y', z')$ and $x' \vee x'' = (y'', z'')$. By quasisupermodularity,

$$u(x') - u(x' \wedge x'') \geq (>)\, 0 \Rightarrow u(x' \vee x'') - u(x'') \geq (>)\, 0,$$

which is the single crossing property of $u(y, z)$ in (z, y). Again by quasisupermodularity,

$$u(x'') - u(x' \wedge x'') \geq (>)\, 0 \Rightarrow u(x' \vee x'') - u(x') \geq (>)\, 0,$$

which is the single crossing property of $u(y, z)$ in (y, z). ■

We now prove the monotone comparative statics theorem for the general context of lattices. We prove the general ordinal version of the theorem, due to Milgrom and Shannon (1994). Theorems 2 and 5 are special cases of Theorem 6. The the proof of Theorem 5 would have been written in a much shorter and transparent way, had we used quasisupermodularity instead.

In the case where the optimization is constrained, if we add to the condition $t_1 < t_2$ in Theorem 6 that the constraint set shifts according to the Veinott set-relation, then we obtain that quasisupermodularity and the single crossing property are not only sufficient, but also necessary to conclude that the argmax is Veinott increasing.

Theorem 6. (Comparative statics on lattices, Milgrom and Shannon, 1994.) *Let X be a lattice, T be a poset, and $u(x,t) : X \times T \to \mathbb{R}$.*

If $u(x,t)$ is quasisupermodular in x on X for every t in T, and has the single crossing property in (x,t) on $X \times T$, then the argmax B_t of $u(x,t)$ over X is a Veinott-increasing correspondence; that is, for every $t_1 < t_2$ in T, for every $x_1 \in B_{t_1}$ and every $x_2 \in B_{t_2}$, $x_1 \wedge x_2 \in B_{t_1}$ and $x_1 \vee x_2 \in B_{t_2}$.

Proof: Take any $x_1 \in B_{t_1}$ and any $x_2 \in B_{t_2}$. By optimality,

$$u(x_1 \wedge x_2, t_1) \leq u(x_1, t_1),$$

and so by quasisupermodularity

$$u(x_2, t_1) \leq u(x_1 \vee x_2, t_1).$$

Because $x_2 \leq x_1 \vee x_2$ and $t_1 < t_2$, by the single crossing property

$$u(x_2, t_2) \leq u(x_1 \vee x_2, t_2).$$

Thus, by optimality, $x_1 \vee x_2 \in B_{t_2}$. For the other part, by optimality,

$$u(x_1 \vee x_2, t_2) \leq u(x_2, t_2).$$

Hence by quasisupermodularity

$$u(x_1, t_2) \leq u(x_1 \wedge x_2, t_2),$$

and by the single crossing property,

$$u(x_1, t_1) \leq u(x_1 \wedge x_1, t_1).$$

Hence, by optimality, $x_1 \wedge x_2 \in B_{t_1}$. ■

Corollary 3. (Increasing extremal selections). *Let X be a lattice and T be a poset. Let $u(x,t) : X \times T \to \mathbb{R}$ be quasisupermodular in x on X for every t in T, and have the single crossing property in (x,t) on $X \times T$. If B_t has either a least element $a_*(t)$ or a greatest element $a^*(t)$ for every $t \in T$ (or both), then this element in an increasing functions.*

Proof: Take any $t_1 < t_2$. Consider $a_*(t)$. Take any $c \in B_{t_1}$. By the previous theorem, $c \wedge a_*(t_2) \in B_{t_1}$. Hence,

$$a_*(t_1) \leq c \wedge a_*(t_2) \leq a_*(t_2).$$

Analogously, for any $b \in B_{t_2}$, by the previous theorem $a^*(t_1) \vee b \in B_{t_2}$, and so

$$a^*(t_1) \leq a^*(t_1) \vee b \leq a^*(t_2).$$

■

The conditions assuring that every B_t has a least and a greatest element are those making it nonempty. We need a further result. This discussion applies to the previous Sections as well.

Definition 9. (Complete lattice). Let X be a nonempty lattice. X is a complete lattice if for every nonempty subset $S \subseteq X$, $\inf S \in X$ and $\sup S \in X$.

What we care about here is that, on setting $S = X$, a complete lattice has a least and a greatest element. We need the following important topological result.

