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A Primal Divisia Technical Change Index Based on the Output Distance Function

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A Primal Divisia Technical Change Index Based on the Output Distance Function*

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ABSTRACT

We derive a primal Divisia technical change index based on the output distance function and further show the validity of this index from both economic and axiomatic points of view. In particular, we derive the primal Divisia technical change index by total differentiation of the output distance function with respect to a time trend. We then show that this index is dual to the Jorgenson and Griliches (1967) dual Divisia total factor productivity growth (TFPG) index when both the output and input markets are competitive; dual to the Diewert and Fox (2008) markup-adjusted revenue-share based dual Divisia technical change index when market power is limited to output markets; dual to the Denny *et al.* (1981) and Fuss (1994) cost-elasticity-share based dual Divisia TFPG index when market power is limited to output markets and constant returns to scale is present; and also dual to a markup-and-markdown adjusted Divisia technical change index when market power is present in both output and input markets. Finally, we show that the primal Divisia technical change index satisfies the properties of identity, commensurability, monotonicity, and time reversal. It also satisfies the property of proportionality in the presence of path independence, which in turn requires separability between inputs and outputs and homogeneity of subaggregator functions.

JEL classification: C43; D24; O47.
Keywords: Output distance function; Divisia technical change index; Imperfect competition; Axiomatic properties; Path independence.

1 Introduction

Measures of productivity have long enjoyed a great deal of interest among economists. One of the most popular and influential methods of productivity measurement is the Divisia-type total factor productivity growth (TFPG) or technical change index.[1] First derived by Solow (1957) within a single-output framework and later generalized by Jorgenson and Griliches (1967) to a multiple-output framework, the conventional dual Divisia TFPG index is calculated as the observed revenue-share-weighted output growth minus the observed cost-share-weighted input growth.[2] While having enjoyed great popularity, the conventional dual Divisia TFPG index is restricted in the sense that it is obtained under the assumptions of price/marginal cost proportionality and constant returns to scale. Insightfully noting this problem, Caves and Christensen (1980) replace the observed revenue shares with cost elasticities; Denny *et al.* (1981) and Fuss (1994) with cost-elasticity shares; and Diewert and Fox (2008) with markup-adjusted revenue shares (marginal revenue shares). The three resulting indexes are thus appropriate in the presence of imperfect competition, thus also being compatible with increasing returns to scale.

Despite the differences in technology/firm behavior assumptions and economic functions used, a feature that the aforementioned indexes share in common is that they all require complete information on both output and input prices. In many situations, however, such information is unavailable, inaccurate or distorted. For example, for many goods, such as free goods, intangible assets, and new commodities introduced in the target period, monetary prices do not exist. In many other situations where market failures (such as, for example, monopoly power and externalities) or governmental interference (such as, for example, tariffs, taxation, subsidizing, and regulation) are present, the observed prices differ from the economic prices. In those situations, a primal Divisia TFPG/technical change index, which relies only on quantity information, is desirable.

With the popularity of the output distance function, recent studies have replaced the observed revenue and cost shares in the conventional Divisia TFPG index with the elasticities of the output distance function with respect to outputs and inputs, respectively. See, for example, Orea (2002) and Lovell (2003). However, none of these studies have theoretically shown that this primal TFPG index is a valid productivity index under different market structures, nor have they derived the properties of this index. For example, Orea (2002)

[1] In this paper we make a distinction between two concepts: technical change and total factor productivity growth (TFPG). The former, which is used in deriving the primal Divisia technical change index in this paper, refers to the rate of shift in a transformation function [Solow (1957, p. 312)], while the latter refers to the rate of change of an index of outputs divided by an index of inputs [Jorgenson and Griliches (1967, p. 253)]. Technical change can reduce to TFPG under appropriate assumptions.

[2] A firm theoretical foundation of the Solow (1957) and Jorgenson and Griliches (1967) dual Divisia TFPG index is provided in the pioneering works of Solow (1957), Jorgenson and Griliches (1967), Richter (1966), Hulten (1973), Star and Hall (1976), Diewert (1976), and Balk (2009).

shows the validity of the distance-elasticity based Divisia TFPG index by simply making a loose analogy between this index and the Denny *et al.* (1981) cost-elasticity-share based Divisia TFPG index. This is apparently unsatisfactory from a theoretical perspective.

The purpose of this study is to fill this gap. In particular, we have three objectives: (a) to theoretically derive an output-distance-function based primal Divisia technical change index; (b) to formally demonstrate the validity of this index from an economic point of view; and (c) to formally discuss its validity from an axiomatic point of view. Generally speaking, there are two major approaches to index numbers — the 'economic' approach and the 'axiomatic' approach. The economic approach, widely used in many influential studies such as, for example, Diewert (1976) and Barnett (1980), assumes that quantities (or prices) arise from the optimizing behavior of economic agents and explores how closely an index approximates some 'true' index based on economic theory. See Diewert (1981a) and Neary (2004). By contrast, the axiomatic approach, a tradition dating back to Fisher (1922) and Frisch (1930), treats prices and quantities as independent variables and assesses the extent to which an index satisfies certain desirable, though not mutually consistent, properties. See Diewert (1992) and Balk (2008) for an excellent overview of this approach.

In deriving the primal Divisia technical change index, we follow Solow (1957) and differentiate a transformation function (in particular, an output distance function in our case) totally with respect to time. We show that technical change is equal to a distance-elasticity-based technical change index, which we call 'primal Divisia technical change index.' We also derive the restrictions on the output and input weights implied by the regularity conditions of the output distance function.

We then show that the primal Divisia technical change index is valid under different market structures. To this end, we consider three cases: (i) both the output and input markets are perfectly competitive, (ii) market power is limited to output markets, and (iii) market power is present in both output and input markets. For the first case, we solve the competitive profit maximization problem and show that the primal Divisia technical change index is equal to the Jorgenson and Griliches (1967) dual Divisia TFPG index. For the second case, we slightly modify the Diewert and Fox (2008) monopolistic profit maximization framework by using the output distance function as the technology constraint. We show that the primal Divisia TFPG index is equal to the Diewert and Fox (2008) markup-adjusted revenue-share (marginal revenue shares) based dual Divisia technical change index, and that it is also equal to the Denny *et al.* (1981) and Fuss (1994) cost-elasticity-share based dual Divisia TFPG index under constant returns to scale. For the third case, we show, by using a more generalized profit maximization framework, that the primal Divisia technical change index is equal to a markup adjusted output growth index minus a markdown adjusted input growth index multiplied by the degree of returns to scale.

Finally, we discuss the validity of the primal Divisia technical change index from an axiomatic point of view. We show that it satisfies the properties of identity, commensurability (dimensional invariance), monotonicity, and time reversal, and that it also satisfies the proportionality property if technology is constant returns to scale and its corresponding cumulative index is path independent. We further show that the necessary and sufficient conditions for path independence of the primal Divisia technical change index are that the output distance function is separable between inputs and outputs and that the output subaggregator function is linearly homogenous in outputs and the input subaggregator function is homogeneous of degree of ε (the degree of local constant returns to scale) in inputs.

We note that Balk (1998, Chapter 3) exploits the duality between the cost function and the input distance function and derives a Malmquist productivity index based on the input distance function. Our paper, however, is different from Balk's (1998) in the following three ways. First, our index is a Divisia technical change index whereas Balk's (1998) is a Malmquist productivity index. Second, our Divisia technical change index is based on the output distance function whereas Balk's (1998) Malmquist productivity index is based on the input distance function. Third, and more importantly, our focus is on the examination of the theoretical properties of the primal output-distance-function based Divisia technical change index whereas Balk (1998) focuses on the decomposition of the input-distance-function based Malmquist productivity index.

The rest of the paper is organized as follows. In Section 2, we discuss the output distance function. In Section 3 we derive the output-distance-function based Divisia technical change index. In Section 4, we show that this index is valid under different market structures. In Section 5, we show that the primal Divisia TFPG index satisfies certain desirable axiomatic properties, and in Section 6 we discuss the issue of path independence. The final section concludes the paper.

