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Abstract: In the case of input demand systems based on specification of technology by a Translog cost function, it is common to estimate either a system of share equations alone, or to supplement them by the cost function. By adding up, one of the share equations is excluded. In this paper it is argued that a system of n-1 share equations is essentially incomplete, whereas if the n-1 share equations are supplemented by the cost function the implied error structure is inadmissible. Similarly, if the technology is specified by a normalized quadratic cost function, it is common to estimate either a system of n-1 demand equations alone, or to supplement them by the cost function. In both cases, the implied error structure is again inadmissible.

Keywords: Cost Function; Input demands; Share equations; Translog; Normalized Quadratic; Error specification.

JEL Classification: C30, D24.

1. Introduction

In production theory, energy economics, agricultural economics etc. it is common to specify the technology by a cost function, and by far the majority of empirical applications employ either a Translog cost function or a normalized quadratic cost function. Estimation by maximum likelihood or SUR methods is often justified by an appeal to the analogous results in consumer demand estimation. In the case of consumer demand systems with exogenous total expenditure, it is well known that estimation of a system of n expenditure equations or n share equations leads to a complete, but degenerate, system specification, and deleting an arbitrary equation leads to no loss of information. Parameter estimates are invariant to the deleted equation. (See, for example, Barten (1969), Powell (1969), or Bewley (1986). In the case of input demand systems based on specification of technology by a Translog cost function, it is common to estimate either a system of share equations alone, or to supplement them by the cost function. By adding up, one of the share equations is excluded. However, because the level of cost is endogenous rather than exogenous, the analogy with consumer demand system is misleading. In this paper it is argued that a system of *n*-1 share equations is essentially incomplete, whereas if the *n*-1 share equations are supplemented by the cost function the implied error structure is inadmissible with the implicit assumptions of the estimation method. In the case of input demand systems based on specification of technology by a normalized quadratic cost function, it is common to estimate either a system of *n*-1 demand equations alone, or to supplement them by the cost function. In either of these cases it is shown that the implied error structure is inadmissible. Alternative estimation strategies for the two specifications are suggested. More generally, the Translog and normalized quadratic are just two typical applications, and the results extend to cost function specification in general, and to technology specified by profit functions or revenue functions.

2. The Typical Translog and Normalized Quadratic Specifications

To introduce ideas, consider the case of production theory where technology is modelled by the cost function, and the two prime examples of functional forms used are the Translog (TL) cost function, and the Normalized Quadratic (NQ) cost function. The typical procedures used to derive estimating equations are as follows.

Define n inputs $x = \left[x_1, x_2, \ldots, x_n\right]$ with prices $w = \left[w_1, w_2, \ldots, w_n\right]$, m outputs $y = \left[y_1, y_2, \ldots, y_m\right]$ and v fixed factors $z = \left[z_1, z_2, \ldots, z_v\right]$. For the TL the cost function may be specified as

$$\begin{aligned}\ln C(w,y,z) &= \alpha_0 + \sum_{i=1}^{n}\alpha_i \ln w_i + \sum_{k=1}^{m}\beta_k \ln y_k \\ &+ \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\ln w_i \ln w_j + \tfrac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{m}\beta_{kl}\ln y_k \ln y_l \\ &+ \sum_{i=1}^{n}\sum_{k=1}^{m}\delta_{ik}\ln w_i \ln y_k + \sum_{g=1}^{v}\psi_g \ln z_g + \sum_{i=1}^{n}\sum_{g=1}^{v}\rho_{ih}\ln w_i \ln z_g \\ &+ \sum_{k=1}^{m}\sum_{g=1}^{v}\varphi_{kg}\ln y_k \ln z_g + \tfrac{1}{2}\sum_{g=1}^{v}\sum_{h=1}^{v}\psi_{gh}\ln z_g \ln z_h.\end{aligned} \tag{2.1}$$

Applying (the logarithmic form of) Shephard's Lemma gives the system of input demand equations in share form

$$S_i(w,y,z) = \alpha_i + \sum_{j=1}^{n}\alpha_{ij}\ln w_j + \sum_{k=1}^{m}\delta_{ik}\ln y_k + \sum_{g=1}^{v}\rho_{ig}\ln z_g \tag{2.2}$$

$$i = 1, \ldots, n$$

where the cost shares $S_i(w,y,z)=\dfrac{w_i X_i(w,y,z)}{C(w,y,z)}$, and the $X_i(w,y,z)$ denote the cost minimizing input demand equations[1]. Regularity conditions on the cost function (2.1) lead to the following common restrictions on the system (2.2):