Theorem 7. (Topological characterization of completeness). *A lattice is compact in its interval topology if and only if it is complete.*

Proof: Birkhoff, 1967, Ch. X, Theorem 20. ■

Assume then that X is compact in its interval topology and that $u(x,t)$ is upper semicontinuous in x for every t. Hence every B_t is nonempty and compact. Furthermore, by theorem 6, on setting $t_1 = t_2$, we see that every B_t is indeed a lattice (a sublattice of X). Thus, by Theorem 7, every B_t is a complete lattice, and so it has a least and a greatest element.

For the previous Sections, hence, just assume that the chain X and the product of chains $Y \times Z$ are compact in their interval topology, and apply the same argument.

7. Games with Strategic Complementarities

Games with strategic complementarities (GSC) are essentially games where each player's payoff is (quasi)supermodular in own strategies and has (the single crossing property) increasing differences in each pair (x, y), where x is any own strategy and y is any profile of opponents' strategies.

In view of the differential characterization of increasing differences, it is clear that GSC are appealing as far as checking conditions on payoffs is concerned. However, and more importantly, GSC are relevant because of their nice properties. Namely, in a GSC Nash equilibria exists, the Nash set is a complete lattice, and furthermore the extremal Nash equilibria have well-determinate comparative statics properties and are rationalizable.

Theorem 6 represents the first of two fundamental results in the construction of games with strategic complementarities. The second fundamental result is Theorem 8 below, a fixpoint theorem due independently to Veinott (1992) and Zhou (1994), and which is an extension to correspondences of the famous fixpoint theorem of Tarski. Thus, GSC are a merge of new comparative statics results and new fixpoint results.

We report here Veinott-Zhou fixpoint theorem in a slightly different, more general version due to Calciano (2009). We have chosen here to use results from Calciano (2009) to make the paper entirely self-contained, as will become clear in the sequel of this section.

In this section, contrary to the rest of the paper, we have decided to omit some proofs, namely those of Theorems 8 and 9. This is because these proofs are more advanced and not so consistent to the scope of this paper, which remains to clarify the notion of complementarity that lies at the heart of modern industrial organization.

Some more notation is needed. Let X be a poset and associate to a correspondence $F : X \to X$ the two sets:

$$A := \{x \in X : \exists y \in F(x) : x \leq y\};$$
$$B := \{x \in X : \exists y \in F(x) : y \leq x\}.$$

Set A is the set of elements x of X at which F jumps above the diagonal. Set B is the dual of A. The fixpoint set of F is a subset of the intersection of A and B. We say that correspondence $F : X \to X$ has a greatest (least) element if for every $x \in X$, the set $F(x)$ has a greatest (least) element.

Theorem 8. (Veinott 1992, Zhou 1994, Calciano 2009). *Let X be a complete lattice and $F : X \to X$ be a correspondence. If F is Veinott-increasing and has both a greatest and a least element, then:*
(a) $\vee A$ and $\wedge B$ are, respectively, the greatest and least fixpoint of F;
(b) the fixpoint set of F is a complete lattice.

In the original theorems of Veinott and Zhou, the correspondence F is assumed to be subcomplete-sublattice-valued. In other words, in these theorems F is assumed to be such that, for every $x \in X$, the infimum and supremum in X of all subsets of $F(x)$ lie in $F(x)$. In Theorem 8 we required this property to hold only for set $F(x)$ itself.

The generalization contained in Calciano (2009) is indeed wider, since it concerns the increasingness notion as well.

All the remarks above apply to the next theorem too, which is the fundamental comparative statics result for the study of the behavior of extremal Nash equilibria.

Theorem 9. (Topkis 1998, Calciano 2009). *Let X be a complete lattice, T be a poset and $F : X \times T \to X$ be a correspondence. If F is Veinott-increasing in (x, t) and has both a greatest and a least element, then:*
(a) for every t in T the correspondence $F(., t)$ has a greatest and a least fixpoint, and these fixpoints are increasing in t;
(b) if, in addition, $\vee F(x, t') < \wedge F(x, t'')$ for every x in X and every $t' < t''$ in T, then both the least and greatest fixpoints of $F(., t)$ are strictly increasing in t.

We now finally introduce GSC. Let Γ be a game over a players set I and let (X, u) be its normal form, where

$$X := \prod_{i \in I} X_i$$

is the space of strategy profiles, and

$$u := (u_i)_{i \in I}$$

is the vector of players' payoffs $u_i : X \to \mathbb{R}$. We do not put any restriction on strategies. The normal form may be in pure strategies, in mixed strategies, in correlated strategies. Furthermore, a continuum of pure strategies is perfectly allowed. Let

$$F : X \to X$$

be the joint best reply correspondence of the game.