2 The Output Distance Function

We first define the production technology and the output distance function on which our primal Divisia technical change index is based. Consider the case of a multi-input, multi-output production technology where producers use the $1\times N$ input vector $\boldsymbol{x} = (x_1, \cdots, x_N) \geq \mathbf{0}_N$ to produce the $1 \times M$ output vector $\boldsymbol{y} = (y_1, \cdots, y_M) \geq \mathbf{0}_M$. Following Diewert and Fox (2010), the production technology at time t can be described by the technology set

$$P^t(\boldsymbol{x}) = \{\boldsymbol{y} : \boldsymbol{y} \text{ is producible from } \boldsymbol{x}\}$$

which satisfies a set of axioms including closedness, nonemptiness, boundedness, positiveness, and disposability of outputs. It should be noted here that in order to allow for increasing returns to scale, the production technology is not assumed to be convex.

An output distance function can then be defined as in Shephard (1970),

$$D_o^t(\boldsymbol{y}, \boldsymbol{x}) = \inf_{\epsilon} \left\{ \epsilon > 0 : \frac{\boldsymbol{y}}{\epsilon} \in P^t(\boldsymbol{x}) \right\}. \tag{1}$$

It seeks the largest proportional increase in the observed output vector possible given that the expanded vector must still be an element of the original output set. Consistent with the properties satisfied by the production technology, the output distance function is non-decreasing and linearly homogeneous in outputs, and non-increasing in inputs. Since the production technology is not assumed to be convex, the usual assumptions that $D_o^t(\boldsymbol{y}, \boldsymbol{x})$ is convex in outputs and quasi-convex in inputs are dropped. See Diewert and Fox (2010) for a detailed proof that $D_o^t(\boldsymbol{y}, \boldsymbol{x})$ is well defined provided that $P^t(\boldsymbol{x})$ satisfies the properties mentioned in the previous paragraph.

To facilitate the calculation of technical change below, we follow the common practice in this literature and make two assumptions. First, we assume that the influence of technology is through the exogenous time variable t; second, we assume that the outputs and inputs are functions of time and differentiable. With these assumptions, the output distance function in (2) can be rewritten as $D_o(\boldsymbol{y}(t),\boldsymbol{x}(t),t)$. Thus, as in Solow (1957), technical progress for technology is due to the t variable in $D_o(\boldsymbol{y}(t),\boldsymbol{x}(t),t)$; i.e., with positive technical change, we expect $\partial D_o(\boldsymbol{y}(t),\boldsymbol{x}(t),t)/\partial t$ to be negative. Furthermore, we assume throughout this paper that production is efficient at all points in time so that we have

$$D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t) = 1. \tag{2}$$

This assumption is consistent with the optimization problems set out in Section 4.

3 The Primal Divisia Technical Change Index

3.1 The Derivation of the Primal Divisia Technical Change Index

Having defined the output distance function, we now turn to the derivation of the primal (or more accurately, output distance function) Divisia technical change index. In doing so, we follow Solow (1957) and differentiate a transformation function (the output distance function in our case) totally with respect to time. In particular, we define the *instantaneous rate of technical change* at time t, TC, to be:

$$TC = -\frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial t}. \tag{3}$$

Differentiating (2) totally with respect to time we obtain

$$\sum_{m=1}^{M} \frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial y_m} \frac{dy_m}{dt} + \sum_{n=1}^{N} \frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial x_n} \frac{dx_n}{dt} + \frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial t} = 0$$

which, when combined with (3), yields

$$TC = \sum_{m=1}^{M} \frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial y_m} \frac{dy_m}{dt} + \sum_{n=1}^{N} \frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial x_n} \frac{dx_n}{dt}$$

$$= \sum_{m=1}^{M} \frac{\partial \ln D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial \ln y_m} \frac{d \ln y_m}{dt} + \sum_{n=1}^{N} \frac{\partial \ln D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial \ln x_n} \frac{d \ln x_n}{dt}$$

$$= \sum_{m=1}^{M} \phi_m \dot{y}_m - \sum_{n=1}^{N} \varphi_n \dot{x}_n. \tag{4}$$

In (4), we assume that $y_m(t) > 0$ for all m and $x_n(t) > 0$ for all n. $\dot{y}_m$ and $\dot{x}_n$ are the growth rates for output m and input n, respectively. That is, $\dot{y}_m = d \ln y_m/dt$ and $\dot{x}_n = d \ln x_n/dt$. Finally, the output growth weights, ϕ_m, and the input growth weights, φ_n, are defined in terms of logarithmic derivatives of the output distance function as follows:

$$\phi_m = \frac{\partial \ln D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial \ln y_m}; \tag{5}$$

$$\varphi_n = -\frac{\partial \ln D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial \ln x_n}. \tag{6}$$

In what follows, we call the term on the right hand side of (4) 'output distance function Divisia technical change index' or simply 'primal Divisia technical change index.'

3.2 The Restrictions on the Output and Input Growth Weights

A distinctive feature of the primal Divisia technical change index, as can be seen from (5) and (6), is that its output and input growth weights are defined in terms of distance elasticities. Thus, to complete the definition of the primal Divisia technical change index, we further need to examine the restrictions on its output and input growth weights, implied by the regularity conditions of the output distance function.

We first examine the restrictions on the output growth weights. From Section 2, we know that $D_o(\boldsymbol{y}(t),\boldsymbol{x}(t), t)$ is linearly homogeneous and non-decreasing in $\boldsymbol{y}$ (with an appropriate free disposal assumption on $\boldsymbol{y}$). Formally, these two properties can be written as

$$\sum_{m=1}^{M} \frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial y_m} y_m = 1; \tag{7}$$

$$\frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial y_m} \geq 0,\ m = 1, \cdots, M. \tag{8}$$

Noting that $D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t) = 1$ [see (2)], (7) can be written as

$$\sum_{m=1}^{M} \frac{\partial \ln D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial \ln y_m} = 1.$$

Furthermore, since $D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)$ and y_m ($m = 1, \cdots, M$) are non-negative, (8) can be rewritten as

$$\frac{\partial \ln D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial \ln y_m} \geq 0,\ m = 1, \cdots, M.$$

Thus, the output growth weights, ϕ_m, satisfy the following restrictions:

$$\phi_m \geq 0, \text{ for } m = 1, \cdots, M, \text{ and } \sum_{m=1}^{M} \phi_m = 1. \tag{9}$$

We now turn to the restrictions on the input growth weights, φ_n. We know that $D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)$ is non-increasing in $\boldsymbol{x}$ (with an appropriate free disposal assumption on $\boldsymbol{x}$). Formally, we have

$$-\frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial x_n} \geq 0,\ n = 1, \cdots, N$$

which can be further written as

$$-\frac{\partial \ln D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial \ln x_n} \geq 0,\ \ n = 1, \cdots, N$$

since $D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)$ and x_n ($n = 1, \cdots, N$) are non-negative. Thus, the input growth weights, φ_n, satisfy the following restriction:

$$\varphi_n \geq 0,\ n = 1, \cdots, N. \tag{10}$$

If the overall technology is a cone, so that we have constant returns to scale in production, then it can be shown that $D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)$ is homogeneous of degree minus one in the components of $\boldsymbol{x}$ so that we have for all $\boldsymbol{x} >> \mathbf{0}_N$, $y >> \mathbf{0}_M$, and $\kappa > 0$:

$$D_o(\boldsymbol{y}(t), \kappa \boldsymbol{x}(t), t) = \kappa^{-1} D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t).$$

Differentiating both sides of the above equation with respect to κ and setting $\kappa = 1$ leads to the following identity:

$$\sum_{n=1}^{N} \frac{\partial D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t), t\right)}{\partial x_n} x_n = -D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t), t\right) = -1.$$

Thus, in the case of a constant returns to scale technology at time t, the input growth weights will also satisfy the following, additional to (10), restriction:

$$\sum_{n=1}^{N} \varphi_n = 1. \tag{11}$$

In the case of an increasing returns to scale technology, the input growth weights, φ_n, will sum to a number greater than one and in the case of a decreasing returns to scale technology, they will sum to a number less than one.

3.3 The Special Case of Hicks Neutral Technical Change

Compared with the Solow (1957) Divisia TFPG index, which is obtained under the assumption of Hicks neutral technical change, the Divisia technical change index, defined by (4)–(6), is quite general in the sense that it is obtained without making a priori assumption about the nature of technical change. Thus, it is worth examining whether the Divisia technical change index reduces to the right answer when technical change happens to be Hicks neutral.