Symmetry: $\alpha_{ij}=\alpha_{ji}$

Homogeneity: $\sum_{j=1}^{n}\alpha_{ij}=0; i=1,\dots,n$

Adding up:

$$\sum_{j=1}^{n}\alpha_i=1;\sum_{i=1}^{n}\alpha_{ij}=0,\ j=1,\dots,n;\sum_{i=1}^{n}\delta_{ik}=0,k=1,\dots,m;\sum_{i=1}^{n}\rho_{ig}=0,g=1,\dots,v.$$

In fact these symmetry restrictions follow from the twice continuous differentiability of the cost function (2.1), so a complete set of parameter restrictions would add the further symmetry restrictions $\beta_{kl}=\beta_{lk};\psi_{gh}=\psi_{hg}$. Both the homogeneity and the adding up restrictions on system (2.2) follow from the homogeneity of degree one in *w* of the cost function. Restrictions imposed by monotonicity of the cost function (that the $X_i(w,y,z)$ be nonnegative, which also implies that the cost function is nonnegative) and the concavity of the cost function (that the Hessian matrix of the cost function i.e. the matrix $\dfrac{\partial X_i(w,y,z)}{\partial w_j}$ be negative semidefinite) are usually not imposed a priori, but may be checked for a particular sample ex post. In typical empirical applications (some references here), a subset of *n*-1 shares from the system (2.2) is estimated. Sometimes this system is augmented by the cost function (2.1), in which case the additional symmetry restrictions become relevant. Applications of the Translog function are too numerous to list. Examples of papers that estimates the share system alone are Binswanger (1974) and Fuss (1977) , while an example of a share system complemented by the cost function is Sickles and Streitwieser (1998).

In the case of the NQ, define $C'(w,y,z)$ and $w'=\left[w'_1,w'_2,...w'_{n-1}\right]$ as the total variable cost and a subvector of input prices, both normalized by the price of the n^{th} input. The cost function in NQ form typically has a representation like the following expression (typical reference?):

[1] At this stage, the notation is that lower case letters denote exogenous variables, while uppercase letters represent the functions representing the decision variables as functions of the givens, the exogenous variables. At the estimation stage there will also be observed values of endogenous variables, and this distinction will be discussed further in section 3.

$$C'(w,y,z)=\alpha_0+\sum_{i=1}^{n-1}\alpha_i w'_i+\sum_{k=1}^{m}\beta_k y_k+\tfrac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\alpha_{ij}w'_i\,w'_j$$
$$+\tfrac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{m}\beta_{kl}y_k y_l+\sum_{i=1}^{n-1}\sum_{k=1}^{m}\delta_{ik}w'_i\,y_k+\sum_{g=1}^{v}\psi_g z_g \quad (2.3)$$
$$+\sum_{i=1}^{n-1}\sum_{g=1}^{v}\rho_{ig}w'_i\,z_g+\sum_{k=1}^{m}\sum_{g=1}^{v}\varphi_{kg}y_k z_g+\tfrac{1}{2}\sum_{g=1}^{v}\sum_{h=1}^{v}\psi_{gh}z_g z_h$$

where for simplicity the notation for parameters is approximately as in the TL specification, but of course the individual parameters here have a different interpretation. A system of cost-minimizing input quantity equations can be derived by applying Shephard's lemma:

$$X_i\left(w,y,z\right)=\frac{\partial C}{\partial w_i}=\frac{\partial C}{\partial C'}\frac{\partial C'}{\partial w_i'}\frac{\partial w_i'}{\partial w_i}=w_n\frac{\partial C'}{\partial w_i'}\frac{1}{w_n}=\frac{\partial C'}{w'_i}$$
$$=\alpha_i+\sum_{j=1}^{n-1}\alpha_{ij}w'_j+\sum_{k=1}^{m}\delta_{ik}y_k+\sum_{g=1}^{v}\rho_{ig}z_g \quad (2.4)$$
$$i=1,\ldots,n-1.$$

The homogeneity in prices condition is maintained by the normalization process, the symmetry of price effects is satisfied by the restrictions $\alpha_{ij}=\alpha_{ji}$ and the global concavity in prices of this cost function is equivalent to the restriction that the matrix of parameters $\left[\alpha_{ij}\right]_{n-1\times n-1}$ be negative semi-definite, (which, not being a function of variables, can be imposed in estimation by means of the Cholesky decomposition). Monotonicity of the cost function in input prices, which corresponds to nonnegativity of the input demands, is usually not imposed a priori, but may be checked for a particular sample ex post. Again, note that no other symmetry type restrictions occur in the equation system (2.4), because none of the parameters β_{kl},ψ_{gh} appears in these equations, but the cost function parameters would also satisfy $\beta_{kl}=\beta_{lk}$ and $\psi_{gh}=\psi_{hg}$.

