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Working Paper n° 5, 1998

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SOMMARIO

Default Bayes Factors for One-Sided Hypothesis Testing *

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November 6, 1997

Abstract

Bayesian hypothesis testing for non-nested hypotheses is studied, using various "default" Bayes factors, such as the fractional Bayes factor, the median intrinsic Bayes factor and the encompassing and expected intrinsic Bayes factors. The different default methods are first compared with each other and with the p-value in normal one-sided testing, to illustrate the basic issues. General results for one-sided testing in location and scale models are then presented. The default Bayes factors are also studied for specific models involving multiple hypotheses. In most of the examples presented we also derive the intrinsic prior; this is the prior distribution which, if used directly, would yield answers (asymptotically) equivalent to those for the given default Bayes factor.

Some key words and phrases: Bayes factor, fractional Bayes factor, intrinsic Bayes factor, model comparison, one-sided hypothesis testing, multiple hypothesis testing.

1 Introduction

Bayesian testing and model selection has been undergoing extensive development because of recent advances in the creation of "default" Bayes factors that can be used in the absence of subjective prior information. Two very general such default Bayes factors are the fractional Bayes factor of O'Hagan (1995) and the intrinsic Bayes factor of Berger and Pericchi (1996). These

*This research was supported by the National Science Foundation (USA), Grant DMS-9303556, and by Murst and CNR (Italy).

methodologies have been applied, with great success, in a variety of settings, and have undergone extensive study in situations involving testing of nested hypotheses or models.

In non-nested hypothesis testing problems, such as in the one-sided testing of $H_1 : \theta < 0$ versus $H_2 : \theta > 0$, there has been less interest in use of these new default Bayes factors. One reason is that, in such situations, it is often felt to be legitimate to perform a default Bayesian analysis directly with standard noninformative prior distributions. Thus, in one-sided testing of a normal mean θ, it would be common to use the noninformative prior $\pi(\theta) = 1$, and directly compute the Bayes factor of H_1 to H_2. A variety of arguments can be given which suggest that this is reasonable from a Bayesian perspective, at least with large sample sizes and data that is not too extreme. (In contrast, one cannot directly use noninformative priors to compute Bayes factors for nested hypotheses; this was the main motivation for development of the new default methodologies mentioned above.) A related reason why Bayesians have had less interest in the one-sided testing situation is that it is perceived that classical methods, such as the p-value, are reasonable for such problems. Indeed, for one-sided testing of a normal mean, it is known that the p-value yields essentially the same answer as does direct Bayesian analysis with a noninformative prior.

Study of the new default methodologies for one-sided (and related) hypothesis testing problems is of interest, however, for several reasons. First, the legitimacy of using noninformative priors directly has only been established in rather simple situations, whereas the new default Bayesian methodologies appear to be very widely applicable. Hence verification of the success of the new methodologies in this domain can greatly broaden the practical use of Bayes factors therein. Note that, in one-sided testing, the Bayesian Information Criterion (BIC) or Schwarz criterion (Schwarz, 1978) can not be directly applied (although Haughton and Dudley, 1992, and Kass and Vaidyanathan, 1992, give generalizations which can apply). Related to this is the value of understanding the properties of the new methodologies in non-nested situations. For instance, in nested situations, these methodologies typically yield Bayes factors which are approximate Bayes factors for (proper) default prior distributions (named "intrinsic priors" in Berger and Pericchi, 1996). Understanding the extent to which this is true for non-nested testing is of interest in helping to judge the domains in which the new methodologies can be successfully applied. Indeed, study of the new default Bayes factor methodologies in one-sided testing will be seen to be useful for comparing their relative success overall, leading to eventual recommendations as to which should be used, and when. Recent references which provide examples of the use of default Bayes factors in one-sided testing include Dmochowski (1994), Lingham and Sivaganesan (1996) and Moreno (1997).

Perhaps the most important reason for consideration of the default Bayesian methods in one-sided testing is that they can lead to substantially different (and arguably better) answers than classical methods. As this seems to contradict the "common wisdom" mentioned above, it is worthwhile to somewhat expand on the point initially. The fact that direct use of noninformative priors often yields answers equivalent to the p-value in one-sided testing is thoroughly discussed in Casella and Berger (1987). In particular, they show, for location parameter problems, that the p-value is the lower bound of the posterior probability of H_1, over reasonable classes of prior densities. (For instance, in the above one-sided testing of a normal mean, this would hold for the class of all symmetric unimodal prior densities.) Coincidentally, this lower bound also arises from direct use of noninformative priors.

It is natural to ask, however, if the lower bound is the best evidential summary to provide. In Bayesian terms, if it were the case that the posterior probability of H_1 were some number between 0.05 and 0.5, depending on assumptions, would it really be reasonable to report 0.05 as the evidential summary (which the p-value or the non-informative prior effectively do), especially as this is the answer which is least favorable to (the often 'privileged') H_1? This point was dramatically illustrated in the discussion by Morris of the Casella and Berger article (Morris, 1987), wherein a reasonable practical situation was considered and it was demonstrated that the lower bound is a misleading measure of the evidence against H_1. From a Bayesian perspective, Morris's argument was essentially that typical prior beliefs will concentrate closer to the dividing line between the hypotheses (zero in the one-sided testing problem mentioned above), and that using a prior distribution which is extremely diffuse is thus unreasonable, at least for small or moderate sample sizes. (It can be shown that, for large sample sizes, use of diffuse priors for one-sided testing is satisfactory when the data is not too extreme.) Interestingly, we will see that the new default Bayesian methodologies do produce answers which are not as extreme as the "standard" (classical or Bayesian) answers.

In Section 2, we introduce the four default Bayes factors that will be studied, and illustrate their computation for the one-sided normal testing problem. The four default Bayes factors considered are the *fractional Bayes* *factor (FBF), the median intrinsic Bayes factor (MIBF), the encompassing intrinsic Bayes factor (EIBF), and the expected encompassing intrinsic Bayes factor (EEIBF).* Note that there are a variety of other possible implementations of the intrinsic Bayes factor methodology of Berger and Pericchi (1996). Indeed, we initially considered the full range of IBFs that could be used for this problem, but we eventually studied, in depth, only the three that seemed most suitable for one-sided testing; interestingly, the "standard" arithmetic and geometric IBFs are not particularly suitable here (see Dmochowski, 1994).

Section 2.5 introduces the key notion of intrinsic priors. These are prior distributions which would yield Bayes factors equivalent to the studied default Bayes factors, in a certain asymptotic sense. Intrinsic priors are very useful for detecting "biases" or other inadequacies of default Bayes factors. Furthermore, they can be used directly as default priors in computing Bayes factors; this may be especially useful for very small sample sizes. In Section 2.6, we compare the use of the various Bayes factors with each other and with the standard Bayesian answer (which is also the p-value) for the one-sided normal testing situation. This indicates the magnitude of the differences that can result from use of the new default methods. In Section 3 we consider the general one-sided testing situation for location models, scale models, and specific location-scale models. Interesting differences among the various default Bayes factors are found.

Section 4 considers a multiple hypothesis testing situation, namely testing $H_1 : \theta = 0$ versus $H_2 : \theta < 0$ versus $H_3 : \theta > 0$. It is shown how the default testing methodology can be applied to such problems, and that valuable (and nonstandard) insights arise from such application. Note that p-values are not much use in such multiple hypothesis scenarios. A few concluding remarks are given in Section 5.

2 Default Bayes Factors and Intrinsic Priors

2.1 Notation

The data $\mathbf{X} = (X_1, \ldots, X_n)$ has density $f(\mathbf{x}|\theta)$, with respect to Lebesgue measure, with θ being a k-dimensional unknown parameter vector in $\Re^k$. Hypotheses $H_i : \theta \in \Theta_i$, $i = 1, \ldots, q$, are under consideration, where the Θ_i are "ordered" such as in one-sided testing of $H_1 : \theta \leq \theta_0$ versus $H_2 : \theta > \theta_0$.

Letting $\pi_i(\theta)$ denote the prior density of θ (with respect to Lesbegue measure) under H_i, the Bayes Factor (BF) for model H_j versus model H_i is

$$B_{ji} = \frac{\int_{\Theta_j} f(\mathbf{x}|\theta)\pi_j(\theta)d\theta}{\int_{\Theta_i} f(\mathbf{x}|\theta)\pi_i(\theta)d\theta} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})}, \tag{1}$$

where $m_i(\mathbf{x}), m_j(\mathbf{x})$ are the marginal distributions under H_i and H_j, respectively. When $\pi_i(\theta)$ is a noninformative (default) prior distribution, say $\pi_i^N(\theta)$, equation (1) becomes

$$B_{ji}^N(\mathbf{x}) = \frac{\int_{\Theta_j} f(\mathbf{x}|\theta)\pi_j^N(\theta)d\theta}{\int_{\Theta_i} f(\mathbf{x}|\theta)\pi_i^N(\theta)d\theta} = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})}. \tag{2}$$

Definition: A subset, $\mathbf{x}(l)$, of the data is a *minimal training sample* if $m_i^N(\mathbf{x}(l)) < \infty$, $i = 1, \ldots, q$, and no subset of $\mathbf{x}(l)$ yields finite marginals. Let $\{\mathbf{x}(l), l = 1, \ldots, L\}$ denote the collection of all minimal training samples.