In particular, for the case of Hicks neutral technical change, the output distance function would have the following representation for times $t \geq t_0$:

$$D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t), t\right) = D_o\left(\frac{\boldsymbol{y}}{\alpha(t)}, \boldsymbol{x}(t), t_0\right); \tag{12}$$

$$\alpha(t_0) = 1 \tag{13}$$

where $\alpha(t)$ is a shift factor for the output distance function. Noting that the observed period t_0 output and input vectors, $\boldsymbol{y}(t_0)$ and $\boldsymbol{x}(t_0)$, are efficient, the output distance function at time t_0 can be written as

$$D_o\left(\frac{\boldsymbol{y}(t_0)}{\alpha(t_0)}, \boldsymbol{x}(t_0), t_0\right) = D_o\left(\boldsymbol{y}(t_0), \boldsymbol{x}(t_0), t_0\right) = 1. \tag{14}$$

Now suppose the input vector in period t were $\boldsymbol{x}(t_0)$. Using the period t technology, it can be seen that $\alpha(t)\boldsymbol{y}(t_0)$, $\boldsymbol{x}(t_0)$ will be on the efficient frontier for period t. Thus, we will have:

$$D_o\left(\alpha(t)\boldsymbol{y}(t_0), \boldsymbol{x}(t_0), t\right) = D_o\left(\frac{\alpha(t)\boldsymbol{y}(t_0)}{\alpha(t)}, \boldsymbol{x}(t_0), t_0\right) = 1$$

by (14). Thus, if $\alpha(t)$ is a monotonic increasing function of t, we will have positive Hicks neutral technical change over time.

We now determine what the general formula for technical change, (3), looks like at time t_0 when we have Hicks neutral technical change:

$$\begin{aligned} TC(t_0) &= -\frac{\partial D_o(\boldsymbol{y}(t_0), \boldsymbol{x}(t_0), t_0)}{\partial t} \\ &= -\frac{\partial D_o(\boldsymbol{y}(t_0)/\alpha(t_0), \boldsymbol{x}(t_0), t_0)}{\partial t} \\ &= -\sum_{m=1}^{M} \frac{\partial D_o(\boldsymbol{y}(t_0)/\alpha(t_0), \boldsymbol{x}(t_0), t_0)}{\partial y_m} y_m(t_0) \left(\frac{-d\alpha(t_0)/dt}{[\alpha(t_0)]^2} \right) \\ &= \sum_{m=1}^{M} y_m(t_0) \frac{\partial D_o(\boldsymbol{y}(t_0), \boldsymbol{x}(t_0), t_0)}{\partial y_m} \frac{d\alpha(t_0)}{dt} \\ &= D_o(\boldsymbol{y}(t_0), \boldsymbol{x}(t_0), t_0) \frac{d\alpha(t_0)}{dt} \\ &= \frac{d\alpha(t_0)}{dt} \end{aligned} \tag{15}$$

where the third last equality is obtained by using (13), the second last equality by (7), and the last equality by (2). Thus, our general measure of technical change picks up the right answer when technical change happens to be Hicks neutral.

4 The Primal Divisia Technical Change Index under Different Market Structures

As is well known in the literature, for the index numbers to provide meaningful estimates of productivity or productivity growth, certain assumptions about the underlying market structure (or behavior of producers) and production technology must be maintained. As noted above, the conventional Jorgenson and Griliches (1967) dual Divisia TFPG index is obtained under the assumption of perfect competition and constant returns to scale. The three more general indexes, namely, the Caves and Christensen (1980) cost-elasticity-based dual TFPG index, the Denny *et al.* (1981) and Fuss (1994) cost-elasticity-share-based dual

TFPG index, and the Diewert and Fox (2008) markup-adjusted-revenue-share-based technical change index, are obtained assuming the presence of imperfect competition in output markets. We note that, within the cost function framework, Balk (2009) derives a similar Divisia index as that in Denny *et al.* (1981) and Fuss (1994). That index, however, like the Denny *et al.* (1981) and Fuss (1994) indexes, is a dual one.

Different from the aforementioned Divisia indexes, the primal Divisia technical change index, defined by (4)–(6), is obtained through total differentiation of the output distance function and without making any assumptions about market structure. Thus, it is very interesting to examine the relationship between the primal Divisia technical change index and the aforementioned indexes under different market structures. Specifically, in what follows we consider three special cases. In the first case, we assume that both output and input markets are perfectly competitive. In the second case, we assume that market power is limited to output markets and that input markets are perfectly competitive. In the third case, the assumption of perfectly competitive input markets is further relaxed. As we shall see below, the primal technical change index is closely linked to the aforementioned well-known indexes.

4.1 Perfect Competition

We first consider the simple case where both output and input markets are perfectly competitive. As is well known, the assumption of perfect competition is not compatible with increasing returns to scale at the firm level, as marginal cost pricing leads in this case to negative profits. See, for example, Small (1999) and Hall (1988). Thus, here we follow Solow (1957, p. 313) and Jorgenson and Griliches (1967, p. 253) and assume that the transformation function (i.e., the output distance function in our particular case) is characterized by constant returns to scale.

In order to simplify the notation, the argument t in outputs and inputs is dropped in this section. We assume that producers solve the following competitive profit maximization problem:

$$\pi = \max_{\{\boldsymbol{y},\boldsymbol{x}\}} \left\{ \sum_{m=1}^{M} p_m y_m - \sum_{m=1}^{N} w_n x_n : D_o\left(\boldsymbol{y}, \boldsymbol{x}, t\right) = 1 \right\} \tag{16}$$

where the output distance function is used to represent the technology constraint. For details regarding the duality between the profit function and the output distance function under the assumption of prefect competition, see Färe and Primont (1995), Färe and Grosskopf (2000), and Chambers and Färe (1993).

The first-order conditions corresponding to outputs, y_m, $m = 1, \cdots, M$, are

$$p_m = \mu \frac{\partial D_o\left(\boldsymbol{y}, \boldsymbol{x}, t\right)}{\partial y_m}, \quad m = 1, \cdots, M \tag{17}$$

where μ is the Lagrange multiplier. Multiplying both sides of equation (17) with $y_m/D_o(\boldsymbol{y},\boldsymbol{x},t)$ and rearranging yields

$$\frac{p_m y_m}{D_o(\boldsymbol{y},\boldsymbol{x},t)} = \mu\frac{\partial \ln D_o(\boldsymbol{y},\boldsymbol{x},t)}{\partial \ln y_m}, \quad m = 1,\cdots,M. \tag{18}$$

Summing up the M equations in (18) yields

$$\sum_{i=1}^{M}\frac{p_i y_i}{D_o(\boldsymbol{y},\boldsymbol{x},t)} = \mu\sum_{i=1}^{M}\frac{\partial \ln D_o(\boldsymbol{y},\boldsymbol{x},t)}{\partial \ln y_i}. \tag{19}$$

Noting that

$$\sum_{i=1}^{M}\frac{\partial \ln D_o(\boldsymbol{y},\boldsymbol{x},t)}{\partial \ln y_i} = 1$$

by the linear homogeneity of the output distance function in outputs [see (7)], we divide (18) by (19) to obtain

$$\frac{\partial \ln D_o(\boldsymbol{y},\boldsymbol{x},t)}{\partial \ln y_m} = \frac{p_m y_m}{\sum\limits_{i=1}^{M} p_i y_i}, \quad m = 1,\cdots,M. \tag{20}$$

Applying a similar procedure to the N inputs, we obtain

$$\frac{\partial \ln D_o(\boldsymbol{y},\boldsymbol{x},t)/\partial \ln x_n}{\sum\limits_{j=1}^{N}\partial \ln D_o(\boldsymbol{y},\boldsymbol{x},t)/\partial \ln x_j} = \frac{w_n x_n}{\sum\limits_{j=1}^{N} w_j x_j}, \quad n = 1,\cdots,N. \tag{21}$$

In the context of the output distance function, local returns to scale at time t, $\varepsilon(t)$, can be formally defined following Caves, Christensen, and Diewert (1982, p. 1402):

$$\varepsilon(t) = -\sum_{n=1}^{N}\frac{\partial D_o(\boldsymbol{y},\boldsymbol{x},t)}{\partial x_n}x_n. \tag{22}$$

Noting that the production technology is characterized by constant returns to scale [i.e., $\varepsilon(t) = 1$] in this case, (21) can be further written as

$$-\frac{\partial \ln D_o(\boldsymbol{y},\boldsymbol{x},t)}{\partial \ln x_n} = \frac{w_n x_n}{\sum\limits_{j=1}^{N} w_j x_j}. \tag{23}$$

Combining (20) and (23), (4) can be further written as

$$TC = \sum_{m=1}^{M} \phi_m \dot{y}_m - \sum_{n=1}^{N} \varphi_n \dot{x}_n$$

$$= \sum_{m=1}^{M} \frac{p_m y_m}{\sum_{i=1}^{M} p_i y_i} \dot{y}_m - \sum_{n=1}^{N} \frac{w_n x_n}{\sum_{j=1}^{N} w_j x_j} \dot{x}_n. \tag{24}$$

According to (24), under the assumptions of perfect competition and constant returns to scale, the primal Divisia technical change index is equal to a Divisia real output growth index minus a Divisia real input growth index. Noting that the right hand side of (24) is just the Jorgenson and Griliches (1967) dual Divisia TFPG index, (24) implies that the primal Divisia technical change index and the Jorgenson and Griliches (1967) dual Divisia TFPG index are dual to each other under the assumptions of perfect competition and constant returns to scale.

