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Influence Diagnostics in GARCH Processes[1]

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Abstract: Influence diagnostics have become an important tool for statistical analysis since the seminal work by Cook (1986). In this paper we present a curvature-based diagnostic to assess local influence of minor perturbations on the modified likelihood displacement in a regression model. Using the proposed diagnostic, we study the local influence in the GARCH model under two perturbation schemes which involve, respectively, model perturbation and data perturbation. We find that the curvature-based diagnostic often provides more information on the local influence being examined than the slope-based diagnostic, especially when the GARCH model is under investigation. An empirical study involving GARCH modeling of percentage daily returns of the NYSE composite index illustrates the effectiveness of the proposed diagnostic and shows that the curvature-based diagnostic may provide information that cannot be uncovered by the slope-based diagnostic. We find that the effect or influence of each observation is not invariant across different perturbation schemes, thus it is advisable to study the local influence under different perturbation schemes through curvature-based diagnostics.

Key words: normal curvature, modified likelihood displacement, GARCH models.

JEL Classification: C32 and C52.

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1 Introduction

Influence analysis studies how relevant perturbations affect specified key results. In a recent paper, Critchley, Atkinson, Lu and Biazi (2002) indicated that the motivating ideas behind influence analysis include the following aspects, namely the stability (minor perturbations have small effects), the warning (minor perturbations have large effects), and the robustness (large perturbations have small effects) of the estimated model. Influence diagnostics have become an important tool for statistical analysis. An important approach to influence analysis is the geometric approach proposed by Cook (1986), where a perturbation scheme is introduced into the postulated model through a perturbation vector with the same dimension as the vector of observations, and the influence is studied via the graph of the likelihood displacement versus the perturbation vector. In influence analysis, important questions are the choices of the perturbation scheme, the particular aspect of an analysis to monitor, and the method of measurement. The possible answers for these separate questions can lead to a variety of different diagnostics, such as the local influence on the transformation parameter in a Box-Cox transformation model discussed by Lawrance (1988), the influence of regression coefficients in generalized linear models investigated by Thomas and Cook (1989), the influence on the smoothing parameter in spline smoothing addressed by Thomas (1991), the generalization of Cook's approach and the influence on the maximum likelihood estimate (MLE) of any parameter in a regression model presented by Wu and Luo (1993a), and the influence analysis in semiparametric mixed models discussed by Fung, Zhu, Wei and He (2002) among many others.

The autoregressive conditional heteroscedasticity (ARCH) model of Engle (1982) and the generalized ARCH (GARCH) model of Bollerslev (1986) have proven to be very successful in capturing the volatility of financial time series [2]. These models enjoy great success in studying the volatility evolution of financial time series. However, they are unable to incorporate effects of outliers commonly occurring in observed time series, and the standard estimation approaches typically are not very robust to such extreme observations (see for example Bera and Higgins (1993) for detailed comments). Consequently, in empirical studies, researchers may find that a small number of observations often have a strong influence on the results of statistical procedures. This phenomenon is undesirable because it may imply that the inference reflects more the particularities of the specific dataset under analysis rather than the relationship between the variates of interest. Moreover, influential observations may also result in a poor choice of model specification. On the other hand, they can provide important clues for improving the model. Recently van Dijk, Franses and Lucas (1999) investigated the properties of the Lagrange multiplier (LM) test for ARCH and GARCH processes in the presence of additive outliers. They showed that additive outliers, which might be interpreted as both a particular deviation from conditional normality and a misspecification in the conditional mean equation, have adverse effects on the size and power of the LM test if neglected. It seems important to address the problem of assessing influence of minor perturbations on various aspects in GARCH models.

To detect outliers in GARCH processes, Hotta and Tsay (1998) presented a sequential procedure which involved adding an additional parameter to an observation at a time, and then applying the Lagrange multiplier test to check the significance of the added parameter. If the parameter is found be significant at a certain location, the corresponding observation is regarded as an outlier. This simple sequential intervention analysis is easy

[2] See the recent surveys by Bollerslev, Chou and Kroner (1992) and Bera and Higgins (1993) for more details on the GARCH family.

to implement, and is effective in some applications. However, it is flawed by the "masking phenomenon" which refers to the fact that several outliers or influential observations may act together, and any individual's effect might be "masked" by the other outliers or influential observations nearby [3].

Influence analysis is different to sequential intervention analysis in that minor perturbations are introduced to the postulated model through a perturbation vector with the same dimension as the vector of observations, and the influence of the minor perturbations associated with the observations can be examined simultaneously. Hence the influence-analysis approach is immune to the masking phenomenon. One remarkable feature of the GARCH process is the clustering of high/low frequencies, which make it possible that influential observations and outliers might be acting together, rather than being isolated far away. Thus, influence analysis seems more useful than sequential intervention analysis in assessing influences in GARCH models.

The purpose of this paper is to assess the influence of minor perturbations under several perturbation schemes in the GARCH model. The rest of this paper is organised as follows. Section 2 presents a brief review of the geometric approaches to the assessment of local influence in linear regression models. In Section 3 we derive the normal curvature on the influence graph which is formed by the modified likelihood displacement and the perturbation vector, and calculate the normal curvature in a linear regression model. Using the curvature-based diagnostic derived in Section 3, we examine the local influence under the model and data perturbation schemes in the GARCH model in Section 4. Section 5 presents an empirical study to illustrate the effectiveness of the proposed diagnostic. Concluding remarks are made in Section 6.

[3] The masking phenomenon has been well discussed in statistical literature, see Atkinson (1986, 1994), Cook (1986), and Lawrance (1995) for more details.

2 Geometric Approaches to Local Influence

2.1 Cook's Normal Curvature

Given a postulated model and a data set with sample size n, we denote the log-likelihood for the postulated model by $L(\theta)$, where θ is a $p \times 1$ vector of parameters. A perturbation scheme is introduced into the model through the $n \times 1$ vector ω which is restricted to a certain open subset Ω ($\in \mathbf{R}^n$) representing the set of relevant perturbations (For instance, ω might be used to induce minor perturbations to the observed response vector in a linear regression model). Let $L(\theta|\omega)$ be the log-likelihood corresponding to the perturbed model for a given $\omega \in \Omega$, and assume that there is a point $\omega_0 \in \Omega$ such that $L(\theta|\omega_0) = L(\theta)$ for all θ. Assume that the lifted line passing through ω_0 is represented by

$$\omega = \omega_0 + a\ell, \tag{1}$$

where the scalar a measures the magnitude of the perturbation in the $n \times 1$ direction ℓ. Let $\hat{\theta}$ and $\hat{\theta}_\omega$ represent the maximum likelihood estimators under $L(\theta)$ and $L(\theta|\omega)$, respectively. To assess the influence of varying ω throughout Ω, Cook (1986) defined the likelihood displacement,