2.2 The Fractional Bayes Factor

The fractional Bayes factor (FBF) (see O'Hagan, 1995) of model H_j versus model H_i is defined as

$$B_{ji}^F = B_{ji}^N(\mathbf{x}) \cdot \frac{\int_{\Theta_i} L^b(\theta)\pi_i^N(\theta)d\theta}{\int_{\Theta_j} L^b(\theta)\pi_j^N(\theta)d\theta}, \tag{3}$$

where $L(\theta) = f(x_1, \ldots, x_n|\theta)$ is the likelihood function and b specifies a "fraction" of the likelihood which, in a certain sense, is to be used as a prior density. The main difficulty in the implementation of the fractional Bayes factor is that of the choice of the fraction b. One frequently suggested choice (see the examples in O'Hagan, 1995, and the discussion by Berger and Mortera of O'Hagan, 1995) is $b = m/n$, where m is the size of the minimal training sample, assuming this is well defined. We will primarily use this choice in examples, although we will see indications that even smaller values of b may be required when n is small.

Example 1.

Let $\mathbf{X} = (X_1, \ldots, X_n)$, with X_i i.i.d. $N(\theta, 1)$. Suppose we are interested in testing $H_1 : \theta \leq 0$ vs. $H_2 : \theta > 0$. Taking $\pi_1^N(\theta) = 1_{(-\infty,0)}(\theta)$ and $\pi_2^N(\theta) = 1_{(0,\infty)}(\theta)$, the standard noninformative priors, we have

$$\begin{aligned} B_{21}^N(\mathbf{x}) &= \frac{\int_0^\infty (2\pi)^{-n/2} \exp\{-[n(\bar{x}-\theta)^2 + s^2]/2\}d\theta}{\int_{-\infty}^0 (2\pi)^{-n/2} \exp\{-[n(\bar{x}-\theta)^2 + s^2]/2\}d\theta} \\ &= \Phi(\sqrt{n}\bar{x})/[1 - \Phi(\sqrt{n}\bar{x})], \end{aligned} \tag{4}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2$ and Φ is the standard normal c.d.f. Simple calculations then yield

$$B_{21}^F = B_{21}^N \cdot \frac{[1 - \Phi(\sqrt{bn}\bar{x})]}{\Phi(\sqrt{bn}\bar{x})}. \tag{5}$$

The minimal training sample size is $m = 1$, so the common choice of b is $b = 1/n$, in which case

$$B_{21}^F = B_{21}^N \cdot \frac{[1 - \Phi(\bar{x})]}{\Phi(\bar{x})}. \tag{6}$$

2.3 The Median Intrinsic Bayes Factor

The median intrinsic Bayes factor (MIBF) was developed in Berger and Pericchi (1996, 1997b) and is defined as

$$B_{ji}^{MI} \quad = \quad B_{ji}^N(\mathbf{x}) \cdot \text{Me}[B_{ij}^N(\mathbf{x}(l))], \tag{7}$$

where Me denotes the median, here to be taken over all the training sample Bayes factors. When each $x(l)$ is a real variable and $B_{ij}^N(\cdot)$ is monotonic, as is often the case in one-sided testing, then $\text{Me}[B_{ij}^N(\mathbf{x}(l))] = B_{ij}^N(\text{Me}[\mathbf{x}(l)])$, an attractive simplification.

Example 1 (continued). Here, a minimal training sample is a single observation, x_i. From (4), $B_{12}^N(x_i) = [\Phi(x_i)]^{-1} - 1$, which is monotonic in x_i, so that the MIBF is simply

$$B_{21}^{MI} = B_{21}^N \cdot (\frac{1}{\Phi(\text{Me}[x_i])} - 1). \tag{8}$$

2.4 The Encompassing and Expected Intrinsic Bayes Factors

2.4.1 The Encompassing Arithmetic Intrinsic Bayes Factor

Arithmetic IBF's (see Berger and Pericchi, 1996) are often not suitable for non-nested situations, especially when testing one-sided hypotheses as here (see Dmochowski, 1994). An attractive alternative, given in Berger and Pericchi (1995, 1996), is to embed the competing models in a larger encompassing model, say H_0, so that all the H_i are nested within H_0. The encompassing arithmetic intrinsic Bayes factor (EIBF) is then defined as

$$B_{ji}^{EI} \quad = \quad B_{ji}^N(\mathbf{x}) \cdot \frac{\sum_{l=1}^{L} B_{i0}^N(\mathbf{x}(l))}{\sum_{l=1}^{L} B_{j0}^N(\mathbf{x}(l))}. \tag{9}$$

Example 1 (continued). The natural encompassing model is $H_0 : \theta \in \Re^1$ with corresponding default prior distribution $\pi_0^N(\theta) = 1$. A minimal training sample is still a single observation, x_i, and easy computation yields

$$B_{10}^N(x_i) = 1 - \Phi(x_i), \quad B_{20}^N(x_i) = \Phi(x_i),$$

$$B_{21}^{EI} = B_{21}^N \cdot \frac{\sum_{i=1}^n (1 - \Phi(x_i))}{\sum_{i=1}^n \Phi(x_i)}.$$

2.4.2 The Encompassing Expected Intrinsic Bayes Factor

With a small sample size, the summations in (9) can be quite unstable. A natural idea, therefore, is to replace them by their expectation under the encompassing model H_0, evaluated at the MLE, $\hat{\theta}$, under H_0. This results in the encompassing expected IBF (EEIBF) which, when the X_i are exchangeable, is given by

$$B_{ji}^{EEI} = B_{ji}^N(\mathbf{x}) \cdot \frac{E_{\hat{\theta}}^{H_0}[B_{i0}^N(\mathbf{X}(l))]}{E_{\hat{\theta}}^{H_0}[B_{j0}^N(\mathbf{X}(l))]}. \tag{10}$$

Example 1(continued). Noting that the MLE under H_0 is $\hat{\theta} = \bar{x}$, the EEIBF is

$$B_{21}^{EEI} = B_{21}^N \cdot \frac{E_{\hat{\theta}}^{H_0}[1 - \Phi(X_i)]}{E_{\hat{\theta}}^{H_0}[\Phi(X_i)]} = B_{21}^N \cdot \frac{1 - \Phi(\bar{x}/\sqrt{2})}{\Phi(\bar{x}/\sqrt{2})}. \tag{11}$$

Interestingly, this equals the fractional Bayes factor with $b = 1/(2n)$.

2.5 Intrinsic Priors

It is of considerable interest to determine if default Bayes factors behave similarly to real Bayes factors and, if so, which prior distributions, called *intrinsic priors*, would yield (an approximate) equivalence. This can provide considerable insight into the behaviour of the default Bayes factors, especially in detecting "biases" of the default Bayes factors towards one of the hypotheses. In addition, intrinsic priors can be directly used to compute Bayes factors, and can be especially useful in this regard when the sample size is very small.

The issue was explored in Berger and Pericchi (1996, 1997a), but primarily for nested models. With non-nested models and, in particular, one-sided testing, it is appropriate to slightly change the question, and ask if default Bayes factors are equivalent to some "real" posterior odds ratio. To be more precise, suppose we are testing $H_1 : \theta \in \Theta_1$ versus $H_2 : \theta \in \Theta_2$ and let $\pi(\theta)$ denote the prior density on $\Theta = \Theta_1 \cup \Theta_2$. The $\pi_i(\theta)$ defined earlier would thus be $\pi_i(\theta) = \pi(\theta)1_{\Theta_i}(\theta)/\text{pr}(H_i)$, where $\text{pr}(H_i) = \int_{\Theta_i} \pi(\theta)d\theta$ is the prior probability that H_i is true. The posterior odds ratio of H_2 to H_1 is then

$$PO_{21}(x) = \frac{\int_{\Theta_2} f(\mathbf{x}|\theta)\pi(\theta)d\theta}{\int_{\Theta_1} f(\mathbf{x}|\theta)\pi(\theta)d\theta}.$$

We now ask if there is a prior density $\pi^I(\theta)$, the *intrinsic prior*, for which a default Bayes factor, B, asymptotically corresponds to PO_{21}, in the sense that $B/PO_{21} \to 1$ as the sample size goes to infinity. (Note that this is a stronger requirement than, say, the asymptotic equivalence of BIC, which is only concerned with equivalence up to an unspecified constant multiple). This question can be answered as in Berger and Pericchi (1996, 1997a). Writing $\pi_i^I(\theta) = \pi^I(\theta)1_{\Theta_i}(\theta)$, and representing a default Bayes factor as $B_{21} = B_{21}^N \cdot B^{CF}$, where B^{CF} is referred to as the *correction factor*, an intrinsic prior is defined as a solution, $\pi^I(\theta)$, such that the $\pi_i^I(\theta)$ are continuous on Θ_i, to the equations

$$\frac{\pi_2^I(\theta)\pi_1^N(\psi_1(\theta))1_{\Theta_2}(\theta)}{\pi_1^I(\psi_1(\theta))\pi_2^N(\theta)} = B_2^*(\theta) \qquad (12)$$

$$\frac{\pi_1^I(\theta)\pi_2^N(\psi_2(\theta))1_{\Theta_1}(\theta)}{\pi_2^I(\psi_2(\theta))\pi_1^N(\theta)} = \frac{1}{B_1^*(\theta)}, \qquad (13)$$

where, for $i = 1, 2$,

$$B_i^*(\theta) = \lim_{n\to\infty} B^{CF} \text{ under } H_i$$

and (with $i \neq j$, both being 1 or 2)

$$\psi_i(\theta) = \lim_{n\to\infty} \hat{\theta}_i \text{ under } H_j\ ,$$

with $\hat{\theta}_i$ being the MLE under H_i. If $\psi_i(\theta)$ is not in Θ_i, define $\pi_i^I(\psi_i)$ by continuity. The needed technical conditions are primarily the existence of the above quantities and limits.