With regards to the calculation of technical change, when all the output and input prices and quantities are observed, the use of the Jorgenson and Griliches (1967) dual Divisia TFPG index is more convenient, since it does not involve any econometric estimation. However, when price information is unavailable or inaccurate, the estimation of the output distance function is required in order to obtain the elasticities of the output distance function with respect to outputs and inputs, needed for the calculation of the primal Divisia technical change index.

4.2 Market Power in Output Markets

In the second case, we assume that market power is limited to output markets. This assumption is more appealing than the assumption of perfect competition for the following two reasons. First, imperfect competition is widely regarded to be an important feature of the economy. See, for example, Diewert and Fox (2008), Hall (1988), Basu and Fernald (1997), and Hulten (2009). Second, it is compatible with internal or firm level increasing returns to scale, which has been shown by recent studies to play a very important role in explaining productivity growth in many industries. See, for example, Diewert and Fox (2008) and Hall (1988).

In particular, we assume that the producers solve the Diewert and Fox (2008) monopolistic profit maximization problem

$$\max_{\boldsymbol{y}} \ \pi = \left\{ \sum_{m=1}^{M} p_m (y_m) \, y_m - C(\boldsymbol{y}, \boldsymbol{w}, t) \right\} \tag{25}$$

where $p_m(y_m)$ is the inverse demand function, and $C(\boldsymbol{y}, \boldsymbol{w}, t)$ is obtained from the following first-stage competitive cost-minimization problem

$$C(\boldsymbol{y}, \boldsymbol{w}, t) = \min_{\boldsymbol{x}} \{\boldsymbol{w}'\boldsymbol{x}\colon\ D_o(\boldsymbol{y}, \boldsymbol{x}, t) = 1\}. \tag{26}$$

Discussion of two-stage profit-maximization problem can also be found in Diewert (1981b, p. 29) and Chambers (1988, p. 121). The duality between the output distance function and the cost function in (26) is discussed in Färe and Primont (1990) and Primont and Sawyer (1993).

The first-order conditions corresponding to problem (25) are

$$p_m(1-\eta_m) = \frac{\partial C(\boldsymbol{y}, \boldsymbol{w}, t)}{\partial y_m}, \quad m = 1, \cdots, M \tag{27}$$

where

$$\eta_m = -\frac{\partial p_m(y_m)}{\partial y_m}\frac{y_m}{p_m} \geq 0. \tag{28}$$

Rearranging (27), we obtain

$$(1-\eta_m) = \frac{\partial C(\boldsymbol{y}, \boldsymbol{w}, t)/\partial y_m}{p_m} \tag{29}$$

where the term on the right hand side is the reciprocal of the markup for output m. Here, markup is defined as the price of output over marginal cost (that is, $p/[\partial C(\boldsymbol{y}, \boldsymbol{w}, t)/\partial y]$ in the case of a single output), as is common in this literature — see, for example, Hall (1988). Thus, according to (29), $(1-\eta_m)$ is an indirect measure of the markup for output m.

Applying the envelope theorem to (26) with respect to the mth output, we obtain:

$$\frac{\partial C(\boldsymbol{y}, \boldsymbol{w}, t)}{\partial y_m} = -\lambda\frac{\partial D_o(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial y_m} \tag{30}$$

where λ is the Lagrange multiplier for the cost-minimization problem in (26). Substituting (30) into (27) we obtain:

$$p_m(1-\eta_m) = -\lambda\frac{\partial D_o(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial y_m}, \quad m = 1, \cdots, M. \tag{31}$$

Multiplying both sides of (31) by $y_m/D_o(\boldsymbol{y}, \boldsymbol{x}, t)$ we obtain:

$$\frac{p_m(1-\eta_m)y_m}{D_o(\boldsymbol{y}, \boldsymbol{x}, t)} = -\lambda\frac{\partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial \ln y_m}, \quad m = 1, \cdots, M. \tag{32}$$

Summing up the M equations in (32) we obtain:

$$\sum_{i=1}^{M} \frac{p_i (1-\eta_i) y_i}{D_o(\boldsymbol{y}, \boldsymbol{x}, t)} = -\lambda \sum_{i=1}^{M} \frac{\partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial \ln y_i}. \tag{33}$$

Noting that $\sum_{i=1}^{M} \partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t) / \partial \ln y_i = 1$, by the linear homogeneity of the output distance function in outputs, and dividing (32) by (33) yields:

$$\frac{p_m (1-\eta_m) y_m}{\sum_{i=1}^{M} p_i (1-\eta_i) y_i} = \frac{\partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial \ln y_m}, \quad m = 1, \cdots, M. \tag{34}$$

Since $(1-\eta_m)$ is an indirect measure of the markup for output m [see (29)], (34) implies that the elasticity of the output distance function with respect to the mth output is equivalent to a markup-adjusted revenue share of the mth output, which, as shown by Diewert and Fox (2008), can be used for aggregating real output growth in the presence of market power in output markets.

Applying a procedure, similar to that used in (31)–(34), to the cost minimization problem in (26), we obtain

$$\frac{\partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t) / \partial \ln x_n}{\sum_{j=1}^{N} \partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t) / \partial \ln x_j} = \frac{w_n x_n}{\sum_{j=1}^{N} w_j x_j}, \quad n = 1, \cdots, N \tag{35}$$

which can be further written as

$$-\frac{\partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial \ln x_n} = \left[-\sum_{j=1}^{N} \frac{\partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial \ln x_j} \right] \frac{w_n x_n}{\sum_{j=1}^{N} w_j x_j}$$

or

$$\varphi_n = \varepsilon(t) \frac{w_n x_n}{\sum_{j=1}^{N} w_j x_j} \tag{36}$$

where the input growth weights, φ_n $(n = 1, \cdots, N)$, are defined by (6) and the local returns to scale, $\varepsilon(t)$, by (22). According to (36), the input growth weight, φ_n, is equal to the product of the degree of returns to scale, $\varepsilon(t)$, and the input cost share, $w_n x_n / \sum_{j=1}^{N} w_j x_j$.

Combining (34) and (36), (4) yields:

$$TC = \sum_{m=1}^{M} \phi_m \dot{y}_m - \sum_{n=1}^{N} \varphi_n \dot{x}_n$$

$$= \sum_{m=1}^{M} \frac{p_m (1-\eta_m) y_m}{\sum_{i=1}^{M} p_i (1-\eta_i) y_i} \dot{y}_m - \varepsilon(t) \sum_{n=1}^{N} \frac{w_n x_n}{\sum_{j=1}^{N} w_j x_j} \dot{x}_n. \tag{37}$$

According to (37), in the presence of imperfect competition in the output markets, the primal Divisia technical change index is equal to a markup-adjusted Divisia real output growth index minus a Divisia real input growth index multiplied by the degree of returns to scale. Put it differently, if there are increasing returns to scale (i.e., $\varepsilon > 1$) and if there is input growth, then the output growth rate will be greater than can be explained by simply adding up input growth and technical progress, due to the multiplication effect of increasing returns to scale on input growth.