$$LD(\omega) = 2[L(\hat{\theta}) - L(\hat{\theta}_\omega)], \tag{2}$$

and showed that a graph of $LD(\omega)$ versus ω contains essential information on the influence of the perturbation scheme in question. This influence graph is regarded as the geometric surface spanned by

$$\alpha(\omega) = [\omega_1, \omega_2, \cdots, \omega_n, LD(\omega)]', \tag{3}$$

and the local influence of minor perturbations on $LD(\omega)$ can be examined through the directions along which $LD(\omega)$ achieves large local changes at $\omega = \omega_0$. Then the normal curvature of $LD(\omega)$ at $\omega = \omega_0$ on $\alpha(\omega)$ was introduced to measure the influence, and the normal curvature is expressed as

$$C_\ell = 2|\ell' \ddot{F} \ell| = 2|\ell' \left(\Delta' \ddot{L}^{-1} \Delta\right) \ell|,$$

where $\|\ell\| = 1$, $\ddot{F} = \partial^2 L(\theta|\omega)/\partial\omega\partial\omega'$, $\Delta = \partial^2 L(\theta|\omega)/\partial\theta\partial\omega'$ and $-\ddot{L}$ is the observed information matrix. Cook (1986) concluded that the direction vector, associated with the maximum normal curvature at $\omega = \omega_0$ on $\alpha(\omega)$, indicates how to perturb the postulated model to obtain the greatest local change in $LD(\omega)$ and is the most important diagnostic for assessing the influence of minor perturbations on the likelihood displacement.

2.2 Second-order Approach to Local Influence

When only a scalar parameter in a regression model is of interest, Wu and Luo (1993a) evaluated the maximum curvature and the associated directional vector, called the second-order approach, to examine the local influence. Assume that minor perturbations are introduced to a postulated model through a perturbation vector denoted by ω which has the same expression as (1). Let ξ denote the scalar parameter of interest, then the maximum likelihood estimate (MLE) of ξ under $L(\theta|\omega)$, denoted as ξ_ω, can be regarded as a surface with Euclidean coordinates,

$$\alpha(\omega) = [\omega_1, \omega_2, \cdots, \omega_n, \xi_\omega]', \tag{4}$$

which has the same meaning as the influence graph defined by (3) and is referred to as the influence graph. The normal curvature at $\omega = \omega_0$ on the influence graph is

$$C_\ell = \frac{\ell' \ddot{\xi}_\omega \ell}{\left[1 + \dot{\xi}'_\omega \dot{\xi}_\omega\right]^{1/2} \ell' \left[I + \dot{\xi}_\omega \dot{\xi}'_\omega\right] \ell}, \tag{5}$$

where $\dot{\xi}_\omega = \partial\xi_\omega/\partial\omega$, $\ddot{\xi}_\omega = \partial^2\xi_\omega/\partial\omega\partial\omega'$, and I is the $n \times n$ identity matrix. Large local change occurs at $\omega = \omega_0$ in the direction along which the normal curvature is maximised.

Let $A = \ddot{\xi}_\omega$ and $B = (1+\dot{\xi}_\omega'\dot{\xi}_\omega)^{1/2}(I+\dot{\xi}_\omega\dot{\xi}_\omega')$, then the maximum normal curvature and the associated direction are, respectively, the largest eigenvalue and the associated eigenvector of the characteristic equation $|A-\lambda B| = 0$. The direction vector of the maximum curvature is often referred to as the second-order diagnostic or the second-order approach to local influence.

A special feature of this geometric approach is that the first derivative of ξ_ω with respect to ω is not zero and contains useful information on the local influence worthy of examination. According to the arguments about influence in Lawrance (1988), the MLE ξ_ω is most sensitive to minor perturbations in the direction that makes the slope or gradient, $\dot{\xi}_\omega$, greatest at the null point. It is not the value of the slope, but the direction of the maximum slope that is important and that forms the main diagnostic. Actually, Cook's curvature result is specific to a subset of parameters. When the subset contains only one parameter, Wu and Luo (1993a) argued that the direction of maximum slope on the influence graph (4) is the same as the direction of the maximum curvature on Cook's likelihood displacement surface. This argument has the same spirit to that of Lawrance (1988), who pointed out that when just one parameter is being considered, the direction of maximum slope is the basis of Cook's presentation, and there is no need to use a likelihood displacement measure. The directional vector of the maximum slope on the influence graph (4) is often referred to as the first-order diagnostic or the first-order approach to local influence.

In addition to the first-order diagnostic, the second-order approach provides a method that is applicable for finding a number of directions of local maximum curvature on the influence graph (4). The cosines of these selected directions help us to choose the possibly influential observations in the postulated model. The second-order diagnostic is a useful extension of Cook's normal curvature in influence diagnostics and has several applications. Using the second-order approach, Wu and Luo (1993b) discussed the local influence on the residual sum of squares and the multiple potential in regression models, and Lee and Zhao (1996) assessed the local influence on Pearson's goodness-of-fit statistic in generalized linear models. Zhang and Tse (2001) applied the second-order approach to the assessment of local influence on the bandwidth selection through cross validation in kernel smoothing. Zhang (2002) investigated the local influence on the Lagrange multiplier test for heteroskedasticity in GARCH models.

2.3 Modified Likelihood Displacement

As discussed above, the first derivative of $LD(\omega)$ with respect to ω is zero, and cannot be used to examine the local influence. When the dimension of the perturbation vector (or the vector of observations) increases, the computation of the normal curvature may become increasingly costly. With this in mind, Billor and Loynes (1993) defined the modified likelihood displacement,

$$LD^*(\omega) = -2[L(\hat{\theta}) - L(\hat{\theta}_\omega|\omega)], \tag{6}$$

based on which the influence surface is defined as

$$\alpha^*(\omega) = [\omega_1, \omega_2, \cdots, \omega_n, LD^*(\omega)]\,. \tag{7}$$

The first derivative of $LD^*(\omega)$ with respect to ω is

$$\frac{\partial LD^*(\omega)}{\partial a} = \ell' \left[\frac{\partial LD^*(\omega)}{\partial \omega}\right] = 2\ell' \left[\frac{\partial L(\hat{\theta}_\omega|\omega)}{\partial \omega} + \frac{\partial \theta}{\partial \omega}\frac{\partial L(\theta|\omega)}{\partial \theta'}\right].$$

When evaluated at $\theta = \hat{\theta}$ and $\omega = \omega_0$, it becomes

$$\frac{\partial LD^*(\omega)}{\partial a} = \ell' \left[2\frac{\partial L(\theta|\omega)}{\partial \omega}\right].$$

Large local change of $LD^*(\omega)$ at the null point is associated with the maximum slope or gradient which occurs in the direction of the unity vector

$$\ell_s = \frac{\partial L(\theta|\omega)/\partial\omega}{\|\partial L(\theta|\omega)/\partial\omega\|}.$$

Hence the components of ℓ_s indicate the influence of small perturbations on $LD^*(\omega)$ at the null point on $\alpha^*(\omega)$.