Example 1(continued). For B_{21}^F in (5) and the choice $b = m^*/n$,

$$\begin{aligned} B_i^*(\theta) &= \lim_{n\to\infty}([1 - \Phi(\sqrt{bn}\bar{x})]/\Phi(\sqrt{bn}\bar{x})) \\ &= \frac{1 - \Phi(\sqrt{m^*}\theta)}{\Phi(\sqrt{m^*}\theta)}, \quad \text{for } \theta \in \Theta_i. \end{aligned}$$

Under H_1 and H_2, the MLE's are, respectively, $\hat{\theta}_1 = min\{\bar{x}, 0\}$ and $\hat{\theta}_2 = max\{\bar{x}, 0\}$, so that

$$\begin{aligned} \psi_1(\theta) &= \lim_{n\to\infty, H_2} min\{\bar{x}, 0\} = 0, \\ \psi_2(\theta) &= \lim_{n\to\infty, H_1} max\{\bar{x}, 0\} = 0. \end{aligned}$$

Hence the intrinsic prior equations become (recalling that $\pi_i^N(\theta) = 1$)

$$\begin{aligned} \frac{\pi_2^I(\theta) 1_{\Theta_2}(\theta)}{\pi_1^I(0)} &= \frac{1 - \Phi(\sqrt{m^*}\theta)}{\Phi(\sqrt{m^*}\theta)} 1_{\Theta_2}(\theta) \\ \frac{\pi_1^I(\theta) 1_{\Theta_1}(\theta)}{\pi_2^I(0)} &= \frac{\Phi(\sqrt{m^*}\theta)}{1 - \Phi(\sqrt{m^*}\theta)} 1_{\Theta_1}(\theta). \end{aligned}$$

The unique proper solution to these equations is

$$\pi^I(\theta) = \begin{cases} c[(\Phi(\sqrt{m^*}\theta)^{-1} - 1]^{-1} & \text{if } \theta < 0 \\ c[(\Phi(\sqrt{m^*}\theta)^{-1} - 1] & \text{if } \theta > 0, \end{cases} \tag{14}$$

where c is chosen so that $\pi(\theta)$ is proper. Note that this is a scale family of priors, with scale factor $(m^*)^{-1/2}$. Hence different choices of b (or m^*) can give answers varying over a wide range. For comparisons, we will choose $m^* = 1$ (the minimal training sample size).

Nearly identical analysis shows that the intrinsic prior for B_{21}^{MI} is as in (14) with $m^* = 1$ (agreeing with the FBF), while for B_{21}^{EI} or B_{21}^{EEI} the intrinsic prior is as in (14) with $m^* = 1/2$.

An interesting feature of these intrinsic priors is that they yield $\text{pr}(H_1) = \text{pr}(H_2) = 1/2$. They are thus "balanced" between the two hypotheses, suggesting that the default Bayes factors are similarly balanced. (This is, of course, directly ascertainable by symmetry here, but comparison of the "intrinsic" $\text{pr}(H_i)$ in later examples will prove enlightening.)

2.6 Discussion and Comparison

As in Section 2.5, it is convenient to represent a default Bayes factor as $B_{21} = B_{21}^N \cdot B^{CF}$, where we refer to B^{CF} as the "correction factor." Table 1 summarizes the four default Bayes factors we have considered (with $b = 1/n$ for the FBF) and gives their corresponding intrinsic priors (up to normalizing constants). The two distinct intrinsic priors are graphed in Figure 1. Note

Table 1: *Summary of default Bayes Factors and intrinsic priors for one-sided normal testing.*

	B^{CF}	**Intrinsic Prior**	$\frac{\mathbf{pr}(H_2)}{\mathbf{pr}(H_1)}$
FBF	$(\Phi(\bar{x}))^{-1} - 1$	$\begin{cases} (\Phi(\theta))^{-1} - 1 & \text{for } \theta > 0 \\ ((\Phi(\theta))^{-1} - 1)^{-1} & \text{for } \theta < 0 \end{cases}$	1
MIBF	$(\Phi(\mathrm{Me}[x_i]))^{-1} - 1$	as above	1
EIBF	$\frac{\sum(1-\Phi(x_i))}{\sum \Phi(x_i)}$	$\begin{cases} (\Phi(\theta/\sqrt{2}))^{-1} - 1 & \text{for } \theta > 0 \\ ((\Phi(\theta/\sqrt{2}))^{-1} - 1)^{-1} & \text{for } \theta < 0 \end{cases}$	1
EEIBF	$(\Phi(\bar{x}/\sqrt{2}))^{-1} - 1$	as above	1

that the intrinic prior for the EIBF and EEIBF is somewhat more diffuse than that for the FBF and MIBF.

Table 2 presents numerical comparisons of the FBF, EEIBF, and p-values, for various values of n and various data values. The data values, $\sqrt{n}\bar{x} = 1.645, 2.326$ and 3.09 are those which would yield p-values (against H_1) of 0.05, 0.01 and 0.001, respectively. The MIBF and EIBF cannot be listed in Table 2, as they do not depend only on $\bar{x}$ (as do the p-value, the FBF and the EEIBF). Table 3 gives the mean and standard deviation of the MIBF in (8), computed for various values of n from 200 simulations of data corresponding to $\sqrt{n}\bar{x} = 1.645$; note that, conditional on $\sqrt{n}\bar{x}$, the distribution of $\mathbf{x}$ does not depend on θ. The MIBF is not defined for $n = 1$. For convenience of comparison with the p-values, we report B_{12} in Tables 2 and 3 instead of B_{21}.

Note first that, for large n, the default Bayes factors essentially agree with each other and with the p-value. (Strictly speaking, the equivalence alluded to in Section 1 is that between the p-value and the posterior probability of H_1; on a "Bayes factor scale", one should thus look at $p/(1-p)$ but, for the small values of p considered here, there is little difference.) For smaller values of n, however, the differences with the p-value are quite pronounced. For instance, when $p = 0.05$ and $n = 5$, the FBF$= 0.175$, the EEIBF$= 0.122$ and the MIBF is approximately $.182 \pm .077$, which imply considerably less evidence than is commonly associated with a p-value of 0.05. (Recall that 0.05 is then also the posterior probability of H_1 under a diffuse symmetric prior). Hence use of the new default Bayesian tests can

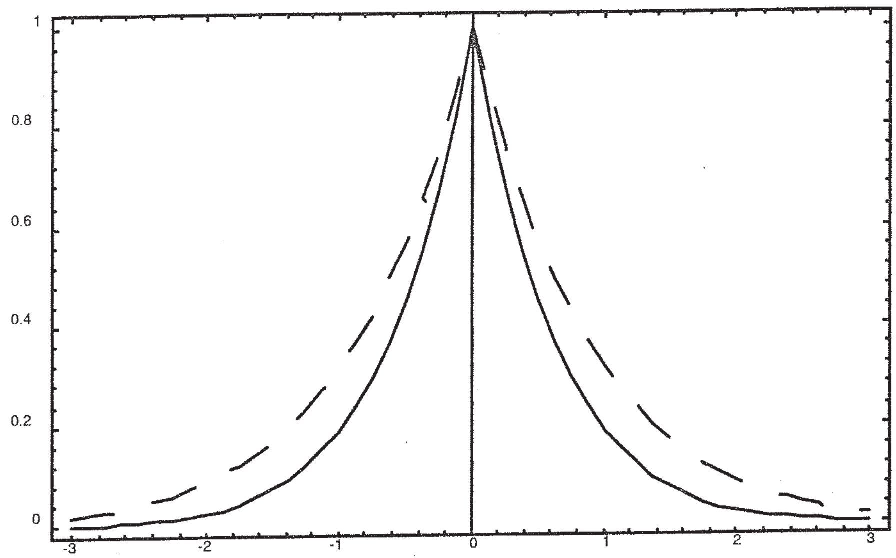

Figure 1: *Intrinsic priors for normal one-sided testing* (FBF & MIBF —, EIBF & EEIBF - -)

make a significant difference. Note that the FBF and (mean) MIBF are very similar, as could be expected since they have the same intrinsic prior; the EEIBF gives consistently smaller values, but the differences are not great.

The FBF, with $b = 1/n$, is clearly unreasonable when $n = 1$, since then $B_{21}^F = 1$ no matter what the data. (A smaller value of b should be used when $n = 1$, and probably with $n = 3$, although it is not clear how b should be chosen with smaller n.) The MIBF is not even defined for $n = 1$. Indeed, default Bayes factors must be used with caution when the sample size is extremely small.