Definition 10. (Games with strategic complementarities (GSC)). A game Γ has strategic complementarities if:
(i) each individual strategy space X_i is a complete lattice;
(ii) F is nonempty and has both a greatest and a least element;
(iii) F is Veinott-increasing.

The most important property of a GSC is the following.

Proposition 1. (Non-emptiness and structure of equilibrium set). *The Nash set of a GSC is a nonempty complete lattice*

Proof: Apply Theorem 8 to the joint best reply F of the game. ■

Let $[\underline{e}, \overline{e}]$ be the Nash set of a GSC, where $\underline{e}$ and $\overline{e}$ are the least and greatest Nash equilibria, respectively. The following result describes the comparative statics of $\underline{e}$ and $\overline{e}$.

Proposition 2. (COMPARATIVE STATICS OF EQUILIBRIA). *Let T be a partially ordered set and $(\Gamma_t)_{t \in T}$ be a collection of games with strategic complementarities. Let $F : X \times T \to X$ associate, to every $t \in T$, the joint best reply $F(., t)$ of game Γ_t.*

If F is Veinott-increasing in (x, t) and has both a greatest and a least element, then:
(a) the least and greatest Nash equilibria, $\underline{e}(t)$ and $\overline{e}(t)$, are increasing in t;
(b) if, in addition, $\vee F(x, t') < \wedge F(x, t'')$ for every x in X and every $t' < t''$ in T, then $\underline{e}(t)$ and $\overline{e}(t)$ are strictly increasing in t.

Proof: Apply Theorem 9 to the joint best reply F of the game. ■

Definition 11. ((QUASI)SUPERMODULAR GAME). A game Γ is (quasi) supermodular if:
(i) every X_i is a lattice compact in its interval topology;
(ii) every individual payoff function $u_i(x_i, x_{-i})$ is upper semi-continuous in own strategies x_i for every opponents' strategy profile x_{-i};
(iii) every individual payoff function is (quasi)supermodular in own strategies x_i for every opponents' strategy profile x_{-i}, and has (the single crossing property) increasing differences in (x_i, x_{-i}).

For a game Γ, being quasisupermodular is a sufficient condition for being a GSC.

Proposition 3. *A quasisupermodular game, hence a supermodular game, is a GSC.*

Proof: The topological assumptions make the joint best reply F have a greatest and a least element, as we have seen in Theorem 7. Quasisupermodularity and the single crossing property make it be Veinott-increasing, as we have seen in Theorem 6. ■

We recall that pure strategy x_i of player i is *serially undominated* if it survives the iterative process of removing from the game strongly dominated strategies (see Milgrom and Roberts, 1990, for details). It is well-known that only serially undominated strategies can be rationalizable, and only serially udominated strategies can be played (can receive positive probability) in a pure Nash equilibrium, in a mixed Nash equilibrium, or in a correlated equilibrium.

The following result says that the set of serially undominated strategies of a GSC is contained in its Nash set. Hence if the equilibrium is unique, a GSC is dominance-solvable.

Proposition 4. (Dominance solvability). *Let Γ be a supermodular game. For each player i, there exists least and greatest rationalizable strategies, $\underline{x}_i$ and $\overline{x}_i$. Furthermore, the strategy profiles $(\underline{x}_i, i \in I)$ and $(\overline{x}_i, i \in I)$ are Nash equilibria.*

Proof: This is Milgrom and Roberts (1990)'s main theorem, theorem 5. ■

8. Applications to Cournot and Bertrand competition

This section contains examples of how to use supermodular games in applications. The applications concern basic IO models, and are presented here for the sake of illustrating some aspects of the techniques that are less clear and not so used in the applied literature but that, we believe, can be useful and conductive of new applications.

We start from basic problems in IO and transform them into supermodular games. To save space and notation, we treat every application as an individual decision problem. The extension to games is immediate, just consider every revenue function treated below as the payoff of a typical player. We remark that there is no implicit assumption of symmetry; we are just avoiding to index players.