Billor and Loynes (1993) employed the direction of maximum slope of $LD^*(w)$ at the null point on $\alpha^*(\omega)$ to examine the local influence of a chosen perturbation scheme. However, when influence graphs have considerable nonlinearity, influential components may have a nearly zero local slope and would not be detected by the first-order diagnostic (see, for example, Cadigan and Farrell, 2002). As argued by Wu and Luo (1993a), the curvature-based diagnostic can provide the information that the first-order diagnostic fails to provide. It presents a method that is applicable for finding a number of local maximum curvature directions and is strongly recommended.

3 Normal Curvature under $LD^*(\omega)$

3.1 Normal Curvature

We investigate the normal curvature on the influence graph formed by the modified likelihood displacement and the perturbation vector. Let the perturbation vector be expressed in the same way as (1), then we have the following theorem.

Theorem 1: The normal curvature of $LD^*(\omega)$ at the null point on the influence surface $\alpha^*(\omega)$ is

$$C_\ell = \frac{\ell'\ddot{F}\ell}{\left[1+\dot{F}'\dot{F}\right]^{1/2}\ell'\left[I+\dot{F}\dot{F}'\right]\ell}, \tag{8}$$

where $\dot{F}$ and $\ddot{F}$ are, respectively, defined by

$$\dot{F} = \frac{2\,\partial L(\theta|\omega)}{\partial \omega}, \tag{9}$$

$$\ddot{F} = 2\left\{\frac{\partial^2 L(\theta|\omega)}{\partial\omega\partial\omega'} - \left[\frac{\partial^2 L(\theta|\omega)}{\partial\theta\partial\omega'}\right]' \left[\frac{\partial^2 L(\theta|\omega)}{\partial\theta\partial\theta'}\right]^{-1} \frac{\partial^2 L(\theta|\omega)}{\partial\theta\partial\omega'}\right\}, \tag{10}$$

with all the derivatives evaluated at the MLE of θ and $\omega = \omega_0$.

Proof: See the appendix.

Let $A = \ddot{\xi}_\omega$ and $B = (1 + \dot{\xi}'_\omega \dot{\xi}_\omega)^{1/2}(I + \dot{\xi}_\omega \dot{\xi}'_\omega)$. According to the discussions on normal curvature in Cook (1986) and Wu and Luo (1993a), the normal curvature is maximised at the null point along the direction, denoted as ℓ_c after normalization, which is the eigenvector associated with the largest eigenvalue of the characteristic equation, $|A - \lambda B| = 0$. Hence the direction vector of maximum normal curvature, ℓ_c, may indicate the local influence of the perturbation vector in the postulated model. Scatter plots of the components of ℓ_c are often useful in locating the influential observations.

3.2 Normal Curvature in a Linear Regression Model

Let the postulated model be the linear regression

$$Y = X\beta + \varepsilon,$$

where Y is the $n \times 1$ vector of responses, X is the $n \times p$ non-stochastic design matrix, β is a $p \times 1$ vector of parameters, and the errors $\varepsilon \sim N(0, \sigma^2 I)$. For simplicity, we assume that σ^2 is known.

3.2.1 Case-weight Perturbation

Assume that the perturbation vector, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)'$, is introduced as case weights in the log-likelihood function,

$$L(\beta|\omega) = -\frac{1}{2\sigma^2}\sum_{t=1}^{n} \omega_t (y_t - x'_t\beta)^2,$$

where ω_t and y_t are, respectively, the t-th components of ω and Y, x_t' is the t-th row of X, and the null point is the $n \times 1$ vector of ones. Computing the normal curvature at the null point on the influence graph (7), we have

$$\dot{F} = -\frac{1}{\sigma^2} e^2, \qquad \ddot{F} = \frac{2}{\sigma^2} D(e) X (X'X)^{-1} X' D(e),$$

where $e = (e_1, e_2, \cdots, e_n)'$ is the vector of ordinary least squares residuals, $e^2 = (e_1^2, e_2^2, \cdots, e_n^2)'$, and $D(e) = \text{diag}(e_1, e_2, \cdots, e_n)$. The direction diagnostic found by the first-order approach is simply the squared residuals, which is indeed a diagnostic for checking model adequacy in linear regression models (see Cook and Weisberg (1982) for more details).

When we study the normal curvature on Cook's influence graph (3), we have

$$\ddot{F} = \frac{2}{\sigma^2} D(e) X (X'X)^{-1} X' D(e),$$

while the first derivative of $LD(\omega)$ with respect to ω is zero.

3.2.2 Data Perturbation

Assume that the perturbation vector, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)'$, is introduced to the response $Y = (y_1, y_2, \cdots, y_n)'$. The relevant part of the log-likelihood under this perturbation scheme is

$$L(\beta|\omega) = -\frac{1}{2\sigma^2} \sum_{t=1}^{n} (y_t + \omega_t - x_t'\beta)^2,$$

where the null point is the $n \times 1$ vector of zeros. When we study the normal curvature on the influence graph (7), we have

$$\dot{F} = -\frac{2}{\sigma^2} e, \qquad \ddot{F} = 2\left[I_n + \frac{1}{\sigma^2} X (X'X)^{-1} X'\right].$$

The direction vector found by the first-order approach is simply the vector of ordinary least squares residuals, which is not a reasonable and effective diagnostic. Thus, the second-order approach is of great importance under this perturbation scheme.

When we study the normal curvature on Cook's influence graph (3), we have

$$\ddot{F} = \frac{2}{\sigma^2} X(X'X)^{-1}X.$$

while the first derivative of $LD(\omega)$ with respect to ω is zero.

4 Local Influence in GARCH Processes

Assume that the postulated model is a GARCH(p, q) process expressed as

$$\begin{aligned} y_t &= \varepsilon_t, \\ \varepsilon_t &= \sqrt{h_t}\,\eta_t, \qquad \eta_t \sim N(0,1), \\ h_t &= a_0 + \sum_{i=1}^{p} a_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} b_j h_{t-j}, \end{aligned} \tag{11}$$

where h_t is the variance of ε_t conditional on the information available at time t. Let θ denote the vector of all parameters in the GARCH model, then the logarithm of the quasi-likelihood function is

$$L(\theta) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n} \log\ h_t - \frac{1}{2}\sum_{t=1}^{n} \frac{y_t^2}{h_t}, \tag{12}$$

while the MLE of θ can be obtained through numerical computation. Assume that the perturbation vector ω is expressed in the same way as (1), and let $L(\theta|\omega)$ denote the logarithm of the quasi-likelihood function when the perturbation vector is introduced to the GARCH model. Then we investigate the normal curvature at the null point on the influence graph defined in (7).

Billor and Loynes (1993) divided perturbation schemes into two main groups. One is the model perturbation that means modifying assumptions underlying the model, while the other is the data perturbation. The reason for considering data perturbation is that there may exist measurement errors or outliers in the data. We follow this partition and consider two types of perturbation schemes, namely data perturbation and model perturbation.