One option, for small n, is to use the actual Bayes factor with respect to the intrinsic prior derived from a default Bayes factor, rather than the default Bayes factor itself. To see the effects of this choice, and also to help judge the extent to which default Bayes factors are reasonable in one-sided testing for small n, Table 4 presents the Bayes factors corresponding to the two intrinsic priors given in Table 1, for various n and for data such that the p-value is 0.05. (The results for p-values of 0.01 and 0.001 were very similar, and hence are not presented here.) The first row of Table 4 corresponds to the intrinsic prior for the FBF (and also the MIBF), and should be compared with the first column of Table 2, which gives the corresponding values for the FBF itself. Likewise, the second row of Table 4 should be compared with the second column of Table 2.

One interesting observation is that the actual Bayes factors derived from

Table 2: *Comparison of the FBF and EEIBF for* $p = 0.05, 0.01$ *and* 0.001, *and various* n *for testing* $H_1 : \theta \leq 0$ *vs.* $H_1 : \theta > 0$.

n	$p = 0.05$		$p = 0.01$		$p = 0.001$	
	B_{12}^F	B_{12}^{EEI}	B_{12}^F	B_{12}^{EEI}	B_{12}^F	B_{12}^{EEI}
1	1.00	.377	1.00	.192	1.00	.068
3	.255	.157	.103	.049	.026	.009
5	.175	.122	.058	.034	.011	.005
7	.144	.107	.043	.028	.007	.004
9	.128	.098	.036	.025	.006	.003
25	.089	.076	.021	.017	.0027	.0020
100	.068	.063	.015	.013	.0016	.0014
200	.063	.060	.0131	.0122	.0014	.0013
400	.060	.058	.0121	.0115	.0013	.0012
∞	.053	.053	.010	.010	.0010	.0010

Table 3: *Mean and standard deviation of the MIBF from 200 simulations of data corresponding to* $p = 0.05$ *and various* n.

n	3	5	7	9	25	100
mean	.276	.182	.152	.134	.091	.068
std. dev.	.137	.077	.062	.047	.019	.007

the intrinsic priors are smaller than the corresponding default Bayes factors themselves. For $n < 5$, the difference is significant, and reinforces the above warning concerning use of the default Bayes factors for very small n. Note, also, that the EEIBF values are considerably closer to their corresponding intrinsic prior Bayes factors than are the FBF values, which suggests that the EEIBF may be more representative of an actual Bayes factor. Finally, use of the intrinsic prior Bayes factor directly may appeal to many, especially for very small n.

Table 4: *Bayes factors corresponding to the intrinsic priors of Table 1 for data corresponding to* $p = 0.05$ *and various* n.

n	1	3	5	7	9	25	100	200	400	∞
F & M	.239	.145	.120	.107	.099	.078	.065	.061	.058	.053
E & EE	.173	.112	.096	.088	.084	.070	.061	.058	.057	.053

3 General One-Sided Testing

In this section, we consider a variety of testing problems of the form $H_1 : \theta < \theta_0$ versus $H_2 : \theta > \theta_0$, or of the form $H_1 : \theta = \theta_0$ versus $H_2 : \theta > \theta_0$. In particular, we study the general location parameter model (Section 3.1), the exponential model (Section 3.2), and testing of a normal mean with unknown variance (Section 3.3). Each scenario reveals interesting phenomena.

3.1 Location Parameter Models

3.1.1 One-Sided Testing

One-sided testing of location models is of interest because the various default Bayes factors are quite simple (for any density) and intrinsic priors are accessible for IBFs. Let $\mathbf{X} = (X_1, \ldots, X_n)$ with X_i i.i.d. $f(x_i - \theta)$. Suppose we are interested in testing $H_1 : \theta < \theta_0$ vs. $H_2 : \theta > \theta_0$. Taking $\pi_1^N(\theta) = 1_{(-\infty,\theta_0)}(\theta)$ and $\pi_2^N(\theta) = 1_{(\theta_0,\infty)}(\theta)$, the standard noninformative priors, and noting that a single observation, x_i, is a minimal training sample, yields

$$B_{12}^N(x_i) = \frac{\int_{-\infty}^{\theta_0} f(x_i - \theta)d\theta}{\int_{\theta_0}^{\infty} f(x_i - \theta)d\theta} = (F(x_i - \theta_0))^{-1} - 1, \tag{15}$$

where $F(x)$ is the c.d.f. of $f(x)$. Writing $L(\theta) = \prod_{i=1}^n f(x_i - \theta)$ and

$$B_{21}^N(\mathbf{x}) = \frac{\int_{\theta_0}^{\infty} L(\theta)d\theta}{\int_{-\infty}^{\theta_0} L(\theta)d\theta},$$

the different default Bayes factors are as follows:

Fractional Bayes Factor: With the choice $b = 1/n$,

$$B_{21}^F = B_{21}^N(\mathrm{x}) \cdot \frac{\int_{-\infty}^{\theta_0} L^{1/n}(\theta)d\theta}{\int_{\theta_0}^{\infty} L^{1/n}(\theta)d\theta}. \tag{16}$$

Useful general expressions for the intrinsic priors of the FBF do not appear to be available here.

Median Intrinsic Bayes Factor: Since (15) is monotonic in x_i, the MIBF is

$$B_{21}^{MI} = B_{21}^N \cdot \{[F(\mathrm{Me}[x_i] - \theta_0)]^{-1} - 1\}. \tag{17}$$

By essentially the same derivation as in Example 1 in Section 2.5, one can show that the intrinsic prior for the MIBF is given by

$$\pi^I(\theta) = \begin{cases} [F(\theta - \theta_0)]^{-1} - 1 & \text{if } \theta > \theta_0 \\ \\ ([F(\theta - \theta_0)]^{-1} - 1)^{-1} & \text{if } \theta \leq \theta_0, \end{cases} \tag{18}$$

assuming the MLE and the median converge to θ as $n \to \infty$, and that $\pi^I(\theta)$ is integrable at θ_0. This is typically proper (up to a normalizing constant which will not affect the Bayes factor) and will be symmetric about θ_0 if F is symmetric.

Encompassing and Expected IBF: It is easy to see that the encompassing IBF is

$$B_{21}^{EI} = B_{21}^N \cdot \frac{\sum_{i=1}^n (1 - F(x_i - \theta_0))}{\sum_{i=1}^n F(x_i - \theta_0)}, \tag{19}$$

and the expected version of (19) is

$$B_{21}^{EEI} = B_{21}^N \cdot ([E_0(F(X + \hat{\theta} - \theta_0))]^{-1} - 1),$$

where E_0 is the expectation w.r.t. $X \sim f(x)$ and $\hat{\theta}$ is the MLE for θ. The intrinsic prior here is as in (18), but with $F(\theta - \theta_0)$ replaced by $E_0[F(X + \theta - \theta_0)]$.

3.1.2 One-Sided Testing of a Precise Hypothesis

Substantially different results are obtained in the location model if one tests

$$H_1 : \theta = \theta_0 \text{ versus } H_2 : \theta > \theta_0.$$

This is more closely related to two-sided testing of a precise hypothesis, for which the Bayesian answers are known to differ markedly from the answers for standard one-sided testing.

The technical analysis for this problem is almost the same as that in Section 3.1.1, but the general answers are not as simple. Hence we will present the results only for the case of a normal location model, with the purpose of showing the considerable difference in practical answers that results. Without loss of generality, we assume the X_i are normal with mean θ and variance 1.

The results of the analysis are summarized in Table 5. Note that, here,

$$\begin{aligned} B_{21}^N(\mathbf{x}) &= \frac{\int_{\theta_0}^{\infty} \prod_{i=1}^n (2\pi)^{-1/2} \exp\{-(x_i-\theta)^2/2\} d\theta}{\prod_{i=1}^n (2\pi)^{-1/2} \exp\{-(x_i-\theta_0)^2/2\}} \\ &= \frac{\Phi(\sqrt{n}(\bar{x}-\theta_0))}{\sqrt{n}\phi(\sqrt{n}(\bar{x}-\theta_0))}, \end{aligned}$$

where Φ and ϕ refer to the standard normal c.d.f. and density, respectively. Recall also that the default Bayes factors are the product of B_{21}^N and B^{CF}. (The only nontrivial derivation was that for the MIBF, the key detail of which is given in Appendix 1.)

Table 5: *Summary of default Bayes Factors and intrinsic priors for one-sided normal testing of a precise hypothesis.*

	$\mathbf{B^{CF}}$	**Intrinsic Prior**	$\mathbf{pr}(H_2)/\mathbf{pr}(H_1)$
FBF	$\frac{\phi(\bar{x}-\theta_0)}{\Phi(\bar{x}-\theta_0)}$	$\frac{\phi(\theta-\theta_0)}{\Phi(\theta-\theta_0)}$	$\log 2$
MIBF	$\frac{\phi(\text{Me}[x_i]-\theta_0)}{\Phi(\text{Me}[x_i]-\theta_0)}$	$\frac{\phi(\theta-\theta_0)}{\Phi(\theta-\theta_0)}$	$\log 2$
EAIBF	$\frac{\sum \phi(x_i-\theta_0)}{\sum \Phi(x_i-\theta_0)}$	$\frac{\phi((\theta-\theta_0)/\sqrt{2})}{\sqrt{2}\Phi((\theta-\theta_0)/\sqrt{2})}$	$\log 2$
EEIBF	$\frac{\phi((\bar{x}-\theta_0)/\sqrt{2})}{\sqrt{2}\Phi((\bar{x}-\theta_0)/\sqrt{2})}$	$\frac{\phi((\theta-\theta_0)/\sqrt{2})}{\sqrt{2}\Phi((\theta-\theta_0)/\sqrt{2})}$	$\log 2$

The column $\text{pr}(H_2)/\text{pr}(H_1)$ in Table 5 has the same interpretation as before; the default Bayes factor corresponds (asymptotically, at least) to a posterior odds ratio for a proper prior with $\text{pr}(H_2)/\text{pr}(H_1)$ as the prior odds. This ratio is $\log 2 = 0.693$, which indicates that the default Bayes factors are modestly biased in favor of H_1.