The right-hand side of (37) is the markup-adjusted revenue-share based dual Divisia technical change index proposed by Diewert and Fox (2008), expressed in a discrete time Törnqvist form. To see this, let $\sigma_m = p_m (1-\eta_m) y_m / \sum_{i=1}^{M} p_i (1-\eta_i) y_i$ and $\delta_n = w_n x_n / \sum_{j=1}^{N} w_j x_j$. Let

$$\ln Q_T \left(\boldsymbol{p}^{t-1}, \boldsymbol{p}^t, \boldsymbol{y}^{t-1}, \boldsymbol{y}^t\right) = \frac{1}{2} \sum_{m=1}^{M} \left(\sigma_m^{t-1} + \sigma_m^t\right) \left(\ln y_m^t - \ln y_m^{t-1}\right)$$

denote the dual Törnqvist index of output growth and

$$\ln Q_{T^*} \left(\boldsymbol{w}^{t-1}, \boldsymbol{w}^t, \boldsymbol{x}^{t-1}, \boldsymbol{x}^t\right) = \frac{1}{2} \sum_{n=1}^{N} \left(\delta_n^{t-1} + \delta_n^t\right) \left(\ln x_n^t - \ln x_n^{t-1}\right)$$

the dual Törnqvist index of input growth. In Diewert and Fox (2008), $\ln Q_T \left(\boldsymbol{p}^{t-1}, \boldsymbol{p}^t, \boldsymbol{y}^{t-1}, \boldsymbol{y}^t\right)$ and $\ln Q_{T^*} \left(\boldsymbol{w}^{t-1}, \boldsymbol{w}^t, \boldsymbol{x}^{t-1}, \boldsymbol{x}^t\right)$ are related to each other through

$$\ln Q_{T^*} = -\widetilde{\tau} + \widetilde{\rho} \times \ln Q_T \tag{38}$$

where $\widetilde{\tau}$ is a measure of exogenous rate of cost reduction and $\widetilde{\rho}$ is the reciprocal of returns to scale, i.e., $\widetilde{\rho} = 1/\varepsilon$. Multiplying both sides of (38) by ε and rearranging yields[3]

$$TC = \ln Q_T - \varepsilon \times \ln Q_{T^*}$$

[3] Within the cost function framework, technical change is the product of the dual rate of cost diminution and the dual rate of returns to scale; i.e., $TC = \widetilde{\tau} \times \varepsilon$. See Feng and Serletis (2008) and Ohta (1974).

which is just (37). That is, the primal Divisia technical change index and the Diewert and Fox (2008) Divisia technical change index are dual to each other in the presence of imperfect competition in output markets.

In addition to being dual to the Diewert and Fox (2008) markup-adjusted revenue-share based Divisia technical change index, the primal Divisia technical change index is also dual to the Denny *et al.* (1980) and Fuss (1994) cost-elasticity-share based TFPG index under the assumption of constant returns to scale. This can be shown by using the same monopolistic profit-maximization framework as set out above. In particular, multiplying both sides of (30) by $y_m / D_o(\boldsymbol{y}, \boldsymbol{x})$ yields the following:

$$\frac{\partial C(\boldsymbol{y}, \boldsymbol{w}, t)}{\partial y_m} \frac{y_m}{D_o(\boldsymbol{y}, \boldsymbol{x}, t)} = -\lambda \frac{\partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial \ln y_m}, \quad m = 1, \cdots, M. \tag{39}$$

Summing up the M equations in (39) we obtain:

$$\sum_{i=1}^{M} \frac{\partial C(\boldsymbol{y}, \boldsymbol{w}, t)}{\partial y_i} \frac{y_i}{D_o(\boldsymbol{y}, \boldsymbol{x}, t)} = -\lambda \sum_{i=1}^{M} \frac{\partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial \ln y_i}. \tag{40}$$

Dividing (39) by (40) yields:

$$\frac{y_m \partial C(\boldsymbol{y}, \boldsymbol{w}, t) / \partial y_m}{\sum_{i=1}^{M} y_i \partial C(\boldsymbol{y}, \boldsymbol{w}, t) / \partial y_i} = \frac{\partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial \ln y_m}, \quad m = 1, \cdots, M. \tag{41}$$

Dividing the numerator and denominator of the left-hand side of (41) by $C(\boldsymbol{y}, \boldsymbol{w}, t)$ yields:

$$\frac{\partial \ln C(\boldsymbol{y}, \boldsymbol{w}, t) / \partial \ln y_m}{\sum_{i=1}^{M} \partial \ln C(\boldsymbol{y}, \boldsymbol{w}, t) / \partial \ln y_i} = \frac{\partial \ln D_o(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial \ln y_m}, \quad m = 1, \cdots, M \tag{42}$$

where the term on the left-hand side is the Denny *et al.* (1981) and Fuss (1994) cost-elasticity share, used for aggregating output growth in the presence of market power in output markets.

Under the assumption of constant returns to scale [i.e., $\varepsilon(t) = 1$], (36) reduces to

$$\varphi_n = \frac{w_n x_n}{\sum_{j=1}^{N} w_j x_j}. \tag{43}$$

Combining (42) and (43), (4) yields:

$$\begin{aligned} TC &= \sum_{m=1}^{M} \phi_m \dot{y}_m - \sum_{n=1}^{N} \varphi_n \dot{x}_n \\ &= \sum_{m=1}^{M} \frac{\partial \ln D_o(\boldsymbol{y}, \boldsymbol{x})}{\partial \ln y_m} \dot{y}_m - \sum_{n=1}^{N} \frac{w_n x_n}{\sum_{j=1}^{N} w_j x_j} \dot{x}_n \end{aligned} \tag{44}$$

where the right hand side is just the Denny *et al.* (1981) and Fuss (1994) cost-elasticity-share based TFPG index. That is, under the assumptions of imperfect competition in output markets and constant returns to scale, the primal Divisia technical change index is dual to the Denny *et al.* (1981) and Fuss (1994) cost-elasticity-share based TFPG index.

The finding that our primal Divisia TFPG index is dual both to the Diewert and Fox (2008) markup-adjusted revenue-share based Divisia technical change index and to the Denny *et al.* (1981) and Fuss (1994) cost-elasticity-share based Divisia TFPG index is not surprising in that the latter two are actually equal to each other under constant returns to scale. This can be shown again by using the same monopolistic profit-maximization framework as set out above. In particular, multiplying both sides of (27) by $y_m/C(\boldsymbol{y}, \boldsymbol{w}, t)$ and rearranging yields:

$$\frac{p_m(1-\eta_m)\,y_m}{C(\boldsymbol{y}, \boldsymbol{w}, t)} = \frac{\partial \ln C(\boldsymbol{y}, \boldsymbol{w}, t)}{\partial \ln y_m}, \quad m = 1, \cdots, M. \tag{45}$$

Summing up the M equations in (45) yields:

$$\sum_{i=1}^{M} \frac{p_i(1-\eta_i)\,y_i}{C(\boldsymbol{y}, \boldsymbol{w}, t)} = \sum_{i=1}^{M} \frac{\partial \ln C(\boldsymbol{y}, \boldsymbol{w}, t)}{\partial \ln y_i}. \tag{46}$$

Dividing (45) by (46) yields:

$$\frac{p_m(1-\eta_m)\,y_m}{\sum_{i=1}^{M} p_i(1-\eta_i)\,y_i} = \frac{\partial \ln C(\boldsymbol{y}, \boldsymbol{w}, t)/\partial \ln y_m}{\sum_{i=1}^{M} \partial \ln C(\boldsymbol{y}, \boldsymbol{w}, t)/\partial \ln y_i}, \quad m = 1, \cdots, M \tag{47}$$

where the term on the left-hand side is the Diewert and Fox (2008) markup-adjusted revenue share for output m and the one on the right-hand side is the Denny *et al.* (1981) and Fuss (1994) cost-elasticity share for output m. Noting that the product of the Divisia index of input growth and the degree of returns to scale in (36) reduces to the cost share for input n in (43), equation (47) implies that the Diewert and Fox (2008) markup-adjusted revenue-share based technical change index and the Denny *et al.* (1981) and Fuss (1994) cost-elasticity-share based Divisia TFPG index are equal to each other under constant returns to scale.

With regards to the calculation of technical change, when all the output and input prices and quantities are observed, both the primal Divisia technical change index and the Diewert and Fox (2008) dual technical change index can be used. In the special case of constant returns to scale, the Denny *et al.* (1981) and Fuss (1994) cost-elasticity-share based TFPG index can also be used. However, when price information is unavailable or inaccurate, only the primal Divisia technical change index is appropriate. It should be noted here that all three indexes are parametric. In particular, the use of the primal Divisia technical change index proposed in this study involves estimating an output distance function in order to obtain the elasticities of the output distance function with respect to outputs and inputs; the use of the Diewert and Fox (2008) markup-adjusted revenue-share based technical change index entails regressing a Törnqvist output index on a Törnqvist input index in order to calculate markups; and the use of the Denny *et al.* (1981) and Fuss (1994) cost-elasticity-share based TFPG index requires estimating a cost function in order to obtain cost elasticities. This is in contrast with the conventional Jorgenson and Griliches (1967) Divisia TFPG index, the use of which does not involves estimating any technology or economic function, essentially being nonparametric.

