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Minimum Variance Unbiased Maximum Likelihood Estimation of the Extreme Value Index

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MINIMUM VARIANCE UNBIASED MAXIMUM LIKELIHOOD ESTIMATION OF THE EXTREME VALUE INDEX

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SUMMARY

New results for ratios of extremes from distributions with a regularly varying tail at infinity are presented. If the appropriately normalized order statistic $X_{(n-j+1)}$ from distribution with tail exponent δ, $(\delta > 0)$ converges weakly to $X^*_{(j)}$ then:

(i) For $\rho = 1/\delta$, the ratio $X^*_{(j)}/X^*_{(j+k)}$ is distributed like $\{U(j,k)\}^{-\rho}$ where $U(j,k)$ has density

$$B(u{:}j,k) = B^{-1}(j,k)u^{j-1}(1-u)^{k-1},\ \ 0 \le u \le 1$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, $a > 0$, $b > 0$, $\Gamma(\cdot)$ denoting the gamma function,

(ii) Consecutive ratios of extremes $X^*_{(1)}/X^*_{(2)}, X^*_{(2)}/X^*_{(3)} \cdots X^*_{(m)}/X^*_{(m+1)} \cdots$ are independently distributed.

(iii) The maximum likelihood estimator (mle) based on the first k ratios is

$$\hat{\rho} = k^{-1} \sum_{m=1}^{k} m \log\{X^*_{(m)}/X^*_{(m+1)}\}$$

(iv) The mle $\hat{\rho}$ is unbiased, has variance ρ^2/k (the Cramer-Rao minimum variance bound) and is asymptotically normally distributed.

(v) It has moments of all orders and moment generating function

$$M_\rho(\theta) = (1-\rho\theta/k)^{-k},\ \theta>0.$$

KEYWORDS: Tail-index, Minimum variance unbiased, Maximum likelihood, Asymptotically normal

JEL CLASSIFICATION: C13

1. INTRODUCTION AND MOTIVATION

Considerable research effort has been devoted over the last thirty years to estimation of the exponent of regular variation, alternatively described depending on context, as the tail-index or the extreme value index.
Popular estimators based on independent random samples such as those due to Hill (1975), Pickands (1975) and maximum likelihood estimators of Smith (1987) and Drees *et al* (2004) are consistent, asymptotically normal (i.e. biased) estimators. They are beset by an inherent dilemma; large variability when the number of extremes is too low, large bias when it is too high. A survey of bias problems is contained in Beirlant *et al*. (1999). Bias reduction of tail-index estimators, including maximum likelihood estimators has generally been approached by applying second order properties of regularly varying functions to the tail-quantile function of the distribution. Invariably the bias depends on the unknown exponent, and is distribution-specific (e.g. Drees *et al*. (2004), Teugels and Vanroelen (2004)). Another approach focuses on optimal threshold selection, trading off bias reduction against variability (e.g. Matthys, (2001)).

'*Nowadays, . . . applications of extreme value theory can be seen in a large variety of fields such as hydrology, engineering, economics, astronomy and finance. The estimation of the extreme value index is the first and main statistical challenge'* (Teugels and Vanroelen, (2004).

2. MAIN RESULTS

The class **F** of distributions F(·) have a regularly varying tail with index δ (equivalently belong to the maximum domain of attraction of the Frechet distribution, (e.g. Embrechts et al. (1997), p.131)) if $1\text{-}F(x) = L(x)x^{-\delta}$, $x > 0$, $\delta > 0$.
The function L(x) is slowly varying at infinity (see for instance Feller, (1971), p.276).

Theorem 1 (distribution of a ratio of extremes)

Denote by $X_{(1)}, X_{(2)}, \cdot\ \cdot\ \cdot X_{(n)}$ ascending order statistics of common parent F ε **F**.
The variables $X^*_{(1)}, X^*_{(2)}, \cdots$ are descending Frechet extremes, i.e $X^*_{(j)}$ is the weak limit of normalized order statistic $X_{(n-j+1)}/v_n$ from the parent F ε **F**, the normalizing sequence $\{v_n\}$obtained from the parent *tail-quantile function*, satisfying $n[1-F(v_n)]=1$,
(see for instance, David and Nagaraja, (1993), Chapter 10).