Table 6: *Comparison of FBF and EEIBF for* $p = 0.05, 0.01$ *and* 0.001, *and various* n.

n	$p = 0.05$		$p = 0.01$		$p = 0.001$	
	B_{12}^F	B_{12}^{EEI}	B_{12}^F	B_{12}^{EEI}	B_{12}^F	B_{12}^{EEI}
1	1	.662	1	.353	1	.128
3	.613	.625	.262	.215	.069	.041
5	.613	.690	.221	.215	.045	.036
7	.641	.752	.213	.225	.039	.035
9	.671	.813	.214	.236	.037	.036
25	.901	1.17	.256	.317	.037	.044
100	1.56	2.11	.412	.546	.055	.072
200	2.13	2.91	.546	.741	.072	.096
400	2.94	4.03	.746	1.02	.096	.131
∞	∞	∞	∞	∞	∞	∞

To see that testing $H_1 : \theta = \theta_0$ versus $H_2 : \theta > \theta_0$ is considerably different than testing $H_1 : \theta \leq \theta_0$ versus $H_2 : \theta > \theta_0$, Table 6 presents the analogue of Table 2. The p-value is computed against H_1. (For convenience of comparison with the p-value and with Table 2, Table 6 gives B_{12} instead of B_{21}.) For large n, the results in Table 2 and Table 6 are completely different, B_{12} approaching the p-value in the former case and going to ∞ in the latter. Even for small n, the results differ by about a factor of 5. (Again, the MIBF is not presented as it does not depend solely on $\bar{x}$; in a simulation not reported here, it was found that, for moderate values of n, the average MIBF was between the FBF and the EEIBF whereas, for large values of n, the average MIBF was close to the FBF.)

A final interesting observation concerning Table 6 is that the default Bayes factors initially decrease with n, and then increase. While this may seem to be odd behavior, it can be verified that many real Bayes factors behave in the same way for this problem.

3.2 Exponential Model

Since scale parameter models can be reduced to location models by log transformations, there is no need to present general expressions for scale models. We thus devote this section to a more careful study of the exponential model, so as to "test" the various default Bayes factors in a highly nonsymmetric situation. Numerous interesting insights emerge.

Let $\mathbf{X} = (X_1, \ldots, X_n)$ with X_i i.i.d. with exponential density $f(x_i|\theta) = \theta e^{-x_i\theta}$, $x_i > 0, \theta > 0$. Suppose we are interested in testing

$$H_1 : \theta < \theta_0 \;\text{ vs. }\; H_2 : \theta > \theta_0.$$

The usual noninformative prior distribution, $\pi^N(\theta) = 1/\theta$, yields

$$B_{21}^N(\mathbf{x}) = \frac{\int_{\theta_0}^{\infty} \theta^{n-1} e^{-n\bar{x}\theta} d\theta}{\int_0^{\theta_0} \theta^{n-1} e^{-n\bar{x}\theta} d\theta} = [\gamma(n, n\bar{x}\theta_0)]^{-1} - 1, \tag{20}$$

where $\gamma(\alpha, x) = (\Gamma(\alpha))^{-1} \int_0^x \xi^{\alpha-1} e^{-\xi} d\xi$ is the incomplete Gamma function.

Fractional Bayes Factor: It is easy to see that, with the choice $b = 1/n$,

$$B_{21}^F = B_{21}^N \cdot (e^{\theta_0 \bar{x}} - 1). \tag{21}$$

Median IBF: The minimal training sample is $m = 1$, and $B_{12}^N(x_i) = e^{\theta_0 x_i} - 1$ is monotonic in x_i. Thus the MIBF is

$$B_{21}^{MI} = B_{21}^N \cdot (e^{\theta_0 \mathrm{Me}[x_i]} - 1). \tag{22}$$

Encompassing and Expected IBF: Utilizing the natural encompassing model $H_0 : \theta \in \Re^+$, computation yields,

$$B_{21}^{EI} = B_{21}^N \cdot \frac{\sum_1^n (1 - e^{-\theta_0 x_i})}{\sum_1^n e^{-\theta_0 x_i}}. \tag{23}$$

Under H_0, the MLE for θ is $1/\bar{x}$ and computation yields, for the expected version of (23),

$$B_{21}^{EEI} = B_{21}^N \cdot \; \theta_0 \bar{x}. \tag{24}$$

Intrinsic Priors: In Appendix 2, the intrinsic priors for the above default Bayes factors are derived, up to irrelevant normalizing constants. They are summarized in Table 7, where the "correction factors", B^{CF}, are also given.

The most interesting aspect of Table 7 is the last column, which gives

$$\frac{\mathrm{pr}(H_2)}{\mathrm{pr}(H_1)} = \frac{\int_{\theta_0}^{\infty} \pi^I(\theta) d\theta}{\int_0^{\theta_0} \pi^I(\theta) d\theta}.$$

For the EIBF and EEIBF, this ratio is 1, indicating that these default Bayes factors are "balanced" between H_1 and H_2. For the FBF and (to a lesser

Table 7: *Summary of default Bayes Factors and intrinsic priors for the exponential model.*

	B^{CF}	Intrinsic Prior	$\frac{pr(H_2)}{pr(H_1)}$
FBF	$e^{\theta_0 \bar{x}} - 1$	$\begin{cases} \theta^{-1}(e^{\theta_0/\theta} - 1) & \text{for } \theta > \theta_0 \\ (e-1)\theta^{-1}(e^{\theta_0/\theta} - 1)^{-1} & \text{for } \theta < \theta_0 \end{cases}$	2.67
MIBF	$e^{\theta_0 \text{Me}[x_i]} - 1$	$\begin{cases} \theta^{-1}(2^{\theta_0/\theta} - 1) & \text{for } \theta > \theta_0 \\ \theta^{-1}(2^{\theta_0/\theta} - 1)^{-1} & \text{for } \theta < \theta_0 \end{cases}$	1.46
EIBF	$\frac{\sum(1-e^{-\theta_0 x_i})}{\sum e^{-\theta_0 x_i}}$	$\begin{cases} \theta_0/\theta^2 & \text{for } \theta > \theta_0 \\ 1/\theta_0 & \text{for } \theta < \theta_0 \end{cases}$	1
EEIBF	$\theta_0 \bar{x}$	as above	1

extent) the MIBF, however, there appears to be considerable imbalance. The FBF corresponds to posterior odds from a (proper) prior which gives 2.67 times as much mass to H_2 as to H_1. This considerable apriori advantage to the alternative hypothesis would not seem acceptable in a "default" analysis. Yet this imbalance is "hidden" within the FBF (for this problem). The clear indication here would thus be to use the EIBF or EEIBF; the MIBF is also not too bad in terms of "balance". Figure 2 shows the graph of the three different intrinsic priors from Table 7, with $\theta_0 = 1$ for convenience. Note that the intrinsic prior for the FBF has a large discontinuity at $\theta = \theta_0$.

Numerical Comparisons: As in Example 1, we compare the FBF and EEIBF for various n, and for data-values corresponding to p-values of $p = 0.05$, $p = 0.01$ and $p = 0.001$. (Again, direct comparison of the p-value with the MIBF and EIBF is not possible, as they do not depend solely on $\bar{x}$.)

The p-value against H_1 is given by

$$p = \text{pr}_{\theta_0}(\bar{X} < \bar{x}) = \gamma(n, n\bar{x}/\theta_0).$$

The p-value against H_2 is given by $p = 1 - \gamma(n, n\bar{x}/\theta_0)$. Because of the "imbalance" in some of the default Bayes factors, Tables 8 and 9 separately give the comparisons for these two cases. We report $B_{12} = 1/B_{21}$ in Table 8 for easier comparison with the p-value.