4.1 Data Perturbation

Assume that a minor perturbation is added to each of the observations,

$$z_t = y_t + \omega_t,$$

for $t = 1, 2, \cdots, n$, where $\{z_t\}$ is the observed data, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)'$ is the perturbation vector with the null point ω_0 being the $n \times 1$ vector of zeroes. Under this perturbation scheme, the log-likelihood is

$$L(\theta|\omega) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\log\ h_t(\omega) - \frac{1}{2}\sum_{t=1}^{n}\frac{z_t^2}{h_t(\omega)}, \tag{13}$$

where

$$h_t(\omega) = a_0 + \sum_{i=1}^{p} a_i(y_{t-i} + \omega_{t-i})^2 + \sum_{j=1}^{q} b_j h_{t-j}(\omega).$$

The derivatives required for the calculation of $\dot{F}$ and $\ddot{F}$ in the normal curvature (8) are

$$\begin{aligned}
\frac{\partial^2 L(\theta|w)}{\partial\theta\partial\theta'} &= \sum_{t=1}^{n}\left[\frac{1}{2h_t^2} - \frac{z_t^2}{h_t^3}\right]\frac{\partial h_t}{\partial\theta}\frac{\partial h_t}{\partial\theta'}, \\
\frac{\partial L(\theta|w)}{\partial w} &= \sum_{t=1}^{n}\left[\frac{z_t^2}{2h_t^2} - \frac{1}{2h_t}\right]\frac{\partial h_t}{\partial w} - \sum_{t=1}^{n}\frac{z_t}{h_t}\frac{\partial z_t}{\partial w}, \\
\frac{\partial^2 L(\theta|w)}{\partial\theta\partial w'} &= \sum_{t=1}^{n}\left[\frac{z_t^2}{2h_t^2} - \frac{1}{2h_t}\right]\frac{\partial^2 h_t}{\partial\theta\partial w'} + \sum_{t=1}^{n}\left[\frac{1}{2h_t^2} - \frac{z_t^2}{h_t^3}\right]\frac{\partial h_t}{\partial\theta}\frac{\partial h_t}{\partial w'} + \sum_{t=1}^{n}\frac{z_t}{h_t^2}\frac{\partial h_t}{\partial\theta}\frac{\partial z_t}{\partial w'}, \\
\frac{\partial^2 L(\theta|w)}{\partial w\partial w'} &= \sum_{t=1}^{n}\left[\frac{1}{2h_t^2} - \frac{z_t^2}{h_t^3}\right]\frac{\partial h_t}{\partial w}\frac{\partial h_t}{\partial w'} + \sum_{t=1}^{n}\left[\frac{z_t^2}{2h_t^2} - \frac{1}{2h_t}\right]\frac{\partial^2 h_t}{\partial w\partial w'} + 2\sum_{t=1}^{n}\frac{z_t}{h_t^2}\frac{\partial z_t}{\partial w}\frac{\partial h_t}{\partial w'} \\
&\quad - \sum_{t=1}^{n}\frac{1}{h_t}\frac{\partial z_t}{\partial w}\frac{\partial z_t}{\partial w'},
\end{aligned}$$

where all the derivatives are computed at the MLE of θ and $\omega = \omega_0$. Both the slope- and curvature-based diagnostics are much more complicated than those under the linear regression model.

4.2 Model Perturbation

The quasi-likelihood function given in (12) is based on the basic assumption that the standardized error process, $\{\eta_t = \varepsilon_t/\sqrt{h_t}\}$, is an *iid* standard Gaussian process. However, the existence of influential observations may have a strong effect on the fitting of the model.

4.2.1 Innovative Perturbation

We may introduce the perturbation vector ω to the postulated model through the case weights in the quasi-likelihood function,

$$L(\theta|\omega) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\log\ h_t - \frac{1}{2}\sum_{t=1}^{n}\omega_t\frac{y_t^2}{h_t}, \tag{14}$$

where the null point ω_0 is the $n \times 1$ vector of ones. This kind of perturbation is regarded as an innovative perturbation, since the volatility of η_t is perturbed. The derivatives required for the calculation of $\dot{F}$ and $\ddot{F}$ in the normal curvature (8) are

$$\begin{aligned}
\frac{\partial^2 L(\theta|\omega)}{\partial\theta\partial\theta'} &= \sum_{t=1}^{n}\left[\frac{1}{2h_t^2} - \frac{z_t^2}{h_t^3}\right]\frac{\partial h_t}{\partial\theta}\frac{\partial h_t}{\partial\theta'}, \\
\frac{\partial L(\theta|\omega)}{\partial\omega} &= \left(-\frac{y_1^2}{2h_1}, -\frac{y_2^2}{2h_2}, \ldots, -\frac{y_n^2}{2h_n}\right)', \\
\frac{\partial^2 L(\theta|\omega)}{\partial\theta\partial\omega'} &= \left(\frac{y_1^2}{2h_1^2}\frac{\partial h_1}{\partial\theta}, \frac{y_2^2}{2h_2^2}\frac{\partial h_2}{\partial\theta}, \ldots, \frac{y_n^2}{2h_n^2}\frac{\partial h_n}{\partial\theta}\right), \\
\frac{\partial^2 L(\theta|w)}{\partial w\partial w'} &= 0_n,
\end{aligned}$$

where 0_n is the $n \times n$ matrix with all elements zero, and all the derivatives are computed at the MLE of θ and $\omega = \omega_0$. We find that the first-order diagnostic is merely the vector of squared standardized errors, which is a diagnostic only for quick and easy checking.

4.2.2 Additive Perturbation

Assume that the perturbation vector is introduced to the postulated model by adding a perturbation to the level of the standardized errors,

$$\frac{\varepsilon_t}{\sqrt{h_t}} + \omega_t \sim N(0,1).$$

Under this perturbation scheme, the quasi-likelihood function becomes,

$$L(\theta|\omega) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\log\ h_t - \frac{1}{2}\sum_{t=1}^{n}\left(\frac{y_t}{\sqrt{h_t}} + \omega_t\right)^2, \tag{15}$$

where the null point ω_0 is an n vector of zeros. This kind of perturbation may be regarded as an additive perturbation, since it has no effect on the volatility. The derivatives required for the calculation of $\dot{F}$ and $\ddot{F}$ in the normal curvature (8) are

$$\begin{aligned}
\frac{\partial^2 L(\theta|\omega)}{\partial\theta\partial\theta'} &= \sum_{t=1}^{n}\left[\frac{1}{2h_t^2} - \frac{z_t^2}{h_t^3}\right]\frac{\partial h_t}{\partial\theta}\frac{\partial h_t}{\partial\theta'},\\
\frac{\partial L(\theta|\omega)}{\partial\omega} &= \left(-\frac{y_1}{\sqrt{h_1}}, -\frac{y_2}{\sqrt{h_2}}, \dots, -\frac{y_n}{\sqrt{h_n}}\right)',\\
\frac{\partial^2 L(\theta|\omega)}{\partial\theta\partial\omega'} &= \left(\frac{y_1}{2h_1^{3/2}}\frac{\partial h_1}{\partial\theta}, \frac{y_2}{2h_2^{3/2}}\frac{\partial h_2}{\partial\theta}, \dots, \frac{y_n}{2h_n^{3/2}}\frac{\partial h_n}{\partial\theta}\right),\\
\frac{\partial^2 L(\theta|w)}{\partial w\partial w'} &= -I,
\end{aligned}$$

where all the derivatives are computed at the MLE of θ and $\omega = \omega_0$. We find that the first-order diagnostic is merely the vector of standardized errors, which is indeed a naive diagnostic employed for quick checking only.