Then for any F ε **F**,

$$\left[\frac{X^*_{(j)}}{X^*_{(j+k)}}\right] = \{U^*(j,k)\}^{-\rho}$$

where $U^*(j,k)$ has beta density $\mathbf{B}(u{:}j,k) = B^{-1}(j,k)u^{j-1}(1-u)^{k-1}$, $(0 \le u \le 1)$ and where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, $(a > 0, b>0)$, $\Gamma(\cdot)$ representing the gamma function.

Proof (see Appendix, Note 1)

Theorem 2 (independence of 'consecutive' ratios of extremes)

Using the same notation as in Theorem 1

For any F ε **F**,

$$\left[\frac{X^*_{(1)}}{X^*_{(1+j)}}\right] = \left[\frac{X^*_{(1)}}{X^*_{(2)}}\right] \times \left[\frac{X^*_{(2)}}{X^*_{(3)}}\right] \times \cdots \left[\frac{X^*_{(j)}}{X^*_{(j+1)}}\right]$$

where the right side is a product of *independent* $\{U(m,1)\}^{-\rho}$ variables, the U(m,1) having beta density $\mathbf{B}(u{:}m,1) = mu^{m-1}, (0 \leq u \leq 1)$.

Proof (see Appendix, Note 1)

Theorem 3 (minimum variance maximum likelihood estimation based on ratios of extremes)

(i) The maximum likelihood estimator $\hat{\rho}$ based on the k observed ratios $Y_m = X^*_{(m)}/X^*_{(m+1)}$ m = 1,2, · · · k is given by

$$\hat{\rho} = k^{-1} \sum_{m=1}^{k} m \log \{X^*_{(m)}/X^*_{(m+1)}\}$$

(ii) It is unbiased and with variance ρ^2/k and is asymptotically normally distributed. The variance ρ^2/k is the Cramer-Rao minimum variance bound for ρ.

(iii) It has moment generating function $M_\rho(\theta) = (1-\rho\theta/k)^{-k}, \theta > 0$.

Proof

(i) Note that $L = \prod_{m=1}^{k} f_m(y_m)$ while $\ell = \ln L$

$$= \text{const.} + k\ln\delta - \sum_{m=1}^{m} (m\delta + 1) \ln y_m$$

since $X^*_{(m)}/X^*_{(m)}$ has distribution of $\{U(m,1)\}^{-\rho}$ where U(m,1) has density $\mathbf{B}(y_m{:}m,1)$; i.e. $f_m(y_m) = m\delta(y_m)^{-m\delta-1}$

Thus

$$\frac{\partial \ell}{\partial \delta} = k\rho - \sum_{m=1}^{m} m \ln y_m$$

i.e. $$\hat{\rho} = k^{-1} \sum_{m=1}^{k} m \ln \{X^*_{(m)}/X^*_{(m+1)}\}$$

Note that $E[-\frac{\partial^2 \ell}{\partial \delta^2}] = k\rho^2$

(ii) Since $Y_m = X^*_{(m)}/X^*_{(m+1)}$ is distributed like $\{U(m,1)\}^{-\rho}$ where U(m,1) has density $B(u{:}m,1) = mu^{m-1}$, (integer $m \geq 1$),

$$E[\{m\ln(X^*_{(m)}/X^*_{(m+1)})\}^j] = \Gamma(j+1)\rho^j, \text{ (integer } j \geq 1)$$

Note that $I_j = E[\{m\ln(X^*_{(m)}/X^*_{(m+1)})\}^j]$

$= (-\rho)^j E[\{m\ln U(m,1)\}^j]$ since $X^*_{(m)}/X^*_{(m+1)}$ is distributed like $\{U(m,1)\}^{-\rho}$