Again note that the default Bayes factors are numerically similar to the p-value only for large n. For smaller n, they indicate that there is less

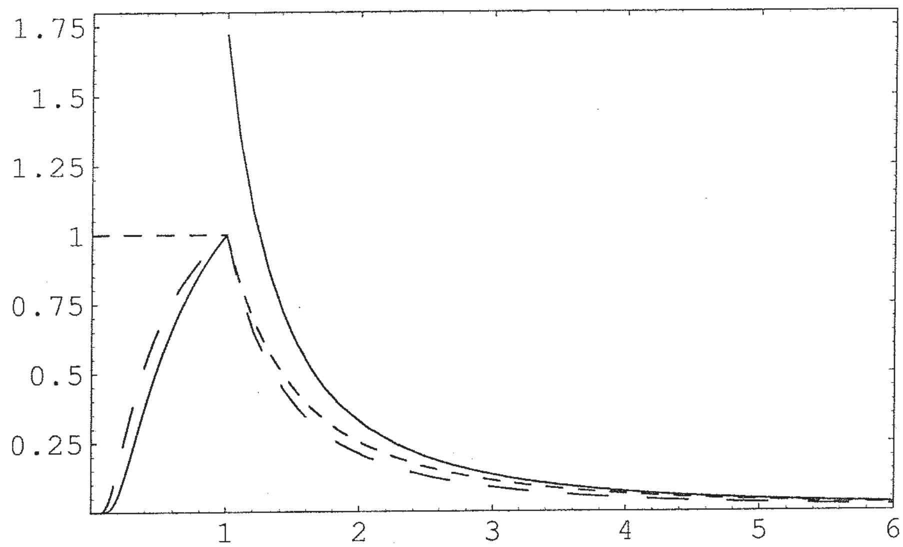

Figure 2: *Intrinsic priors for exponential one-sided testing* (FBF —, MIBF – –, EIBF & EEIBF - - -)

evidence against an hypothesis having a small p-value than is commonly perceived.

Next, note that the "bias" of the FBF in favor of H_2, that was indicated by the study of the intrinsic priors, also appears in the numerical study. Since the FBF tends to be larger for small n, the effect of the "bias" for small n is masked in Table 8 but is magnified in Table 9. For large n, the bias is clearer; thus for $n = 400$ (a large enough value that one would expect equivalence of the p-value and $B/(1+B)$), both tables show the FBF to be biased in favor of H_2, while the EEIBF exhibits no such bias.

Finally, we again investigated the extent to which the default Bayes factors correspond to the actual Bayes factor with respect to their intrinsic priors, given in Table 7. Tables 10 and 11 present the Bayes factors induced by the fractional, median, and expected intrinsic priors, for various n and data such that the p-value is 0.05. (Again, the results for p-values of 0.01 and 0.001 were very similar and, hence, are not presented here.) The first two rows of Table 10 correspond to the intrinsic priors for the FBF and the EEIBF, respectively, and should be compared with the first two columns of Table 9, which give the corresponding values for the FBF and EEIBF. Likewise, the first two rows of Table 11 should be compared with the first two columns of Table 9. In Tables 10 and 11, we also present the actual Bayes factors arising from use of the intrinsic prior for the MIBF, although corresponding values for the MIBF itself could not be given in Tables 8 and

Table 8: *Comparison of the FBF and EEIBF when testing $H_1 : \theta \leq \theta_0$ vs. $H_2 : \theta > \theta_0$ for the exponential model and selected p-values (against H_1).*

	$p = 0.05$		$p = 0.01$		$p = 0.001$	
n	B_{12}^F	B_{12}^{EEI}	B_{12}^F	B_{12}^{EEI}	B_{12}^F	B_{12}^{EEI}
1	1.00	1.03	1.00	1.01	1.00	1.00
3	.168	.193	.065	.070	.158	.0158
5	.109	.133	.035	.039	.0063	.0068
7	.088	.122	.026	.030	.0041	.0046
9	.077	.101	.021	.026	.0032	.0037
25	.052	.076	.013	.017	.0016	.0020
100	.040	.063	.009	.013	.0009	.0014
200	.037	.059	.008	.012	.0008	.0013
400	.035	.057	.007	.011	.0007	.0012
∞	.031	.053	.00059	.010	.00058	.001

9.

For $n < 5$, the differences between the default Bayes factors and the corresponding intrinsic prior Bayes factors are significant, and reinforce the previous warning concerning use of the default Bayes factors for very small n. Also, the EEIBF values are considerably closer to their corresponding intrinsic prior Bayes factors than are the FBF values (especially when comparing Tables 9 and 11), which again suggests that the EEIBF may be more representative of an actual Bayes factor.

3.3 Normal One-sided Testing with Unknown Variance

For the general location-scale model, closed form expressions for the default Bayes factors are not typically obtainable. We thus consider only the normal model, where closed form expressions can be found and allow interesting comparison of the default Bayes factors.

Let $\mathbf{X} = (X_1, \ldots, X_n)$ with X_i i.i.d. $N(\theta, \sigma^2)$, where both θ and σ^2 are unknown. Suppose we are interested in testing:

$$H_1 : \theta < 0 \text{ vs. } H_2 : \theta > 0.$$

Taking $\pi_1^N(\theta, \sigma) = 1_{(-\infty,0)}(\theta)\sigma^{-1}$ and $\pi_2^N(\theta, \sigma) = 1_{(0,\infty)}(\theta)\sigma^{-1}$, the standard default priors, the minimal training sample size is $m = 2$. Computation

Table 9: *Comparison of the FBF and EEIBF when testing $H_1 : \theta \leq \theta_0$ vs. $H_2 : \theta > \theta_0$ for the exponential model and selected p-values (against H_2).*

n	$p = 0.05$ B_{21}^F	B_{21}^{EEI}	$p = 0.01$ B_{21}^F	B_{21}^{EEI}	$p = 0.001$ B_{21}^F	B_{21}^{EEI}
1	1.00	.158	1.00	.047	1.00	.0069
3	.377	.110	.156	.028	.041	.0037
5	.276	.096	.093	.023	.018	.0029
7	.233	.089	.071	.021	.012	.0026
9	.209	.084	.060	.020	.0095	.0024
25	.150	.071	.036	.015	.0046	.0017
100	.117	.062	.025	.013	.0028	.0013
200	.108	.059	.023	.012	.0024	.0012
400	.103	.057	.021	.011	.0022	.0011
∞	.090	.053	.017	.010	.0017	.0010

yields

$$B_{21}^N(\mathbf{x}) = \frac{\int_0^\infty \int_0^\infty (2\pi)^{-n/2}\sigma^{-(n+1)}e^{-n(\bar{x}-\theta)^2/(2\sigma^2)}e^{-s^2/(2\sigma^2)}d\sigma d\theta}{\int_{-\infty}^0 \int_0^\infty (2\pi)^{-n/2}\sigma^{-(n+1)}e^{-n(\bar{x}-\theta)^2/(2\sigma^2)}e^{-s^2/(2\sigma^2)}d\sigma d\theta} = \frac{\mathcal{T}(t)}{[1-\mathcal{T}(t)]}, \tag{25}$$

where $\bar{x} = (\sum_{i=1}^n x_i)/n$, $s^2 = \sum_{i=1}^n (x_i - \bar{x})^2$, $t = \sqrt{n(n-1)}\bar{x}/s$ and $\mathcal{T}$ is the standard t c.d.f with $(n-1)$ d.f. Also note that

$$\frac{\int_{-\infty}^0 [1 + k(\bar{x}-\theta)^2/s^2)]^{-1}d\theta}{\int_0^\infty [1 + k(\bar{x}-\theta)^2/s^2)]^{-1}d\theta} = \frac{\pi/2 + \arctan(-\sqrt{k}\bar{x}/s)}{\pi/2 - \arctan(-\sqrt{k}\bar{x}/s)}. \tag{26}$$

Fractional Bayes Factor: A simple calculation with $b = 2/n$ and applying (26) with $k = n$ yields

$$B_{21}^F = B_{21}^N \cdot \frac{[\pi/2 + \arctan(-\sqrt{n}\bar{x}/s)]}{[\pi/2 - \arctan(-\sqrt{n}\bar{x}/s)]}. \tag{27}$$

Median IBF: A minimal training sample is $\mathbf{x}(l) = (x_1(l), x_2(l))$ so, using (26) with $k = 2$ and obvious notation,

$$B_{12}^N(\mathbf{x}(l)) = \frac{\pi/2 + \arctan(-\sqrt{2}\bar{x}(l)/s(l))}{\pi/2 - \arctan(-\sqrt{2}\bar{x}(l)/s(l))}.$$

Table 10: *Bayes factors induced by the intrinsic priors of Table 7 for data corresponding to a p-value of* 0.05 *(against* H_1*).*

n	1	3	5	7	9	25	100	200	400	∞
F	.113	.070	.061	.056	.053	.044	.037	.035	.033	.031
E	.198	.109	.093	.085	.080	.068	.060	.058	.056	.053
M	.191	.116	.100	.092	.087	.073	.063	.060	.057	.053

Table 11: *Bayes factors induced by the intrinsic priors of Table 7 for data corresponding to a p-value of* 0.05*(against* H_2*).*

n	1	3	5	7	9	25	100	200	400	∞
F	.488	.290	.233	.203	.186	.140	.112	.105	.098	.090
E	.147	.102	.089	.083	.079	.068	.060	.058	.056	.053
M	.244	.147	.120	.106	.098	.077	.063	.060	.057	.053

Note that $\bar{x}(l)/s(l) = w_l/\sqrt{2}$, where $w_l = (x_1(l) + x_2(l))/|x_1(l) - x_2(l)|$. Since

$$\frac{\pi/2 + \arctan(y)}{\pi/2 - \arctan(y)} = \frac{1}{(0.5 + \pi^{-1}\arctan(y))^{-1} - 1}$$

is monotonic in y, the MIBF can be written as

$$B_{21}^{MI} \quad = \quad B_{21}^{N} \cdot \frac{[\pi/2 + \arctan(-\text{Me}[w_l])]}{[\pi/2 - \arctan(-\text{Me}[w_l])]}. \tag{28}$$

Encompassing IBF: The encompassing model is $H_0 : \theta \in \Re^1$ and, with the default prior $\pi^N(\theta) = 1$, it is easy to show that

$$B_{10}^N(\mathbf{x}(l)) = \frac{\pi s}{\sqrt{k}}(\frac{\pi}{2} + \arctan(-w_l)), \;\; B_{20}^N(\mathbf{x}(l)) = \frac{\pi s}{\sqrt{k}}(\frac{\pi}{2} - \arctan(-w_l)).$$

Noting that there are $L = n(n-1)/2$ training samples, it follows that

$$B_{21}^{EI} = B_{21}^{N} \cdot \frac{\sum_{l=1}^{L}[\pi/2 + \arctan(-w_l)]}{\sum_{l=1}^{L}[\pi/2 - \arctan(-w_l)]}.$$

As we have been unable to express the expected version in closed form, we do not discuss it.