In most empirical applications, the system of *n*-1 demands (specifically excluding the equation for the numeraire) of the system (2.4) is estimated. Sometimes this system is also augmented by the cost function (2.3), but rarely by the demand function for the numeraire input *n*.

The implied equation for the numeraire commodity, $X_n(w,y,z)$ can be derived as follows. By definition,

$$C(w,y,z)\equiv\sum_{i=1}^{n}w_i X_i(w,y,z)$$

which implies that

$$C' = \frac{C}{w_n} \equiv \sum_{i=1}^{n} w_i' X_i(w, y, z)$$

and hence

$$X_n(w, y, z) \equiv C'(w, y, z) - \sum_{i=1}^{n-1} w_i' X_i(w, y, z)$$

where the dependence on w, y, z is to remind us that this is a relation among functions, as well as among variables. Note in passing the asymmetry of this expression; that this is an identity involving the *quantity* of the n^{th} factor, but the (normalized) *expenditures* on the other n-1 inputs, together with normalized cost. However, the implications of this inherent asymmetry are probably easier to see in terms of the implied cost function, rather than the normalized cost function. The implied structure of the corresponding (un-normalized) cost function is

$$\begin{aligned} C(w, y, z) &= \alpha_0 w_n + \sum_{i=1}^{n-1} \alpha_i w_i + \sum_{k=1}^{m} \beta_k y_k w_n + \tfrac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{ij} w_i w_j' \\ &+ \tfrac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{m} \beta_{kl} y_k y_l w_n + \sum_{i=1}^{n-1} \sum_{k=1}^{m} \delta_{ik} w_i y_k + \sum_{i=1}^{n-1} \sum_{g=1}^{v} \rho_{ig} w_i z_g \\ &+ \sum_{g=1}^{v} \psi_g z_g w_n + \sum_{k=1}^{m} \sum_{g=1}^{v} \varphi_{kg} y_k z_g w_n + \tfrac{1}{2} \sum_{g=1}^{v} \sum_{h=1}^{v} \psi_{gh} z_g z_h w_n \end{aligned} \tag{2.5}$$

which emphasises the extremely asymmetric treatment of the n^{th} input. Applying Shephard's Lemma then gives the n input demand equations directly:

$$X_i(w, y, z) = \frac{\partial C(w, y, z)}{w_i} = \alpha_i + \sum_{j=1}^{n-1} \alpha_{ij} w_j' + \sum_{k=1}^{m} \delta_{ik} y_k + \sum_{g=1}^{v} \rho_{ig} z_g \qquad i = 1, \ldots, n-1$$

$$\begin{aligned} X_n(w, y, z) = \frac{\partial C(w, y, z)}{w_n} &= \alpha_0 + \sum_{k=1}^{m} \beta_k y_k - \tfrac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{ij} w'_i \, w_j' \\ &+ \tfrac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{m} \beta_{kl} y_k y_l + \sum_{k=1}^{m} \sum_{g=1}^{v} \varphi_{kg} y_k z_g + \sum_{g=1}^{v} \psi_g z_g + \tfrac{1}{2} \sum_{g=1}^{v} \sum_{k=1}^{v} \psi_{gh} z_g z_h \end{aligned}$$

which further emphasises the asymmetric treatment of input n. Examples of empirical applications of the NQ functional form include Shumway (1983) and Moschini (1988).

3. Estimation Issues

These two alternative "standard" specifications, which account for the majority of empirical applications in areas as diverse as agricultural economics and modelling energy demand, raise a number of questions. The two particular functional forms, TL and NQ, are merely two common alternative specifications of the technology as characterized by the cost function. So apart from mathematical simplicity, why carry out estimation based on shares in one case, and quantities demanded in another? The appropriate choice of transformation of endogenous variables for estimation purposes should be based on the implied statistical properties of the error terms, which are usually introduced after this specification stage. Is it even appropriate to estimate a set of *demand* equations such as (2.4)? Is it logical, or even necessary, to delete the *n*th equation of such a system? Can, or should, the cost function be appended to either of the systems (2.2) or (2.4)?