$= (-\rho)^j(-j)E[\{m\ln U(m,1)\}^{j-1}]$

$= j\rho(-\rho)^{j-1} E[\{m\ln U(m,1)\}^{j-1}]$

$= j\rho I_{j-1}$ while $I_0 = 1$

leading to $E[\{m\ln(X^*_{(m)}/X^*_{(m+1)})\}^j] = \Gamma(j+1)\rho^j$, (integer $j \geq 1$)

In particular, $E[\{m\ln(X^*_{(m)}/X^*_{(m+1)})\}] = \rho$

$E[\{m\ln(X^*_{(m)}/X^*_{(m+1)})\}^2] = 2\rho^2$ (so that $\mathrm{Var}[m\ln(X^*_{(m)}/X^*_{(m+1)})] = \rho^2$

$E[\{m\ln(X^*_{(m)}/X^*_{(m+1)})\}^3] = 6\rho^3$

From these results,

$$E[\hat{\rho}] = k^{-1}E[\sum_{m=1}^{k} m \ln \{X^*_{(m)}/X^*_{(m+1)}\}] = \rho$$

$$\mathrm{Var}[\hat{\rho}] = k^{-2}\,\mathrm{Var}[\sum_{m=1}^{k} m \ln \{X^*_{(m)}/X^*_{(m+1)}\}] = \rho^2/k$$

Since $E[\{m\ln(X^*_{(m)}/X^*_{(m+1)})\}^3] = 6\rho^3 < \infty$, asymptotic normality follows for example from Liapunov's Central Limit Theorem, (Rao, (1973), p.127).

Since $E[-\frac{\partial^2 \ell}{\partial \delta^2}] = k\rho^2$, the Cramer-Rao minimum variance bound for $g(\delta) = \delta^{-1}$ is given by

$\mathrm{Var}[\hat{\rho}] \geq \{g'(\delta)\}/ E[-\frac{\partial^2 \ell}{\partial \delta^2}] = \rho^4/k\rho^2 = \rho^2/k$,

(iii) Recalling the independence of the ratios $\{X^*_{(m)}/X^*_{(m+1)}\}$, $m = 1,2, \cdots k$, the moment generating function of $\hat{\rho}$ is

$$E[\exp(\theta\hat{\rho})] = E[\prod_{m=1}^{k}\{X^*_{(m)} / X^*_{(m+1)}\}^{m\vartheta/k}]$$

$$= (1-\rho\theta/k)^{-k}$$

Since $E[\{\{X^*_{(m)}/X^*_{(m+1)}\}^{m\theta/k}] = E[\{U(m,1)\}^{-\rho m\theta/k}]$ where $U(m,1)$ has density $B(u:m,1) = mu^{m-1}$, $(0 \leq u \leq 1)$, its expectation is $(1-\rho\theta/k)^{-1}$.

This completes the proof.

3. SUMMARY AND CONCLUSIONS

Fundamental new results for heavy-tailed extremes are formulated in terms of ratios of extremes. Such ratios are non-distribution specific, being independent of normalizing constants.
Individual ratios in the form $\{X^*_{(j)}/X^*_{(j+k)}\}$ are distributed like powers of beta variates.
'Consecutive' ratios $X^*_{(1)}/X^*_{(2)}$, $X^*_{(2)}/X^*_{(3)}$, $\cdots$ are independently distributed.
These results affect considerable simplification in dealing with samples of extremes.
One consequence is that unbiased estimators of the tail-index are available.
The maximum likelihood estimator $\hat{\rho} = k^{-1} \sum_{m=1}^{k} m \ln \{X^*_{(m)}/X^*_{(m+1)}\}$ based on k observed ratios is itself unbiased. It achieves minimum variance ρ^2/k among unbiased estimators, and is asymptotically normal. Moreover $\hat{\rho}$ has moments of all orders, with moment generating function $M_\rho(\theta) = (1-\rho\theta/k)^{-k}$.