From the two schemes for model perturbation, we find that the first-order diagnostic or the slope-based diagnostic cannot provide all the information on the local influence being examined, and the second-order diagnostic or the curvature-based diagnostic is required to understand the local influence uncovered by the underlying perturbation schemes.

In order to calculate the slope and curvature of $LD^*(\omega)$ at the null point on the influence graph, we need to compute the derivative of h_t with respect to θ, which depends on the analytical form of h_t. If the underlying return series is an ARCH(p) process, we may obtain the analytical form of $\partial h_t/\partial\theta$. However, if the return series is a GARCH(p, q) process, a recursive procedure is needed to calculate $\partial h_t/\partial\theta$. Given the GARCH($p, q$) model defined by (11), we define the following derivatives of h_t with respect to the components of θ,

$$d_t = \frac{\partial h_t}{\partial a_0}, \qquad e_{i,t} = \frac{\partial h_t}{\partial a_i}, \qquad f_{j,t} = \frac{\partial h_t}{\partial b_j},$$

where $i = 1, \cdots, p$; $j = 1, \cdots, q$ and $t = 1, 2, \cdots, n$. To calculate these derivatives, the following recursive equations may be used,

$$d_t = 1 + \sum_{k=1}^{q} b_k d_{t-k}, \qquad e_{i,t} = \varepsilon_{t-i}^2 + \sum_{k=1}^{q} b_k e_{i,t-k}, \qquad f_{j,t} = h_{t-j} + \sum_{k=1}^{q} b_k f_{j,t-k},$$

where the initial values are, respectively, $d_{1-k} = 1.0$, $e_{i,1-k} = y_0^2$ and $f_{j,1-k} = h_0$ for $i = 1, \cdots, p$ and $j, k = 1, \cdots, q$, and all the derivatives are evaluated with b_k ($k = 1, \cdots, q$) being replaced by their MLEs derived under $L(\theta)$.

5 An Application to the Percentage Daily Returns on the NYSE Index

In this section we apply the influence diagnostics discussed in Section 4 to the percentage daily returns on the NYSE composite index. The sample, denoted as $\{y_t\}$, consists of 1255 observations from 2 January 1997 to 31 December 2001, and is plotted in Figure 1. First, we consider testing the null hypothesis that $\{y_t\}$ is an *iid* white noise process against the alternative hypothesis of a GARCH(1,1) specification. The LM statistic is 1244.86 for $q = 1$ which favours the alternative hypothesis.

Second, we examine the local influence on the modified likelihood displacement $LD^*(\omega)$ under the model perturbation and data perturbation schemes. A summary of the relevant results is given in Table 1. Regarding the innovative model perturbation scheme, we plot the components of the direction vector of the slope-based diagnostic in Figure 2, and the components of the vector of the curvature-based diagnostic in Figure 3. Under this perturbation scheme, the first-order diagnostic is merely the vector of squared standardized errors, which shows that the 206th return (corresponding to 27 October 1997), the 416th and 418th returns (corresponding to the 27th and 31st of August 1997, respectively), the 828th (corresponding to 14 April 2000) and the 1182nd (corresponding to 17 September 2001) returns have strong effects on $LD^*(\omega)$. The curvature-based diagnostic reconfirms the strong effects of these points much more obviously than the slope-based diagnostic. Moreover, the curvature-based diagnostic shows that the 207th (corresponding to 28 October 1997) and the 434th (corresponding to 17 September 1998) returns have strong influence, which is not revealed by the slope-based diagnostic. The findings warn that the assumption of a Gaussian distribution for the standardized error process is questionable. As the perturbation vector is introduced to the GARCH(1,1) model through the variance of the standardized errors, we have doubts about the homoskedasticity of the standardized error process.

Under the additive model perturbation scheme, the components of the direction vector of the slope-based diagnostic is plotted in Figure 4, which provides similar information to that uncovered by the slope-based diagnostic under the innovative model perturbation scheme, because these two slope-based diagnostics have the same meaning. However, the vector of the curvature-based diagnostic, whose components are plotted in Figure 5, shows that only the 418th and 434th observations have strong effects on $LD^*(\omega)$. All the other influential points uncovered under the innovative model perturbation scheme have no obvious effect when the perturbation is introduced to the mean of the standardized errors.

Regarding the two model perturbation schemes, we may conclude that the 206th, 207th, 416th, 418th, 434th, 828th and 1182nd observations are influential on $LD^*(\omega)$ under the innovative perturbation, while only the 418th and 434th observation is influential on $LD^*(\omega)$ under the additive perturbation. We notice that the effect of each observation on the modified likelihood displacement is not invariant across perturbation schemes.

Table 1. Summary of the Slope- and Curvature-based Diagnostics

Date	Serial No.	Model Perturbation				Data Perturbation	
		Innovative		Additive			
		slope	curvature	slope	curvature	slope	curvature
20/10/97	201	-0.0207	-0.0016	-0.0365	-0.0357	0.0217	**0.1195**
21/10/97	202	-0.0353	-0.0476	-0.0160	-0.0297	0.0405	**0.1807**
22/10/97	203	-0.0009	-0.0010	0.0075	0.0002	-0.0126	-0.0469
23/10/97	204	-0.0439	0.0231	0.0531	0.0130	-0.0907	**-0.3145**
24/10/97	205	-0.0056	-0.0074	0.0189	-0.0076	-0.0545	**-0.1750**
27/10/97	206	**-0.4875**	**-0.4215**	**0.1771**	-0.0534	**0.1618**	0.0086
28/10/97	207	-0.0343	**-0.1274**	-0.0470	0.0982	-0.0283	0.0449
26/08/98	415	-0.0123	0.0281	0.0198	-0.0075	-0.0260	**-0.1260**
27/08/98	416	**-0.1845**	**0.2391**	**0.1089**	-0.0278	0.0402	0.0443
28/08/98	417	-0.0080	-0.0080	0.0227	-0.0365	-0.0116	-0.0239
31/08/98	418	**-0.2056**	**-0.3459**	**0.1150**	**-0.1841**	0.0660	-0.0675
16/09/98	433	-0.0019	0.0042	-0.0110	0.0248	-0.0031	0.0127
17/09/98	434	-0.0554	**0.1510**	-0.0597	**0.1279**	-0.0339	0.0357
12/04/00	826	-0.0149	0.0362	0.0310	-0.0025	-0.0269	**-0.1554**
13/04/00	827	-0.0282	0.0505	0.0426	-0.0053	-0.0492	**-0.2506**
14/04/00	828	**-0.3077**	**0.2184**	**0.1407**	-0.0581	**0.1326**	0.0278
07/09/01	1180	-0.0422	-0.0406	0.0521	-0.0076	-0.0120	**-0.1240**
10/09/01	1181	-0.0010	-0.0015	-0.0080	0.0054	0.0066	0.0310
17/09/01	1182	**-0.2169**	**-0.1939**	**0.1181**	-0.0584	0.0963	-0.0139