4.3 Market Power in Both Output and Input Markets

We now turn to the third case where market power is present in both output and input markets. Like that of monopolistic power, the assumption of monopsonistic power is also compatible with increasing returns to scale. See Small (1999) for a detailed discussion. In particular, we assume that the producers solve the following profit-maximization problem:

$$\max_{\{\boldsymbol{y},\boldsymbol{x}\}} \pi = \left\{ \sum_{m=1}^{M} p_m(y_m)\, y_m - \sum_{n=1}^{N} w_n(x_n)\, x_n,\ D_o(\boldsymbol{y},\boldsymbol{x},t) = 1 \right\}$$

where $p_m(y_m)$ is the inverse demand function for output m, as mentioned earlier, and $w_n(x_n)$ is the inverse supply function for input n.

Using a similar mathematical derivation as above, the primal Divisia TFPG index in this case can be shown to be:

$$TC = \sum_{m=1}^{M} \phi_m \dot{y}_m - \sum_{n=1}^{N} \varphi_n \dot{x}_n$$

$$= \sum_{m=1}^{M} \frac{p_m(1-\eta_m)\, y_m}{\sum_{i=1}^{M} p_i(1-\eta_i)\, y_i} \dot{y}_m - \varepsilon(t) \sum_{n=1}^{N} \frac{w_n(1-\zeta_n)\, x_n}{\sum_{j=1}^{N} w_j(1-\zeta_j)\, x_j} \dot{x}_n \tag{48}$$

where η_m and $\varepsilon(t)$ are defined as above and

$$\zeta_n = -\frac{\partial w(x_n)}{\partial x_n}\frac{x_n}{w_n} \geq 0$$

which is an indirect measure of monopsonistic markdown. According to (48), when market power is present in both output and input markets, the primal Divisia technical change index is equal to a markup adjusted output growth index minus a markdown adjusted input growth index multiplied by the degree of returns to scale, $\varepsilon(t)$. The right-hand side of (48) is a generalization of the Diewert and Fox (2008) markup-adjusted revenue-share based

Divisia technical change index, in the sense that imperfect competition in input markets is also allowed for by replacing input prices, w_n $(n = 1, \cdots, N)$, by markdown-adjusted input prices, $w_n(1 - \zeta_n)$ $(n = 1, \cdots, N)$.

In summary, the duality or equality between the primal Divisia technical change index and the aforementioned well known indexes [i.e., the Jorgenson and Griliches (1967) dual Divisia TFPG index, the Diewert and Fox (2008) markup-adjusted revenue-share (marginal revenue shares) based dual Divisia technical change index, and the Denny *et al.* (1981) and Fuss (1994) cost-elasticity-share based dual Divisia TFPG index] establish the validity of the former index under different market structures.

5 Axiomatic Properties of the Primal Divisia Technical Change Index

In this section we examine whether the primal Divisia technical change index satisfies certain desirable properties. In other words, in addition to justifying the use of the primal Divisia technical change index under different market structures from an economic point of view, we further examine its validity from an axiomatic point of view. Noting that under constant returns to scale the primal Divisia technical change index reduces to a Divisia total factor productivity growth (TFPG) index [see (11), (24), (37) and (48)], the desirable properties that a TFPG index should satisfy can thus also be used to assess the primal Divisia technical change index.

There is a general consensus among researchers that a TFPG index should satisfy five desirable properties: identity, commensurability (dimensional invariance), monotonicity, time reversal, and proportionality. See, for example, Diewert (1992), Diewert and Nakamura (2003), and Orea (2002). The identity property states that if outputs and inputs do not change, then the TFPG/technical change index should remain unchanged. It is apparent from (4) that the primal Divisia technical change index satisfies this property, since both $\dot{y}_m$ and $\dot{x}_n$ are zero when the outputs and inputs do not change.

The commensurability property requires that the TFPG/technical change index be independent of the units of measurement of quantities (prices). The primal Divisia technical change index satisfies this property by construction. More specifically, the distance elasticities and output/input growth rates in (4) are expressed in elasticity or semi-elasticity forms, thus rendering the primal Divisia technical change index independent of the units of measurement of the quantities.

The monotonicity property requires that the TFPG/technical change index be non-decreasing in the output vector and non-increasing in the input vector. An examination of

(4) reveals that the monotonicity property is satisfied if:

$$\frac{\partial \ln D_o(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial \ln y_m(t)} \geq 0;$$

$$\frac{\partial \ln D(\boldsymbol{y}(t), \boldsymbol{x}(t), t)}{\partial \ln x_n(t)} \leq 0$$

which are the monotonicity conditions of the output distance function, since outputs, inputs, and $D_o(\boldsymbol{y}(t),\boldsymbol{x}(t), t)$ are all non-negative.

The time reversal property states that, for two periods 0 and 1, $T(1)/T(0) = T(0)/T(1)$, where T is a cumulative index obtained by integrating the TFPG/technical change index.[4] That is, if the data for periods 0 and 1 are interchanged, then the resulting cumulative index should equal the reciprocal of the original cumulative index. For the case of the primal Divisia technical change index, its corresponding cumulative index for time 1 relative to time 0 can be obtained by integrating (4) as follows:

$$\frac{T(1)}{T(0)} = \frac{\exp\left\{\int_0^1 \left[\sum_{m=1}^M \phi_m \dot{y}_m\right] dt\right\}}{\exp\left\{\int_0^1 \left[\sum_{n=1}^N \varphi_n \dot{x}_n\right] dt\right\}}. \tag{49}$$

Similarly, its corresponding cumulative index for time 0 relative to time 1 can be written as:

$$\frac{T(0)}{T(1)} = \frac{\exp\left\{\int_1^0 \left[\sum_{m=1}^M \phi_m \dot{y}_m\right] dt\right\}}{\exp\left\{\int_1^0 \left[\sum_{n=1}^N \varphi_n \dot{x}_n\right] dt\right\}}. \tag{50}$$

The time reversal property then requires the following:

$$\frac{T(1)}{T(0)} = \frac{T(0)}{T(1)}. \tag{51}$$

It should be noted here that the interchange in data between periods 0 and 1 only reverses the direction of the curve for the line integral without changing the curve itself. See Balk (2005). From a fundamental property of line integrals, the opposite direction rule, we have:

$$\int_0^1 \left[\sum_{m=1}^M \phi_m \dot{y}_m\right] dt = -\int_1^0 \left[\sum_{m=1}^M \phi_m \dot{y}_m\right] dt \tag{52}$$

[4] In the case of a TFPG index, the cumulative index is a cumulative index of total factor productivity.

and

$$\int_0^1 \left[\sum_{n=1}^{N} \varphi_n \dot{x}_n\right] dt = -\int_1^0 \left[\sum_{n=1}^{N} \varphi_n \dot{x}_n\right] dt. \tag{53}$$

Combining (49), (50), (52), and (53), it can be shown that (51) holds.

Finally, the proportionality property states that whenever $(\boldsymbol{x}(1),\boldsymbol{y}(1)) = (\vartheta\boldsymbol{x}(0), \theta\boldsymbol{y}(0))$, where time 0 is treated as the base period, the cumulative index, T, for time 1 relative to time 0 should be equal to θ/ϑ. For the case of the primal Divisia technical change index, the proportionality property requires that its corresponding cumulative index satisfies the following:

$$\frac{T(1)}{T(0)} = \frac{\theta}{\vartheta}, \quad \text{if } (\boldsymbol{x}(1), \boldsymbol{y}(1)) = (\vartheta\boldsymbol{x}(0), \theta\boldsymbol{y}(0)). \tag{54}$$

As we shall see in the next section, the proportionality property holds if the technology is constant returns to scale (i.e. when the primal Divisia technical change index reduces to a TFPG index) and the primal Divisia technical change index is path independent [i.e, the index depends only on the beginning points, ($\boldsymbol{x}(0)$ and $\boldsymbol{y}(0)$), and the end points, ($\boldsymbol{x}(1)$ and $\boldsymbol{y}(1)$) of the line integrals in (49)]. However, the primal Divisia TFP index is path independent only under certain restrictive conditions.