APPENDIX

Note 1: **Theorem 1** (The distribution of a ratio of extremes)

Denote by $X_{(1)}, X_{(2)}, \cdots X_{(n)}$ ascending order statistics of common parent $F \varepsilon \mathbf{F}$.

The variables $X^*_{(1)}, X^*_{(2)}, \cdots$ are descending Frechet extremes, i.e $X^*_{(j)}$ is the weak limit of normalized order statistic $X_{(n-j+1)}/v_n$ from the parent F ε **F**, the normalizing sequence $\{v_n\}$obtained from the parent *tail-quantile function*, satisfying $n[1-F(v_n)]=1$, (see for instance, David and Nagaraja, (1993), Chapter 10).

Then for any F ε **F**,

$$\left[\frac{X^*_{(j)}}{X^*_{(j+k)}}\right] = \{U^*(j,k)\}^{-\rho}$$

where $U^*(j,k)$ has beta density $\mathbf{B}(u{:}j,k) = B^{-1}(j,k)u^{j-1}(1-u)^{k-1}$, $(0 < u < 1)$ and where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, $(a >0, b>0)$, $\Gamma(\cdot)$ representing the gamma function.

Proof:

For the order statistics from common distribution F ε **F** and corresponding density $f(\cdot)$ the joint density of $(X_{(n-j-k+1)}, X_{(n-j+1)})$, $(j < k)$ denoted by $f^{\#}(x_{(n-j-k+1)},x_{(n-j+1)})$ is

$$\mathbf{B}(u{:}j,k) \times \mathbf{B}(v{:}j+k,n-j-k+1) = B^{-1}(j, k)u^{j-1}(1-u)^{k-1} \times B^{-1}(j+k,n-j-k+1)\, v^{j+k-1}(1-v)^{n-j-k}$$

i.e. that of *independent* random variables U and V deriving from transformations (see for instance, Arnold et al. (1993), Chapter 2).

(i) $u(x_{(n-j-k+1)}, x_{(n-j+1)}) = \{1-F(x_{(n-j+1)})\}/\{1-F(x_{(n-j-k+1)})\}$ u ε [0,1]

(ii) $v(x_{(n-j-k+1)}, x_{(n-j+1)}) = x_{(n-j-k+1)}$

Note that $\left|\frac{\partial(u,v)}{\partial(x_{(n-j-k+1)},x_{(n-j+1)})}\right| = \{f(x_{(n-j+1)})/(1-F(x_{(n-j-k+1)})\}\times f(x_{(n-j-k+1)})$

where '|·|' represents absolute value.

The implication of the transformation:

$$u(x_{(n-j-k+1)}, x_{(n-j+1)}) = \{1-F(x_{(n-j+1)})\}/\{1-F(x_{(n-j-k+1)})\}$$

for F ε **F**, i.e. $1-F(x) = L(x)x^{-\delta}$ when n is large, can be determined as follows:

Put: (i) $x_{(n-j+1)} = v_n x^*_{(j)}$

(ii) $x_{(n-j-k+1)} = v_n x^*_{(j+k)}$

Note that for $1-F(x) = L(x)x^{-\delta}$, $v_n^{\delta} = nL(v_n)$

and $u = \{1-F(x_{(n-j+1)})\}/\{1-F(x_{(n-j-k+1)})\}$

i.e. $u = L(v_n x^*_{(j)}) v_n^{-\delta} (x^*_{(j)})^{-\delta} / L(v_n x^*_{(j+k)}) v_n^{-\delta} (x^*_{(j+k)})^{-\delta}$

$= (x^*_{(j)})^{-\delta} / (x^*_{(j+k)})^{-\delta}$

using for example $L(v_n x^*_{(j)})/ L(v_n) \to 1$ as $n \to \infty$.