Note: Large components of each diagnostic appear in bold. Their corresponding observations have a strong effect on the influence graph.

We now assess the local influence under the data perturbation scheme. Figure 6 plots the components of the direction vector of the slope-based diagnostic and shows that the 206th and 828th returns are influential on $LD^*(\omega)$. This finding is consistent with what we have obtained under the innovative model perturbation scheme. However, the vector of the curvature-based diagnostic whose components are plotted in Figure 7 shows that these two observations have no strong influence on $LD^*(\omega)$. Moreover, the curvature-based diagnostic shows that two patches of observations are influential on the modified likelihood displacement. The first patch contains the 201st, 202nd, 204th and 205th observations (corresponding to the 20th, 21st, 23rd and 24th of October 1997), while the second patch contains the 826th and 827th observations (corresponding to the 12th and 13th of April 2000, respectively). As the perturbation vector is introduced to the underlying model by perturbing each observation, large values of the absolute components of this diagnostic vector might mean "typing errors" or "outliers" at the corresponding locations. Hence the evidence suggests that the influential observations uncovered under the model perturbation schemes are not typing errors or outliers.

The influential observations indicated by the curvature-based diagnostic under the data perturbation scheme might be interpreted as "abnormal noises" to the observed return series. They exert a strong effect on the modified likelihood displacement, but no obvious effect on the mean and variance of the standadized errors. From Table 1 and Figure 1, we observe that the return series experienced one or two "jumps" just one day after the influential observations indicated under the data perturbation scheme. Taking the first patch of abnormal noises as an example, we would not be surprised to find out some kind of linkage between the patch of abnormal noise and the jump thereafter through some other analytical tools. It is quite natural to guess that the NYSE composite index might have experienced a series of shocks during the period from the 16th to the 22nd of October 1997 (this period is exactly one week). These shocks exerted strong effects on $LD^*(\omega)$ when the the data perturbation scheme is under investigation, and they had no effect on the mean and variance of the standardized errors during these days. However, these shocks did produce a strong effect on the variance of the standardized error for 23 October 1997, the day that immediately followed the shocks. Eraker, Johannes and Polson (2002) referred to this phenomenon during the above-mentioned period as "market stress" when they studied the stochastic volatility in the return series of S&P 500 index. A similar situation occurred during the second patch of abnormal noises and the day immediately following.

The influence analysis in this section warns that the specification of a GARCH(1,1) model may not be appropriate for the percentage daily returns on the NYSE composite index, because the model cannot incorporate the effects of the influential observations which are found not to be outliers. In order to incorporate these influential observations, we add a dummy variable, denoted as d_t, to the conditional mean equation of the GARCH(1,1) model (11),

$$\begin{aligned} y_t &= \delta d_t + \sqrt{h_t}\, \eta_t, \\ h_t &= a_0 + a_1 \varepsilon_{t-1}^2 + b_1 h_{t-1}, \end{aligned} \tag{16}$$

where $\eta_t \sim N(0,1)$, δ is the parameter attached to the dummy, and d_t takes values of one at the locations associated with influential observations and zero elsewhere (as indicated by the curvature-based diagnostic with the innovative perturbation scheme in Table 1). The quasi-MLEs of the parameters are obtained through numerical calculation, and the estimates of the parameters under both (11) and (16) are summarized in Table 2. The estimated standardized errors are plotted in Figure 8 which provides a strong evidence on the Gaussian assumption of the standardized error process. The estimated standardized errors under the model (11) is actually the diagnostic vector whose components have been sequentially plotted in Figure 4.

To compare the distributional property of the standardized errors obtained from each model, we calculated their skewness and kurtosis, which are, respectively,

$$\mu_s = \frac{\mu_3}{\mu_2^{3/2}}, \qquad \mu_k = \frac{\mu_4}{\mu_2^2},$$

where $\mu_i = E(R-\mu)^i$ for $i = 2, 3, 4$ with $\mu = E(R)$ and R representing the standardized error. The skewness and kurtosis of the estimated standardized errors for both models are summarized in Table 3. The skewness and kurtosis of the estimated standardized errors obtained from model (11) are both far away from those of standard Gaussian errors. However, when a dummy variable is added to the GARCH(1,1) model defined by (11), the skewness and kurtosis of the estimated standardized errors are much nearer to those of standard Gaussian errors than those of model (11). Given that there are only 7 non-zero values of the dummy variable, the improvement in skewness and kurtosis is clearly remarkable and demonstrates just how influential those 6 observations are. Model (16) is plainly doing a much better job of modeling the data than model (11).

Table 2. The Estimates of the Parameters

model	$\hat{a}_0$	$\hat{a}_1$	$\hat{b}_1$	$\hat{\delta}$
model (11)	0.0754	0.8293	0.1097	——
model (16)	0.0623	0.8428	0.1039	-2.7401

Table 3. Basic Statistics of the Standardized Errors

model	mean	standard deviation	skewness	kurtosis
model (11)	0.0175	1.0002	-0.5569	5.2388
model (16)	0.0292	0.9856	-0.1748	3.6532

6 Conclusion

This paper investigates the problem of assessing local influence in a GARCH model based on the modified likelihood displacement by using a curvature-based diagnostic. We find that the curvature-based diagnostic often provides more accurate information on the local influence being examined than the slope-based diagnostic does. The empirical study of the previous section clearly illustrates the effectiveness of the curvature-based diagnostic which identified 7 influential observations in the set of 1255 data points. The standardized errors from the GARCH(1,1) model with a dummy variable for these 7 observations are much more Gaussian-like (as measured by skewness and kurtosis) than those from the GARCH(1,1) model that ignores the importance of these 7 observations. The use of curvature-based diagnostics to identify problem observations has plainly resulted in a better model being fitted. Moreover, we find that the effect of each observation on the modified likelihood displacement is not invariant across perturbation schemes. Thus, it is advisable to study the local influence under different perturbation schemes through curvature-based diagnostics.