Intrinsic Priors: In Appendix 3, the intrinsic prior for the FBF is shown to be

$$\pi^I(\theta,\sigma) = \begin{cases} \sqrt{1+\frac{\theta^2}{\sigma^2}} \cdot \frac{[\pi/2+\arctan(-\theta/\sigma)]}{[\pi/2-\arctan(-\theta/\sigma)]} & \text{for } \theta > 0 \\ \sqrt{1+\frac{\theta^2}{\sigma^2}} \cdot \frac{[\pi/2-\arctan(-\theta/\sigma)]}{[\pi/2+\arctan(-\theta/\sigma)]} & \text{for } \theta < 0. \end{cases} \tag{29}$$

Interestingly, this is not a proper density (even conditionally on σ). For large $|\theta|$, it behaves like the constant $1/\pi$.

It appears to be the case that the intrinsic priors for the other default Bayes factors cannot be expressed in closed form. However, one can show that, for the EIBF, the intrinsic prior behaves like $O(\frac{1}{|\theta|/\sigma})$ for large θ. Interestingly, this is just barely improper.

An Example: Cicirelli and Smith (1985) considered data on cyclic adenosine monophosphate (cAMP) content in oocytes extracted from 4 female frogs. One batch of oocytes was treated with progesterone and one not. The data (paired differences) were (0.78, 1.07, 0.11, 0.74).

Let Δ be the mean difference in cAMP content between the control and treated batch. It is desired to test the hypothesis $H_1 : \Delta < 0$ versus $H_2 : \Delta > 0$. Computation yields:

$p = .022$	$B_{12}^F = .126$	$B_{12}^M = .228$	$B_{12}^{EI} = .161$

While the p-value would appear to indicate significant evidence against H_1, the default Bayes factors suggest that the evidence is, at best, quite modest. This could thus lead to a quite different practical conclusion. Note, however, that n=4, which is a small enough sample size that one might be concerned with at least the FBF and the MIBF. (Also, the intrinsic prior for the FBF is suspect, being nearly constant as the mean moves away from zero; this is probably why the B_{12}^F is closer to the p-value than the other default Bayes factors.) As intrinsic priors for the MIBF or EIBF are not readily available here, one cannot resort to computing the Bayes factor for an intrinsic prior. The best option would thus seem to be use of B_{12}^{EI}. (Of course, in this type of situation, one should probably try to obtain a subjective prior distribution.)

4 Multiple hypotheses

A simple, but important, problem of multiple hypothesis testing consists in choosing between three alternative hypotheses such as

$$H_1 : \theta = \theta_0 \text{ vs. } H_2 : \theta < \theta_0 \text{ vs. } H_3 : \theta > \theta_0. \tag{30}$$

For instance, if θ refers to the difference of mean effects between a new drug and a placebo, $H_1 : \theta = 0$ would correspond to no effect, while $H_2 : \theta < 0$ and $H_3 : \theta > 0$ would correspond to positive and negative effects, respectively. (See Bertolino *et al.*, 1995, for a robust Bayesian analysis of multiple hypothesis testing.)

Table 12: *Summary of correction factors for the default Bayes factors in multiple hypothesis testing.*

	$\mathbf{B_{21}^{CF}}$	$\mathbf{B_{31}^{CF}}$	$\mathbf{B_{32}^{CF}}$
FBF	$\frac{\phi(\bar{x})}{1-\Phi(\bar{x})}$	$\frac{\phi(\bar{x})}{\Phi(\bar{x})}$	$\frac{1-\Phi(\bar{x})}{\Phi(\bar{x})}$
MIBF	$\frac{\phi(\text{Me}[x_i])}{1-\Phi(\text{Me}[x_i])}$	$\frac{\phi(\text{Me}[x_i])}{\Phi(\text{Me}[x_i])}$	$\frac{1-\Phi(\text{Me}[x_i])}{\Phi(\text{Me}[x_i])}$
EIBF	$\frac{\sum \phi(x_i)}{\sum(1-\Phi(x_i))}$	$\frac{\sum \phi(x_i)}{\sum \Phi(x_i)}$	$\frac{\sum(1-\Phi(x_i))}{\sum \Phi(x_i)}$
EEIBF	$\frac{\phi(\bar{x}/\sqrt{2})}{\sqrt{2}(1-\Phi(\bar{x}/\sqrt{2}))}$	$\frac{\phi(\bar{x}/\sqrt{2})}{\sqrt{2}\Phi(\bar{x}/\sqrt{2})}$	$\frac{1-\Phi(\bar{x}/\sqrt{2})}{\Phi(\bar{x}/\sqrt{2})}$

We confine attention to the simple problem of testing a normal mean with known variance. Thus it suffices to consider $\mathbf{X} = (X_1, \ldots, X_n)$ with X_i i.i.d. $N(\theta, 1)$, and we are interested in testing (30). Taking $\pi_2^N(\theta) = 1_{(-\infty,0)}(\theta)$ and $\pi_3^N(\theta) = 1_{(0,\infty)}(\theta)$, the results in Example 1 of Section 2 and the results in Section 3.1.2 immediately yield:

$$B_{21}^N(\mathbf{x}) = \frac{1-\Phi(\sqrt{n}\bar{x})}{\sqrt{n}\,\phi(\sqrt{n}\bar{x})}, \quad B_{31}^N(\mathbf{x}) = \frac{\Phi(\sqrt{n}\bar{x})}{\sqrt{n}\,\phi(\sqrt{n}\bar{x})}, \quad B_{32}^N(\mathbf{x}) = \frac{\Phi(\sqrt{n}\bar{x})}{1-\Phi(\sqrt{n}\bar{x})}.$$

Also, with $b = 1/n$ for the FBF, those results yield "correction factors" (recall $B_{ij} = B_{ij}^N \cdot B_{ij}^{CF}$) for the various default Bayes factors as given in Table 12.

Note that all four default Bayes factors have the self-consistency property that $B_{32} = B_{31}/B_{21}$. Unfortunately, they do not have the property of corresponding to Bayes factors with respect to an intrinsic prior. This is because each pairwise intrinsic prior analysis led to different (and incompatible) intrinsic priors.

Numerical Comparison:

Table 13: *Comparison, for the FBF and EEIBF, of the resulting posterior probabilities of the hypotheses* $H_1 : \theta = 0$ *vs.* $H_2 : \theta < 0$ *vs.* $H_3 : \theta > 0$ *for the normal model with* $p = 0.05$ *and various values of* n.

n	P_1^F	P_1^E	P_2^F	P_2^E	P_3^F	P_3^E
1	.333	.325	.333	.185	.333	.490
3	.328	.351	.137	.088	.535	.561
5	.343	.380	.098	.067	.559	.552
7	.358	.404	.081	.057	.561	.538
9	.373	.425	.071	.051	.556	.524
25	.453	.522	.045	.034	.502	.444
100	.593	.665	.026	.020	.381	.315
200	.665	.729	.020	.015	.315	.255
400	.729	.796	.015	.011	.255	.193
∞	1	1	0	0	0	0

When comparing three or more models, it is easier to express the results in terms of the posterior probabilities, P_i, of the models, assuming they have equal prior probabilities: here, $\mathrm{pr}(H_1) = \mathrm{pr}(H_2) = \mathrm{pr}(H_3) = 1/3$. (The $B_{ji} = P_j/P_i$ can easily be reconstructed from the P_i, if desired.) For the FBF and EEIBF, the P_i are given by $P_i^F = (\sum_{j=1}^{3} B_{ji}^F)^{-1}$ and $P_i^E = (\sum_{j=1}^{3} B_{ji}^E)^{-1}$. Table 13 gives the P_i for various values of n and when $\sqrt{n}\bar{x} = 1.645$, corresponding to a one-sided p-value of 0.05 against $H_1 : \theta = 0$. Table 14 gives the values of the posterior probabilities for different p-values against $H_1 : \theta = 0$, when $n = 9$ and $\bar{x}$ is positive. (Here, again, direct comparison of the p-value with the MIBF and the EIBF is not possible, since they do not depend solely on $\bar{x}$. However, the average value of the MIBF in 200 simulations was very close to the FBF for all values of n.)