Thus the u-transformation implies for large n that

(A.1) $$\{U^*(j, k)\}^{-\rho} = \left[\frac{X^*_{(j)}}{X^*_{(j+k)}}\right]$$

where $U^*(j,k)$ has density **B**(u:j, k), independent of $X^*_{(j+k)}$.

Check: The independence can be checked by showing equivalence of the moments of

$\{X^*_{(j)}\}^{\phi}$ and those of $\{X^*_{(j)}/X^*_{(j+k)}\}^{\phi} \times \{X^*_{(j+k)}\}^{\phi}$ on any dense ϕ-set interior to $(0, j-\rho\phi)$, $j-\rho\phi>0$.

Note that $E[\{X^*_{(j)}\}^{\phi}] = \Gamma(j-\rho\phi)/\Gamma(j)$

This is the same as $E\{X^*_{(j)}/X^*_{(j+k)}\}^{\phi} \times E\{X^*_{(j+k)}\}^{\phi} = E[\{U(j,k)\}^{-\rho\phi}] \times E\{X^*_{(j+k)}\}^{\phi}]$

$= \{B(j-\rho\phi,k)/B(j,k)\} \times \Gamma(j+k-\rho\phi)/\Gamma(j+j)$

$= \Gamma(j-\rho\phi)/\Gamma(j)$

This completes the proof of Theorem 1

Note 2: **Theorem 2** (independence of consecutive ratios of extremes)

Using the same notation as in Theorem 1

For any F ε **F**,

(A.2) $$\left[\frac{X^*_{(1)}}{X^*_{(1+j)}}\right] = \left[\frac{X^*_{(1)}}{X^*_{(2)}}\right] \times \left[\frac{X^*_{(2)}}{X^*_{(3)}}\right] \times \cdots \left[\frac{X^*_{(j)}}{X^*_{(j+1)}}\right]$$

where the right side is a product of *independent* $\{U^*(m,1)\}^{-\rho}$ variables, the $U^*(m,1)$ being beta (m,1) random variables, for m = 1,2, · · · j.

Proof

We re-write product (A.2) as

$$\text{(A.3)} \qquad \left[\frac{X^*_{(1)}}{X^*_{(1+j)}}\right]^{-\delta} = \left[\frac{X^*_{(1)}}{X^*_{(2)}}\right]^{-\delta} \times \left[\frac{X^*_{(2)}}{X^*_{(3)}}\right]^{-\delta} \times \cdots \times \left[\frac{X^*_{(j)}}{X^*_{(1+j)}}\right]^{-\delta}$$

Each of the ratios on the right side is a B(m,1) variable, m = 1, 2, · · · j, using Theorem 1. Their *independence* then follows using Kendall and Stuart (1969), Exercise 11.8, and noting that the left side cannot be beta if the betas on the right side are not independent An alternative proof by equivalence of moments on a dense set (as outlined in the proof of Theorem 1) is also available using the uniqueness of moments for distributions (like beta) confined to closed intervals (see for instance Feller (1971) p.227).

In fact Theorem 2 also succumbs to a proof based on extending the proof of Theorem 1 as follows:

Consider the joint distribution of $X_{(n-i-j-k+1)}$, $X_{(n-i-j+1)}$ and $X_{(n-i+1)}$ jointly distributed order statistics with common parent $F(\cdot) \varepsilon F$ with density denoted by $f^{\#}$, i.e.

$$f^{\#}(x_{(n-i-j-k+1)}, x_{(n-i-j+1)}, x_{(n-i+1)})$$

$$= \Gamma(n+1)/\{\Gamma(n-i-j-k+1)\Gamma(k)\Gamma(j)\Gamma(i) \times F(x_{(n-i-j-k+1)})^{n-i-j-k} \times f(x_{(n-i-j-k+1)}) \times$$