Appendix:

Proof of the Theorem 1: The first derivative of $LD^*(\omega)$ with respect to a can be expressed as

$$\frac{\partial LD^*(\omega)}{\partial a} = \ell' \left[\frac{\partial LD^*(\omega)}{\partial \omega}\right] = 2\ell' \left[\frac{\partial L(\theta|\omega)}{\partial \omega}\right],$$

which is denoted by $\ell'\dot{F}$, and the second derivative of $LD^*(\omega)$ with respect to a can be expressed as

$$\frac{\partial^2 LD^*(\omega)}{\partial a^2} = \ell' \left[\frac{\partial^2 LD^*(\omega)}{\partial \omega \partial \omega'}\right] \ell,$$

which is denoted by $\ell'\ddot{F}\ell$. Then we have

$$\begin{aligned} \ddot{F} &= 2\left\{\frac{\partial^2 L(\theta|\omega)}{\partial \omega \partial \omega'} + \frac{\partial^2 L(\theta|\omega)}{\partial \omega \partial \theta'}\frac{\partial \theta}{\partial \omega} + \frac{\partial^2 L(\theta|\omega)}{\partial \omega \partial \theta'}\frac{\partial \theta}{\partial \omega} + \left[\frac{\partial \theta}{\partial \omega}\right]' \frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \theta'}\frac{\partial \theta}{\partial \omega} + \frac{\partial L(\theta|\omega)}{\partial \theta}\frac{\partial^2 \theta}{\partial \omega \partial \omega'}\right\} \\ &= 2\left\{\frac{\partial^2 L(\theta|\omega)}{\partial \omega \partial \omega'} + 2\frac{\partial^2 L(\theta|\omega)}{\partial \omega \partial \theta'}\frac{\partial \theta}{\partial \omega} + \left[\frac{\partial \theta}{\partial \omega}\right]' \frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \theta'}\frac{\partial \theta}{\partial \omega}\right\}. \end{aligned}$$

When evaluated at the maximum likelihood estimate of θ, the first derivative of $L(\theta|\omega)$ with respect to θ is zero, that is,

$$\frac{\partial L(\theta|\omega)}{\partial \theta} = 0.$$

If we differentiate both sides of this equation with respect to ω, and we have

$$\frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \theta'} \frac{\partial \theta}{\partial \omega'} + \frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \omega'} = 0,$$

based on which we obtain

$$\frac{\partial \theta}{\partial \omega'} = -\left[\frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \theta'}\right]^{-1} \frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \omega'}.$$

Substitute this equation into $\ddot{F}$, then we have

$$\ddot{F} = 2\left\{\frac{\partial^2 L(\theta|\omega)}{\partial \omega \partial \omega'} - \left[\frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \omega'}\right]' \left[\frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \theta'}\right]^{-1} \frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \omega'}\right\}.$$

We have obtained the analytical forms of $\dot{F}$ and $\ddot{F}$. According to the discussions on the geometric background of the normal curvature in Cook (1986), Wu and Luo (1993a), Poon and Poon (1999) and Fung and Kwan (1997), we obtain the normal curvature expressed in Theorem 1.

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Figure 1: Percentage Daily Returns of NYSE