To begin to answer these questions, abstract from the issues raised by specific choices of functional forms. The starting point is the theory of duality. Given a primal technology characterised by $V\left(y,z\right)$, where $x \in V\left(y,z\right)$ indicates that, given the vector of fixed factors *z*, the output vector *y* can be produced by the input vector *x*, then the behaviour of a cost minimizing firm can be characterised by the cost function $C(w,y,z)$ which is defined as the minimum cost of producing a given vector of outputs subject to the production technology:

$$C(w,y,z) = \min_{x}\left\{\sum_{i=1}^{n} w_i x_i : x \in V\left(y,z\right)\right\} = \sum_{i=1}^{n} w_i X_i(w,y,z) . \tag{3.1}$$

Then $C(w,y,z)$ satisfies the standard regularity conditions of a cost funtion: non-negative; concave in *w*; non-decreasing in *w*; homogeneous of degree 1 in *w*; increasing in *y*; and $C(w,y,z)$ is said to be dual to the specification of technology *V*. The structure of *C* contains both the structure of technology and the results of optimization, in this case cost minimization. The resulting demand equations are often called Hicksian demands, in order to reinforce the analogy with consumer demand, where they would correspond to income compensated (utility constant) demands. Application of the Envelope Theorem to (3.1) gives Shephard's Lemma:

$$X_i\,(w,y,z) = C_{w_i}(w,y,z) = \frac{\partial C(w,y,z)}{\partial w_i} \tag{3.2}$$

the observable input demand equations. These equations are analogous to the Hicksian demand equations of consumer demand theory (though in this case the output vector *y* is observable, whereas in Hicksian demands the level of utility is unobservable), and provided the cost function is twice continuously differentiable in prices then Young's Theorem implies

$$\frac{\partial X_i}{\partial w_j} = C_{w_i w_j} = C_{w_j w_i} = \frac{\partial X_j}{\partial w_i} \tag{3.3}$$

which is the equivalent of Slutsky symmetry in consumer demand, and here implies the symmetry of the cross price responses of input demand equations.

By the property of homogeneity of degree one of the cost function, Euler's Theorem implies the fundamental identity that

$$C(w, y, z) \equiv \sum_{i=1}^{n} w_i C_{w_i}(w, y, z) \equiv \sum_{i=1}^{n} w_i X_i(w, y, z). \tag{3.4}$$

In the notation implicit in the above, and that has been used so far, the exogenous variables, the variables taken as given in the optimization problem (3.1), have been denoted by lower case letters w, y, z and can be identified directly with the data on these variables. The decision variables in the optimization problem (3.1) have been denoted by capital letters, $C(w, y, z), X_i(w, y, z), S_i(w, y, z), i = 1,\ldots, n$ indicating that they are derived functions of the exogenous variables. The endogenous variables will be identified with the observed data corresponding to these decision variables, and will be denoted by the corresponding lower case letters $c, x_i, s_i, i = 1,\ldots, n$. In practice, of course, the model does not fit the data exactly, and this leads to the specification of an approriate statistical model which corresponds to a parameterized form of the cost function, say $C(w, y, z;\beta)$ (where at this point β simply represents all possible parameters characterising the cost function) which implies the specification of the conditional means, plus a set of errors, which together lead to a set of stochastic equations which might be written most simply as follows:

$$\begin{aligned} c_t &= C(w_t, y_t, z_t;\beta) + v_t \\ x_{it} &= X_i(w_t, y_t, z_t;\beta) + u_{it}, \quad i = 1,\ldots, n \end{aligned} \tag{3.5}$$

where the subscript t denotes variation over a sample. SUR or ML type estimation of this system would implicitly assume that the n+1 vector of errors $\left(u_{1t}, u_{2t},\ldots, u_{nt}, v_t\right)$ be i.i.d with constant (parameterized) variances and covariances. But this would lead to a logical inconsistency. The data will satisfy $c_t \equiv \sum_{i=1}^{n} w_{it} x_{it}$ by construction, and this identity together with the functional identity (3.4) produces an identity involving the error terms and exogenous variables:

$$v_t \equiv \sum_{i=1}^{n} w_{it} u_{it}. \tag{3.6}$$

Since the exogenous variables are arbitrary and time varying, identity (3.6) means that it is impossible for the errors of the cost function, the v_t, to have constant variance or constant covariances with the u_{it}. Thus while the *n*+1 dimensional system (3.5) is in some sense degenerate, in that any *n* dimensional subset implies the (theoretical and statistical) structure of the remaining equation, this degeneracy is data specific, and hence the usual arguments for the invariance of parameter estimates to the deleted equation do not carry through. Estimates of systems of demand equations, such as the NQ model (2.4), with or without the cost function, are based on an internal logical inconsistency!

An analogy with consumer demand systems is illuminating in a number of respects. While demand equations for goods are of fundamental interest in the theory of consumer demand, empirical work in consumer demand rarely (if ever?) estimates demand equations directly. Typically estimation is of expenditure systems, such as the Linear Expenditure system, or share systems, such as the Almost Ideal Demand System AIDS or the Translog. In each case this consists of the use of exogenous variables to transform one set of endogenous variables, the demands, to derived sets of endogenous variables, either to expenditures by multiplying individual demands by their corresponding prices, or further to shares by then dividing expenditures by total expenditure. Individual *demands* are subject to a data-dependent degeneracy. In contrast, individual *expenditures* add identically to (exogenous, in this case) expenditure, and *shares* add identically to unity, and in both cases the error in any one equation can be expressed as a (not data-dependent) linear combination of the remaining *n*-1 errors, and estimates are invariant to the equation deleted. (see, for example, Barten (1969), Powell (1969), Bewley (1986), McLaren(1990))

Based on the insights from this analogy, consider first the case of transforming input demands to expenditures. Then system (3.5) is translated to

$$\begin{aligned} c_t &= C(w_t, y_t, z_t; \beta) + v_t \\ w_{it}x_{it} &= w_{it}X_i(w_t, y_t, z_t; \beta) + u_{it}, \quad i = 1,\ldots,n \end{aligned} \tag{3.7}$$

where it may be noted that all left-hand side variables now have common units of measurement. (As is the case with the notation for parameters, the same notation will be used for errors with quite different statistical properties in different specifications. This avoids a proliferation of notation, and is meant to reinforce the assumption that any errors should have “reasonable” statistical properties.) The restriction on the error terms corresponding to (3.6) is now

$$v_t \equiv \sum_{i=1}^{n} u_{it} \tag{3.8}$$

and now the implicit assumption of SUR or ML type estimation that the *n*+1 vector of errors $\left(u_{1t}, u_{2t}, \ldots, u_{nt}, v_t\right)$ be i.i.d with constant (parameterized) variances and covariances is mathematically possible (though perhaps not empirically legitimate, since the errors have units of measurement of current dollars). At the very least, internal consistency requires that NQ demands be estimated in terms of expenditures, not demands, in order to justify either deleting one (the n^{th}) equation, or else appending the cost function to the n-1 dimensional system.

Consider now the estimation in terms of shares. In this case, the analogy with consumer demand is quite misleading (or enlightening?). In consumer demand systems, transformation to shares involves division of the endogenous variables, expenditures, by an exogenous variable, (predetermined) total expenditure. In maximizing utility subject to a budget constraint, total expenditure is given. But in minimizing cost subject to a production technology, cost is a decision variable and hence endogenous. In particular, dividing by either c or $C(w, y, z)$ is a conceptually different process.

One way to think about this is to return to the set of functions that result from the optimization problem. Logically prior to any issues of estimation, these can be written as the (theoretical and degenerate) system:

$$\begin{aligned}&C(w, y, z)\\ &w_i X_i(w, y, z) \quad i = 1, \ldots, n.\end{aligned} \tag{3.9}$$

This system can be nonlinearly transformed to a related but mathematically equivalent system by dividing each of the last n equations by the first equation:

$$\begin{aligned}&C(w, y, z)\\ &S_i(w, y, z) \quad i = 1, \ldots, n.\end{aligned} \tag{3.10}$$

(The complete sequence of transformations of the demand equations of: multiply by w_i; divide by $C(w, y, z)$; is mathematically equivalent to applying Shephard's Lemma in logarithmic form instead of Shephard's Lemma in normal form.)