$$\{F(x_{(n-i-j+1)})-F(x_{(n-i-j-k+1)})\}^{k-1} \times f(x_{(n-i-j+1)}) \times$$

$$\{F(x_{(n-i+1)})-F(x_{(n-i-j+1)})\}^{j-1} \times f(x_{(n-i+1)}) \times$$

$$\{1-F(x_{(n-i+1)})\}^{i-1}$$

This can be re-written as

$$f^{\#}(x_{(n-i-j-k+1)}, x_{(n-i-j+1)}, x_{(n-i+1)}) =$$

$$\Gamma(n+1)/\{\Gamma(i+j+k)\Gamma(n-i-j-k+1)\} \times F(x_{(n-i-j-k+1)})^{n-i-j-k} \times \{1-F(x_{(n-i-j-k+1)})^{i+j+k-1} \times f(x_{(n-i-j-k+1)}) \times$$

$$\Gamma(i+j+k)/\{\Gamma(k)\Gamma(i+j)\} \times [1-\{(1-F(x_{(n-i-j+1)})/(1-F(x_{(n-i-j-k+1)})\}^{k-1}] \times$$
$$\{(1-F(x_{(n-i-j+1)})/(1-F(x_{(n-i-j-k+1)})\}^{i+j-1} \times f(x_{(n-i-j+1)})/(1-F(x_{(n-i-j-k+1)})) \times$$

$\Gamma(i+j)/\{\Gamma(j)\Gamma(i)\}[1-\{(1-F(x_{(n-i+1)})/(1-F(x_{(n-i-j+1)})^{j-1}]\times$
$\{(1-F(x_{(n-i+1)})/(1-F(x_{(n-i-j+1)})\}^{i-1}\times f(x_{(n-i+1)})/(1-F(x_{(n-i-j+1)}))$

Now make the substitutions:

(i) $u(x_{(n-i-j-k+1)}, x_{(n-i-j+1)}, x_{(n-i+1)}) = (1-F(x_{(n-i+1)})/(1-F(x_{(n-i-j+1)})$

(ii) $v(x_{(n-i-j-k+1)}, x_{(n-i-j+1)}, x_{(n-i+1)}) = (1-F(x_{(n-i-j+1)})/(1-F(x_{(n-i-j-k+1)})$

(iii) $v(x_{(n-i-j-k+1)}, x_{(n-i-j+1)}, x_{(n-i+1)}) = x_{(n-i-j-k+1)}$

Note that

$$\frac{\partial(u,v,w)}{\partial(x_{(n-i-j-k+1)}, x_{(n-i-j+1)}, x_{(n-i+1)})} =$$

$$\{f(x_{(n-i+1)}/(1-F(x_{(n-i-j+1)})\}\times \{f(x_{(n-i-j+1)}/(1-F(x_{(n-i-j-k+1)})\}$$

So $f^{\#}(x_{(n-i-j-k+1)}, x_{(n-i-j+1)}, x_{(n-i+1)})$

$= B^{-1}(n-i-j-k,i+j+k)w^{n-i-j-k}(1-w)^{i+j+k-1}\times$

$B^{-1}(i+j,k)v^{i+j-1}(1-v)^{k-1}\times$

$$B^{-1}(i,j)u^{i-1}(1-u)^{j-1}\times \frac{\partial(u,v,w)}{\partial(x_{(n-i-j-k+1)}, x_{(n-i-j+1)}, x_{(n-i+1)})}$$

showing U,V, and W to be independently distributed.

As in the proof of Theorem 1, consider the implications as $n \to \infty$ when $1-F(x) = L(x)x^{-\delta}$, and