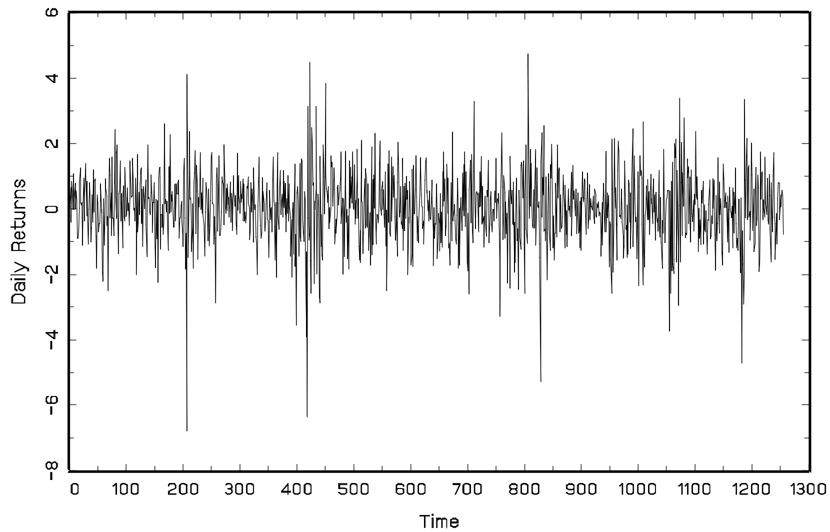

Figure 2: Innovative Model Perturbation: Slope

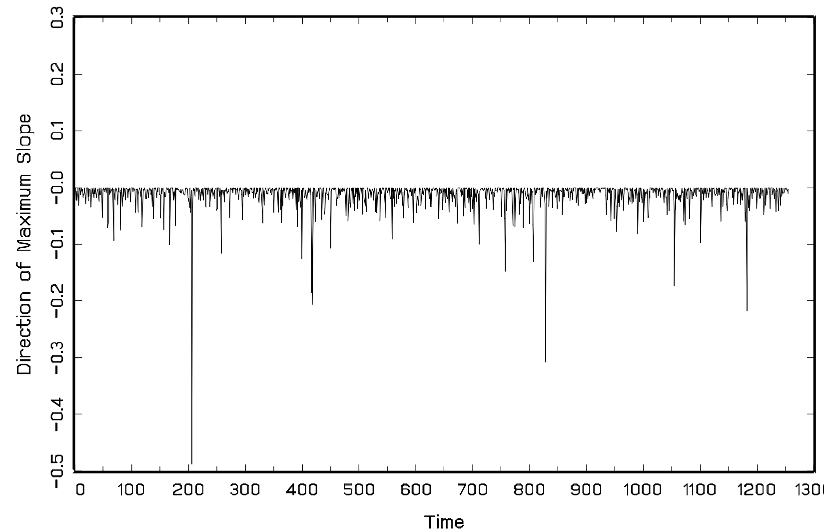

Figure 3: Innovative Model Perturbation: Curvature

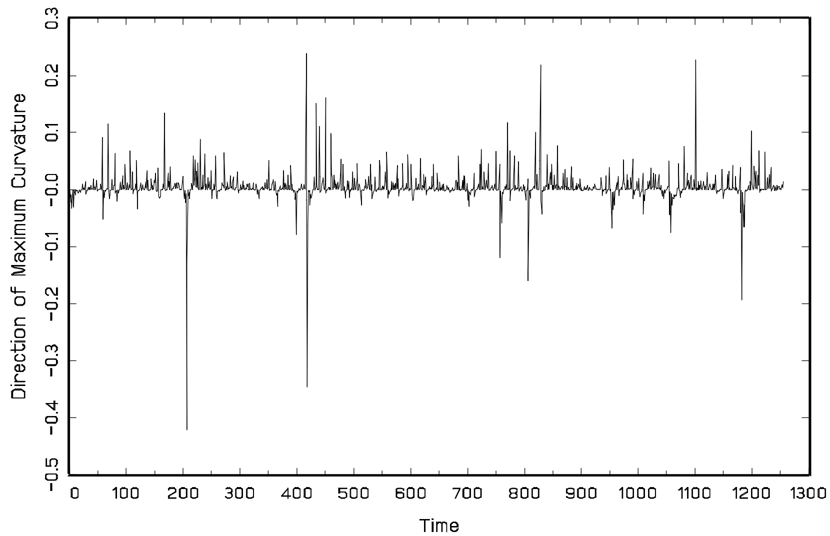

Figure 4: Additive Model Perturbation: Slope

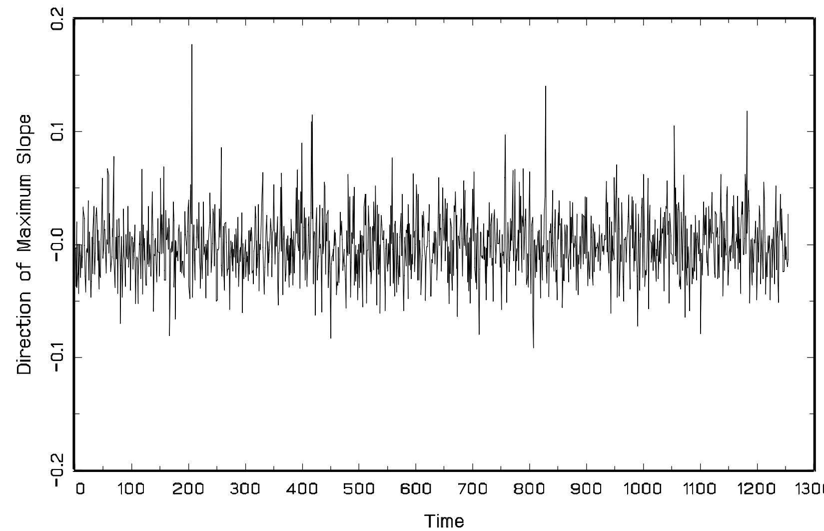

Figure 5: Additive Model Perturbation: Curvature

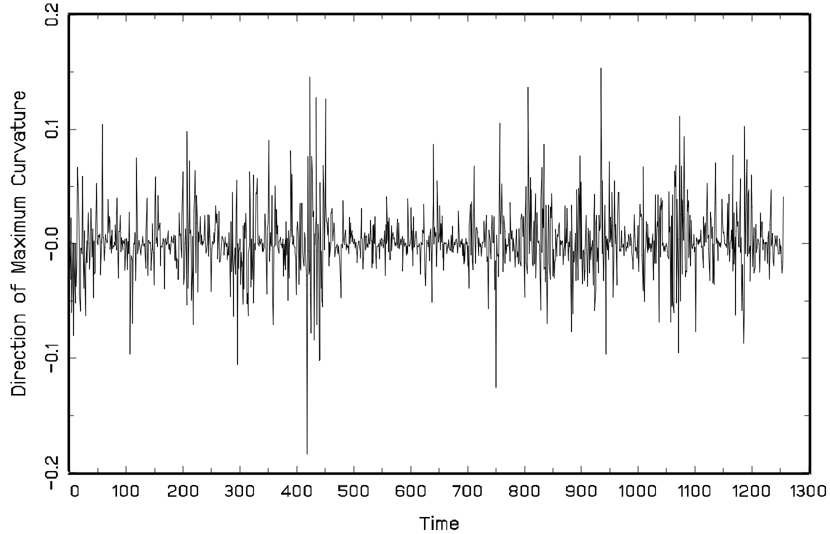

Figure 6: Data Perturbation: Slope

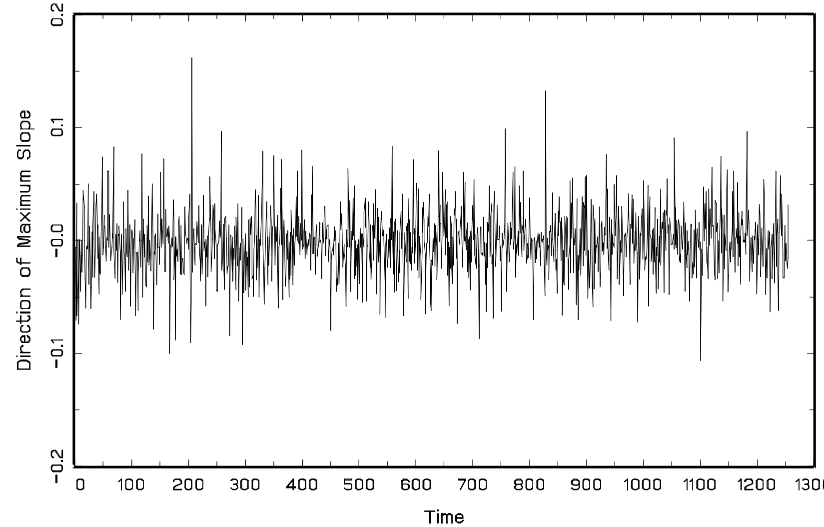

Figure 7: Data Perturbation: Curvature

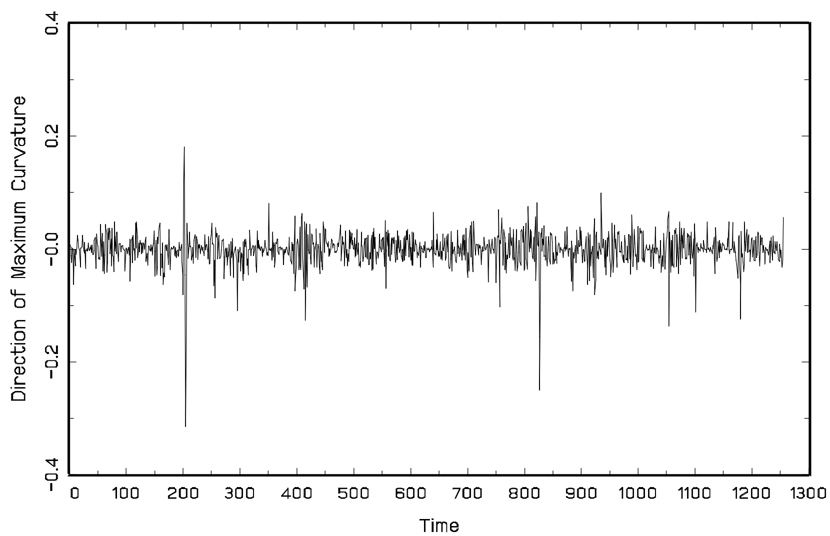

Figure 8: Standardized Errors when Dummy Added