The corresponding empirical form of system (3.10) is

$$\begin{aligned}c_t &= C(w_t, y_t, z_t; \beta) + v_t\\ s_{it} &= S_i(w_t, y_t, z_t; \beta) + u_{it}, \quad i = 1, \ldots, n.\end{aligned} \tag{3.11}$$

In this form, the data will satisfy the identity $\sum_{i=1}^{n} s_{it} \equiv 1$ by construction, and this identity together with the functional identity $\sum_{i=1}^{n} S_i\left(w_t, y_t, z_t\right) \equiv 1$ produces an identity involving the last n of the error terms:

$$\sum_{i=1}^{n} u_{it} \equiv 0\,. \tag{3.12}$$

It is well-known that estimation of the share sub-system of (3.11) will be invariant to the deletion of any one of the *n* share equations. Similarly, estimation of the complete system (3.11) will be invariant to the deletion of any one of the *n* share equations. While this complete system (3.11) may be mathematically logical, it is unlikely to be empirically admissible. The reason for estimating in share form is to transform to a system where the errors are likely to be homoscedastic, by expressing the left hand variables as shares, bounded between 0 and 1, rather than as variables measured in current dollars. But one equation of the complete system, the cost equation, is still measured in current dollars. It is empirically unsustainable to assume that the errors of the cost function, the v_t, have constant variance or constant covariances with the u_{it}. Estimation of any of the *n*-1 shares as a system on their own may be reasonable, but appending the cost function (either in levels form or in logarithmic form) in order to recover estimates of the additional parameters seems unadvisable. Thus it appears inadvisable to add the cost function to a TL share system.

Returning to the analogy with consumer demand systems, if what is sought is a logically complete system of equations for which an assumption of homoscedasticity of the vector of errors may be empirically admissible, scaling of data should be carried out using an exogenous variable. For the cost function model, obvious candidates are the exogenous variables *w*, *y* and *z*. The variables in *w* are not firm or scale specific and would seem unsuitable on their own. Variables in *y* and some *z*, such as fixed factors, are observation and scale specific; however these variables are real, whereas the variables in the logically complete system are nominal. One option would be to remove price effects by first dividing all of the theoretical equations by an index of the *w*, or some form of generally available price index, such as the CPI or a PPI, and then by some measure of scale, such as an index of the elements of *y* and/or those elements of *z* that are observation specific. However, if the prices of outputs are available, a more obvious procedure is the following. If the prices of outputs are given by the *n* vector *p*, define (observed) revenue as $r_t \equiv \sum_{j=1}^{m} p_{jt} y_{jt}$. This variable has the advantage of being measured in nominal dollars, and being a natural measure of the size of the particular firm.

Now modify system (3.7) to

$$\begin{aligned} \frac{c_t}{r_t} &= \frac{C(w_t, y_t, z_t; \beta)}{r_t} + v_t \\ \frac{w_{it} x_{it}}{r_t} &= \frac{w_{it} X_i(w_t, y_t, z_t; \beta)}{r_t} + u_{it}, \quad i = 1, \ldots, n. \end{aligned} \tag{3.13}$$

This system of *n*+1 equations contains all of the information from the paradigm of cost minimization, and embodies a set of errors that are logically consistent and likely to satisfy the implicit assumptions involved in ML or SUR estimation. Again, the errors satisfy the identity $v_t \equiv \sum_{i=1}^{n} u_{it}$ and estimation can proceed by estimating any *n* dimensional subset of equations. Parameter estimates will be invariant to the equation deleted. Note that the equation corresponding to the cost function has no specific role in system(3.13), such as allowing the estimation of parameters that are unavailable from the other equations. This is obvious since *any* equation in (3.13) can be expressed as a linear combination of the other n equations. The fact that appending the Translog cost function (2.1) to the share system (2.2), or appending the NQ cost function (2.3) to the demand system (2.4), allows the estimation of additional parameters such as the $\alpha_0, \beta_k, \beta_{kl}, \phi_{kg}, \psi_g, \psi_{gh}$ simply reflects the loss of information that occurs when estimating arbitrarily restricted systems such as (2.2) or (2.4).