6 Path Independence of the Primal Divisia Technical Change Index

Path independence has been a very important issue in the literature regarding the Divisia index. Its necessary and sufficient conditions were systematically explored by Hulten (1973) and further discussed in many later works. For example, Balk (2005) establishes the necessary and sufficient conditions for path independence of the Divisia price index in the consumer context. Since the primal Divisia technical change index is different from the conventional Divisia quantity and price indexes examined in previous studies, here we formally derive the necessary and sufficient conditions for path independence. It should be noted here that path independence of the primal Divisia technical change index requires that both the line integral for inputs

$$\int_0^1 \left[\sum_{n=1}^{N} \varphi_n \dot{x}_n\right] dt \tag{55}$$

and the line integral for outputs

$$\int_0^1 \left[\sum_{m=1}^{M} \phi_m \dot{y}_m\right] dt \tag{56}$$

be independent of their respective curves for integration. In this section use the output distance function defined in (2) and drop the subscript t for the output distance function.

The necessary and sufficient conditions for path independence of the two line integrals in (55) and (56) (or equivalently, the primal Divisia technical change index) can be obtained by modifying Corollary 1 of Hulten (1973).[5]

Corollary.

(i) *The output distance function, $D_o(\boldsymbol{y}(t), \boldsymbol{x}(t))$, is weakly separable into a function of the M outputs, $y_1(t), \cdots, y_M(t)$, and a function of the N inputs, $x_1(t), \cdots, x_N(t)$. Formally, there exist continuously differentiable functions g, f, and h such that*

$$D_o(y_1(t), \cdots, y_M(t), x_1(t), \cdots, x_N(t)) = g\Big(f(y_1(t), \cdots, y_M(t)), h(x_1(t), \cdots, x_N(t))\Big)$$

$$= g\Big(f(\boldsymbol{y}(t)), h(\boldsymbol{x}(t))\Big) \tag{57}$$

(ii) *f is homogeneous of degree one in y_m ($m = 1, \cdots, M$) and h is homogeneous of degree $\varepsilon(t)$ in x_n ($n = 1, \cdots, N$), where $\varepsilon(t)$ is the local returns to scale at time t defined by* (22).

Proof.

Sufficiency. We will concentrate our attention on the proof that conditions (i) and (ii) imply path independence of the line integral for inputs defined in (55). The proof that conditions (i) and (ii) imply path independence of the line integral for outputs defined in (56) is analogous.

To prove that conditions (i) and (ii) imply path independence of the line integral for inputs, we first show that $\varphi_n = \partial \ln h(\boldsymbol{x}(t)) / \partial \ln x_n(t)$. More specifically, condition (i)

[5] It should be noted that condition (iii) in Hulten (1973), the price normal uniqueness condition, is redundant for the case of the primal Divisia technical change index and thus it is ignored here. This is because market prices have been replaced with the elasticities of outputs and inputs and thus condition (iii) in Hulten (1973) always holds for the case of the primal Divisia technical change index.

implies that

$$\varphi_n = -\frac{\partial \ln D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)}{\partial \ln x_n(t)}$$

$$= -\frac{\partial D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)}{\partial x_n(t)} \frac{x_n(t)}{D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)}$$

$$= -\frac{\partial D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)}{\partial h(x(t))} \frac{\partial h(x(t))}{\partial x_n(t)} \frac{x_n(t)}{D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)}$$

$$= -\frac{\partial D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)}{\partial h(x(t))} \frac{\partial h(x(t))}{\partial x_n(t)} \frac{x_n(t)}{D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)} \frac{h(x(t))}{h(x(t))}$$

$$= -\left[\frac{\partial h(x(t))}{\partial x_n(t)} \frac{x_n(t)}{h(x(t))}\right] \left[\frac{\partial D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)}{\partial h(x(t))} \frac{h(x(t))}{D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)}\right]. \tag{58}$$

On the other hand, the homogeneity of degree $\varepsilon(t)$ of $h(x(t))$ in inputs [that is, the second part of condition (ii)] implies that the output distance function is homogenous of degree minus one in $h(x(t))$. This can be shown as follows. Multiplying both sides of (22) by $D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)$, we get

$$\varepsilon(t) D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right) = -\sum_{n=1}^{N} \frac{\partial D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)}{\partial x_n} x_n \tag{59}$$

which implies that $D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)$ is homogenous of degree $-\varepsilon(t)$ in inputs. This in turn implies that

$$g\left(f\left(\boldsymbol{y}(t)\right), h\left(\rho \boldsymbol{x}(t)\right)\right) = \rho^{-\varepsilon} g\left(f\left(\boldsymbol{y}(t)\right), h\left(\boldsymbol{x}(t)\right)\right) \tag{60}$$

where ρ is a scalar. The homogeneity of degree $\varepsilon(t)$ of $h(x(t))$ in inputs implies that

$$g\left(f\left(\boldsymbol{y}(t)\right), h\left(\rho \boldsymbol{x}(t)\right)\right) = g\left(f\left(\boldsymbol{y}(t)\right), \rho^{\varepsilon} h\left(\boldsymbol{x}(t)\right)\right). \tag{61}$$

Equations (60) and (61) imply that $D_o\left(y_1(t), \cdots, y_M(t), x_1(t), \cdots, x_N(t)\right) = g\left(f\left(\boldsymbol{y}(t)\right), h\left(\boldsymbol{x}(t)\right)\right)$ is homogenous of degree minus one in $h(x(t))$. Thus, we have

$$\frac{\partial D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)}{\partial h} \frac{h(x(t))}{D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)} = \frac{\partial D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right)}{\partial h} h(x(t)) = -1 \tag{62}$$

where the first equality is obtained by noting that $D_o\left(\boldsymbol{y}(t), \boldsymbol{x}(t)\right) = 1$. Substituting (62) into (58) yields

$$\varphi_n = \frac{\partial h(x(t))}{\partial x_n(t)} \frac{x_n(t)}{h(x(t))} = \frac{\partial \ln h\left(\boldsymbol{x}(t)\right)}{\partial \ln x_n(t)} \tag{63}$$

according to which φ_n can be expressed in terms of the elasticities of the function h with respect to inputs.

With (63) in mind, the line integral for outputs in (55) can be written as

$$\int_0^1 \left[\sum_{n=1}^{N} \varphi_n \dot{x}_n\right] dt = \int_0^1 \left[\sum_{n=1}^{N} \frac{\partial \ln h(\boldsymbol{x}(t))}{\partial \ln x_n(t)} \frac{d \ln x_n(t)}{dt}\right] dt$$

$$= \int_0^1 \left[\sum_{n=1}^{N} \frac{d \ln h(\boldsymbol{x}(t))}{dt}\right] dt$$

$$= \ln h(\boldsymbol{x}(1)) - \ln h(\boldsymbol{x}(0)) \tag{64}$$

according to which the line integral in (55) is path independent in that it depends only on the beginning point [i.e., $\boldsymbol{x}(0)$] and the end point [i.e., $\boldsymbol{x}(1)$] of the curve for integration. It also implies that $\ln h(\boldsymbol{x}(t))$ is an eligible potential function for this line integral.

In a similar manner, we can show that the line integral in (56) is also path independent in that it depends only on the beginning point, $\boldsymbol{y}(0)$, and the end point, $\boldsymbol{y}(1)$, of the curve for integration, and that $\ln f(\boldsymbol{y}(t))$ is an eligible potential function for the latter line integral. This pair of path independence thus implies that the primal Divisia technical change index is path independent.

Necessity. We will show that path independence implies conditions (i) and (ii). We first show that path independence implies condition (i). As already noted, path independence of the primal Divisia technical change index implies that both the line integral for outputs and that for inputs are path independent. With this in mind, condition (i) can then be established by contradiction, as in Hulten (1973). To see this, suppose that the output distance function is not weakly separable in the manner implied by (57). Then either the output distance function is not separable for any partition of the $M + N$ variables, or it is weakly separable for a different partition of the variables, say, $(K, M+N-K)$. In the former case, there is clearly no potential function defined on R^N for the line integral for inputs (or no potential function defined on R^M for the line integral for outputs); hence the line integral for inputs (or that for outputs) is path dependent, by Hulten's (1973) Potential Function Theorem. In the latter case, there is still no potential function on R^N for inputs (or R^M for outputs), although there are two potential functions, with one defined on R^K and the other defined on R^{M+N-K}. Therefore, the line integral for inputs (or that for outputs) is again path dependent. Thus, by contradiction, the output distance function is weakly separable in the manner implied by (57).