The most striking feature of these numerical results is the dramatic difference, for any of the default Bayes factors, in the conclusions concerning H_1 and those concerning H_2 and H_3. For instance, in Table 14, one sees that the evidence against H_1 grows only slowly as the p-value decreases; even with the quite small p-value of 0.005, the (default) posterior probabil-

Table 14: *Comparison, for the FBF and EEIBF, of the resulting posterior probabilities of the hypotheses $H_1 : \theta = 0$ vs. $H_2 : \theta < 0$ vs. $H_3 : \theta > 0$ for the normal model with $n = 9$ and various p-values.*

p	P_1^F	P_1^E	P_2^F	P_2^E	P_3^F	P_3^E
.05	.373	.425	.071	.051	.556	.524
.025	.279	.313	.050	.035	.672	.652
.01	.171	.187	.029	.019	.800	.794
.005	.111	.118	.018	.012	.871	.871
.001	.035	.034	.005	.003	.960	.962
.0005	.021	.020	.003	.002	.976	.979
.0001	.006	.005	.0008	.0004	.993	.995
.00005	.003	.003	.0005	.0002	.996	.997

ities of H_1 still exceed 0.1. In contrast, one can quickly rule out H_2 or H_3 (depending on the sign of $\bar{x}$) as the p-value drops. Thus, from Table 14, one sees that the (default) posterior probability of H_2 is quite small when the p-value is 0.005 (though still roughly three times larger than the p-value). The implications of this in, say, the drug testing scenario mentioned at the beginning of the section is that it is much easier to determine the direction of a treatment effect, given that there is an effect, than to actually establish that the effect is different from zero. This important understanding needs to become more widely recognized in practice.

5 Conclusions

The most important conclusion is that use of the default tests does provide less extreme and arguably better answers in one-sided testing that do the p-value or standard Bayes factors computed with noninformative priors. The reason for this is best evidenced by the nature of the corresponding intrinsic priors (in, *e.g.*, Figure 1); they will typically concentrate more mass near the boundary of the hypotheses, as is typically reasonable in practice. One might object that such intrinsic priors appear to be rather arbitrary (even though they are, in a sense, inferred from the expected scale of the data) and that subjective elicitation of the prior makes more sense. We would not disagree, but would observe that most users seem to prefer default methods, and that the current standard default method is likewise arbitrary, choosing to use that prior distribution which is *least favorable* to the null hypothesis (among, say, all symmetric unimodal priors). Using the prior which is least favorable to the null would seem to violate the standard perception of how testing should operate.

We feel that all the studied default Bayes factors performed reasonably well, and can be useful in general one-sided testing problems, unless the sample size is extremely small. Nevertheless, comparisons among the default procedures revealed interesting differences.

(i) The EEIBF would appear to be the best procedure. It is satisfactory for even very small sample sizes, as is indicated by its not differing greatly from the corresponding intrinsic prior Bayes factor. Also, it was "balanced" between the two hypotheses, even in the highly nonsymmetric exponential model. It may be somewhat more computationally intensive than the other procedures, although its computation through simulation is virtually always straightforward.

(ii) The FBF was typically quite inadequate for very small sample sizes (although this could perhaps be corrected by a better choice of the "fraction" b) and evidenced considerable "bias" towards one of the hypotheses in nonsymmetric situations. Such apriori bias is a significant concern, and suggests that the FBF not be used in clearly nonsymmetric testing situations.

(iii) The MIBF and EIBF are typically satisfactory, with performance between that of the EEIBF and the FBF. Their "bias" was typically moderate, and their small sample size performance reasonable. One can object that they do not depend on sufficient statistics for the problem, but see Berger and Pericchi (1997c) for indications that this may provide "wrong model" robustness. Note that computation of the MIBF and EIBF is typically straightforward. The satisfactory nature of the MIBF here lends support to the argument in Berger and Pericchi (1997b) that the MIBF provides the best *single* general purpose default model selection and hypothesis testing tool, apparently working well with both nested and non-nested models or hypotheses, with virtually any distributions, and even with small sample sizes.

The default testing methodology was shown to be directly applicable to testing of multiple models, and to reveal important differences from standard testing. In particular, when testing $H_1 : \theta = 0$ versus $H_2 : \theta < 0$ versus $H_3 : \theta > 0$, it was seen that there is a crucial difference between the typical strength of evidence against H_1 and against the other two hypotheses.

Appendix 1. *Technical details from Section 3.1.2.*

Here we prove the monotonicity of $\phi(y)/\Phi(y)$, needed to derive the MIBF in Section 3.1.2. Note that

$$\frac{d}{dy}[-y\Phi(y) - \phi(y)] \quad = \quad -\Phi(y) < 0.$$

Also, $\lim_{y\to-\infty}[-y\Phi(y) - \phi(y)] = 0$, while $\lim_{y\to+\infty}[-y\Phi(y) - \phi(y)] = -\infty$, so that $[-y\Phi(y) - \phi(y)] < 0$. Hence

$$\frac{d}{dy}\log\frac{\phi(y)}{\Phi(y)} = \frac{1}{\Phi(y)}[-y\Phi(y) - \phi(y)] < 0,$$

so that $\frac{\phi(y)}{\Phi(y)}$ is monotonically decreasing. (Note that, for $y < 0$, the monotonicity condition is basically a "monotone failure rate " condition; in establishing this condition for other models, one can sometimes take advantage of known results about monotonic failure rate.)

Appendix 2 *Technical details from Section 3.2*

Derivation of the intrinsic prior for the FBF: Since $\lim_{n\to\infty} B^{CF} = e^{\theta_0/\theta} - 1$ and the MLE's under H_1 and H_2 are $\hat{\theta}_1 = min\{\theta_0, 1/\bar{x}\}$, $\hat{\theta}_2 = max\{\theta_0, 1/\bar{x}\}$, respectively, we obtain the intrinsic prior equations

$$\frac{\pi_2^I(\theta)\theta 1_{\{\theta>\theta_0\}}}{\pi_1^I(\theta_0)\theta_0} = e^{\theta_0/\theta} - 1, \quad \frac{\pi_1^I(\theta)\theta 1_{\{\theta<\theta_0\}}}{\pi_2^I(\theta_0)\theta_0} = (e^{\theta_0/\theta} - 1)^{-1},$$

yielding the solution given in Table 7.
Derivation of the intrinsic prior for the MIBF: First note that the median, m^*, of the exponential distribution is $m^* = \theta^{-1}\log 2$. Hence, $\lim_{n\to\infty}(\exp\{\theta_0\mathrm{Me}[x_i]\} - 1) = 2^{\theta_0/\theta} - 1$, and the result follows as for the FBF.
Derivation of the intrinsic prior for the EIBF and EEIBF: Note that

$$\lim_{n\to\infty} B^{CF} \quad = \quad \frac{E_\theta(1 - e^{-\theta_0 X})}{E_\theta(e^{-\theta_0 X})} = \frac{\theta_0}{\theta},$$

and solving the resulting intrinsic prior equations yields the result in Table 7. The derivation of the intrinsic prior for the EEIBF is essentially identical.

Appendix 3 *Technical details from Section 3.3*

Derivation of the intrinsic prior for the FBF: Clearly

$$B_2^*(\theta,\sigma) = \lim_{n\to\infty} B^{CF} = \frac{\pi/2 + \arctan(-\theta/\sigma)}{\pi/2 - \arctan(-\theta/\sigma)}$$

and $B_1^*(\theta,\sigma)$ is the inverse of this expression. Also, the MLE's under H_1 and H_2 are respectively, $\hat{\theta}_1 = min\{\bar{x}, 0\}$, $\hat{\sigma}_1^2 = \frac{1}{n}\sum_1^n (x_i - \hat{\theta}_1)^2$ and $\hat{\theta}_2 = max\{\bar{x}, 0\}$, $\hat{\sigma}_2^2 = \frac{1}{n}\sum_1^n (x_i - \hat{\theta}_2)^2$. Note that, for $i \neq j$,

$$\lim_{n\to\infty, H_i} \hat{\sigma_j}^2 = E_{H_i}[\frac{1}{n}\sum_1^n X_l^2] = \theta^2 + \sigma^2.$$

Recalling that $\pi_1^N(\theta,\sigma) = (1/\sigma)1_{\{\theta<0\}}$ and $\pi_2^N(\theta,\sigma) = (1/\sigma)1_{\{\theta>0\}}$, we obtain the intrinsic prior equations

$$\frac{\pi_2^I(\theta,\sigma)\sigma 1_{\{\theta>0\}}(\theta)}{\pi_1^I(0,\sqrt{\theta^2+\sigma^2})\sqrt{\theta^2+\sigma^2}} = \frac{\pi/2 + \arctan(-\theta/\sigma)}{\pi/2 - \arctan(-\theta/\sigma)}$$

$$\frac{\pi_1^I(\theta,\sigma)\sigma 1_{\{\theta<0\}}(\theta)}{\pi_2^I(0,\sqrt{\theta^2+\sigma^2})\sqrt{\theta^2+\sigma^2}} = \frac{\pi/2 - \arctan(-\theta/\sigma)}{\pi/2 + \arctan(-\theta/\sigma)}.$$

It is easy to check that (29) is a solution to these equations.

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