4. Further Issues with the Estimation of NQ Systems

The above issues are further confused in the case of the NQ system, because of the naturally asymmetric treatment of the n^{th} factor. In fact, there are always n possible specific NQ specifications, according to which of the n inputs is treated as the numeraire. Applying the reasoning above, an econometrically compatible system of n+1 equations is given by the cost function plus the input expenditure equations:

$$\begin{aligned} w_i x_i &= \alpha_i w_i + \sum_{j=1}^{n-1} \alpha_{ij} w'_j w_i + \sum_{k=1}^{m} \delta_{ik} y_k w_i + \sum_{g=1}^{v} \rho_{ig} z_g w_i \quad i = 1,\ldots,n-1 \\ w_n x_n &= \alpha_0 w_n + \sum_{k=1}^{m} \beta_k y_k w_n - \tfrac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{ij} w_i w'_j + \tfrac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{m} \beta_{kl} y_k y_l w_n \\ &\quad + \sum_{k=1}^{m} \sum_{g=1}^{v} \varphi_{kg} y_k z_g w_n + \sum_{g=1}^{v} \psi_g z_g w_n + \tfrac{1}{2} \sum_{g=1}^{v} \sum_{h=1}^{v} \psi_{gh} z_g z_h w_n. \end{aligned} \qquad (3.14)$$

The above system highlights a more fundamental asymmetry in the treatment of inputs in the NQ model[2]. Comparing the first n-1 expenditure equations with the expenditure equation for input n, it is clear that all higher order interaction terms in the cost function have in fact been loaded on the n^{th} factor. This probably explains why attempts to estimate systems of n-1 demand equations plus the cost function usually fail. There appears no logical reason why the n^{th} factor should be singled out for this responsibility. One possible response is to note that while the standard specification of the NQ cost function appears general in regard to its specification in terms of the variables *w*, *y*, and *z*, it is in fact highly constrained in regards to its treatment of the technical characteristics of the n^{th} factor relative to its treatment of all other factors. More logically, given the structure of the first n-1 expenditure equations it would seem that a reasonable set of n input demand equations, still using the n^{th} price as a numeraire, would be to specify the system

[2] Symmetric treatments of the NQ system do exist, but they introduce a number of unidentified additional parameters. See Diewert and Wales (1987).

$$
\begin{aligned}
w_i x_i &= \alpha_i w_i + \sum_{j=1}^{n-1} \alpha_{ij} w_j' w_i + \sum_{k=1}^{m} \delta_{ik} y_k w_i + \sum_{g=1}^{v} \rho_{ig} z_g w_i \quad i = 1,\ldots,n-1 \\
w_n x_n &= \alpha_0 w_n - \tfrac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{ij} w_i w_j' + \sum_{k=1}^{m} \beta_k y_k w_n + \sum_{g=1}^{v} \psi_g z_g w_n
\end{aligned} \tag{3.15}
$$

with the implied cost function

$$
\begin{aligned}
C(w_t, y_t, z_t; \beta) = \alpha_0 w_n &+ \sum_{i=1}^{n-1} \alpha_i w_i + \sum_{k=1}^{m} \beta_k y_k w_n + \tfrac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{ij} w_i w_j' \\
&+ \sum_{i=1}^{n-1} \sum_{k=1}^{m} \delta_{ik} w_i y_k + \sum_{i=1}^{n-1} \sum_{g=1}^{v} \rho_{ig} w_i z_g + \sum_{g=1}^{v} \psi_g z_g w_n .
\end{aligned} \tag{3.16}
$$

This is a complete n+1 equation system which is both economically and statistically degenerate – any n dimensional subset could be estimated, and estimates would be invariant to the equation deleted. Of course, the units of measurement of all equations are now (current) dollars, and homoscedasticity may require scaling all equations by a common measure of price (such as w_n, or more reasonably, a measure of revenue). This system would be a parsimonious system that may be preferable to a completely symmetric treatment, such as using a generalized Barnett or generalized McFadden system. It is also interesting to note the potential simplification in notation that is suggested by this form. (α_0 becomes α_n , β_k becomes δ_{nk} , ψ_g becomes ρ_{ng})

5. Conclusion

This paper has considered a number of theoretical issues that arise in the statistical specification of a number of commonly used empirical specifications based on a cost function specification of technology and optimizing behaviour, with special emphasis on the Translog and Normalized Quadratic functional forms. By analogy, similar issues arise when specifications are based on alternative representations of technology, such as profit functions and revenue functions. A companion paper will illustrate all of the issues raised in this paper by estimating all of the specifications considered for a large dataset comprising pseudo-micro level observations on Australian broadacre agriculture.

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