8.1. Monotone comparative statics on a chain. Bertrand competition with linear demand curves.

A firm with constant marginal cost $c > 0$ choose own scalar price $p \in [c, +\infty)$, against the pricing of its competitors $\boldsymbol{p}_{-i} \in \mathbb{R}^n_+$, to maximize:

$$\Pi\left(p, \boldsymbol{p}_{-i}\right) = (p - c)\, D\left(p, \boldsymbol{p}_{-i}\right),$$

where the demand function for firm's product is

$$D\left(p, \boldsymbol{p}_{-i}\right) = \alpha + \beta + p \sum_{j=1}^{n} \gamma_j p^j_{-i},$$

with $\beta, \gamma > 0$.

Direct computation shows that Π has increasing differences in $\left(p, \boldsymbol{p}_{-i}\right)$.

8.2. **Monotone comparative statics on a chain. Cournot duopoly with substitute products.** A firm choose quantity (capacity) $q \in [0, +\infty)$ to maximize

$$\Pi(q, q_{-i}) = qP(q, q_{-i}) - C(q),$$

against its opponent decision on its own capacity $q_{-i} \in [0, +\infty)$. $P(q, q_{-i})$ is the inverse demand function, and $C(q)$ is the cost function.

We make the following assumptions:
1.$P(q, q_{-i})$ is decreasing in q_{-i}, hence the products are *gross substitutes* (we use uncompensated demand functions). This is typical in Cournot duopoly.
2.$P(q, q_{-i})$ has decreasing differences in (q, q_{-i}), meaning that own demand is less elastic when opponent's production is higher: the decrease of own price following an increase in own production is decreasing as opponent's production is higher.

As such, the problem in not supermodular. We make it supermodular by **selection of the order**. Given the total order $([0, +\infty), \leq)$, consider its **dual order** $\left([0, +\infty), \leq^d\right)$, where

$$\forall a, b \in [0, +\infty), a \leq^d b \Leftrightarrow b \leq a.$$

Consider the new decision problem where q is chosen in $[0, +\infty)$ and q_{-i} is now chosen in the dual $[0, +\infty)^d$, which is $[0, +\infty)$ endowed with $\leq^d$

Claim: $P(q, q_{-i})$ has increasing differences in (q, q_{-i}) on $[0, +\infty) \times [0, +\infty)^d$.
Proof: Pick any $\left(q', q'_{-i}\right), \left(q'', q''_{-i}\right)$ in $[0, +\infty) \times [0, +\infty)^d$ such that $q' \leq q''$ and $q'_{-i} \leq^d q''_{-i}$. Hence, $q''_{-i} \leq q'_{-i}$. So, *in the original decision problem*, by decreasing differences of P in (q, q_{-i}),

$$P\left(q'', q''_{-i}\right) - P\left(q', q''_{-i}\right) \geq P\left(q'', q'_{-i}\right) - P\left(q', q'_{-i}\right),$$

which is exactly increasing difference of P in (q, q_{-i}) on $[0, +\infty) \times [0, +\infty)^d$, i.e. the first difference of P in q increases as q_{-i} increased in $[0, +\infty)^d$.

Claim: $\Pi(q, q_{-i})$ has increasing differences in (q, q_{-i}) on $[0, +\infty) \times [0, +\infty)^d$.
Proof: Pick any $\left(q', q'_{-i}\right), \left(q'', q''_{-i}\right)$ in $[0, +\infty) \times [0, +\infty)^d$ such that $q' \leq q''$ and $q'_{-i} \leq^d q''_{-i}$. By increasing differences of P on $[0, +\infty) \times [0, +\infty)^d$,

rearranging terms,

$$P\left(q'', q'_{-i}\right) - P\left(q'', q''_{-i}\right) \leq P\left(q', q'_{-i}\right) - P\left(q', q''_{-i}\right).$$

Since, in the original problem, $P\left(q, q_{-i}\right)$ is decreasing in q_{-i} on $[0, +\infty)$, by the same argument as in Claim 1 above $P\left(q, q_{-i}\right)$ is now increasing in q_{-i} on the dual order $[0, +\infty)^d$. Hence, in the inequality above, both differences on l.h.s. and r.h.s. are negative. Since $0 \leq q' \leq q''$, can multiply as follows:

$$q''\left[P\left(q'', q'_{-i}\right) - P\left(q'', q''_{-i}\right)\right] \leq q'\left[P\left(q', q'_{-i}\right) - P\left(q', q''_{-i}\right)\right].$$

The inequality above is what we needed.

Comparative statics conclusion: The set of optimal capacities for the firm is increasing in q'_{-i} on the dual $[0, +\infty)^d$, hence it is decreasing in q'_{-i} on $[0, +\infty)$. As the opponent increases production, its optimal for the firm to decrease own production. Indeed, product substitution implies that by increasing production, the opponent lowers its price and steals consumers from the firm, which needs to lower own production to keep profits.