(i') $x_{(n-i-j-k+1)} = v_n x^*_{(i+j+k)}$

(ii') $x_{(n-i-j+1)} = v_n x^*_{(i+j)}$

(iii') $x_{(n-i-j+1)} = v_n x^*_{(i)}$

As outlined in the proof of Theorem 1, under these changes the transformation

$u(x_{(n-i-j-k+1)}, x_{(n-i-j+1)}, x_{(n-i+1)}) = (1-F(x_{(n-i+1)})/(1-F(x_{(n-i-j+1)})$

becomes

$u(x^*_{(i+j+k)}, x^*_{(i+j)}, x^*_{(i)}) = \{x^*_{(i)}/x^*_{(i+j)}\}^{-\delta}$

while the transformation

$v(x_{(n-i-j-k+1)}, x_{(n-i-j+1)}, x_{(n-i+1)}) = (1-F(x_{(n-i-j+1)})/(1-F(x_{(n-i-j-k+1)})$

becomes

$u(x^*_{(i+j+k)}, x^*_{(i+j)}, x^*_{(i)}) = \{x^*_{(i+j)}/x^*_{(i+j+k)}\}^{-\delta}$

These transformations imply that

$\{X^*_{(i)}/X^*_{(i+j)}\}$ has the distribution of $\{U(i,j)\}^{-\rho}$ where U(i,j) has beta density B(u:i,j)

and is distributed independently of

$\{X^*_{(i+j)}/X^*_{(i+j+k)}\}$ which has the distribution of $\{V(i+j,k)\}^{-\rho}$ where V(i+j,k) has beta density B(v:i+j,k).

Thus the dependent set of extremes $\{X^*_{(i)}, X^*_{(i+j)}, X^*_{(i+j+k)}\}$ is transformed to the independent random variables $\{(X^*_{(i)}/X^*_{(i+j)})^{-\delta}, (X^*_{(i+j)}/X^*_{(i+j+k)})^{-\delta}, X^*_{(i+j+k)}\}$ the first two random variables having densities B(u:i,j) and B(v:i+j,k) respectively.

A special case of Theorem 2 then follows on putting i = j = k = 1;

i.e. $(X^*_{(1)}/X^*_{(2)})$, $(X^*_{(2)}/X^*_{(3)})$ are independently distributed with distributions like $\{U(1,1)\}^{-\rho}$, $\{U(2,1)\}^{-\rho}$ where U(m,1) has beta density B(u:m,1), m = 1, 2.

Evidently the mechanism extends to k ratios and Theorem 2 follows.

ARNOLD, B. BALAKRISHNAN, N. AND NAGARAJA H. (1993) *A First Course in Order Statistics,* J.Wiley

BIERLANT, J. DIERCKX, G. GOEGEBEUR, Y. AND MATTHYS, G. (1999) Tail-index estimation and an exponential regression model, *Extremes*, **2**, 177-200

DAVID, H.A. AND NAGARAJA, H.N. (2003) *Order Statistics*, 3rd. Edition, Wiley-Interscience

DREES, H. FERREIRA, A. AND DE HAAN, L. (2004) On maximum likelihood estimation of the extreme value index, *Annals of Applied Probability* **14**, 1179-1201

EMBRECHTS, P. KLUPPELBERG, C. AND MIKOSCH, T. (1997) *Modelling Extremal Events for Insurance and Finance* Springer-Verlag.

FELLER, W. (1971) *An Introduction to Probability Theory and Its Applications, Volume II*, J. Wiley

HILL, B.M. A simple general approach to inference about the tail of a distribution, *Ann. Statist*. **3**, 119-131

KENDALL, M.G. AND STUART, A.(1969) The Advanced Theory of Statistics Vol. 1. Griffin.

MATTHYS, G. (2001) Estimating the extreme-value index and high quantiles: bias reduction and optimal threshold selection. Doctoral Thesis, Katholieke Universiteit Leuven.

PICKANDS, J. (1975) Statistical inference using extreme order statistics, *Ann. Statist*, **3**, 119-131

SMITH, R.L. (1987) Estimating the tails of probability distributions, *Ann. Statist*. **15**, 1174-1207

RAO, C.R. (1973) *Linear Statistical Inference and Its Applications*, J.Wiley.

TEUGELS, J.L. AND VANREOLEN, G. (2004) Box-Cox transformations and heavy-tailed distributions, in *'Stochastic Methods and Their Applications'*. *Journal of Applied Probability*, Special Volume, 41A, 211-227