We now turn to show that path independence of the primal Divisia technical change index implies condition (ii). Again, we will concentrate our attention on the proof that path independence of the line integral in (55) implies that $h(\boldsymbol{x}(t))$ is homogenous of degree $\varepsilon(t)$ in inputs. In particular, the path independence of the line integral in (55) implies that there exists a potential function, denoted by $\Phi(\boldsymbol{x}(t))$, for this line integral. Define $h(\boldsymbol{x}(t)) = \exp(\Phi(\boldsymbol{x}(t)))$. It can be shown that

$$\Phi_n(\boldsymbol{x}(t)) = \frac{\partial \ln h(\boldsymbol{x}(t))}{\partial x_n(t)} = \frac{h_n(\boldsymbol{x}(t))}{h(\boldsymbol{x}(t))} \tag{65}$$

where the subscript n for $\Phi(\boldsymbol{x}(t))$ indicates the partial derivative with respect to x_n, that is, $\Phi_n(\boldsymbol{x}(t)) = \partial\Phi(\boldsymbol{x}(t))/\partial x_n(t)$. By Hulten's (1973) Potential Function Theorem, we can verify that

$$\Phi_n(\boldsymbol{x}(t)) = -\frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t))}{\partial x_n(t)}. \tag{66}$$

Combining (65) and (66) yields:

$$\frac{h_n(\boldsymbol{x}(t))}{h(\boldsymbol{x}(t))} = -\frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t))}{\partial x_n(t)}. \tag{67}$$

Multiplying both sides of (67) by x_n, and summing over N gives

$$\begin{aligned}\sum_{n=1}^{N} h_n(\boldsymbol{x}(t))\, x_n &= \left[-\sum_{n=1}^{N} \frac{\partial D_o(\boldsymbol{y}(t), \boldsymbol{x}(t))}{\partial x_n(t)} x_n(t)\right] h(\boldsymbol{x}(t)) \\ &= \varepsilon(t) \cdot h(\boldsymbol{x}(t))\end{aligned} \tag{68}$$

where the last equality is obtained by using the definition of the local returns to scale in (22). By Euler's homogeneous function theorem, (68) implies that $h(\boldsymbol{x}(t))$ is positive homogeneous of degree $\varepsilon(t)$ in inputs. In a similar manner, we can show that path independence of the primal Divisia technical change index implies that $f(\boldsymbol{y}(t))$ is linearly homogenous in outputs, that is

$$\sum_{m=1}^{M} f_m(\boldsymbol{y}(t))\, y_m = f(\boldsymbol{y}(t)) \tag{69}$$

where the subscript m for $f(\boldsymbol{y}(t))$ indicates the partial derivative with respect to y_m. Therefore, (68) and (69) together establish condition (ii). ■

Conditions (i) and (ii) can be understood at a more intuitive level. In particular, the input and output separability condition in (i) guarantees the existence of the output-distance-function based input and output indexes. The linear homogeneity of $f(\boldsymbol{y}(t))$ in outputs guarantees that if all the outputs change proportionally, the output index changes by the same factor of proportionality, and the homogeneity of degree $\varepsilon(t)$ of $h(\boldsymbol{x}(t))$ in inputs ensures that if all the inputs change proportionally by a factor ϑ the input index changes by a factor ϑ^{ε}. Intuitively, the homogeneity property of h means that if there are increasing returns to scale (i.e. $\varepsilon > 1$) and if there is input growth, then the output growth rate will be greater than can be explained by simply adding up input growth and technical progress, due to the multiplication effect of increasing returns to scale on input growth.

Having proved that conditions (i) and (ii) are the necessary and sufficient conditions for path independence, we now turn to show that the proportionality property discussed in Section 5 holds if the primal Divisia technical change index is path dependent and $\varepsilon = 1$. Noting that $\ln h(\boldsymbol{x}(t))$ and $\ln f(\boldsymbol{y}(t))$ are the potential functions for $\int_0^1 \left[\sum_{n=1}^{N} \varphi_n \dot{x}_n\right] dt$ and $\int_0^1 \left[\sum_{m=1}^{M} \phi_m \dot{y}_m\right] dt$, respectively (see above), the ratio of $T(1)$ to $T(0)$ can be written as follows:

$$\frac{T(1)}{T(0)} = \frac{\exp\left\{\int_0^1 \left[\sum_{m=1}^{M} \phi_m \dot{y}_m\right] dt\right\}}{\exp\left\{\int_0^1 \left[\sum_{n=1}^{N} \varphi_n \dot{x}_n\right] dt\right\}}$$

$$= \frac{\exp\left\{\ln f(\boldsymbol{y}(1)) - \ln f(\boldsymbol{y}(0))\right\}}{\exp\left\{\ln h(\boldsymbol{x}(1)) - \ln h(\boldsymbol{x}(0))\right\}} = \frac{f(\boldsymbol{y}(1))/f(\boldsymbol{y}(0))}{h(\boldsymbol{x}(1))/h(\boldsymbol{x}(0))}. \tag{70}$$

When $(\boldsymbol{x}(1), \boldsymbol{y}(1)) = (\vartheta\boldsymbol{x}(0), \theta\boldsymbol{y}(0))$, (70) can be further written as:

$$\frac{T(1)}{T(0)} = \frac{f(\theta\boldsymbol{y}(0))/f(\boldsymbol{y}(0))}{h(\vartheta\boldsymbol{x}(0))/h(\boldsymbol{x}(0))} = \frac{\theta f(\boldsymbol{y}(0))/f(\boldsymbol{y}(0))}{\vartheta^{\varepsilon} h(\boldsymbol{x}(0))/h(\boldsymbol{x}(0))} = \frac{\theta}{\vartheta^{\varepsilon}} \tag{71}$$

where the second last equality is obtained by using the homogeneity properties of $f(\boldsymbol{y}(t))$ and $h(\boldsymbol{x}(t))$, stated in condition (ii). According to the definition in (54), (71) shows that the proportionality property holds when the primal Divisia technical change index is path independent and $\varepsilon = 1$ (i.e. constant returns to scale).

It should be noted here, however, that both conditions (i) and (ii) are very restrictive. Specifically, the separability condition in (i) implies that the marginal rate of transformation between any two outputs is independent of all inputs and that the marginal rate of technical substitution between any two inputs is also independent of all outputs. See, for example, Chambers (1988). The linear homogeneity of $f(\boldsymbol{y}(t))$ in outputs and the homogeneity of degree ε of h in inputs in condition (ii) mean that additional structure must be imposed on the output distance function in order to yield path independence for the case of the primal Divisia technical change index.

In summary, the primal Divisia technical change index satisfies the properties of identity, commensurability, monotonicity, and time reversal; and also satisfies the property of proportionality if the primal Divisia TFP index is path independent and the degree of returns to scale is one. Path independence of the primal Divisia TFP index in turn requires separability between inputs and outputs and homogeneity of subaggregator functions.

7 Conclusion

The previous Divisia-type TFPG or technical change indexes have an apparent practical problem in that it requires information on both output and input prices, which is unavailable or distorted in many situations. In those situations it is desirable to use a primal Divisia TFPG or technical change index which requires only quantity information. To this end, in this paper we formally derive the distance-elasticity-based primal Divisia technical change index by differentiating the output distance function totally with respect to a time trend, and show the validity of this index from both economic and axiomatic points of view.

In particular, we show the validity of this primal Divisia technical change index under three different market structures. In the first case, where both the output and input markets are assumed to be perfectly competitive, we exploit the duality between the output distance function and the profit function and show that this index is dual to the Jorgenson and Griliches (1967) dual Divisia TFPG index. In the second case where market power is limited to output markets only, we show that the primal Divisia technical change index is dual to the Diewert and Fox (2008) markup-adjusted revenue-share based technical change index and also dual to the Denny *et al.* (1981) and Fuss (1994) cost-elasticity-share based TFPG index under constant returns to scale. We do so, by solving the Diewert and Fox (2008) monopolistic profit-maximization problem. In the third case where market power is present in both output and input markets, we show that the primal Divisia technical change index is dual to a markup-and-markdown adjusted Divisia technical change index, by solving the monopolistic-monopsonistic profit-maximization problem subject to the technology constraint represented by the output distance function.

We finally show that the primal Divisia technical change index satisfies the properties of identity, commensurability, monotonicity, and time reversal. It also satisfies the property of proportionality under the assumptions of constant returns to scale and path independence, where the latter assumption requires separability between inputs and outputs and homogeneity of subaggregator functions.

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