8.3. **Monotone comparative statics on the product of chains. Pricing and advertising in Bertrand competition with substitute products.** This example is taken from Topkis (1995). A firm with constant marginal cost $c > 0$ chooses its price $p \in [c, +\infty)$, and its advertising effort $a \in [0, +\infty)$, against the pricing of its competitors $\boldsymbol{p}_{-i} \in \mathbb{R}^n_+$. The firm has cost $K(a)$ for advertising. The profit function is:

$$\Pi\left(p, a, \boldsymbol{p}_{-i}\right) = (p - c)\, D\left(p, a, \boldsymbol{p}_{-i}\right) - K(a).$$

We make the following assumptions:
1. The demand function $D\left(p, a, \boldsymbol{p}_{-i}\right)$ is increasing in a.
2. The demand function is increasing in $\boldsymbol{p}_{-i}$. This means that competitors' products are *gross substitutes* to that of the firm (we consider uncompensated demand).
3. The demand function is supermodular in (p, a), meaning that an increase in own price leads to a loss of demand (if demand is decreasing) which is less, if quality is higher.
4. The demand functions has increasing differences in $\left((p, a), \boldsymbol{p}_{-i}\right)$, meaning that (i) the loss of demand for increasing own prices is mitigated when opponents' substitute products are priced higher, since less

consumers are willing to leave own market because substitute products are more expensive, and (ii) the increase in demand for increasing advertising is higher when opponents' substitute products are priced higher, since the firm has then a wider market share.

Claim: The profit function is supermodular in (p, a) on $[c, +\infty) \times [0, +\infty)$ and has increasing differences in $\left((p, a), \boldsymbol{p}_{-i}\right)$ on $[c, +\infty) \times [0, +\infty) \times \mathbb{R}^n_+$.
Proof: To show increasing differences of Π in $\left((p, a), \boldsymbol{p}_{-i}\right)$, pick any $\left(p', a', \boldsymbol{p}'_{-i}\right) \leq \left(p'', a'', \boldsymbol{p}''_{-i}\right)$. Using the definition of increasing differences,

$$\begin{gathered}(p'' - c) D\left(p'', a'', \boldsymbol{p}'_{-i}\right) - K(a'') - \left[(p' - c) D\left(p', a', \boldsymbol{p}'_{-i}\right) - K(a')\right] \\ \leq \\ (p'' - c) D\left(p'', a'', \boldsymbol{p}''_{-i}\right) - K(a'') - \left[(p' - c) D\left(p', a', \boldsymbol{p}''_{-i}\right) - K(a')\right] \\ \Leftrightarrow \\ (p'' - c)\left[D\left(p'', a'', \boldsymbol{p}'_{-i}\right) - D\left(p'', a'', \boldsymbol{p}''_{-i}\right)\right] \\ \leq \\ (p' - c)\left[D\left(p', a', \boldsymbol{p}'_{-i}\right) - D\left(p', a', \boldsymbol{p}''_{-i}\right)\right]\end{gathered}$$

In the last inequality, the two terms in square brackets are nonpositive, due to increasingness of D in $\boldsymbol{p}_{-i}$. Furthermore, the last inequality holds without the multiplying factors $(p'' - c)$ and $(p' - c)$, by the assumption that demand has increasing differences in $\left((p, a), \boldsymbol{p}_{-i}\right)$. Since both $p'' \geq c$ and $p' \geq c$, the last inequality holds.

To show supermodularity of Π in (p, a), take any $\left(p', a', \boldsymbol{p}'_{-i}\right) \leq \left(p'', a'', \boldsymbol{p}''_{-i}\right)$ with $\boldsymbol{p}'_{-i} = \boldsymbol{p}''_{-i}$, and redo the same argument as above, by recalling that the assumption of supermodularity of demand in the vector variable (p, a) is equivalent of it having increasing differences in (p, a). This time, however, we need to use increasingness of demand in a.

Comparative statics conclusion: Optimal prices and advertising levels are increasing in the parameters $\boldsymbol{p}_{-i}$. Note that if also opponents set their advertising level $\boldsymbol{a}_{-i}$, and if own demand D is decreasing with this, then profits do not have increasing differences in $\left((p, a), \left(\boldsymbol{p}_{-i}, \boldsymbol{a}_{-i}\right)\right)$.

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